Submission to Blackboard: Compile your work into ONE file either PDF (preferred) or Word. Do not send images, or Mac Pages files. Do not write your answers on the exam so you do not need to include the exam pages in your submission.

1. [10 points] In a discrete math class of 32 students, what is the largest number of students who *must* receive the same grade if there are only 5 possible grades?

2. [10 points] Use a truth table to demonstrate that these two statements are equivalent:

$$((P \wedge \overline{Q}) \vee (P \wedge Q)) \wedge Q \quad \text{and} \quad (P \wedge Q).$$

3. [10 points] Express $f(x, y, z) = x\overline{y} + y\overline{z}$ in Disjunctive Normal Form.

4. [10 points] Given set $A = \{a, b, c, d\}$ show the equivalence relation, which contains eight ordered pairs, that induces this partition of $A : \{\{a, c\}, \{b, d\}\}$.

5. [10 points] Use Kruskal's algorithm to find a minimum spanning tree in the connected weighted graph shown in Figure 1.

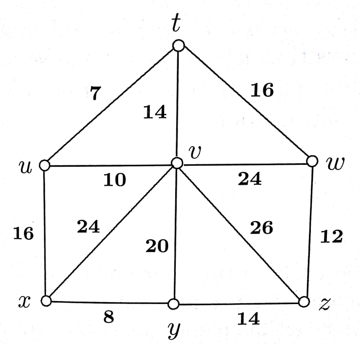


Figure 1: Connected Weighted Graph G.

6. [10 points] Prove that the function $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{1\}$ defined by $f(x) = \dfrac{x}{x-3}$ is bijective.

7. [10 points] Let $J_5 = \{0, 1, 2, 3, 4\}$ and define a function $g : J_5 \times J_5 \to J_5 \times J_5$ as follows: For all $(a, b) \in J_5 \times J_5$

$$g(a, b) = ((5a - 3) \textbf{ mod } 5, (4b + 2) \textbf{ mod } 5)$$

Find $g(3, 4)$.

8. [10 points] How many elements are in the one-dimensional array shown below?

$$A[7], A[8], \ldots, A\left[\left\lfloor \frac{145}{2} \right\rfloor\right]$$

9. [20 points] Define a relation T from $\mathbb{R}$ to $\mathbb{R}$ as follows: $\forall(x, y) \in \mathbb{R} \times \mathbb{R}, xTy \Leftrightarrow y > x + 1$.
   (a) Is $(1, 0) \in T$?
   (b) Is $(0, 1) \in T$?
   (c) Is $(-2, 5) \in T$?
   (d) Is $(-3, -4) \in T$?

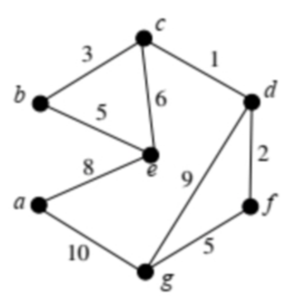
10. [10 points] Let $A = \{0, 1, 2, 3\}$ and define a relation R on A as follows: $R = \{(0, 2), (0, 3), (2, 0), (2, 1)\}$
    (a) Draw the directed graph of R
    (b) Is R reflexive? Explain
    (c) Is R symmetric? Explain
    (d) Is R transitive? Explain

11. [15 points] Let S be the set of all strings of 0's and 1's of length 3 . Define a relation R on S as follows: for all strings s and t in S

$$sRt \iff \text{the two left-most characters of } s \text{ are the same as the two left-most characters of } t$$

    (a) Prove that R is an equivalence relation on S
    (b) Find the distinct equivalence classes of R

12. [20 points] Use Dijkstra's algorithm to find the shortest path from a to d for the following graph. Make a table to show the action of the algorithm.



13. [20 points]
    (a) For all positive integers a and b, show that if $a|b$, then $a \leq b$.
    (b) Prove that for all integers a, b, and c, if $a|b$ and $b|c$, then $a|c$.
    (c) Use the Euclidean Algorithm to find the $\gcd(191, 3769)$.
    (d) Let p, q, r, s, m and n be positive integers such that $p \equiv q(\textbf{ mod } m), r \equiv s(\textbf{ mod } n)$ and $d = \gcd(m, n)$. Show that $p + r \equiv q + s(\textbf{ mod } d)$.

14. [15 points] An investor intends to buy shares of stock in 3 companies chosen from a list of 12 companies. How many different investment options are there if

   (a) equal amounts are invested in each company?

   (b) equal amounts are invested in exactly two companies and the rest in the third?

   (c) different amounts are invested in each company?

15. [10 points] Draw the graph G corresponding to each adjacency matrix:

$$(a)\ A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}; \quad (b)\ A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

16. [15 points] The state digraph of a finite-state automaton A is shown in Figure 2. The states of A are s_0, s_1, s_2, s_3 and the input values are $1, 2, 3$. For the transition function $f\ \ (S \times I) \rightarrow S$, determine $f(s_2, i)$ for $i = 1, 2, 3$.

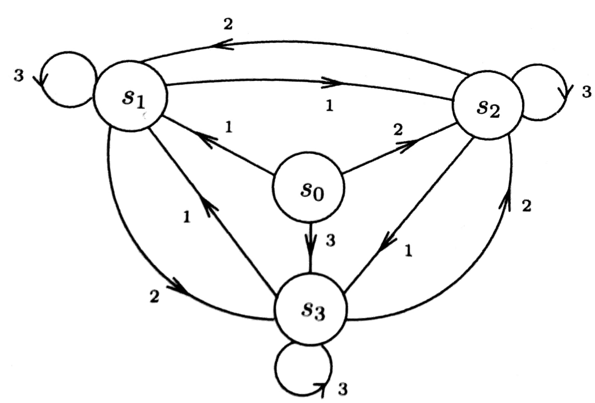


Figure 2: The digraph for Exercise 16

17. [10 points] The figure below is a map of a city made up of six regions denoted A, B, C, D, E, and F. There are 12 bridges that connect the six regions. Is it possible to take a stroll around the city where each bridge is crossed exactly once? Explain your answer in detail (just answering 'yes' or 'no' will not get credit).

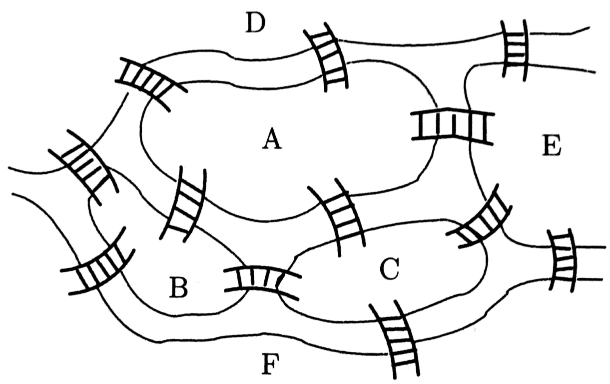


Figure 3: The map of Parisiville, showing six regions and 12 bridges for Exercise 17

18. [15 points] Find the Boolean expression that corresponds to the circuit shown below. A dot indicates a soldering of two wires; wires that cross without a dot are assumed not to touch.

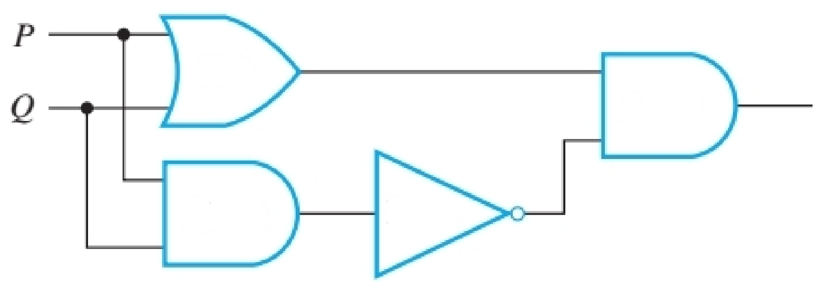


Figure 4: Combinatorial circuit for Exercise 18

19. [20 points] A large pile of coins consists of pennies, nickels, dimes, and quarters (at least 20 of each).
    (a) How many different collections of 20 coins can be chosen?
    (b) How many different collections of 20 coins chosen at random will contain at least 3 coins of each type?

20. [10 points] If a graph has vertices of degrees 1, 1, 2, 3, and 3, how many edges does it have? Why?

21. [20 points]

    (a) Is $\{5\} \in \{1, 3, 5\}$?

    (b) Is $\{5\} \subseteq \{1, 3, 5\}$?

    (c) Is $\{5\} \in \{\{1\}, \{3\}, \{5\}\}$?

    (d) Is $\{5\} \subseteq \{\{1\}, \{3\}, \{5\}\}$?

22. [10 points] Prove by induction: For any real number r except 1, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

23. [10 points] Prove that for all integers a, b, and c, with $a \neq 0$, if $a|b$ and $a|c$, then $a|(bx + cy)$.

24. [30 (bonus)] **Bonus Question** Let n and r be positive integers and suppose $r \leq n$. Prove Pascal's formula

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

    Hint: Write out the expressions for each combination on the rhs and combine terms to show it equals the lhs. Partial credit will be given.