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Exact Diagonalization with Fermionic and Bosonic limits

Notes

We introduce the Anyon-Hubbard model:

$$H^a = -J \sum_j^L \left(a_j^\dagger a_{j+1} + h.c. \right) + \frac{U}{2} \sum_j^L n_j \left(n_j - 1 \right) \tag{1}$$

where J is the tunnelling amplitude connecting two neighboring sites and U is the on-site interaction energy.
Now introducing an exact mapping between anyons and bosons in 1D. We can define the fractional version of a Jordan-Wigner transformation [1] [2] as:

$$a_j = b_j \exp\left(i\theta \sum_{i=1}^{j-1} n_i \right) \tag{2}$$

with the number operator for anyons and bosons given by $n_i = a_i^\dagger a_i = b_i^\dagger b_i$, where we can see in the mapping of bosons with a string operator is equivalent to anyons on 1D.

$$\begin{aligned} H^b = -J \sum_j^L \Bigg(& b_j^\dagger \exp\left(i\theta \sum_{i=1}^{j-1} n_i \right) b_{j+1} \exp\left(i\theta \sum_{i=1}^{j} n_i \right) + b_{j+1}^\dagger \exp\left(i\theta \sum_{i=1}^{j} n_i \right) b_j \exp\left(i\theta \sum_{i=1}^{j-1} n_i \right) \Bigg) \\ & + \frac{U}{2} \sum_j^L n_j \left(n_j - 1 \right) \end{aligned} \tag{3}$$

Rewriting the Hamiltonian 1 in terms of bosonic operators:

$$H^b = -J \sum_j^L \left(b_j^\dagger b_{j+1} e^{i\theta n_j} + h.c. \right) + \frac{U}{2} \sum_j^L n_j \left(n_j - 1 \right) \tag{4}$$

This bosonic Hamiltonian mapped from the Anyon-Hubbard model, describes bosons with a occupation dependent amplitude $J e^{i\theta n_j}$ for hopping processes from right to left $j+1 \to j$.
We will study the two special cases, one for bosons ($\theta = 0$) in this case we recover the Bose-Hubbard Hamiltonian:

$$H^{BH} = -J \sum_j^L \left(b_j^\dagger b_{j+1} + h.c. \right) + \frac{U}{2} \sum_j^L n_j \left(n_j - 1 \right) \tag{5}$$

And the case for quasi-fermions ($\theta = \pi$), with Hamiltonian:

$$H^{QF} = -J \sum_j^L \left(b_j^\dagger b_{j+1} e^{i\pi n_j} + h.c. \right) + \frac{U}{2} \sum_j^L n_j \left(n_j - 1 \right) \tag{6}$$

Writing explicitly the Hamiltonian for 3 particles and 3 sites:

$$\hat{H} = -J\left(\hat{b}_1^\dagger \hat{b}_2 e^{i\theta\hat{n}_1} + e^{-i\theta\hat{n}_1}\hat{b}_2^\dagger \hat{b}_1 + \hat{b}_2^\dagger \hat{b}_3 e^{i\theta\hat{n}_2} + e^{-i\theta\hat{n}_2}\hat{b}_3^\dagger \hat{b}_2 + \hat{b}_3^\dagger \hat{b}_1 e^{i\theta\hat{n}_3} + e^{-i\theta\hat{n}_3}\hat{b}_1^\dagger \hat{b}_3\right) + \frac{U}{2}\left(\hat{n}_1(\hat{n}_1 - 1) + \hat{n}_2(\hat{n}_2 - 1) + \hat{n}_3(\hat{n}_3 - 1)\right) \tag{7}$$

With the kinetic part:

$$H_{kin} = -J\begin{bmatrix} 0 & \sqrt{3}e^{-2i\theta} & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3}e^{2i\theta} & 0 & 1 & 2e^{-i\theta} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 1 & 0 & 0 & \sqrt{2}e^{-i\theta} & 2e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & 2e^{i\theta} & 0 & 0 & \sqrt{2}e^{-i\theta} & 0 & \sqrt{3} & 1 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2}e^{i\theta} & \sqrt{2}e^{i\theta} & 0 & \sqrt{2} & 0 & \sqrt{2} & \sqrt{2}e^{i\theta} & 0 \\ 0 & 0 & 2e^{-i\theta} & 0 & \sqrt{2} & 0 & 0 & 0 & 1 & \sqrt{3}e^{2i\theta} \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & \sqrt{3}e^{-2i\theta} & 0 & 0 \\ 0 & 0 & 0 & 1 & \sqrt{2} & 0 & \sqrt{3}e^{2i\theta} & 0 & 2e^{-i\theta} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}e^{-i\theta} & 1 & 0 & 2e^{i\theta} & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3}e^{-2i\theta} & 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \tag{8}$$

And with the interaction potential:

$$\hat{H}_{pot} = U\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \tag{9}$$

Writing the Hamiltonian as a matrix:

$$H = -J\begin{bmatrix} 3U/J & \sqrt{3}e^{-2i\theta} & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3}e^{2i\theta} & U/J & 1 & 2e^{-i\theta} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 1 & U/J & 0 & \sqrt{2}e^{-i\theta} & 2e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & 2e^{i\theta} & 0 & U/J & \sqrt{2}e^{-i\theta} & 0 & \sqrt{3} & 1 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2}e^{i\theta} & \sqrt{2}e^{i\theta} & 0 & \sqrt{2} & 0 & \sqrt{2} & \sqrt{2}e^{i\theta} & 0 \\ 0 & 0 & 2e^{-i\theta} & 0 & \sqrt{2} & U/J & 0 & 0 & 1 & \sqrt{3}e^{2i\theta} \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 3U/J & \sqrt{3}e^{-2i\theta} & 0 & 0 \\ 0 & 0 & 0 & 1 & \sqrt{2} & 0 & \sqrt{3}e^{2i\theta} & U/J & 2e^{-i\theta} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}e^{-i\theta} & 1 & 0 & 2e^{i\theta} & U/J & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3}e^{-2i\theta} & 0 & 0 & \sqrt{3} & 3U/J \end{bmatrix} \tag{10}$$

Let the fermionic and bosonic limit $\theta \to \pi$ y $\theta \to 0$ respectively:

$$H_{fermionic} = \lim_{\theta\to\pi} -J \begin{bmatrix} 3U/J & \sqrt{3}e^{-2i\theta} & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3}e^{2i\theta} & U/J & 1 & 2e^{-i\theta} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 1 & U/J & 0 & \sqrt{2}e^{-i\theta} & 2e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & 2e^{i\theta} & 0 & U/J & \sqrt{2}e^{-i\theta} & 0 & \sqrt{3} & 1 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2}e^{i\theta} & \sqrt{2}e^{i\theta} & 0 & \sqrt{2} & 0 & \sqrt{2} & \sqrt{2}e^{i\theta} & 0 \\ 0 & 0 & 2e^{-i\theta} & 0 & \sqrt{2} & U/J & 0 & 0 & 1 & \sqrt{3}e^{2i\theta} \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 3U/J & \sqrt{3}e^{-2i\theta} & 0 & 0 \\ 0 & 0 & 0 & 1 & \sqrt{2} & 0 & \sqrt{3}e^{2i\theta} & U/J & 2e^{-i\theta} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}e^{-i\theta} & 1 & 0 & 2e^{i\theta} & U/J & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3}e^{-2i\theta} & 0 & 0 & \sqrt{3} & 3U/J \end{bmatrix} \tag{11}$$

with:

$$\begin{aligned} e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\ e^{2i\pi} &= \cos(2\pi) + i\sin(2\pi) = 1 \\ e^{i\pi} &= \cos(\pi) + i\sin(\pi) = -1 \\ e^{0} &= \cos(0) + i\sin(0) = 1 \end{aligned}$$

In the limit:

$$H_{fermionic} = -J \begin{bmatrix} 3U/J & \sqrt{3}e^{-2i\pi} & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3}e^{2i\pi} & U/J & 1 & 2e^{-i\pi} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 1 & U/J & 0 & \sqrt{2}e^{-i\pi} & 2e^{i\pi} & 0 & 0 & 0 & 0 \\ 0 & 2e^{i\pi} & 0 & U/J & \sqrt{2}e^{-i\pi} & 0 & \sqrt{3} & 1 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2}e^{i\pi} & \sqrt{2}e^{i\pi} & 0 & \sqrt{2} & 0 & \sqrt{2} & \sqrt{2}e^{i\pi} & 0 \\ 0 & 0 & 2e^{-i\pi} & 0 & \sqrt{2} & U/J & 0 & 0 & 1 & \sqrt{3}e^{2i\pi} \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 3U/J & \sqrt{3}e^{-2i\pi} & 0 & 0 \\ 0 & 0 & 0 & 1 & \sqrt{2} & 0 & \sqrt{3}e^{2i\pi} & U/J & 2e^{-i\pi} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}e^{-i\pi} & 1 & 0 & 2e^{i\pi} & U/J & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3}e^{-2i\pi} & 0 & 0 & \sqrt{3} & 3U/J \end{bmatrix} \tag{12}$$

$$H_{fermionico} = -J \begin{bmatrix} 3U/J & \sqrt{3} & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & U/J & 1 & -2 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 1 & U/J & 0 & -\sqrt{2} & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & U/J & -\sqrt{2} & 0 & \sqrt{3} & 1 & 0 & 0 \\ 0 & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & \sqrt{2} & 0 & \sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & -2 & 0 & \sqrt{2} & U/J & 0 & 0 & 1 & \sqrt{3} \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 3U/J & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & \sqrt{2} & 0 & \sqrt{3} & U/J & -2 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2} & 1 & 0 & -2 & U/J & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 & \sqrt{3} & 3U/J \end{bmatrix} \tag{13}$$

and for the bosonic limit:

$$H_{bosonico} = \lim_{\theta\to 0} -J \begin{bmatrix} 3U/J & \sqrt{3}e^{-2i\theta} & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3}e^{2i\theta} & U/J & 1 & 2e^{-i\theta} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 1 & U/J & 0 & \sqrt{2}e^{-i\theta} & 2e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & 2e^{i\theta} & 0 & U/J & \sqrt{2}e^{-i\theta} & 0 & \sqrt{3} & 1 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2}e^{i\theta} & \sqrt{2}e^{i\theta} & 0 & \sqrt{2} & 0 & \sqrt{2} & \sqrt{2}e^{i\theta} & 0 \\ 0 & 0 & 2e^{-i\theta} & 0 & \sqrt{2} & U/J & 0 & 0 & 1 & \sqrt{3}e^{2i\theta} \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 3U/J & \sqrt{3}e^{-2i\theta} & 0 & 0 \\ 0 & 0 & 0 & 1 & \sqrt{2} & 0 & \sqrt{3}e^{2i\theta} & U/J & 2e^{-i\theta} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}e^{-i\theta} & 1 & 0 & 2e^{i\theta} & U/J & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3}e^{-2i\theta} & 0 & 0 & \sqrt{3} & 3U/J \end{bmatrix} \tag{14}$$

$$H_{bosonico} = -J \begin{bmatrix} 3U/J & \sqrt{3} & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & U/J & 1 & 2 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 1 & U/J & 0 & \sqrt{2} & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & U/J & \sqrt{2} & 0 & \sqrt{3} & 1 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} & 0 & \sqrt{2} & 0 & \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 2 & 0 & \sqrt{2} & U/J & 0 & 0 & 1 & \sqrt{3} \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 3U/J & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & \sqrt{2} & 0 & \sqrt{3} & U/J & 2 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 1 & 0 & 2 & U/J & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 & \sqrt{3} & 3U/J \end{bmatrix} \tag{15}$$

References

[1] T. Keilmann, S. Lanzmich, I. McCulloch and M. Roncaglia ”Statistically induced phase transitions and anyons in 1D optical lattices” Nature Communications, 2:361 (2011) doi: 10.1038/ncomms1353

[2] E. P. Wigner and P. Jordan, ”Über das Paulische Äquivalenzverbot” Zeitschrift für Physik (1928).