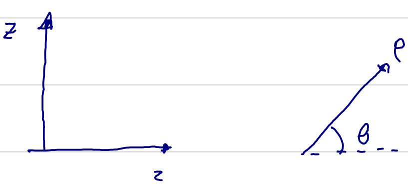
Analytische oplossing II

$$\begin{cases} \nabla\left(r \cdot \sigma \nabla c_o(r,z)\right) = f^1(c_o,c_v)\, r \\ \nabla\left(r \cdot \sigma \nabla c_v(r,z)\right) = f^2(c_o,c_v) \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial r}\left(r \cdot \sigma_r \cdot \frac{\partial c_o}{\partial r}\right) + \frac{\partial}{\partial z}\left(r \cdot \sigma_z \cdot \frac{\partial c_o}{\partial z}\right) = f^1(c_o,c_v)\, r \\ \frac{\partial}{\partial r}\left(r \cdot \sigma_r \frac{\partial c_v}{\partial r}\right) + \frac{\partial}{\partial z}\left(r \cdot \sigma_z \cdot \frac{\partial c_v}{\partial z}\right) = -f^2(c_o,c_v)\, r \end{cases}$$

$$\begin{cases} \sigma_r \frac{\partial c_o}{\partial r} + \sigma_r \frac{\partial^2 c_o}{\partial r^2} \cdot r + r \cdot \sigma_z \frac{\partial^2 c_o}{\partial z^2} = f^1(c_o,c_v) \\ \sigma_r \cdot \frac{\partial c_v}{\partial r} + \sigma_r \frac{\partial^2 c_v}{\partial r^2} \cdot r + r \cdot \sigma_z \cdot \frac{\partial^2 c_v}{\partial z^2} = -f^2(c_o,c_v)\, r \end{cases}$$

$$(r, z) \rightarrow (\rho, \theta)$$



$$r = \rho \cos(\theta) \quad ; \quad z = \rho \sin(\theta)$$

$$dr = d\rho \cos(\theta) - \rho \sin(\theta)\, d\theta \quad ; \quad dz = d\rho \sin(\theta) + \rho \cos(\theta)\, d\theta$$

$$dr^2 = d\rho^2 \cos(\theta) - \sin(\theta)\, d\theta\, d\rho - d\rho\, d\theta \sin(\theta) - \rho \cos(\theta)\, d\theta^2$$

$$dr^2 = d\rho^2 \cos(\theta) - d\theta^2 \rho \cos(\theta) - 2\, d\theta\, d\rho \sin(\theta)$$

$$dz^2 = d\rho^2 \sin(\theta) + d\rho\, d\theta \cos(\theta) + d\theta\, d\rho \cos(\theta) - \rho \sin(\theta)\, d\theta^2$$

$$dz^2 = d\rho^2 \sin(\theta) - \rho \sin(\theta)\, d\theta^2 + 2\, d\theta\, d\rho \cos(\theta)$$

$$c(r,z) = c(\rho,\theta)$$

$$\frac{\partial c}{\partial \theta} = \frac{\partial c}{\partial r}\frac{\partial r}{\partial \theta} + \frac{\partial c}{\partial z}\frac{\partial z}{\partial \theta} = -\rho \frac{\partial c}{\partial r} \cdot \sin(\theta) + \rho \frac{\partial c}{\partial z} \cdot \cos(\theta)$$

$$\frac{\partial c}{\partial \rho} = \frac{\partial c}{\partial r}\frac{\partial r}{\partial \rho} + \frac{\partial c}{\partial z}\frac{\partial z}{\partial \rho} = \frac{\partial c}{\partial r}\cos(\theta) + \frac{\partial c}{\partial z}\sin(\theta)$$

$$\boxed{\begin{aligned} \frac{\partial c}{\partial \theta} &= \rho\left(\frac{\partial c}{\partial z}\cos(\theta) - \frac{\partial c}{\partial z}\sin(\theta)\right) \\ \frac{\partial c}{\partial \rho} &= \frac{\partial c}{\partial z}\cos(\theta) + \frac{\partial c}{\partial z}\sin(\theta) \end{aligned}}$$

$$\frac{\partial^2 c}{\partial \theta^2} = \rho\left[\left(\frac{\partial^2 c}{\partial z \partial z}\frac{\partial z}{\partial \theta} + \frac{\partial^2 c}{\partial z \partial z}\frac{\partial z}{\partial \theta}\right)\cos(\theta)\right.$$

$$\left. - \left(\frac{\partial^2 c}{\partial z \partial z}\frac{\partial z}{\partial \theta} + \frac{\partial^2 c}{\partial z \partial z}\frac{\partial z}{\partial \theta}\right)\sin(\theta)\right] - \rho\frac{\partial c}{\partial z}\sin(\theta) - \frac{\partial c}{\partial z}\cos(\theta)$$

$$\frac{\partial^2 c}{\partial \theta^2} = \rho\left[\left(\frac{\partial^2 c}{\partial z^2}\rho\cos(\theta) - \frac{\partial^2 c}{\partial z \partial z}\rho\sin(\theta)\right)\cos(\theta)\right.$$

$$\left. - \left(-\frac{\partial^2 c}{\partial z^2}\rho\sin(\theta) + \frac{\partial^2 c}{\partial z \partial z}\rho\cos(\theta)\right)\sin(\theta)\right]$$

$$= \frac{\partial^2 c}{\partial z^2}\rho^2\cos(\theta)^2 - \frac{\partial^2 c}{\partial z \partial z}\rho^2\sin(\theta)\cos(\theta) - \frac{\partial^2 c}{\partial z \partial z}\rho^2\sin(\theta)\cos(\theta)$$

$$+ \frac{\partial^2 c}{\partial z^2}\rho^2\sin(\theta)^2 - \rho\frac{\partial c}{\partial z}\sin(\theta) - \rho\frac{\partial c}{\partial z}\cos(\theta)$$

$$\boxed{\begin{aligned} \frac{\partial^2 c}{\partial \theta^2} = {}& \frac{\partial^2 c}{\partial z^2}\rho^2\cos(\theta)^2 - 2\frac{\partial^2 c}{\partial z \partial z}\rho^2\sin(\theta)\cos(\theta) + \frac{\partial^2 c}{\partial z^2}\rho^2\sin(\theta)^2 \\ & - \rho\frac{\partial c}{\partial z}\sin(\theta) - \rho\frac{\partial c}{\partial z}\cos(\theta) \end{aligned}}$$

$$\frac{\partial^2 c}{\partial \rho^2} = \left(\frac{\partial^2 c}{\partial z^2}\frac{\partial z}{\partial \rho} + \frac{\partial^2}{\partial z \partial z}\frac{\partial z}{\partial \rho}\right)\cos(\theta)$$

$$+\left(\frac{\partial^2 c}{\partial z^2}\frac{\partial z}{\partial \rho} + \frac{\partial^2 c}{\partial z \partial z}\frac{\partial z}{\partial \rho}\right)\sin(\theta)$$

$$\boxed{\frac{\partial^2 c}{\partial \rho^2} = \frac{\partial^2 c}{\partial z^2}\cos^2(\theta) + \frac{\partial^2 c}{\partial z^2}\sin^2(\theta) + 2\sin(\theta)\cos(\theta)\frac{\partial^2 c}{\partial z \partial \rho}}$$

$$\boxed{\begin{aligned}
\frac{\partial c}{\partial \theta} &= \rho\left(\frac{\partial c}{\partial z}\cos(\theta) - \frac{\partial c}{\partial z}\sin(\theta)\right)\\
\frac{\partial c}{\partial \rho} &= \frac{\partial c}{\partial z}\cos(\theta) + \frac{\partial c}{\partial z}\sin(\theta)\\
\frac{\partial^2 c}{\partial \rho^2} &= \frac{\partial^2 c}{\partial z^2}\cos^2(\theta) + \frac{\partial^2 c}{\partial z^2}\sin^2(\theta) + 2\sin(\theta)\cos(\theta)\frac{\partial^2 c}{\partial z\,\partial \rho}\\
\frac{\partial^2 c}{\partial \theta^2} &= \frac{\partial^2 c}{\partial z^2}\rho^2\cos(\theta)^2 - 2\frac{\partial^2 c}{\partial z\,\partial z}\rho^2\sin(\theta)\cos(\theta) + \frac{\partial^2 c}{\partial z^2}\rho^2\sin(\theta)^2\\
&\qquad - \rho\frac{\partial c}{\partial z}\sin(\theta) - \rho\frac{\partial c}{\partial z}\cos(\theta)
\end{aligned}}$$

$$\boxed{\begin{aligned}
\sigma_z\frac{\partial c_w}{\partial z} + \sigma_z\frac{\partial^2 c_v}{\partial z^2}\cdot z + z\cdot\sigma_z\frac{\partial^2 c_v}{\partial z^2} &= f^1(c_w, c_v)\,z\\
\sigma_z\cdot\frac{\partial c_v}{\partial z} + \sigma_z\frac{\partial^2 c_v}{\partial z^2}\cdot z + z\cdot\sigma_z\cdot\frac{\partial^2 c_v}{\partial z^2} &= -f^2(c_w, c_v)\,z
\end{aligned}}$$

- $$z\,\sigma\cdot\left(\frac{\partial^2 c}{\partial z^2} + \frac{\partial^2 c}{\partial z^2}\right) = \rho\cos(\theta)\,\sigma\cdot\left(\underbrace{\frac{\partial^2 c}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial c}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 c}{\partial \theta^2}}\right)$$

$$\left|\begin{aligned}
&\frac{\partial^2 c}{\partial z^2}\cos^2(\theta) + \frac{\partial^2 c}{\partial z^2}\sin^2(\theta) + \cancel{2\sin(\theta)\cos(\theta)\frac{\partial^2 c}{\partial z\,\partial \rho}}\\
&+\frac{1}{\rho}\left(\cancel{\frac{\partial c}{\partial z}\cos(\theta)} + \cancel{\frac{\partial c}{\partial z}\sin(\theta)}\right)\\
&+\frac{1}{\rho^2}\left(\frac{\partial^2 c}{\partial z^2}\rho^2\cos(\theta)^2 \cancel{- 2\frac{\partial^2 c}{\partial z\,\partial z}\rho^2\sin(\theta)\cos(\theta)} + \frac{\partial^2 c}{\partial z^2}\rho^2\sin(\theta)^2 \cancel{- \rho\frac{\partial c}{\partial z}\sin(\theta)} \cancel{- \rho\frac{\partial c}{\partial z}\cos(\theta)}\right)
\end{aligned}\right.$$

- $\frac{\partial c}{\partial z}$?

$$\left(-\frac{\partial c}{\partial z}\cos(\theta) + \frac{\partial c}{\partial z}\sin(\theta)\right) + \left(\frac{\partial c}{\partial z}\cos(\theta) + \frac{\partial c}{\partial z}\sin(\theta)\right)$$

$$= \frac{\partial c}{\partial z}\left(\sin(\theta) + \cos(\theta)\right) + \frac{\partial c}{\partial z}\left(\sin(\theta) - \cos(\theta)\right)$$

$$\left(-\frac{\partial c}{\partial z}\cos(\theta)+\frac{\partial c}{\partial z}\sin(\theta)\right)\left(\frac{\partial c}{\partial z}\cos(\theta)+\frac{\partial c}{\partial z}\sin(\theta)\right)$$

$$= -\frac{\partial c}{\partial z}\frac{\partial c}{\partial z}\cos(\theta)^2+\frac{\partial c}{\partial z}\frac{\partial c}{\partial z}\sin(\theta)^2-\frac{\partial c}{\partial z}\frac{\partial c}{\partial z}\cos(\theta)\sin(\theta)$$
$$+\frac{\partial c}{\partial z}\frac{\partial c}{\partial z}\sin(\theta)\cos(\theta)$$

$$\frac{\partial c}{\partial \theta} = \rho\left(\frac{c}{\partial z}\cos(\theta)-\frac{\partial c}{\partial z}\sin(\theta)\right)$$
$$\frac{\partial c}{\partial \rho} = \frac{\partial c}{\partial z}\cos(\theta)+\frac{\partial c}{\partial z}\sin(\theta)$$

$$\left(\frac{\partial c}{\partial \rho}-\frac{\partial c}{\partial z}\sin(\theta)\right)\Big/\cos(\theta) = \frac{\partial c}{\partial z}$$

$$\left(\frac{1}{\rho}\frac{\partial c}{\partial \theta}+\frac{\partial c}{\partial z}\sin(\theta)\right)\Big/\cos(\theta) = \frac{\partial c}{\partial z}$$

$$-\left(\frac{1}{\rho}\frac{\partial c}{\partial \theta}-\frac{\partial c}{\partial z}\cos(\theta)\right)\Big/\sin(\theta) = \frac{\partial c}{\partial z}$$

$$\frac{\frac{\partial c}{\partial \rho}-\frac{\partial c}{\partial z}\sin(\theta)}{\cos(\theta)} = \frac{-1/\rho\,\frac{\partial c}{\partial \theta}+\frac{\partial c}{\partial z}\cos(\theta)}{\sin(\theta)}$$

$$\frac{\partial c}{\partial \rho}\sin(\theta)-\frac{\partial c}{\partial z}\sin^2(\theta) = -1/\rho\,\frac{\partial c}{\partial \theta}\cos(\theta)+\frac{\partial c}{\partial z}\cos^2(\theta)$$

$$\frac{\partial c}{\partial \rho}\sin(\theta)+\frac{1}{\rho}\frac{\partial c}{\partial \theta}\cos(\theta) = \frac{\partial c}{\partial z}$$

$$\left(\frac{\partial c}{\partial \rho}-\frac{\partial c}{\partial z}\cos(\theta)\right)\Big/\sin(\theta) = \frac{\partial c}{\partial z}$$

$$\frac{\frac{\partial c}{\partial \rho}-\frac{\partial c}{\partial z}\cos(\theta)}{\sin(\theta)} = \frac{1/\rho\,\frac{\partial c}{\partial \theta}+\frac{\partial c}{\partial z}\sin(\theta)}{\cos(\theta)}$$

$$\frac{\partial c}{\partial \rho}\cos(\theta)-\frac{\partial c}{\partial z}\cos(\theta)^2 = \frac{1}{\rho}\frac{\partial c}{\partial \theta}\sin(\theta)+\frac{\partial c}{\partial z}\sin^2(\theta)$$

$$\boxed{\frac{\partial c}{\partial \rho}\cos(\theta)-\frac{1}{\rho}\frac{\partial c}{\partial \theta}\sin(\theta) = \frac{\partial c}{\partial z}}$$

$$\boxed{\begin{aligned}&\frac{\partial c}{\partial z} = \frac{\partial c}{\partial \rho}\cos(\theta) - \frac{1}{\rho}\frac{\partial c}{\partial \theta}\sin(\theta)\\ &z\,\sigma\cdot\left(\frac{\partial^2 c}{\partial z^2} + \frac{\partial^2 c}{\partial z^2}\right) = \rho\cos(\theta)\,\sigma\cdot\left(\frac{\partial^2 c}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial c}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 c}{\partial \theta^2}\right)\end{aligned}}$$

$$\sigma\cdot\frac{\partial c_v}{\partial z} + \sigma\cdot z\left(\frac{\partial^2 c_v}{\partial z^2} + \frac{\partial^2 c_v}{\partial z^2}\right) = f^1(c_v, c_i)$$

Assume $\frac{\partial c}{\partial \theta} = 0$!

$$\sigma\cdot\frac{\partial c_v}{\partial \rho}\cos(\theta) + \sigma\cdot\rho\cos(\theta)\left(\frac{\partial^2 c_v}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial c_v}{\partial \rho}\right) = f^1(c_v, c_i)$$

$$\boxed{\begin{aligned}&2\cdot\sigma\cdot\frac{\partial c_i}{\partial \rho}\cos(\theta) + \sigma\cdot\rho\cdot\cos(\theta)\frac{\partial^2 c_i}{\partial \rho^2} = f^1(c_i, c_v)z\\ &2\cdot\sigma\cdot\frac{\partial c_v}{\partial \rho}\cos(\theta) + \sigma\cdot\rho\cos(\theta)\frac{\partial^2 c_v}{\partial \rho^2} = -f^2(c_i, c_v)z\end{aligned}}$$

Best oplossing:

$$\boxed{2\sigma\cdot\frac{\partial c_i}{\partial \rho}\cdot\frac{1}{\rho} + \sigma\cdot\frac{\partial^2 c_i}{\partial \rho^2} = f^1(c_i, c_v)}$$

A. $c_v = \rho^2 - 1 + c_a$, $\quad c_i = -\rho^2 + 1 + c_a \quad \Rightarrow$ CORRECT !

$f^1(c_i, c_v) = 6\sigma_i$, $\quad f^2(c_i, c_v) = 6\sigma_v$

B. $c_i = \rho^2 - c_i c_v \iff c_i = \dfrac{\rho^2}{1 + c_v}$

$c_v = \rho^2 + c_i c_v \iff c_v = \dfrac{\rho^2}{1 - c_i}$

$$\boxed{\begin{aligned} 2\sigma \cdot \frac{\partial C_u}{\partial e} \cdot \frac{1}{e} + \sigma \cdot \frac{\partial^2 C_u}{\partial e^2} &= f^1(C_u, C_v) \\ 2\sigma \cdot \frac{\partial C_v}{\partial e} \cdot \frac{1}{e} + \sigma \cdot \frac{\partial^2 C_v}{\partial e^2} &= -f^2(C_u, C_v) \end{aligned}}$$

B. $C_u = e^2 - C_u C_v \iff C_u = \dfrac{e^2}{1 + C_v}$

$C_v = -e^2 + C_u C_v \iff C_v = \dfrac{-e^2}{1 - C_u}$

$$f^1 = \frac{6\sigma}{1 + C_v} \qquad \frac{\partial f^1}{\partial C_v} = \frac{-6\sigma}{(1 + C_v)^2}$$

$$f^2 = \frac{6\sigma}{1 - C_u} \qquad \frac{\partial f^2}{\partial C_u} = \frac{6\sigma}{(1 - C_u)^2}$$

C. $C_u = e^4 - 1 + C_a$ $\qquad 4e^3 \qquad 12e^2$

$\hookrightarrow 2\sigma 4e^2 + \sigma \cdot 12e^2 = f^1(C_u, C_v)$

$20\sigma e^2 = 20\sigma(-C_v + 1 - C_v^{\text{ank}}) = f^1(C_u, C_v)$

$C_v = -e^2 + 1 + C_a$

$\hookrightarrow f^2(C_u, C_v) = 6\sigma_v$