Un ejemplo clásico para la integral de Riemann :)

$$\boxed{\text{Calcular: } \int_0^{\pi} \frac{\sin^2 rx}{\sin^2 \frac{1}{2}x} dx, \quad r \geq 0}$$

Solución:

Resolveremos la integral mediante sumas de Riemann, recordemos que:

$$\lim_{n\to\infty} \frac{\pi}{n} \sum_{k=1}^{n} \frac{\sin^2(kr\pi/n)}{\sin^2(k\pi/2n)} \tag{1}$$

Ahora note que para $m \geq 2$:

$$1 - \cos mx = 1 - \cos x + \sum_{p=1}^{m-1} [\cos px - \cos(p+1)x]$$

$$\cos px - \cos(p+1)x = 1 - \cos x - \sum_{q=1}^{p} [\cos(q-1)x - 2\cos qx + \cos(q+1)x]$$

$$\cos(q-1)x + \cos(q+1)x = 2\cos qx \cos x$$

Entonces:

$$1 - \cos mx = (1 - \cos x) \left[m + 2 \sum_{p=1}^{m-1} \sum_{q=1}^{p} \cos qx \right] = (1 - \cos x) \left[m + 2 \sum_{j=1}^{m-1} (m-j) \cos jx \right]$$

$$\Rightarrow \frac{\sin^2 \frac{1}{2}mx}{\sin^2 \frac{1}{2}x} = \frac{1 - \cos mx}{1 - \cos x} = m + 2 \sum_{j=1}^{m-1} (m-j) \cos jx, \qquad 0 < \frac{1}{2}x < \pi$$

Con $m = 2r > 0$ y $x = k\pi/n$ se obtiene algo similar a lo buscado en (1):

$$\sum_{k=1}^{n-1} \left[2r + 2 \sum_{j=1}^{2r-1} (2r-j) \cos(\frac{jk\pi}{n}) \right] = 2r(n-1) + 2 \sum_{j=1}^{2r-1} (2r-j) \sum_{k=1}^{n-1} \cos \frac{jk\pi}{n} \tag{2}$$

Usamos la fórmula:

$$\cos x + \cos 2x +' cdots + \cos mx = \frac{\sin(m + \frac{1}{2})x}{2\sin(\frac{1}{2}x)} - \frac{1}{2}$$