An Exploration of the Notion of Equidistance

Final Year Project

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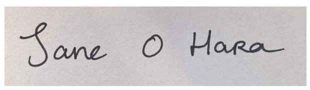
Declaration

We hereby certify that this material, which we now submit for assessment in our programme of study leading to the award of degree, is entirely our own work and has not been taken from the work of others, save and to the extent that such work has been cited and acknowledged within our paper.



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Contents

1 Abstract

In this paper we explore equidistance in an abstract setting. We first recount some historical ideas that allow us to consider equidistance in the settings we move onto. We then define a collection of equidistant points in a normed linear space and go on to show for what conditions the set is convex, and chebyshev. We finally take a look at equidistant points in a metric space and show sufficient conditions to say sets are homeomorphic to intervals in $\mathbb{R}$.

2 Introduction

For many students finishing an undergraduate degree in Mathematics, the leap to reading and writing academic papers can be jarring. This renders many ideas inaccessible to the average graduate. In this paper, we shine a spotlight on a particular notion which falls into this category. We will carefully examine equidistance, or more accurately, equidistant points through various lens'. We have selected this area of study as it is a widely known idea but few have delved into it.

Firstly, we focus on some historical discoveries, most of which are commonly used in mathematics today, and how they led to the development of the notion of equidistance. The starting point for this section is the founding of synthetic geometry. From here, we discuss the development of various concepts like vector and Banach spaces, and we show that the concept of equidistance was being applied in various scenarios long before it was given a formal definition. This section acts as a background to the findings portrayed in the following two sections.

In the second section we pay close attention to normed linear spaces. This includes many examples from the study of vector spaces, which we believe aids in visualising the ideas we highlight. In particular, we define an equiset, a collection of equidistant points, and study it's behaviour in different normed spaces. Some of the concepts that appear here will be foreign to most undergraduates, such as metric projections or Chebyshev sets. O. P. Kapoor and B. B. Panda's paper [1] on equidistant sets in normed linear spaces provided much of the results we show here. Not only that, but we demonstrate a motivation for beginning this study, namely, to study orthogonality when an inner-product isn't readily available. Results on orthogonality are largely drawn from a paper by R. C. James [2]. Examples of equisets are demonstrated and highlighted whenever a relevant theorem is stated.

In the final section we broaden our scope and study equidistance in metric spaces, drawing mainly from a paper by Anthony D. Berard [3]. More precisely, we introduce the unique midpoint property and, in tandem with ideas like connectedness, attempt to characterise sets which are homeomorphic to intervals of $\mathbb{R}$.

Whilst many of the papers and textbooks we reference may be laborious we humbly believe that we have distilled the information in a way that any student with a firm grasp of linear algebra can understand.

3 A History of Equidistance

3.1 Synthetic Geometry

The definition of a set of equidistant points is a co-ordinate free means of creating a relation between those points. Therefore, it involves synthetic geometry. From the founding of Cartesian geometry by Fermat and Descartes in 1636 up until the mid-19th century, geometry was studied on a concrete level using algebraic methods. However Poncelet and Chasles were pioneers in leading the change towards a more abstract form of geometry. They were the founders of synthetic geometry, the study of geometry without the use of co-ordinates or formulae. Synthetic geometry uses axioms instead of algebraic expressions to create relationships between points, lines and planes. Eduardo N. Giovannini provides further reading [4] into synthetic geometry, but some axioms we can consider are:

- Betweenness: A point b is between points a and c if $|a-c| \geq |b-c|$
- Containment: A plane X contains the line l if every point on the line l lies in X
- Congruency: The angle between lines l_1 and l_2 is congruent to the angle between lines l_3 and l_4 if they are the same size

The discoveries of Poncelet and Chasles in these axioms meant that mathematicians after them could move away from concrete objects such as sequence spaces and look at geometry on an altogether more abstract level.

3.2 Vectors

One such mathematician was A. F. Mobius. Mobius' contribution to mathematicians was hugely significant. If we look at some of the motivation behind his work, we can considered the discovery of vectors. Vectors were first discovered by Bernard Bolzano. At this time, synthetic geometry had not yet been defined, but Bolzano was particularly interested in defining operations on points, lines and planes. He published a book outlining these operations in 1804 [5], and this is where we are first introduced to a strong definition of a vector.

In 1827, Mobius published a book studying transformations of lines and conics [6]. Although this work is most noted for introducing barycentric calculus, a study of geometry which considers a point as the center of gravity of certain other points, it featured an early appearance of Bolzano's vectors and did much to develop the concept. Ten years later, Mpbius published another book on statics where he introduced the idea of resolving a vector quantity along two axes. These workings led Bellavitis, in 1832, to define the term equipollent. He said that two line segments can be considered equipollent if they are equal and parallel, which we recognise today as two line segments that represent the same vector.

Another mathematician who was motivated by the work of Möbius was Hermann Grassmann. In 1844 he published the first version of a book studying algebra through abstract quantities. [7] This book (which was republished in a more readable version in 1862) introduced precise structures that are now known as Grassmann algebras. They defined addition, scalar multiplication and multiplication on an algebra. They also consider linear independence and dimension. His findings come very close to the development of an axiomatic theory, which put them well ahead of their time.

3.3 Linear Spaces

Approaching the end of the 19th century, Grassmann's work, stemming from the early work of Möbius, gave motivation to an Italian mathematician, Giuseppe Peano. Thanks to the findings of those before him, in 1888 Peano gave the first axiomatic definition of a linear space, [8], which would not be out of place in modern mathematics. These axioms consider addition, inverses and scalar multiplication of vectors, and for a set X with an element 0, can be considered as follows:

- Addition: For any $x, y \in X$ the sum of $x + y$ must also be in X, for any $x, y, z \in X$:
  - $x + y = y + x$
  - $(x + y) + z = x + (y + z)$
  - $0 + x = x + 0 = x$
  - $(-x) + x = x + (-x) = 0$
- Inverse: For any $x \in X$, we can find the inverse $-x$, which must also be in X.
- Scalar multiplication: For any $x \in X$ and $a \in \mathbb{R}$, we can form the product ax, which must be in X. Also, for $a, b \in \mathbb{R}$:
  - $0(x) = x(0) = 0$
  - $1(x) = x(1) = x$
  - $a(bx) = (ab)x$
- Distributive: For any $x, y \in X$ and $a, b \in \mathbb{R}$:
  - $a(x + y) = ax + ay$
  - $(a + b)x = ax + bx$

Peano called these spaces linear systems, since any vector in the space can be got by completing a linear combination of vectors and scalars in the space. From these axioms, we can also define the spanning set of a vector space, which is a set of vectors that can generate the whole space through linear combinations, and the dimension of a vector space, which is the number of vectors in the smallest possible spanning set, providing further development to concepts Grassmann had outlined in his algebras.

3.4 Metric Spaces

Although it would prove very useful in later years, Peano's axiomatic definition of a linear space did not have an immediate impact. It was not used in any substantial way again until the 1920's, when Banach used it to create his axioms for a Banach space. One area that did see much development during this time, and would become necessary for Banach's discovery, was metric spaces. In his doctoral dissertation in 1906, French mathematician Maurice Fréchet considered a distance function that, in 1914, would be named as a metric space by Felix Haussdorff in his book "General Set Theory" [9]. Fréchet's concept become known by today's definition of a metric space, which is:

Definition 3.1. *Let X be a non-empty set. A metric space (X, d) consists of the set X along with a function $d : X^2 \to \mathbb{R}$ such that:*

1. $d(x, y) \geq 0 \, \forall \, x, y \in X$, $d(x, y) = 0$ *if and only if* $x = y$
2. $d(x, y) = d(y, x) \, \forall \, x, y \in X$
3. $d(x, y) \leq d(x, z) + d(z, y) \, \forall \, x, y, z \in X$

Fréchet developed this concept by building on the work of Hadamard and Volterra concerning functional analysis, although his concept was considered a more abstract study of this field.

To further understand how Banach founded his Banach spaces, we must also consider Cauchy completeness in a metric space. In a paper [10] published in 1817, Bernard Bolzano gave a proof of the intermediate value theorem which inadvertently introduced what we now know as a Cauchy sequence; that is a sequence whose elements become progressively closer and closer to one another. The concept was officially defined and named four years later by Augustin-Louis Cauchy, although it is believed unlikely that Cauchy had read Bolzano's work. This definition involves only metric concepts, so it was very straightforward to generalize it to metric spaces. Bolzano is also credited with creating the completeness axiom in $\mathbb{R}$, which states that any nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound, and therefore converges to a point in $\mathbb{R}$. It is clear from the definition of a Cauchy sequence that it is bounded above, therefore Bolzano's completeness axiom proves that every Cauchy sequence in $\mathbb{R}$ has a least upper bound in $\mathbb{R}$. Since the set of real numbers, together with the metric $d(x, y) = \|x - y\|$, is a metric space, it then became possible to generalise this notion of Cauchy completeness to a definition of a complete metric space. A metric space X is said to be complete if every Cauchy sequence in X converges to a point in X.

3.5 Banach Spaces

As previously mentioned, the work of Peano wasn't revisited again until the 1920's by Stefan Banach. Banach founded modern functional analysis and majorly contributed to measure theory, integration, set theory and orthogonal series. In his dissertation in 1920 he axiomatically defined what become known after his death as a Banach space. It was defined as a real or complex normed vector space that is complete as a metric space under the metric $d(x, y) = \|x - y\|$. Banach managed to create a link between the early works of Fréchet in metric spaces and completeness respectively, with Peano's definition of a real linear space.

3.6 First Appearance of Equidistant Set

In 1957, G. K. Kalisch and E. G. Straus wrote a paper [11] for the Annals of the Brazilian Academy of Sciences. The aim of this paper was to determine different types of sets in Banach spaces. One such set they were examining in Banach spaces were Chebyshev sets. For a set to be Chebyshev, each element in the set must have a unique element of approximation. A more thorough analysis of this definition is given in section 3.1.1, but for now we can think of an element of best approximation as a nearest point, that is the point with the smallest distance to a given point. However, if more than one of these "nearest points" exist, then those points must be of equal distance to the given point.

Kalisch and Straus, when considering the existence of more than one elements of best approximation, found that although this concept of points of equal distance was already being applied in this instance, a set of points of equal distance to a given point had never been given a formal definition. This led them to give the first definition of a set of what they called equidistant points. They said that for a real normed linear space X, and two distinct points x and y in X, the equidistant set E from x and y is the set $E(x, y) = \{p \in X : \|p - x\| = \|p - y\|\}$. This definition forms the basis of what we will explore in the next two sections of this project.

4 Equisets in Normed Linear Spaces

4.1 Some useful ideas

4.1.1 Metric Projections

Definition 4.1. *Let X be a normed space and M be a subset of X with an element $x \in X$. An element $m_0 \in M$ is said to be an element of best approximation of x if:*

$$\|x - m_0\| = inf_{m \in M} \|x - m\|$$

We denote the collection of these:

$$\mathbf{P}_M(x) = \{m_0 \in M | \|x - m_0\| = inf_{m \in M} \|x - m\|\}$$

as the elements of best approximation of M in X.

When our subset M is chebyshev it means that for every element $x \in X$, there is only one element of best approximation in M, and in this case we call our function $\mathbf{P}_M(x)$ the metric projection of x in M [12].

Later in the paper we will encounter the kernel of a set M. We will follow Panda and Kapoor's notation and denote this by M^θ and use it to describe the collection of elements $x \in X$ that have no element of best approximation in M; that is to say $M^\theta = \{x \in X | \mathbf{P}_M(x) = \theta\}$

4.1.2 Orthogonality

It is sufficient in most courses to say that two elements of a vector space are orthogonal when their inner product is zero, but this gives rise to the question of orthogonality outside of inner product spaces. Take our Taxi-cab metric, for example. There is no definable inner product in this space. For the purposes of our paper, where it isn't always assumed that we are dealing with 'nice' norms, we will use a more abstract definition of orthogonality:

Definition 4.2. *Two elements u,v of a normed vector space V are said to be orthogonal if $\|u - v\| = \|u + v\|$. This is called isosceles orthogonality*

Another result need from James' paper [2] is the following:

Theorem 4.1. *For each pair of linearly independent vectors $x, y \in X$, there exists a number $t \in \mathbb{R}$ such that $tx + y \perp x$.*

To prove this we require a lemma whose use is not obvious, and proof is even less so:

Lemma 4.1. *If x and y are elements of a normed linear space X, then*
$lim_{n \to \infty} [\|(n + a)x + y\| - \|nx - y\|] = a\|x\|$

Proof. Firstly $\frac{n}{n+a} + \frac{a}{n+a} = 1$. We can rewrite $\|(n+a)x+y\|$ using the previous identity as $\|nx + \frac{ny}{n+a}\| + a\|x + \frac{y}{n+a}\|$. For $n > 0$ sufficiently large we can say $n + a > 0$, therefore
$\|(n+a)x+y\| - \|nx+y\| = [\|nx + \frac{ny}{n+a}\| - \|nx+y\|] + a\|x + \frac{y}{n+a}\|$ □

And now for the proof of theorem 1:

Proof. let $f(n)$ be a real valued function defined as follows:

$$f(n) = \|x + (nx+y)\| - \|x - (nx+y)\| \equiv \|(n+1)x+y\| - \|(n-1)x+y\|$$

Next take the limit $lim_{n\to\infty} f(n)$:

$$lim_{n\to\infty}[\|(n+1)x+y\| - \|(n-1)x+y\|] = lim_{n\to\infty} f(n+1) = lim_{n\to\infty}[\|(n+2)x+y\| - \|nx+y\|]$$

which, by the previous lemma, we know is equal to $2\|x\|$.
It is also the case that $lim_{n\to-\infty} f(n) = lim_{n\to\infty} f(-n) = -2\|x\|$.
As $f(n)$ is continuous, there must exist a number a such that $f(a) = 0$, in other words, such that $\|x + (ax+y)\| = \|x - (ax+y)\|$. □

4.1.3 Cone

As in Panda and Kapoor's paper [1] we will denote by a cone in X a set K such that $x \in K \to tx \in K$ for every non-negative number $t \in \mathbb{R}$

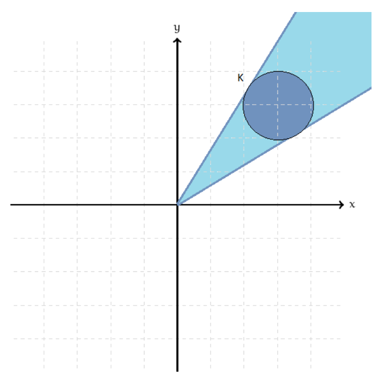

Figure 1: The Cone in Euclidean $\mathbb{R}^2$, note it can be generated by a unit sphere highlighted in dark blue

4.1.4 Extreme Points

To fully explain extreme points we will rely on some ideas from Bankston, McCluskey and Smith's paper [13].

Definition 4.3. *Let X be a vector space, and let $a \in K$ where $K \subseteq X$. We define the facet of a as the set:*

$$F(a) = \{b \in K : s^{-1}a + (1 - s^{-1})b \in K\}$$

for some $s \in \{0, 1\}$

To better understand this definition, we can rewrite the above expression like so: $s^{-1}a + (1 - s^{-1})b = b + s^{-1}(a - b)$. Since $s \in (0, 1)$, we know $t = s^{-1} > 1$. The collection of all of these points forms a ray which points out of a. This can be seen clearly in the following figure.

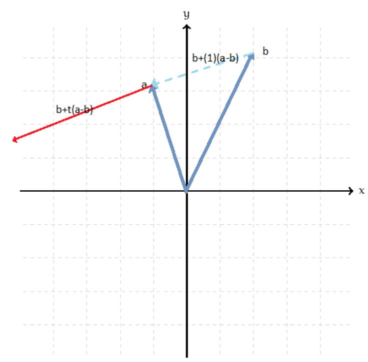

Figure 2: The collection of elements in b+t(a-b)

We can use the example of a tetrahedron T to show what the facet of different points look like (figures 3 and 4). If we select a point e on a face of our tetrahedron, then the collection $F(e)$ will span the entire face, including all vertices and edges. If we look at a a point g along an edge, it is clear that $F(g)$ is simply the entire edge.

From here, we can concisely define the extreme points of a set:

Definition 4.4. *Let K be a convex subset of a normed space X, and $ext(K)$ is the collection $a \in ext(K)$ if and only if $F(a) = \{a\}$.*

In our tetrahedron, we can see that the only points that are in $ext(T)$ are the vertices. This is analogous to saying that a is an extreme point if no closed line segment containing a in its interior lies entirely in K.

4.1.5 Inner-Product Spaces (Day's Result)

A lemma which we will find useful later is the following from Day's paper [14]. It is worth noting that we have altered the wording of this result so as to lessen it's scope at the expense of making it easier to understand. The full result can be found in the original paper.

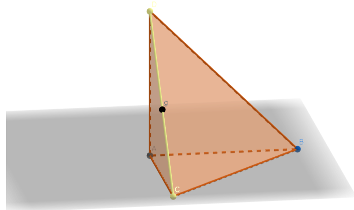

Figure 3: Facet F(g) of a point g along the edge

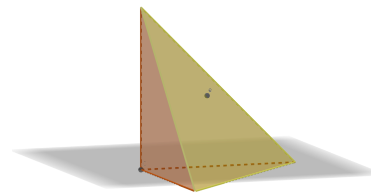

Figure 4: Facet F(e) of a point on a face

Lemma 4.2. *A normed space X is an inner product space if and only if $E(x, y)$ is itself a subspace $\forall x, y \in X$*

The proof of this will be omitted, but this will help explain how a vector space with the taxi-cab norm fails to be an inner-product space.

4.2 Equisets and examples

Definition 4.5. *Let X be a normed vector space, and let $x, y \in X$. A point p is equidistant from x and y if $\|p - x\| = \|p - y\|$. The collection of all such p we will call the equiset of x and y denoted by $E(x, y)$.*

For the majority of this section of the paper we will only discuss equiset of points symmetric about the origin, that is to say equisets of the form $E(-x, x)$. This is not an oversight and we will next show why we can constrain our explorations to this collection of equisets initially.

Equisets will vary greatly depending on the underlying norm, see below an example of an equiset in a Euclidean Space and a less well behaved taxi-cab norm.

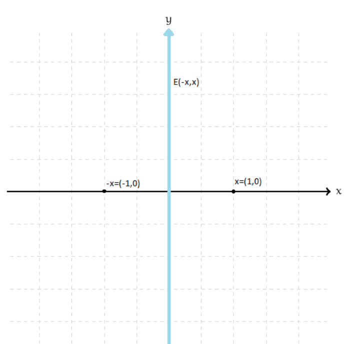

Figure 5: Taxi-Cab1

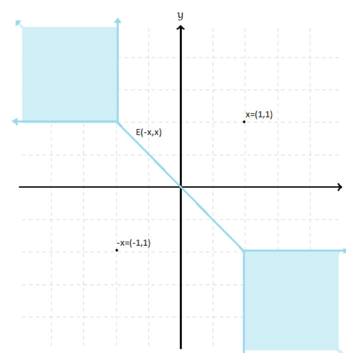

Figure 6: Taxi-Cab2

You will notice that our definition of an equiset bears resemblance to the definition of orthogonality given earlier. It is, in fact, the case that $E(-x,x)$ is the collection of elements orthogonal to x.

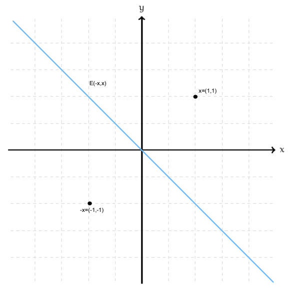

Figure 7: Euclidean Equiset

4.3 Transformations

A simple observation one can see is that equisets of points anywhere in space greatly resembles equisets of points symmetric about the origin. We can show that there exists a continuous bijection between any choice of $E(y,z)$ and an equiset $E(-x,x)$ for $x,y,z \in X$.

Proposition 5. *Let $x, y \in X$ be any two points in a normed linear space, then there exists $z \in X$ and $f : X \to X$ such that f is a homeomorphism between $E(-z,z)$ and $E(x,y)$.*

Proof. Let f be the function $f(x) = x + b$ where $b \in X$. Choose b such that $-z + b = x$ and $z + b = y$. By the axioms of a linear space we know such an element b exists. This function $f(x)$ also preserves distance between any two points it maps, that is to say $\|f(x) - f(y)\| = \|(x + b) - (y + b)\| = \|x - y\|$. For this reason, any element of the equiset $E(-z,z)$ is sent to an element in $E(x,y)$, i.e. for some $p \in E(-z,z)$, $\|p-z\| = \|p+z\|$. Therefore $\|f(p)-f(z)\| = \|f(p) + f(z)\| = \|f(p) - x\| = \|f(p) - y\|$, and so $f(p) \in E(x,y) \, \forall \, p \in E(-z,z)$. This function is injective, as there is only one element b which maps one point to another, and is also surjective, so our function is bijective.
Our function is also continuous. Suppose $P \subseteq X$ is an open subset. The pre-image of this is $f^{-1}(P) = \{w \in X : w+b \in P\}$. For P to be open, any arbitrarily small unit disk must remain in P, as $f(x)$ is a distance preserving map, so any unit disk, say B, to which the function is applied will remain in $f(P)$. To summarise this idea, for any unit disk $B \subset P$ where P is an open subset in X, $f^{-1}(B) \subset f^{-1}(P)$ also. So we have shown our function to be a continuous bijection between two metric spaces, hence it is a homeomorphism. □

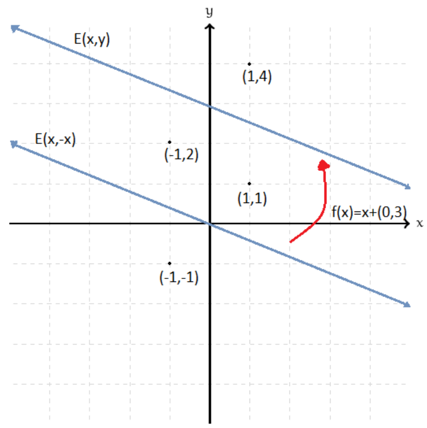

Figure 8: the homeomorphism $f(x) = x + (0,3)$ maps the equiset $E((-1,-1),(1,1))$ to $E((-1,2),(1,4))$

With this in mind we can proceed to explore equisets. However we must take care rephrasing our various theorems and lemmas for $E(x,y)$, for example: if our statement involves "about the origin" we must instead say "about the midpoint $z = \frac{1}{2}(x+y)$".

5.1 Properties of $E(-x,x)$

5.1.1 Convexity

A few simple lemmas regarding convexity of equisets:

Lemma 5.1. *Let x be any point in a two dimensional normed linear space X. If $E(-x,x)$ is convex, then it must be a line through the origin.*

Proof. Let $E(-x,x)$ be convex and let $z \in E(-x,x)$ be non-zero; the existence of such an element has been shown by James in his paper.
Since $0, z \in E(-x,x)$ and the equiset is convex, then $\{tz : |t| \leq 1\} \subset E(-x,x)$.
Next let $y \in E(-x,x)$ be linearly independent from z. We intend to show that such an element cannot exist. $[x]$ must separate y from z or $-y$ from z. Convexity of $E(-x,x)$ implies that the line joining y and z must remain in the equiset, however it intersects $[x]$ at a point other than the origin, reaching our desired contradiction.
Finally, let $\lambda > 1$. From James' paper [2] we know that $\lambda z + tx \in E(-x,x)$ for some number $t \in \mathbb{R}$. As shown earlier, this implies that z and $\lambda z + tx$ are linearly dependent, which is only the case when $t = 0$. And so we have shown that $[z] = E(-x,x)$ and our equiset is a line through the origin. $\square$

This lemma can also be observed in our two primary examples; $E(-x,x)$ is in fact a straight line through the origin when we select the euclidean norm. However when we impose the taxi-cab norm it only remains a line when we choose x such that $E(-x,x)$ is convex, i.e. when x lies on an axis. Next, another useful theorem:

Theorem 5.1. *Let $x \neq 0$ be any point of a normed linear space X. Then $E(-x,x)$ must be a proximal subspace of codimension 1.*

Proof. Let $E(-x,x)$ be convex and let z be any point of X outside of $[x]$. Then $E(-x,x) \cap [x,z]$ is convex, and by the previous lemma it must be a line. Thus if $z \in E(-x,x)$ then $[z] \subset E(-x,x)$. This also means that $E(-x,x)$ is a convex cone symmetric about the origin, therefore it is a subspace. Let $u \in X$. Then either $u = \lambda x$ or, by James's result [2], $u + \lambda x = z \in E(-x,x)$ for some λ. This means that $E(-x,x)$ and x span X together. Therefore $E(-x,x)$ is of codimension 1. Since every equiset is closed, then it follows that $E(-x,x)$ is a closed subspace.
Now let $h \in E(-x,x)$. Then $\|x-h\| = \|x+h\|$, and hence 0 is a nearest point of x in $E(-x,x)$.
If $\alpha \in \mathbb{R}$, then
$$\|\alpha x - h\| = j\alpha j\|x - \alpha^{-1}h\| = j\alpha j\|x + \alpha^{-1}h| = \|\alpha x + h\||$$
for all $h \in E(-x,x)$, and hence 0 is also a nearest point in $E(-x,x)$ to αx. As any $\omega \in X$ has a representation $\omega = \alpha x + h$, where $h \in E(-x,x)$, we have
$$\|\omega - h\| = \|\alpha x\| \leq \|\alpha x - z\|,\ z \in E(-x,x).$$
From here we can simply rewrite the above line as:
$$\|\omega - h\| \leq \|\omega - z - h\|,\ z \in E(-x,x)$$
As $E(-x,x)$ is a subspace, for every $z \in E(-x,x)$, it is also the case that $z+h \in E(-x,x)$, therefore $\|\omega-h\| \leq \|\omega-v\|$ for all $v \in E(-x,x)$. Hence every ω in X has a nearest point in $E(-x,x)$. □

Corollary 5.1. *Let X be a normed vector space. If $E(-x,x)$ is convex for each $x \in X$, then X must be an inner product space.*

The proof of this is immediate from the above theorem and lemma 4.2. We have shown $E(-x,x)$ to be a subspace above which, by lemma 4.2, makes it an inner product space.
This corollary is crucial in demonstrating why $\mathbb{R}^n$ with the taxi-cab norm fails to be an inner product space.

5.1.2 Extreme Points

A single theorem that allows us to characterise an equiset by its extreme points:

Theorem 5.2. *Let $E(-x,x)$ be a convex subset of a normed linear space X with $\|x\| = 1$. Then $E(-x,x)$ is Chebyshev if and only if x is an extreme point of the unit ball of X*

Proof. Let $E(-x,x)$ be a Chebyshev set. As found in theorem 5.1, this is also a subspace. Suppose x is not an extreme point of the unit ball of X. Then there exists points x_1 and x_2 in the unit sphere $S = z \in X : \|z\| = 1$ such that $x = \frac{1}{2}(x_1 + x_2) \in S$ and $F(x) = \{s^{-1}x + (1 - s^{-1})\} \neq \{x\}$. To explain this, consider the following equality: $\|x_1 - x - x\| = \|x_2\| = 1 = \|x_1\| = \|x_1 - x + x\|$. Here, $x_1 - x \in E(-x, x)$, and also $x_2 - x \in E(-x, x)$. Hence $x_1, x_2 \in E(0, 2x)$, since $E(-x, x)$ is a subspace, and $F(x) \in E(0, 2x)$. Since $E(-x, x)$ is Chebyshev and $h \in E(-x, x)$ implies that $\|x - h\| = \|x + h\|$, it is the case that:

$1 = \|X\| = inf\{\|x - h\| : h \in E(-x, x)\}$.

Hence the origin is the nearest point of x in $E(-x, x)$. This also means that the origin is the nearest point of x in $E(0, 2x)$. But $x \in F(x)$ and every point of $F(x)$ has norm 1, contradicting the fact that $E(0, 2x)$ is Chebyshev. Therefore x is an extreme point of the unit ball of X.

To prove the converse, let x be an extreme point of the unit ball. Then 0 is the unique nearest point in $E(-x, x)$ to λx, $\lambda \in \mathbb{R}$. So if $u = z + \lambda x$, and $z \in E(-x, x)$, then z is the unique nearest point of u; therefore $E(-x, x)$ is Chebyshev. □

5.1.3 Metric Projections

Recall that for a subset M of a normed linear space X we denote the kernel of the metric projection by $M^\theta = \{x \in X | \mathbf{P}_M(x) = \theta\}$. A theorem that makes use of the metric projection:

Theorem 5.3. *Let M be the one dimensional span $[x]$ of $x \in X$ where X is a normed linear space. When M is Chebyshev then the following statements hold:*

1. $M^\theta \subset E(-x, x) \implies M^\theta = E(-x, x)$
2. $E(-x, x)$ *is a cone* $\implies M^\theta = E(-x, x)$

To prove this we will need the following lemma, the proof of which can be found at [15] (Lemma 1):

Lemma 5.2. *If $x \in X$ and $M = [x]$ is Chebyshev, then*
$\mathbf{P}_M(E(-x, x)) \subset \{tx : -1 \leq t \leq 1\}$

And now to prove the previous theorem:

Proof. (**1**): For $u \in E(-x, x)$, $\mathbf{P}_M = \alpha x$ with $\|\alpha\| \leq 1$. Since $\|u - x\| = \|u + x\|$, and u has a unique nearest point in $[x]$, we can say $\|\alpha\| \neq 1$, since $\|\alpha\| = 1$ would imply that x and $-x$ are both nearest neighbours of u in $[x]$. We can rewrite any $u \in E(-x, x)$ as $u = u_\theta + \alpha x$ where $u_\theta \in M^\theta$. Also, since $\mu u_\theta \in M^\theta \subset E(-x, x)$, we can say that $u_\theta \perp \mu x$ for all $\mu \in \mathbb{R}$; that is to say that:

$\|u_\theta - \mu x\| = \|u_\theta + \mu x\|$ for $\mu \in \mathbb{R}$.

We can now choose $\mu = 1 - \alpha \in \mathbb{R}$ to render the following statement:

$\|u_\theta + \mu x\| = \|u_\theta - \mu x\| = \|u_\theta + (1 - \alpha)x\| = \|u_\theta - \alpha x + x\|$

(Note: since $\mu = 1 - \alpha$, then $\mu + \alpha = 1$, so we can use this as the constant in front of the second x)

$= \|u_\theta - \alpha x + (\mu + \alpha)x\| = \|u + (1 - 2\alpha)x\|$

(here we've used the fact that $u = u_\theta + \alpha x$).

If we let the above statements equal to an arbitrary constant, say r, then we've shown that the sphere centered at μ of radius r intersects with M in at least three places: x,$-x$, and $(-1+2\alpha)x$. However this cannot be the case as $1-2\alpha = \mu - \alpha = \pm 1$. So either $\alpha = 0$ or $\alpha = +1$, the latter which is also impossible as we've seen above. Hence $\alpha = 0$ and $u \in M^\theta$ for any choice of $u \in E(-x, x)$.

(**2**): Next, let $u \in M^\theta$. There exists a number $t \in \mathbb{R}$ such that $u - tx \in E(-x, x)$. As $\mathbf{P}_M(E(-x, x)) \subset \{ax : \|a\| \leq 1\}$, we can say that $\|t\| \leq 1$. Since $E(-x, x)$ is also a cone, $u - tx \in E(-x, x)$ implies that $\lambda u - \lambda tx \in E(-x, x)$ for an arbitrary $\lambda \in \mathbb{R}$. But $\lambda u \in M^\theta \ \forall \ \lambda \in \mathbb{R}$, so $\|\lambda t\| \leq 1$. This can only be the case when $t = 0$, as if we let $\|t\| < 0$ then we can choose an arbitrary $\lambda \in \mathbb{R}$ such that $\|t\lambda\| < 1$. Hence $M^\theta \subset E(-x, x)$ and by (**1**), proven above, $M^\theta = E(-x, x)$

$\square$

6 Equisets in Metric Spaces

6.1 Motivation

It is easy to assume that much of what we found for normed vector spaces must also apply to metric spaces, as a norm can be thought of as a metric in the sense that $\|x\| = d(x, 0)$ for some metric $d(x, y)$. However, this is not the case. There are many metrics which prove unruly; if we take a metric that is not also a norm then we can see how the behaviour of it's equiset is quite unexpected.

Example 6.1. *In a vector space X we define $\rho(u, v) = \|u\| + \|v\|$ for all distinct points u and $v \in X$ and $\rho(u, u) = 0$ for any $u \in X$ our metric space, $\|x\| = \sqrt{x_1^2 + x_2^2}$ being the euclidean norm that we've seen before. We call this the British-Rail Metric.*

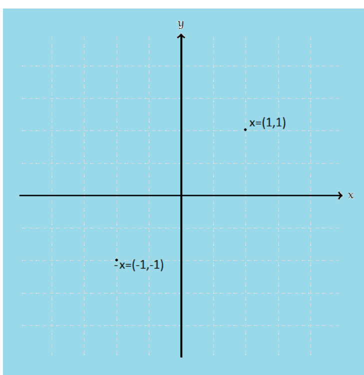

Figure 9: $E(-x, x)$

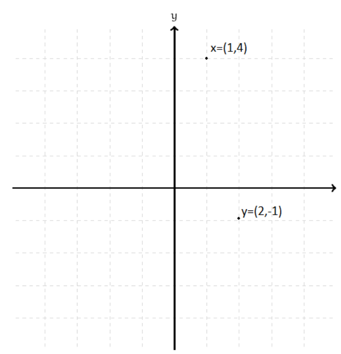

Figure 10: $E(x, y)$,$x \neq y$

It is difficult to squeeze any wisdom from this example, as the set fails to be convex or chebyshev, and can also be empty
When we select points u and v such that $\|u\| = \|v\|$ the equiset $E(u, v) = \{p \in X : \rho(u, p) = \rho(p, v)\}$ simply becomes the entire set less u and v. To help understand this, we can solve $\{p \neq u, v \in X : \|u\| - \|p\| = \|v| - \|p\|\}$ when $\|u\| = \|v\|$, which renders every point $p \in X \backslash \{u, v\}$ to be a solution.
If we select any points u and v where $\|u\| \neq \|v\|$, then the equiset is empty. So if we wish to characterise a metric space X based on its equiset $E(-x, x)$, the findings outlined Panda and Kapoor's paper cannot be used.

It would be more fruitful to try and find metric spaces with conditions that show it to be homeomorphic to intervals of real numbers, as is done in the paper "Characterisation of metric spaces by the use of their midsets:intervals*" By Anthony D. Berard Jr [3].

6.2 Preamble

Let us begin by giving some definitions which will be helpful in navigating equisets in metric spaces:

Definition 6.1. *If (X, d) is a non-trivial metric space and x and y are distinct points in X, the midset of x and y in X is the set $A(x, y) = \{z \in X : d(x, z) = d(z, y)\}$. If each midset in X consists of a unique point, then we say X has the unique midpoint property **(UMP)**.*

As we navigate this section, it proves much more useful to try and characterise metric spaces that have this **UMP** property.

Definition 6.2. *In a subset K in a metrix space X, we say $a \in K$ is a cut point if $K \backslash \{a\} = A \cup B$ for disjoint sets A and B.*

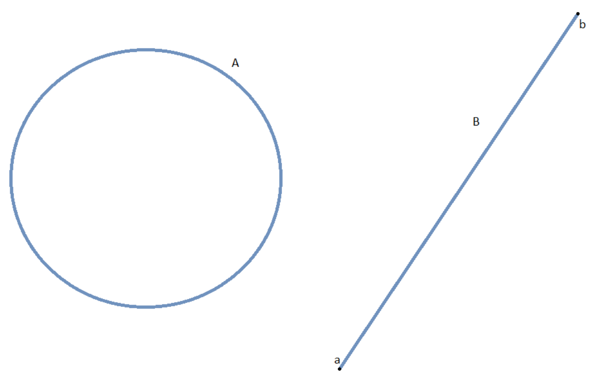

Figure 11: A has no cut points, B has exactly two non-cut points, a and b

Definition 6.3. *A metric space (X,d) is said to be connected if and only if X cannot be written as a disjoint union of non-empty open subsets of X. If X is a connected metric space, then we can say x is a cut point of X if the complement X - $\{x\}$ is disconnected.*

Definition 6.4. *A homeomorphism is a function $f : X \to Y$, where (X, d_X) and (Y, d_Y) are metric spaces, that is continuous, and whose inverse f^{-1} is also continuous with respect to the given metric spaces. If there exists a homeomorphism between X and Y, we say they are homeomorphic to each other and write $X \cong Y$.*

6.3 Examples

Before we explore properties of metric spaces with **UMP**, we will describe some examples of these spaces.

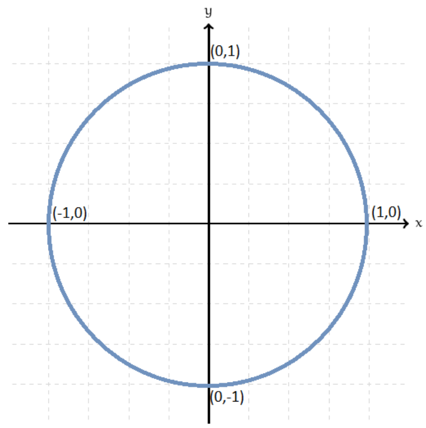

Figure 12: The Set S, also the collection of all elements $x \in \mathbb{R}^2$ such that $\|x\| = 1$ with the euclidean metric

Example 6.2. *Consider the set S in $\mathbb{R}^2$ describes as follows:*
$S^1 = \{(cos(\theta), sin(\theta)) : \theta \in [0, 2\pi]\}$
We denote a metric on this set as $d(x, y) = |cos^{-1}(y_1) - cos^{-1}(x_1)|$, which is simply the arc-length between any two elements.
This is a connected space and has the **UMP** *property. However this space has no cut-points.*
We can define metrics of, say, the unit sphere S^2, but this would fail to have the **UMP**.

Example 6.3. *Next consider the subset of $\mathbb{C}$ consisting of all pure imaginary numbers, that is the set $C = \{a + bi : a = 0, b \in \mathbb{R}\}$ together with the metric $d(x, y) = |x - y|$, where $|x|$ is the modulus of a complex number. This set has the* **UMP**, *and moreover every element is a cut point.*
As in the previous example, we can define a metric on $\mathbb{C}$, but this space will fail to have the **UMP**.

In the proceeding section, it is useful to consider the following sets. For an arbitrary point x in a metric space X and some $\epsilon > 0$, denote the sets $\{q \in X : d(x, q) = \epsilon\}$, $\{q \in X : d(x, q) < \epsilon\}$, $\{q \in X : d(x, q) \leq \epsilon\}$, and $\{q \in X : d(x, q) > \epsilon\}$ by $D(x, \epsilon)$, $B(x, \epsilon)$, $\bar{B}(x, \epsilon)$, and $c\bar{B}(x, \epsilon)$ respectively.

6.4 Properties

Now that we have a foundation to work upon, we can begin our exploration. By the end of this section, we will be be able to characterise homeomorphisms between sets with the **UMP** and intervals of the Real Line. First a useful lemma:

Lemma 6.1. *If X is a connected metric space with UMP, z in X is a non-cut point of X if and only if there are no distinct points x and y in X for which* $d(x, z) = d(y, z)$.

Proof. Let z be a cut point of X. When z is removed from X we are left with two seperate subsets A and B of X such that $X - z = A \cup B$. A and B are non-empty, so let $a \in A$ and $b \in B$. Let $\epsilon = \frac{1}{2} min\{\rho(a, z), \rho(b, z)\}$. Assume there does not exist $x, y \in X$ such that $d(x, z) = d(y, z)$. X has UMP, so there is at most one point x in X such that $d(z, x) = \epsilon$. If we assume that $x \in A$, then $B - B(z, \epsilon) = B - B(z, \epsilon)$ and $X = [A \cup B(z, \epsilon)] \cup [B - B(z, \epsilon)]$. This contradicts the connectedness property of X, therefore z cannot be a cut point if $d(x, z) = d(y, z)$ for $x, y \in X, x \neq y$.
Suppose $d(x, z) = d(y, z)$ for $x, y, z \in X, x \neq y$. Let $A = \{d(x, z) < d(x, z)\}$ and $B = \{d(x, z) > d(y, z)\}$. Then $X/A(x, y) = A \cup B$, where $A(x, y)$ is the midset of x and y in X. This implies that z causes a seperation in X, therefore z is a cut point of X when $d(x, z) = d(y, z)$ for $x, y \in X, x \neq y$. □

Theorem 6.1. *If X is a connected metric space with the* **UMP***, then there exist at most two distinct non-cut points.*

Proof. Suppose there are three distinct non-cut points a,b, and c in X. As our space is connected, none of these points may be the midpoint of the other two points. Without loss of generality say that $d(a, c) < d(b, c) < d(a, b)$. We can make this assertion since, if any of these were equal, then one of the points would be a midpoint. Next let $d(a, c) = \epsilon_1$ and $d(b.c) = \epsilon_2$. Consider the following subset $M = B(b, \epsilon_2) \cap cB(c, \epsilon_1)$. M is non empty as $b \in M$, and also $M \neq X$ since $a \notin M$. M is also open and closed in X, since M has an empty boundary. This would mean that X is not connected, which is a contradiction, Therefore X can have at most two non-cut points. □

Theorem 6.2. *If X is a connected metric space with UMP and a non-cut point z, then the function* $f(x) = d(z, x)\,; f : X \to f(X)$ *is a homeomorphism and f(X) is an interval of* $\mathbb{R}$*.*

Before we prove this theorem, we will show that this function is continuous with an example:

Example 6.4. *Let* X *be* $[1, \infty)$ *with the metric* $d(x, y) = |y - x|$, $1 \in X$ *being a non-cut point. For our function* $f(x) = d(x, 1) = |1 - x|$ *to be continuous, the pre-image* $f^{-1}(V)$ *of any open set* $V \subseteq f(X)$ *must be open in* X*.*
All open sets in $\mathbb{R}$ *are of the form* (a, b)*, so if we take the inverse of this we will find that* $f^{-1}(a, b) = (a - 1, b - 1)$*, which is open. Therefore we've shown that our function is continuous.*

Proof. Since z is a non-cut point in X, meaning $d(z, x) \neq d(z, y)$ for $x, y \in X$, $x \neq y$, $f(x) \neq f(y)$, and therefore f is an injective function. Similarly, $f^{-1}(d(z, x)) \neq f^{-1}(d(z, y))$, so f is a continuous bijective function. X is a connected space, so $f(x)$ must also be connected, and therefore it is an interval. Consider a point $q \in X$.

$$f(q) = \begin{cases} f(x), & \text{if } x = q \\ (-\infty, f(x)) \cap f(X), & \text{if } q < x \\ (f(x), \infty) \cap f(X), & \text{if } q > x \end{cases}$$

Therefore $f(q)$ must be some interval of $\mathbb{R}$. $\square$

Theorem 6.3. *If X is a connected metric space with UMP and X has at least one non-cut pont, then either:*

- *X has exactly one non-cut point and X is homeomorphic to $\mathbb{R}^+$;*

or

- *X has two distinct non-cut points and X is homeomorphic to [0,1].*

To prove this theorem, we must first introduce the following lemma:

Lemma 6.2. *Let X be a connected metric space with UMP and a non-cut point z. Then for $a, b \in X$, with $a < b$, $I(a,b) = \{x \in X : a \leq x \leq b\}$ is a connected metric space with UMP and exactly two non-cut points, a and b.*

Proof. Suppose $I(a,b)$ is not connected. Then $I(a,b) = A \cup B$, where A and B are disjoint, non-empty closed subsets of X. Let $\epsilon_1 = d(z,a)$ and $\epsilon_2 = d(z,b)$. Then $\epsilon_1 < \epsilon_2$. If $a \in A$ and $b \in B$, then $X = [A \cup B(z,\epsilon_1)] \cup [B \cup cB(z,\epsilon_2)]$ represents a separation of the connected set X, while if $a, b \in A$, then $X = [A \cup B(z,\epsilon_1) \cup cB(z,\epsilon_2)] \cup B$ also represents a separation of X. Therefore, $I(a,b)$ must be connected. This connectedness also means that for any pair of points in $I(a,b)$, their midpoint must also be in $I(a,b)$, so $I(a,b)$ has UMP.
Suppose a is a cut point of $I(a,b)$. Then there exists $s, t \in I(a,b)$, with $s < t$, such that a separates s and t. But s separates a and t [16][Theorem 71], therefore a must be a non-cut point of $I(a,b)$. Similarly, b must also be a non-cut point. Therefore, according to theorem 6.1, $I(a,b)$ has exactly two non-cut points, a and b. $\square$

We can now prove Theorem 6.2:

Proof. Let $f : X \to f(X) \subseteq \mathbb{R}^+$ be the homeomorphism from theorem 5.1. Suppose $f(X)$ is unbounded. If X had two non-cut points, then for all $x \in X$, $f(x)$ could be no greater than the distance between the two non-cut points, which would contradict the unboundness of f. The same argument can be made for X having more than two non-cut points. Therefore if f is bounded, then X must have exactly one non-cut point.
Now suppose $f(X)$ is bounded. Let $M = \sup f(x)$. Suppose $q \in X$ such that $f(q) = M$. Then $X = I(z,q)$ and q is a non-cut point of X. If w is

another non-cut point in X and $f(w) < M$, then there must be a $y \in X$ such that $f(w) < f(y)$. But $d(z,w) < d(z.y)$ seperates w from y and z, thereby contradiciting the connectedness of X. So if $f(x)$ is bounded, it has exactly two non-cut points, M and $f^{-1}(M)$. Finally, we can conclude that if X has exactly one non-cut point, $f(X) = [0, M)$ and X is homeomorphic to $\mathbb{R}^+$, while if X has two non-cut points, $f(X) = [0, M]$ and X is homeomorphic to $[0, 1]$. □

Theorem 6.4. *If X is a connected metric space with UMP and no non-cut points, then X is homeomorphic to $\mathbb{R}$.*

Proof. Let $x, y \in X$ be arbitrary points with a unique midpoint z. Denote $x \backslash \{z\} = A \cup B$ as a separation of X. We need to show that $A \cup \{z\}$ and $B \cup \{z\}$ satisfy the first condition of the previous theorem.
To begin, $A \cup \{z\}$ is connected, by [Berard 4, theorem 60]. We will make three assertions about $A \cup \{z\}$.

Assertion 1: $A \cup \{z\}$ has **UMP**.
To show this, select arbitrary points $r, s \in A \cup \{z\}$. Since $r, s \in X$, there exists a unique point $q \in X$ such that $d(r, q) = d(s, q)$. Suppose $q \in B$, and let $X \backslash \{q\} = C \cup D$ be a seperation. So we can write $A \cup \{z\}$ as follows:

$A \cup \{z\} = [(A \cup \{z\}) \cap C] \cup [(A \cup \{z\}) \cap D]$

This implies that $A \cup \{z\}$ is disconnected, which is a contradiction.

Assertion 2: z is a non-cut point of $A \cup \{z\}$.
Suppose z is a cut point of $A \cup \{z\}$. Let $A = A \cup \{x\} \backslash \{z\} = g \cup H$ be a separation. There exists an $\epsilon > 0$ such that if $0 < \delta < \epsilon$, and there exists $g \in G$, $h \in H$, and $b \in B$ such that g,h,b are in $D(z, \delta) = \{x \in X : d(x, z) = \delta\}$. Without loss of generality, let such points be $g_1 \in G$, $h_1 \in H$, and $b_1 \in B$, and let $\epsilon = min\{d(z, g_1), d(z, h_1), d(z, b_1)\}$. Then we can write $G \cup \{z\}$ as follows:

$G \cup \{z\} = [G \cap c\bar{B}(z, \delta)] \cup [(G \cup \{z\}) \cap B(z, \delta)]$

This implies that $G \cup \{z\}$ is comprised of a part that falls within $B(z, \delta)$ and a part that is in the compliment of this ball, i.e. $c\bar{B}(z, \delta)$. This would mean there is a separation of $G \cup \{z\}$, contradicting it's connectedness.

Assertion 3: There is no non-cut points of $A \cup \{z\}$ except z.
Suppose that a point $q \in A \cup \{z\}$ is a non-cut point of $a \cup \{z\}$ and $q \neq z$. However, q is not a non-cut point of X, and so there exists r and s in X such that $d(r, q) = d(q, s)$. Denote by $X \backslash \{q\} = E \cup F$ the usual separation. It is either the case that $A \cup \{z\} \subseteq E$ or $A \cup \{z\} \subseteq F$. Without loss of generality, assume that $A \cup \{z\} \subseteq E$. Then:

$B \cup \{z\} = [E \cap (B \cup \{z\})] \cup [F \cap (B \cup \{z\})]$

is a seperation, which contradicts the seperation of $B \cup \{z\}$.

Finally, we know by the first part of Theorem 5.3 that the set $A \cup \{z\}$ is homeomorphic to $\mathbb{R}^+$, the positive real numbers. We will denote the homeomorphism to be α. Similarly, $B \cup \{z\}$ is homeomorphic to $\mathbb{R}^+$. Denote the matching homeomorphism by β. Next define a function $\gamma : X \to \mathbb{R}$ as follows:

$$\gamma(x) = \begin{cases} \alpha(x) & \text{for } x \in A \cup \{z\} \\ -\beta(x) & \text{for } x \in B \cup \{z\} \end{cases}$$

This is a homeomorphism from X to $\mathbb{R}$.

$\square$

Proving this theorem gives us the following corollary, which explains exactly how the number of non-cut points in a set describes the type of interval we are dealing with:

Corollary 6.1. *If X is a connected metric space with* **UMP**, *then X is an interval:*

1. *if X has two distinct non-cut points X is a closed interval.*
2. *if X has exactly one non-cut point X is a half-open interval*
3. *if X has no non-cut points, X is an open interval*

7 Collaboration Outline

This paper covers three topics, namely history, normed linear spaces, and metric spaces. Jane completed the historical section, Ronan the normed linear space section, and both worked on the metric spaces section. The remaining sections: abstract, introduction, and conclusion, as well as formatting of the paper were done via the combined work of both Ronan and Jane.

8 Conclusion

It is our belief that we have demonstrated, if only scraping the surface, how equidistance can be interpreted, described and studied in an abstract environment. We have shown some of the ideas proposed throughout history that led to the axioms we lean on today.

We have shown how to define an equidistant point in a normed linear space, and how, by considering the collection of these points, we can characterize them based on convexity, chebyshev, and the behaviour of the underlying unit ball.

We have finally demonstrated how metric spaces with the UMP can be homeomorphic to intervals of the Real line, relying on properties such as connectedness, and cut-points.

There are many avenues of further study which we did not highlight here. Equidistance in l^p-spaces, and in wider topological spaces to name but a few.

We hope that the reader finds that this has been a useful exercise, and that they have a firmer grasp on the nature of orthogonality, intervals in $\mathbb{R}$, and the historical background that has led to fundamental ideas.

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