1. Sea $f : U \to \mathbb{R}$ diferenciable en $U \subset \mathbb{R}^m$ y abierta. Suponga $df(a) \neq 0$ para cierto $a \in U$ y considere el vector unitario $u \in \mathbb{R}^m$ tal que $df(a) \cdot u = \max\{df(a) \cdot h; |h| = 1\}$. Sea $v \in \mathbb{R}^m$ tal que $df(a) \cdot v = 0$ muestre que v es perpendicular a u.

Solución:
Se tiene: $df(a) \cdot u \geq df(a) \cdot h, \ \forall h/|h| = 1$. En particular:

$$h = \frac{\nabla f(a)}{|\nabla f(a)|}$$

$$\Rightarrow df(a) \cdot u \geq df(a) \cdot \frac{\nabla f(a)}{|\nabla f(a)|} = \left\langle \nabla f(a), \frac{\nabla f(a)}{|\nabla f(a)|} \right\rangle = |\nabla f(a)|$$

$$df(a) \cdot u = \langle \nabla f(a), u \rangle \leq |\nabla f(a)| \cdot |u| = |\nabla f(a)|$$

$$\therefore df(a) \cdot u = |\nabla f(a)|$$

$$u = \alpha \nabla f(a) \Rightarrow \alpha = \pm 1$$

$$df(a) \cdot v = 0 \ \Rightarrow \ \langle \nabla f(a), v \rangle = 0 \ \Rightarrow v \perp \nabla f(a)$$

$$\nabla f(a) // u \ \Rightarrow u \perp v$$

2. Muestre que $\dfrac{d\mathbf{r}}{ds} \cdot \dfrac{d^2\mathbf{r}}{ds^2} \times \dfrac{d^3\mathbf{r}}{ds^3} = \dfrac{\tau}{\rho^2}$:

$$\frac{d\mathbf{r}}{ds} = \mathbf{T}, \quad \frac{d^2\mathbf{r}}{ds^2} = \kappa\mathbf{N}, \quad \frac{d^3\mathbf{r}}{ds^3} = \kappa\tau\mathbf{B} - \kappa^2\mathbf{T} + \frac{d\kappa}{ds}\mathbf{N}$$

$$= \mathbf{T} \cdot \left(\kappa^2\tau\mathbf{T} + \kappa^3\mathbf{B}\right) = \kappa^2\tau = \frac{\tau}{\rho^2}, \quad \rho = \frac{1}{\kappa}$$

3. Dada la ecuación $x = t, \ y = t^2, \ z = \frac{2}{3}t^3$, calcule la curvatura y la torsión.

   (a)

$$\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}, \quad \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k}$$

$$\frac{ds}{dt} = \sqrt{1 + (2t)^2 + (2t^2)^2} = 1 + 2t^2$$

$$\mathbf{T} = \frac{\mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k}}{1 + 2t^2}$$

$$\frac{d\mathbf{T}}{ds} = \frac{-4t\mathbf{i} + (2 - 4t^2)\mathbf{j} + 4t\mathbf{k}}{(1 + 2t^2)^3}$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{2}{(1 + 2t^2)^2}$$

(b)

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{2t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}}{1 + 2t^2}$$

$$\frac{d\mathbf{B}}{ds} = \frac{4t\mathbf{i} + (4t^2 - 2)\mathbf{j} - 4t\mathbf{k}}{(1 + 2t^2)^3}$$

$$\tau = \frac{2}{(1 + 2t^2)^2}$$