An Exploration of the Notion of Equidistance

Final Year Project

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Overview

1. A Background to Equidistance
2. Equisets in Normed Linear Spaces

Equidistant Set

Equidistant Points

For a real normed linear space X, and two distinct points x and y in X, the equidistant set E for x and y is the set

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2. Why was it made?

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- Involved 11 axioms, including associativity, commutitivity, etc.
- Vectors- Bolzano, 1804
- Synthetic Geometry- Poncelet and Chasles, mid 19th century (creates relations between points, lines, and planes without using co-ordinates)

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Metric Space- Maurice Fréchet (1906)

Let X be a non-empty set. A metric space (X, d) consists of the set X along with a function $d : X^2 \to \mathbb{R}$ such that:

1. $d(x, y) \geq 0 \, \forall \, x, y \in X$, $d(x, y) = 0 \text{ if and only if } x = y$
2. $d(x, y) = d(y, x) \, \forall \, x, y \in X$
3. $d(x, y) \leq d(x, z) + d(z, y) \, \forall \, x, y, z \in X$

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Example of a complete metric space: $\mathbb{R}$ with Euclidean metric $(d(x,y) = \|x-y\|, x, y \in \mathbb{R})$.
Incomplete: $\frac{1}{n}$ in (0,1], $n \in \mathbb{N}$.

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- Looking at Chebyshev sets
- Needed to determine nearest points, or elements of best approximation
- Possible for more than one nearest point to exist, ie. those were points were of equal distance
- Gave the first formal definition of a set of equidistant points

Application of Equidistant Points

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- Works well for countries opposite one another with narrow sea territories, eg. Australia and New Zealand
- Principle of equidistance doesn't work everywhere as the line can be difficult to determine

Equisets in Normed Linear Spaces

- We take a closer at how equisets behave and explain why they have certain properties.
- We will constrain our look to normed vector spaces.

But first some definitions:

Definition

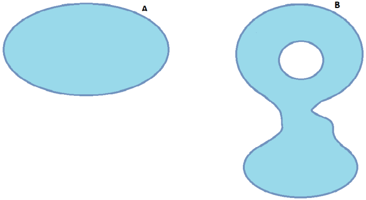
A normed vector space X is a vector space over the Real numbers with a function $\|x\| : X \to \mathbb{R}$ called a norm with the following properties:

- $\|x\| \geq 0 \quad \forall x \in X$ and $\|x\| = 0 \Leftrightarrow x = 0$ the zero vector
- $\|ax\| = |a|\|x\| \quad \forall x \in X$ and scalars a
- $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$

Convexity

Definition

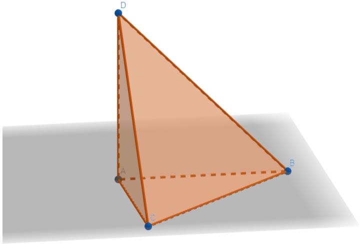
A subset K of a normed vector space X is said to be convex if the 'line' joining any two points remains in K. That is to say $\forall x, y \in K$, $\{(1-t)x + ty : t \in (0,1)\} \subseteq K$



Extreme Point

Definition

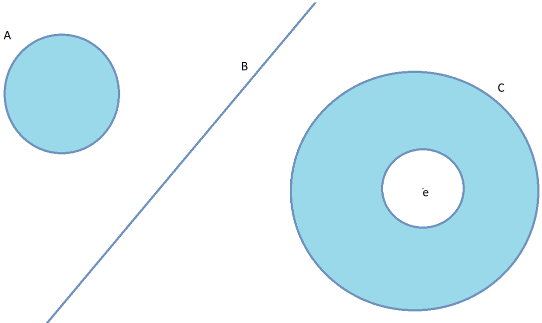
A point a of a set K is said to be an extreme point if there is no $s \in (0,1)$ such that $(as^{-1} + (1-s^{-1})b) \in K$



Chebyshev

Definition

Let X be a normed space and M be a subset of X with an element $x \in X$. An element $m_0 \in M$ is said to be an element of best approximation if:
$\|x - m_0\| = inf_{m \in M} \|x - m\|$
A set X is said to be proximal if any element x has elements of best approximation, and Chebyshev if any element x has a unique element of best approximation.



Equiset

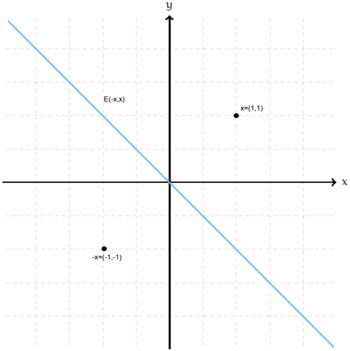
Definition

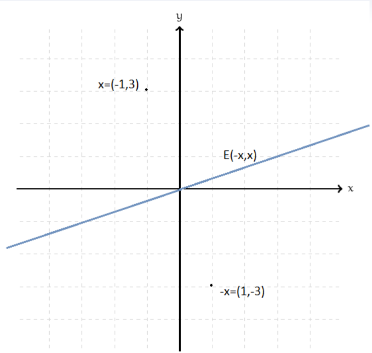
For any two points x and y in a normed vector space X let p be equidistant from x and y if $\|x - p\| = \|y - p\|$. We denote the collection of these values p as $E(x, y)$ and call this the equiset of x and y

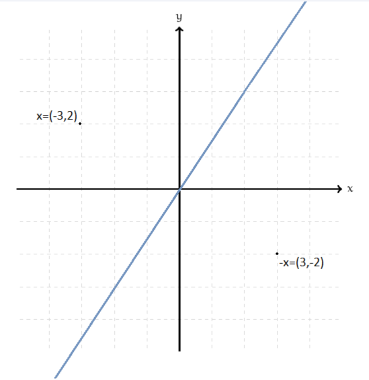
It is useful to simply consider $E(-x, x)$ for the purposes of this presentation

Euclidean Norm

- $X = \mathbb{R}^2, \qquad \|x\| = \sqrt{x_1^2 + x_2^2}$
- $E(-x, x) := \{p \in \mathbb{R}^2 : \|x - p\| = \|x + p\|\}$




y
x
x=(-1,3)
E(-x,x)
-x=(1,-3)


y
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x=(-3,2)
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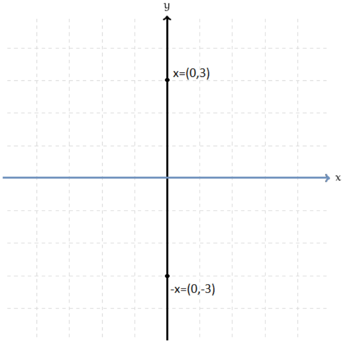
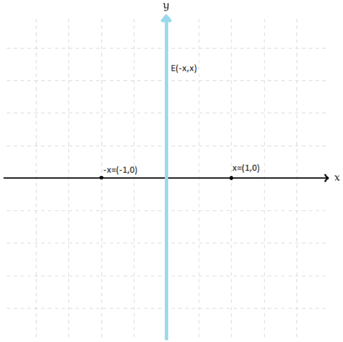
Taxi-cab Norm

In $\mathbb{R}^2$ define the taxi cab norm as $\|x\| = |x_1| + |x_2|$





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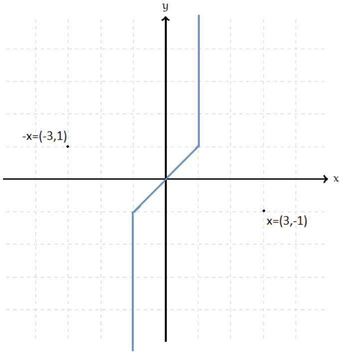

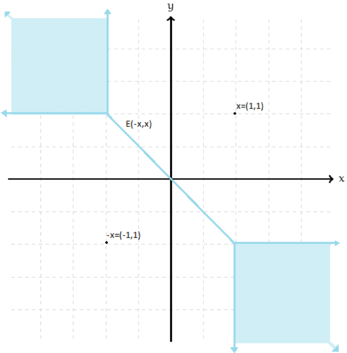

Initial Observations

- The Euclidean Equiset appears very well behaved
- The Taxi-Cab Equiset is a little less so!

A Theorem to Help to Explain This

Theorem

Let $E(-x, x)$ *be a subset of a normed linear space* X *with* $\|x\| = 1$*. Then* $E(-x, x)$ *is Chebyshev if and only if* x *is an extreme point of the unit ball of* X*.*

Firstly, a lemma that we will need to prove this

Lemma

Let $x \neq 0$ be any point of a normed linear space X. If $E(-x, x)$ is convex then it must be a proximal subspace of dimension n-1.

Proof.

Let x be an element of the unit circle in X.

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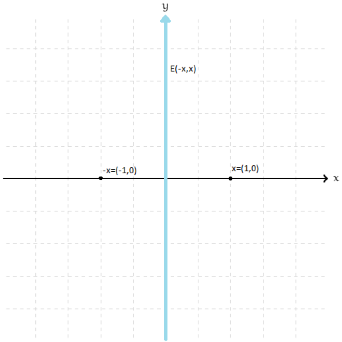
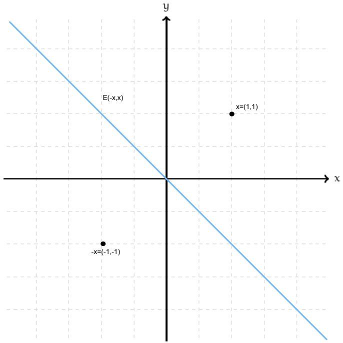
Let x be an element of the unit circle in X. We can see that the origin is the element of best approximation of λx in $E(-x, x)$ for all $\lambda \in \mathbb{R}$. You can see that λx and $z \in E(-x, x)$ form a basis in X, not only that but for all $u = z + \lambda x \in X$ the element of best approximation of u in $E(-x, x)$ is z, showing our set $E(-x, x)$ is Chebyshev. $\square$





Well, what's so useful about this?

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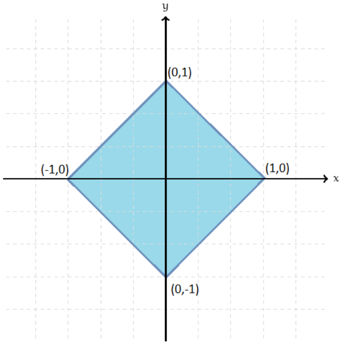
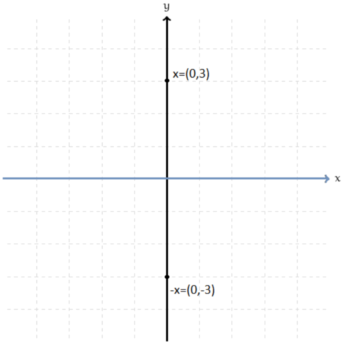
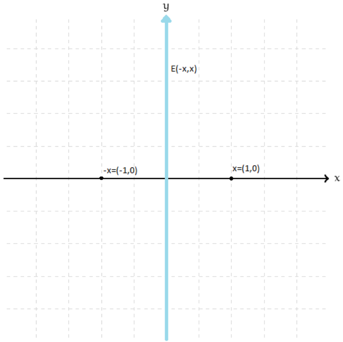
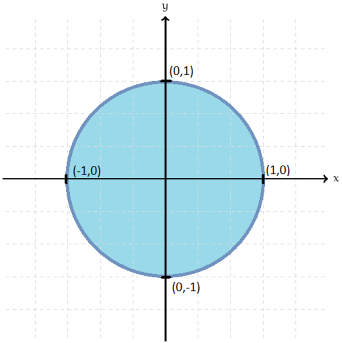
It explains the behaviour of the Taxi-cab norm!

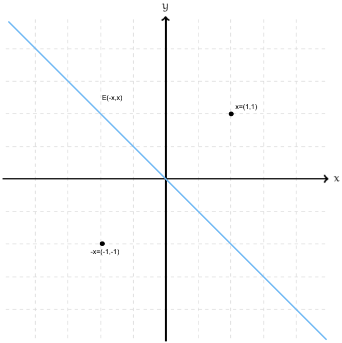






It explains why the equiset created by the euclidean norm is always chebyshev, as every point on the circle is an extreme point.






That's all Folks!