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Another notes

Notes

1 Wave function on Order parameters, 3 sites 3 particles case

Taking a wave function as a superposition of the Fock basis in lexicographic order:

$$|\Psi\rangle = \sum_{i=1}^{D} \alpha_i |\psi_i\rangle \tag{1}$$

where D is the dimension of our Hilbert space generated by the basis vectors:

$$D = \frac{(N+M-1)!}{N!(M-1)!} \tag{2}$$

Where N is the number of particles and M the number of sites, for $N = M = 3$, $D = 10$.

$$|\Psi\rangle = \sum_{i=1}^{10} \alpha_i |\psi_i\rangle \tag{3}$$

Writing our wave function explicitly for this case:

$$\begin{aligned} |\Psi\rangle = \alpha_1|3,0,0\rangle + \alpha_2|2,1,0\rangle + \alpha_3|2,0,1\rangle + \alpha_4|1,2,0\rangle \\ + \alpha_5|1,1,1\rangle + \alpha_6|1,0,2\rangle + \alpha_7|0,3,0\rangle + \alpha_8|0,2,1\rangle + \alpha_9|0,1,2\rangle + \alpha_{10}|0,0,3\rangle \end{aligned} \tag{4}$$

Given the parity order parameter and string order parameters:

$$\mathcal{O}_P = \lim_{|i-j|\to\infty} \langle e^{i\theta \sum_{i\le k<j} \delta\hat{n}_k} \rangle \tag{5}$$

$$\mathcal{O}_S = \lim_{|i-j|\to\infty} \langle \delta\hat{n}_i e^{i\theta \sum_{i\le k<j} \delta\hat{n}_k} \delta\hat{n}_j \rangle \tag{6}$$

with $\delta\hat{n}_j = \hat{n}_i - \rho$ and ρ is the filling factor i.e. the average density per-site, writing explicitly the matrices for $\hat{n}_i$ operators, where we know these matrices are diagonal due the orthogonality of the basis and the operator $\hat{n}_i|\psi\rangle = n_i|\psi\rangle$

$$\hat{n}_1 = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

$$\hat{n}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

$$\hat{n}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \tag{9}$$

or for a more feasible view:

$$\hat{n}_1 = \begin{bmatrix} 3 & & & & & & & & & \\ & 2 & & & & & & & & \\ & & 2 & & & & & & & \\ & & & 1 & & & & & & \\ & & & & 1 & & & & & \\ & & & & & 1 & & & & \\ & & & & & & 0 & & & \\ & & & & & & & 0 & & \\ & & & & & & & & 0 & \\ & & & & & & & & & 0 \end{bmatrix} \tag{10}$$

$$\hat{n}_2 = \begin{bmatrix} 0 &&&&&&&&& \\ & 1 &&&&&&&& \\ && 0 &&&&&&& \\ &&& 2 &&&&&& \\ &&&& 1 &&&&& \\ &&&&& 0 &&&& \\ &&&&&& 3 &&& \\ &&&&&&& 2 && \\ &&&&&&&& 1 & \\ &&&&&&&&& 0 \end{bmatrix} \tag{11}$$

$$\hat{n}_3 = \begin{bmatrix} 0 &&&&&&&&& \\ & 0 &&&&&&&& \\ && 1 &&&&&&& \\ &&& 0 &&&&&& \\ &&&& 1 &&&&& \\ &&&&& 2 &&&& \\ &&&&&& 0 &&& \\ &&&&&&& 1 && \\ &&&&&&&& 2 & \\ &&&&&&&&& 3 \end{bmatrix} \tag{12}$$

making the sum $\sum_{i \leq k < j} \delta\hat{n}_k$ with $\delta\hat{n}_i = \hat{n}_i - \rho$ with $\rho = I$ for an ideal SF.

$$\sum_{1 \leq k < 3} \delta\hat{n}_k = \delta\hat{n}_1 + \delta\hat{n}_2 = \hat{n}_1 - I + \hat{n}_2 - I \tag{13}$$

$$\sum_{1 \leq k < 3} \delta\hat{n}_k = \hat{n}_1 + \hat{n}_2 - 2I \tag{14}$$

writing as a matrix:

$$\sum_{1 \leq k < 3} \delta\hat{n}_k = \begin{bmatrix} 1 &&&&&&&&& \\ & 1 &&&&&&&& \\ && 0 &&&&&&& \\ &&& 1 &&&&&& \\ &&&& 0 &&&&& \\ &&&&& -1 &&&& \\ &&&&&& 1 &&& \\ &&&&&&& 0 && \\ &&&&&&&& -1 & \\ &&&&&&&&& -2 \end{bmatrix} \tag{15}$$

and for the exponential operator can be obtained by exponentiating each entry of the main diagonal:

$$\sum_{1 \leq k < 3} \delta\hat{n}_k = \hat{n}_1 + \hat{n}_2 - 2I \tag{16}$$

writing as a matrix:

$$e^{i\theta\sum_{1\le k<3}\delta\hat{n}_k} = \begin{bmatrix} e^{i\theta} &&&&&&&&& \\ & e^{i\theta} &&&&&&&& \\ && 1 &&&&&&& \\ &&& e^{i\theta} &&&&&& \\ &&&& 1 &&&&& \\ &&&&& e^{-i\theta} &&&& \\ &&&&&& e^{i\theta} &&& \\ &&&&&&& 1 && \\ &&&&&&&& e^{-i\theta} & \\ &&&&&&&&& e^{-2i\theta} \end{bmatrix} \tag{17}$$

for $\theta = \pi$

$$e^{i\pi\sum_{1\le k<3}\delta\hat{n}_k} = \begin{bmatrix} -1 &&&&&&&&& \\ & -1 &&&&&&&& \\ && 1 &&&&&&& \\ &&& -1 &&&&&& \\ &&&& 1 &&&&& \\ &&&&& -1 &&&& \\ &&&&&& -1 &&& \\ &&&&&&& 1 && \\ &&&&&&&& -1 & \\ &&&&&&&&& 1 \end{bmatrix} \tag{18}$$

and

$$(\hat{n}_1 - I)(e^{i\pi\sum_{1\le k<3}\delta\hat{n}_k})(\hat{n}_3 - I) = \begin{bmatrix} 2 &&&&&&&&& \\ & 1 &&&&&&&& \\ && 0 &&&&&&& \\ &&& 0 &&&&&& \\ &&&& 0 &&&&& \\ &&&&& 0 &&&& \\ &&&&&& -1 &&& \\ &&&&&&& 0 && \\ &&&&&&&& 1 & \\ &&&&&&&&& -2 \end{bmatrix} \tag{19}$$

writing equation 4 as a vector:

$$|\Psi\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \\ \alpha_{10} \end{bmatrix} \tag{20}$$

and

$$\langle\Psi| = \begin{bmatrix} \alpha_1^*\alpha_2^*\alpha_3^*\alpha_4^*\alpha_5^*\alpha_6^*\alpha_7^*\alpha_8^*\alpha_9^*\alpha_{10}^* \end{bmatrix} \tag{21}$$

Our objective is maximize the string order parameter with the constraint of parity order parameter equal to zero, also considering the sum of probability amplitude must be equal to 1, for our wave function on superposition $|\Psi\rangle$:

$$\mathcal{O}_P = \lim_{|i-j|\to\infty} \langle e^{i\theta \sum_{i\leq k<j} \delta\hat{n}_k} \rangle = 0 \tag{22}$$

$$\mathcal{O}_P = -\alpha_1^2 - \alpha_2^2 + \alpha_3^2 - \alpha_4^2 + \alpha_5^2 - \alpha_6^2 - \alpha_7^2 + \alpha_8^2 - \alpha_9^2 + \alpha_{10}^2 = 0 \tag{23}$$

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2 + \alpha_6^2 + \alpha_7^2 + \alpha_8^2 + \alpha_9^2 + \alpha_{10}^2 = 1 \tag{24}$$

$$Max\left(\mathcal{O}_S\right) = \lim_{|i-j|\to\infty} \langle \delta\hat{n}_i e^{i\theta \sum_{i\leq k<j} \delta\hat{n}_k} \delta\hat{n}_j \rangle \tag{25}$$

with:

$$\mathcal{O}_S = 2\alpha_1^2 + \alpha_2^2 - \alpha_7^2 + \alpha_9^2 - 2\alpha_{10}^2 \tag{26}$$