



Lecture Notes on Quantum Mechanics

Alpha Version

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1 Review of Preparatory Knowledge

1.1 Classical Mechanics

1. $\delta S \equiv 0$ Principle of Least Action
2. $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} = \frac{\partial \mathcal{L}}{\partial q_\alpha} \quad (\alpha = 1, 2, \dots, D)$ Euler-Lagrange eqs.
3. $\begin{cases} \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \\ \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \end{cases} \quad (\alpha = 1, 2, \dots, D)$ Hamilton's Canonical eqs.
4. $\frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H]_{\text{PB}}$ Poisson Bracket
5. $\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}, t) = 0$ Hamilton-Jacobi eq.

1.2 Newtonian Mechanics

Euclidean Space (Physical Space) \longrightarrow Cartesian (Descartes) Coordinates

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \quad \mathbf{p} = m\mathbf{v} \quad (\text{Momentum \& Force})$$

$$\frac{d\mathbf{L}}{dt} = \mathbf{M}, \quad \mathbf{L} = \mathbf{x} \times \mathbf{p}, \quad \mathbf{M} = \mathbf{x} \times \mathbf{F} \quad (\text{Angular Momentum \& Torque})$$

Mass point \longrightarrow Mass point system \longrightarrow Rigid Body \longrightarrow Inertia $\begin{cases} \text{Mass: } m \\ \text{Inertia of Rotation: } mr^2 \end{cases}$

1.3 Lagrangian Mechanics - Analytical M.

1.3.1 Constraint

x_1, x_2, \dots, x_{3N} are dependent $\implies q_1, q_2, \dots, q_D$ are independent $\left(\frac{\partial q_i}{\partial q_j} = 0, \quad i \neq j \right)$

constraint $\begin{cases} \text{holonomic: } f(x; t) \equiv 0 \\ \text{non-holonomic: } f(x, \dot{x}; t) \equiv 0 \end{cases}$

1.3.2 Coordinates and Space

Generalized Coordinates $q_\alpha(t)$, $\alpha = 1, 2, \dots, D \leq 3N$

Configuration Space (Abstract Space): D -dim

1.3.3 Functional

Example of functionals (function of function):

1. Lagrangian:

$$\mathcal{L}(q(t), \dot{q}(t); t)$$

generalized velocity: $\dot{q} = \frac{dq}{dt}$

2. Hamiltonian:

$$H(q(t), p(t); t)$$

canonical position: $q(t)$; canonical momentum: $p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}$ (in Phase Space)

3. Action:

$$S[q(t)] = \int_{t_1}^{t_2} \mathcal{L}(q(t), \dot{q}(t); t) dt$$

Dimensions of these functionals:

$$[\mathcal{L}] = [H] = [\text{Energy}], [S] = [\text{Energy}] \cdot [\text{Time}] = [\text{Length}] \cdot [\text{Momentum}] = [\hbar]$$

Reduced Planck's Constant: $\hbar = \frac{h}{2\pi}$

1.3.4 Difference & Differential & Variation

$$f(x_2) - f(x_1) = \underbrace{\Delta f}_{\text{difference}} \xrightarrow[\Delta x = x_2 - x_1]{\Delta x \rightarrow 0} \underbrace{df}_{\text{differential}}, \text{ variation: } \delta q(t) = q_2(t) - q_1(t)$$

$$f(q, \dot{q}; t) : \begin{cases} \Delta f = \frac{\partial f}{\partial q} \Delta q + \frac{\partial f}{\partial \dot{q}} \Delta \dot{q} \\ df = \frac{\partial f}{\partial q} dq + \frac{\partial f}{\partial \dot{q}} d\dot{q} \\ \delta f = \frac{\partial f}{\partial q} \delta q + \frac{\partial f}{\partial \dot{q}} \delta \dot{q} \quad (\delta q(t_1) = \delta q(t_2) = 0; \delta t = 0) \end{cases}$$

1.3.5 Principle of Least Action (Hamilton's Principle)

$$\boxed{\delta S \equiv 0} \implies \text{dynamical equations}$$

1.3.6 Euler-Lagrange Equations

$$\boxed{\frac{\partial \mathcal{L}}{\partial q_\alpha} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \right), \quad \alpha = 1, 2, \dots, D}$$

D 2nd-order ODE

$$\begin{aligned}
0 \equiv \delta S &= \delta \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}; t) dt = \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q} \right) dt \\
&= \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \delta q \right] dt \\
&= \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \right] \delta q dt + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q \Big|_{t_1}^{t_2}}_{=0} \implies \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \equiv 0
\end{aligned}$$

1.4 Hamiltonian Mechanics - Analytical M.

1.4.1 Legendre Transformations

$$\begin{cases} p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}, \alpha = 1, 2, \dots, D \\ H(q, p; t) = \sum_\alpha p_\alpha \dot{q}_\alpha - \mathcal{L}(q, \dot{q}; t) \end{cases}$$

$$\implies \frac{\partial H}{\partial \dot{q}_\alpha} = p_\alpha - \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \equiv 0 \implies H \text{ is } \dot{q} \text{ - independent, } \mathcal{L} \text{ is } p \text{ - independent}$$

1.4.2 Hamilton's Canonical Equations of Motion

$$\boxed{
\begin{cases} \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \\ \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \end{cases} \quad \alpha = 1, 2, \dots, D
}$$

2D 1st-order ODE

$$\begin{aligned}
0 \equiv \frac{\partial \mathcal{L}}{\partial p_\alpha} &= \dot{q}_\alpha - \frac{\partial H}{\partial p_\alpha} \implies \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}; \dot{p}_\alpha = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \right) = \frac{\partial \mathcal{L}}{\partial q_\alpha}, \frac{\partial H}{\partial q_\alpha} = -\frac{\partial \mathcal{L}}{\partial q_\alpha} \implies \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \\
dH &= \frac{\partial H}{\partial t} dt + \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp = d(p\dot{q} - \mathcal{L}(q, \dot{q}; t)) \\
&= (dp)\dot{q} + p d\dot{q} - \frac{\partial \mathcal{L}}{\partial t} dt - \frac{\partial \mathcal{L}}{\partial q} dq - \frac{\partial \mathcal{L}}{\partial \dot{q}} d\dot{q} = \dot{q} dp - \dot{p} dq - \frac{\partial \mathcal{L}}{\partial t} dt \\
&\implies \dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q} \\
0 \equiv \delta S &= \delta \int (p\dot{q} - H(q, p; t)) dt \implies \dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q}
\end{aligned}$$

1.4.3 Poisson Bracket (Classical Canonical Commutator)

$$[A, B]_{\text{PB}} = \sum_{\alpha} \left(\frac{\partial A}{\partial q_{\alpha}} \frac{\partial B}{\partial p_{\alpha}} - \frac{\partial A}{\partial p_{\alpha}} \frac{\partial B}{\partial q_{\alpha}} \right)$$

$$\Rightarrow \text{Quantum Canonical Commutator } [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$\forall f(q, p; t), \quad \frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{\alpha} \left(\frac{\partial f}{\partial q_{\alpha}} \dot{q}_{\alpha} + \frac{\partial f}{\partial p_{\alpha}} \dot{p}_{\alpha} \right) = \frac{\partial f}{\partial t} + \sum_{\alpha} \left(\frac{\partial f}{\partial q_{\alpha}} \frac{\partial H}{\partial p_{\alpha}} - \frac{\partial f}{\partial p_{\alpha}} \frac{\partial H}{\partial q_{\alpha}} \right) = \frac{\partial f}{\partial t} + [f, H]_{\text{PB}}$$

$$\text{if } f(q, p; t) = q_{\alpha} \Rightarrow \frac{dq_{\alpha}}{dt} = \frac{\partial H}{\partial p_{\alpha}}, \text{ if } f(q, p; t) = p_{\alpha} \Rightarrow \frac{dp_{\alpha}}{dt} = -\frac{\partial H}{\partial q_{\alpha}}$$

1.4.4 Fundamental Commutation Relations in CM

$$[q_{\alpha}, p_{\beta}]_{\text{PB}} = \delta_{\alpha\beta} \longleftrightarrow [\hat{q}_{\alpha}, \hat{p}_{\beta}] = i\hbar\delta_{\alpha\beta}$$

$$[q_{\alpha}, q_{\beta}]_{\text{PB}} = [p_{\alpha}, p_{\beta}]_{\text{PB}} \equiv 0 \longleftrightarrow [\hat{q}_{\alpha}, \hat{q}_{\beta}] = [\hat{p}_{\alpha}, \hat{p}_{\beta}] \equiv 0$$

$$[q_{\alpha}, p_{\beta}] = \sum_{\gamma} \left(\frac{\partial q_{\alpha}}{\partial q_{\gamma}} \frac{\partial p_{\beta}}{\partial p_{\gamma}} - \frac{\partial q_{\alpha}}{\partial p_{\gamma}} \frac{\partial p_{\beta}}{\partial q_{\gamma}} \right) = \delta_{\alpha\gamma} \delta_{\beta\gamma} - 0 = \delta_{\alpha\beta}$$

$$[q_{\alpha}, q_{\beta}] = \sum_{\gamma} \left(\frac{\partial q_{\alpha}}{\partial q_{\gamma}} \frac{\partial q_{\beta}}{\partial p_{\gamma}} - \frac{\partial q_{\alpha}}{\partial p_{\gamma}} \frac{\partial q_{\beta}}{\partial q_{\gamma}} \right) = \delta_{\alpha\gamma} \cdot 0 - 0 \cdot \delta_{\beta\gamma} = 0$$

1.4.5 Properties of Poisson Bracket

1. $[A, B]_{\text{PB}} = -[B, A]_{\text{PB}}$
2. $[A + B, C]_{\text{PB}} = [A, C]_{\text{PB}} + [B, C]_{\text{PB}}, [A, B + C]_{\text{PB}} = [A, B]_{\text{PB}} + [A, C]_{\text{PB}}$
3. $[A, BC]_{\text{PB}} = [A, B]_{\text{PB}}C + B[A, C]_{\text{PB}}, [AB, C]_{\text{PB}} = [A, C]_{\text{PB}}B + A[B, C]_{\text{PB}}$
4. $[A, B^n]_{\text{PB}} = n[A, B]_{\text{PB}}B^{n-1}$
5. $[A, f(B)]_{\text{PB}} = [A, B]_{\text{PB}} \frac{\partial f}{\partial B}$
6. $[L_i, L_j]_{\text{PB}} = \epsilon_{ijk} L_k$
7. $[L_i, \mathbf{L}^2]_{\text{PB}} = 0$
8. $[L_i, x_j]_{\text{PB}} = \epsilon_{ijk} x_k$
9. $[L_i, p_j]_{\text{PB}} = \epsilon_{ijk} p_k$

$$\begin{aligned}
[L_i, L_j]_{\text{PB}} &= [\epsilon_{iab}x_ap_b, \epsilon_{jcd}x_cp_d]_{\text{PB}} = \epsilon_{iab}\epsilon_{jcd} \left(x_a \underbrace{[p_b, x_c]_{\text{PB}}}_{=-\delta_{bc}} p_d + x_c \underbrace{[x_a, p_d]_{\text{PB}}}_{=\delta_{ad}} p_b \right) \\
&= \epsilon_{iab}\epsilon_{jca}x_cp_b - \epsilon_{iab}\epsilon_{jbd}x_ap_d = (\delta_{bj}\delta_{ic} - \delta_{bc}\delta_{ij})x_cp_b - (\delta_{ij}\delta_{ad} - \delta_{id}\delta_{aj})x_ap_d \\
&= x_ip_j - \delta_{ij}(\mathbf{x} \cdot \mathbf{p}) + \delta_{ij}(\mathbf{x} \cdot \mathbf{p}) - x_jp_i = x_ip_j - x_jp_i = \epsilon_{ijk}L_k
\end{aligned}$$

1.4.6 Hamilton-Jacobi Equation

$$\boxed{\frac{\partial S}{\partial t} + H(q, \frac{\partial H}{\partial q}; t) = 0}$$

$$\begin{aligned}
\delta S &= \int_{t_0}^t \mathcal{L}(q, \dot{q}; t) dt = \int_{t_0}^t \left[\frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \delta q \right] \\
&= \int_{t_0}^t \underbrace{\left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right)}_{=0 \text{ (physical path)}} \delta q dt + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q(t) - \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q(t_0)}_{=0 \text{ (fixed)}} = \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q = p \delta q
\end{aligned}$$

$$\begin{cases} \delta S = p \delta q \\ \delta S[q(t), t] = \frac{\partial S}{\partial q} \delta q \end{cases} \implies p = \frac{\partial S}{\partial q}, \begin{cases} \frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q} \dot{q} \\ \frac{dS}{dt} = \mathcal{L} = p\dot{q} - H \end{cases} \implies \frac{\partial S}{\partial t} + H = 0$$

1.4.7 Phase Space (Dynamics)

phase point $(q, p) \longrightarrow$ state

$$\int \rho(q, p; t) d^D q d^D p = 1$$

ρ : probability

$$\boxed{\frac{\partial \rho}{\partial t} = [H, \rho]_{\text{PB}}} \text{ Liouville Theorem}$$

$$0 = \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + [\rho, H]_{\text{PB}} \xrightarrow{QM} \frac{\partial \hat{\rho}}{\partial t} = \frac{1}{i\hbar} [\hat{H}, \hat{\rho}]$$

1.4.8 Regular Lagrangian

$$\left| \frac{\partial^2 \mathcal{L}}{\partial q_i \partial q_j} \right| \neq 0 \implies \begin{cases} \mathcal{L} = T - V \\ H = T + V \end{cases}$$

2 Classical Electrodynamics

2.1 Vector Analysis

$$\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3 = \sum_i v_i \mathbf{e}_i = v_i \mathbf{e}_i \text{ (Einstein convention), } i = 1, 2, 3$$

2.1.1 Generalized Stokes' Theorem

$$\boxed{\int_{\Omega} d\omega = \int_{\partial\Omega} \omega}$$

1. Newton-Leibniz formula

$$\int_a^b df = f(b) - f(a)$$

2. Green's theorem

$$\oint f dx + g dy = \iint \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy$$

3. Stokes' theorem

$$\begin{aligned} \oint \mathbf{F} \cdot d\mathbf{l} &= \iint (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \\ \Rightarrow \oint \psi d\mathbf{l} &= \iint d\mathbf{S} \times \nabla \psi \end{aligned}$$

4. Gauss-Ostrogradsky's theorem

$$\begin{aligned} \oiint \mathbf{F} \cdot d\mathbf{S} &= \iiint (\nabla \cdot \mathbf{F}) dV \\ \Rightarrow \begin{cases} \oiint \psi d\mathbf{S} = \iiint (\nabla \psi) dV \\ \oiint d\mathbf{S} \nabla \mathbf{A} = \iiint (\nabla \times \mathbf{A}) dV \end{cases} \end{aligned}$$

2.1.2 Helmholtz's theorem

For all continuous differentiable \mathbf{F} ,

$$\boxed{\mathbf{F} = \mathbf{F}_{\perp} + \mathbf{F}_{\parallel}}$$

1. transverse component

$$\nabla \cdot \mathbf{F}_{\perp} = 0 \quad \& \quad \mathbf{F}_{\perp} = \nabla \times \mathbf{A}$$

2. longitudinal component

$$\nabla \times \mathbf{F}_{\parallel} = 0 \quad \& \quad \mathbf{F}_{\parallel} = -\nabla \phi$$

For magnetic field \mathbf{B}

$$\nabla \cdot \mathbf{B} \equiv 0$$

For electric field \mathbf{E}

$$\begin{cases} \nabla \times \mathbf{E} = 0 & \text{for static electric field} \\ \nabla \times \mathbf{E} \neq 0 & \text{for AC electric field} \end{cases}$$

2.2 Experimental Laws and Maxwell's Equations

2.2.1 Coulomb's Law

$$\begin{aligned} \mathbf{E}(\mathbf{x}) &= \int \frac{\rho(\mathbf{x}')(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} dV' = -\nabla\phi, \quad \phi = \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} dV' \\ \implies \boxed{\nabla \cdot \mathbf{E} = 4\pi\rho(\mathbf{x})} & \quad \text{Gauss' Law} \end{aligned}$$

2.2.2 Biot-Savart's Law

$$\mathbf{B} = \int \frac{\mathbf{j}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} dV' = \nabla \times \mathbf{A}(\mathbf{x})$$

3 Fundamental Postulates (Axioms)

4 SOME IS

1. State
2. Observable (Operation)
3. Measurement
4. Evolution
5. Identical Particals and Symmetrization

4.1 State

system \longleftrightarrow state vector

ket: $|\psi\rangle$

bra: $\langle\psi|$

$$(|\psi\rangle)^\dagger = \langle\psi|$$

$$[(|\psi\rangle)^\dagger]^\dagger = (\langle\psi|)^\dagger = |\psi\rangle$$

\dagger : adjoint or Hermitian conjugate

4.2 Observable

physical quantity \longleftrightarrow observable \longleftrightarrow operation $\hat{A} \begin{cases} \text{linear} \\ \text{Hermitian} \end{cases}$

4.3 Measurement

4.3.1 Measurement + Outcomes

$$\hat{A} |\psi\rangle = A_n |\psi_n\rangle \quad (\text{Eigenvalue equation})$$

\hat{A} : operation (q-number); A_n : eigenvalue (c-number); ψ_n : eigenstate

4.3.2 Probability of Measurement Outcomes

$$\begin{aligned} |\psi\rangle &= \sum_n c_n |\psi_n\rangle, \quad c_n = \langle \psi_n | \psi \rangle \\ P(A_n) &= |c_n|^2 = |\langle \psi_n | \psi \rangle|^2 \\ \langle \hat{A} \rangle &= \langle \psi | \hat{A} | \psi \rangle = \sum_n P(A_n) A_n = \sum_n |\langle \psi_n | \psi \rangle|^2 A_n \end{aligned}$$

4.3.3 State Collapse

$$|\psi\rangle \xrightarrow{\hat{A}\text{-measurement}} |\psi_n\rangle$$

4.4 Evolution (Dynamics)

4.4.1 Schrödinger Picture

state: change; observable: no change

$$\begin{cases} i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \\ |\psi(t=0)\rangle = |\psi_0\rangle \end{cases} \quad \text{Schrödinger Equation}$$

where $\hat{H} = \hat{T} + \hat{V} = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}(\hat{\mathbf{x}})$

representation: coordinate system

$\hat{\mathbf{x}}$ -representation (position):

$$\hat{\mathbf{x}} |\mathbf{x}\rangle = \mathbf{x} |\mathbf{x}\rangle, \quad |\psi\rangle \longrightarrow \langle \mathbf{x} | \psi \rangle = \psi(\mathbf{x}) \rightarrow \text{wave-function}$$

$$\implies \boxed{i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t) = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{x}, t) + \hat{V}(\mathbf{x}, t) \psi(\mathbf{x}, t)}$$

$\hat{\mathbf{p}}$ -representation (momentum): $\hat{\mathbf{p}} |\mathbf{p}\rangle = \mathbf{p} |\mathbf{p}\rangle$

4.4.2 Heisenberg Picture

state: no change; observable: change

$$\boxed{\frac{d\hat{F}}{dt} = \frac{\partial \hat{F}}{\partial t} + \frac{1}{i\hbar} [\hat{F}, \hat{H}]} \quad \text{Heisenberg Equation}$$

4.5 Identical Particles and Symmetrization

4.5.1 Identical Particles

$$\begin{cases} \text{boson: } s = 0, 1, 2, \dots & \text{Bose-Einstein Statistics} \\ \text{fermion: } s = \frac{1}{2}, \frac{3}{2}, \dots & \text{Fermi-Dirac Statistics} \end{cases} \xrightarrow[\text{reduce}]{\text{classical}} \text{Maxwell-Boltzmann Statistics}$$

4.5.2 Symmetrization

$$\begin{cases} \text{Boson: symmetric} \\ \text{Femion: anti-symmetric} \end{cases}$$

5 State Vector

5.1 Euclidean Space \mathbb{R}^3

$$\begin{aligned} \mathbf{v} &= v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3 \\ \mathbf{e}_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \mathbf{v} &= \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \mathbf{v}^T = (v_1 \quad v_2 \quad v_3) \end{aligned}$$

1. vector addition: $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$
2. scalar multiplication: $\begin{cases} a(\mathbf{v}_1 + \mathbf{v}_2) = a\mathbf{v}_1 + a\mathbf{v}_2 \\ (a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v} \end{cases} \quad \forall a, b \in \mathbb{R}$
3. inner product (dot product): $\mathbf{v}_1^T \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot \mathbf{v}_1 = \mathbf{v}_2^T \mathbf{v}_1$
4. dyadic: $\mathbf{v}_1 \mathbf{v}_2 \neq \mathbf{v}_2 \mathbf{v}_1$
5. basis: $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \longrightarrow \text{orthonormal} \quad \mathbf{e}_i \mathbf{e}_i = \mathbf{I} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$
6. $\mathbf{v}_1 \cdot \mathbf{v}_2 = r \in \mathbb{R}$
 $\mathbf{v} \cdot \mathbf{v} = r \in \mathbb{R}, r \geq 0$ (equality iff $\mathbf{v} = \mathbf{0}$)

5.2 State Vector Space \mathbb{H} (Hilbert Space)

$$|\psi\rangle = \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix}, \quad \langle\psi| = (\cdots \quad \cdots \quad \cdots)$$

1. vector addition: $|\psi_1\rangle + |\psi_2\rangle = |\psi_2\rangle + |\psi_1\rangle$
2. scalar multiplication: $\begin{cases} c(|\psi_1\rangle + |\psi_2\rangle) = c|\psi_1\rangle + c|\psi_2\rangle \\ (c + c')|\psi\rangle = c|\psi\rangle + c'|\psi\rangle \end{cases} \quad \forall c \in \mathbb{C}$
3. inner product: $\langle\psi_1|\psi_2\rangle = (\langle\psi_2|\psi_1\rangle)^*$
4. outer product: $|\psi\rangle\langle\phi| \neq |\phi\rangle\langle\psi|$

$$(|\psi\rangle\langle\phi|)^\dagger = (\langle\phi|)^\dagger(|\psi\rangle)^\dagger = |\phi\rangle\langle\psi|$$

5. basis: for Hermitian operation $\begin{cases} \hat{A}|\psi_n\rangle = A_n|\psi_n\rangle & \text{discrete} \\ \hat{A}|a\rangle = a|a\rangle & \text{continuous} \end{cases}$

$$\text{discrete: } \begin{cases} \langle\psi_n|\psi_m\rangle = \delta_{nm} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases} \\ \sum_n |\psi_n\rangle\langle\psi_n| = \hat{I} \end{cases}$$

$$\text{continuous: } \begin{cases} \langle a|a'\rangle = \delta(a - a') = \begin{cases} \infty, & a = a' \\ 0, & a \neq a' \end{cases} \\ \int |a\rangle\langle a| da = \hat{I} \end{cases}$$

6. $\langle\psi_1|\psi_2\rangle = c \in \mathbb{C}$
 $(\langle\psi_1|\psi_2\rangle)^* = \langle\psi_2|\psi_1\rangle$
 $\langle\psi|\psi\rangle = \|\psi\|^2 = r \in \mathbb{R}, r \geq 0$ (equality iff $|\psi\rangle = \mathbf{0}$)

6 Operator

$$\hat{A}|\phi\rangle = |\psi\rangle, \quad \hat{A}^{-1}|\psi\rangle = |\phi\rangle$$

6.1 Linear Operator

$$\forall |\psi_1\rangle, |\psi_2\rangle \in \mathbb{H}, a, b \in \mathbb{C}, \hat{A}(a|\psi_1\rangle + b|\psi_2\rangle) = a\hat{A}|\psi_1\rangle + b\hat{A}|\psi_2\rangle$$

then, \hat{A} is a linear operator

$$\hat{A}(a|\psi_1\rangle + b|\psi_2\rangle) = a^*\hat{A}|\psi_1\rangle + b^*\hat{A}|\psi_2\rangle$$

then, \hat{A} is an anti-linear operation

6.1.1 Two Kinds of Linear Operators

$$\begin{cases} \text{Hermitian operator: } \hat{A}^\dagger = \hat{A} \\ \text{Unitary operator: } \hat{U}^\dagger = \hat{U}^{-1} \end{cases}$$

6.1.2 Rules

$$\forall |\psi\rangle, \text{ if } \hat{A}|\psi\rangle = \hat{B}|\psi\rangle \iff \hat{A} = \hat{B}$$

$$1. \hat{A} + \hat{B} = \hat{B} + \hat{A}$$

$$\text{def: } \forall |\psi\rangle, \hat{A}|\psi\rangle + \hat{B}|\psi\rangle = (\hat{A} + \hat{B})|\psi\rangle$$

$$2. (\hat{A} + \hat{B}) + \hat{C} = \hat{A} + (\hat{B} + \hat{C})$$

$$3. \hat{A}\hat{B} \neq \hat{B}\hat{A}$$

$$\text{def commutator: } [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \longleftrightarrow [A, B]_{\text{PB}} \longleftrightarrow \frac{1}{i\hbar} [\hat{A}, \hat{B}]$$

$$4. \text{ adjoint (Hermitian conjugate): } (\hat{A}|\phi\rangle)^\dagger = \langle\phi| \hat{A}^\dagger \iff \langle\phi| \hat{A}^\dagger |\psi\rangle = (\langle\psi| \hat{A} |\phi\rangle)^* \\ \implies \text{Hermitian operation: } \hat{A}^\dagger = \hat{A} \text{ (Hermitian matrix)} \implies A_{ij} = A_{ji}^*$$

6.1.3 Properties

$$1. \text{ Identity: } \forall |\psi\rangle, \exists \hat{I}, \hat{I}|\psi\rangle = |\psi\rangle$$

$$2. \text{ Inverse: } \hat{A}|\psi\rangle = |\phi\rangle \implies |\psi\rangle = \hat{A}^{-1}|\phi\rangle, \hat{A}\hat{A}^{-1} = \hat{A}^{-1}\hat{A} = \hat{I} \\ (\hat{A}_1\hat{A}_2 \cdots \hat{A}_n)^{-1} = \hat{A}_n^{-1} \cdots \hat{A}_2^{-1}\hat{A}_1^{-1}$$

$$3. (\hat{A}_1\hat{A}_2 \cdots \hat{A}_n)^\dagger = \hat{A}_n^\dagger \cdots \hat{A}_2^\dagger \hat{A}_1^\dagger$$

7 Overview of QM

ROSE P... M

8 Roles: State Vector, Operator

8.0.1 Operator

$$\begin{cases} \text{Observable} \\ \text{Transformation} \end{cases} \quad \begin{cases} \text{Hermitian: } \hat{A}^\dagger = \hat{A} \\ \text{Unitary: } \hat{A}^\dagger = \hat{A}^{-1} \end{cases}$$

8.0.2 State

$$\begin{cases} \text{ket } |\psi\rangle \text{ (vector): } (|\psi\rangle)^\dagger = \langle\psi| \\ \text{bra } \langle\psi| \text{ (dual vector): } (\langle\psi|)^\dagger = |\psi\rangle \end{cases}$$

8.0.3 Eigenspectrum

$$\begin{cases} \text{discrete (bound state)} \\ \text{continuous (unbound state)} \end{cases}$$

8.1 Properties

Wave Particle Duality

8.2 Problems

8.2.1 Eigenvalue Problem

$$\hat{A}|\psi\rangle = \lambda|\psi\rangle \begin{cases} \hat{A}|u_n\rangle = A_n|u_n\rangle \begin{cases} \langle u_n|u_m\rangle = \delta_{nm} \\ \sum_n |u_n\rangle\langle u_n| = \hat{I} \end{cases} \\ \hat{A}|a\rangle = a|a\rangle \begin{cases} \langle a|a'\rangle = \delta(a-a') \\ \int da |a\rangle\langle a| = \hat{I} \end{cases} \end{cases}$$

8.2.2 Evolution Problem

$$\begin{cases} i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \text{ Schrödinger eq.} \\ \frac{d\hat{F}}{dt} = \frac{\partial \hat{F}}{\partial t} + \frac{1}{i\hbar} [\hat{F}, \hat{H}] \text{ Heisenberg eq.} \end{cases}$$

8.3 Picture

8.3.1 Schrödinger Picture

$$\frac{d\hat{F}}{dt} = \frac{\partial \hat{F}}{\partial t} \equiv 0$$

8.3.2 Heisenberg Picture

$$\frac{d}{dt} |\psi(t)\rangle \equiv 0$$

8.4 Particles

$$\begin{cases} \text{Boson: } s=\text{integer} \longrightarrow \text{Bose-Einstein Statistics} \\ \text{Femion: } s=\text{half-integer} \longrightarrow \text{Fermi-Dirac Statistics} \end{cases} \xrightarrow{\hbar \rightarrow 0} \text{Maxwell-Boltzmann Statistics}$$

8.5 Perturbation Theories

$$\begin{cases} \text{Non-degenerate} \\ \text{Degenerate} \end{cases}$$

8.6 Angular Momentum and Magnetic Moment

$$\begin{cases} \text{Orbital Angular Momentum} \\ \text{Spin Angular Momentum} \end{cases} \Longleftrightarrow \begin{cases} \text{Orbital Magnetic Moment} \\ \text{Intrinsic Magnetic Moment} \end{cases}$$

8.7 Mechanics

$$\begin{cases} \text{Wave Mechanics: wavefunction} \longleftrightarrow \text{state vector} \\ \text{Matrix Mechanics: matrix forms of} \begin{cases} \text{state vector} \\ \text{operator: } n \times n \text{ matrix} \end{cases} \end{cases}$$

8.8 Theorem of Hermitian Operator

$$\hat{A}^\dagger = \hat{A}, \quad \hat{A} |\psi_n\rangle = A_n |\psi_n\rangle \quad (\text{non-degenerate, discrete})$$

1. eigenvalue $A_n \in \mathbb{R}$
2. eigenstate \longrightarrow orthonormal and complete basis

$$\begin{aligned} \langle \psi_n | \hat{A}^\dagger | \psi_n \rangle &= (\langle \psi_n | \hat{A} | \psi_n \rangle)^* = A_n^* \langle \psi_n | \psi_n \rangle = \langle \psi_n | \hat{A} | \psi_n \rangle = A_n \langle \psi_n | \psi_n \rangle \\ &\implies A_n^* = A_n \implies A_n \in \mathbb{R} \end{aligned}$$

$$\begin{cases} \hat{A} |\psi_n\rangle = A_n |\psi_n\rangle \\ \hat{A} |\psi_m\rangle = A_m |\psi_m\rangle \end{cases} \implies \langle \psi_n | \hat{A}^\dagger | \psi_m \rangle = (\langle \psi_m | \hat{A} | \psi_n \rangle)^* = A_n^* (\langle \psi_m | \psi_n \rangle)^* = A_n^* \langle \psi_n | \psi_m \rangle$$

$$\begin{aligned} \langle \psi_n | \hat{A} | \psi_m \rangle &= A_m \langle \psi_n | \psi_m \rangle \implies (A_n - A_m) \langle \psi_n | \psi_m \rangle = 0 \\ n \neq m &\implies A_n \neq A_m \implies \langle \psi_n | \psi_m \rangle = 0 \implies |\psi_n\rangle \perp |\psi_m\rangle \end{aligned}$$

Remark: Degenerate,

$$\hat{A} |\psi_n^i\rangle = A_n |\psi_n^i\rangle \quad (i = 1, 2, \dots, \underset{\text{degeneracy}}{g_n}) \implies |\psi_n^i\rangle \neq c |\psi_n^j\rangle \quad (\text{linear independent})$$

$$|\psi_n^i\rangle \xrightarrow{\text{Gram-Schmidt Orthogonalization}} |\phi_n^i\rangle \left\{ \begin{array}{l} \langle \phi_n^i | \phi_m^j \rangle = \delta_{nm} \delta_{ij} \\ \sum_{n,i} |\phi_n^i\rangle \langle \phi_n^i| = \hat{1} \end{array} \right.$$

9 Wave Mechanics & Matrix Mechanics

9.1 Wave Mechanics: wavefunction

$$\text{wavefunction} \left\{ \begin{array}{l} \text{matter wave (Schrödinger)} \\ \text{probabilistic wave (Born): } |\Psi(\mathbf{x}, t)|^2 \sim \text{probability density} \end{array} \right.$$

wavefunction is the projection of state vector in representation
basis

$$|\psi\rangle = \hat{1} |\psi\rangle = \left\{ \begin{array}{l} \left(\sum_n |u_n\rangle \langle u_n| \right) |\psi\rangle = \sum_n c_n |u_n\rangle, \quad c_n = \langle u_n | \psi \rangle \quad \text{discrete} \\ \left(\int da |a\rangle \langle a| \right) |\psi\rangle = \int da \psi(a) |a\rangle, \quad \psi(a) = \langle a | \psi \rangle \quad \text{continuous} \end{array} \right.$$

c_n is the wavefunction in $\{|u_n\rangle\}$ -representation, $\psi(a)$ is the wavefunction in $\{|a\rangle\}$ -representation.

$$|\psi\rangle = \sum_n c_n |\psi_n\rangle, \quad c_n = \langle \psi_n | \psi \rangle, \quad \hat{A} |\psi\rangle = A_n |\psi_n\rangle$$

$$\text{continuous} \left\{ \begin{array}{l} \mathbf{x}\text{-representation: } \hat{\mathbf{x}} |\mathbf{x}\rangle = \mathbf{x} |\mathbf{x}\rangle \left\{ \begin{array}{l} \langle \mathbf{x} | \mathbf{x}' \rangle = \delta(\mathbf{x} - \mathbf{x}') \\ \int d^3x |\mathbf{x}\rangle \langle \mathbf{x}| = \hat{1} \end{array} \right. \\ \mathbf{p}\text{-representation: } \hat{\mathbf{p}} |\mathbf{p}\rangle = \mathbf{p} |\mathbf{p}\rangle \left\{ \begin{array}{l} \langle \mathbf{p} | \mathbf{p}' \rangle = \delta(\mathbf{p} - \mathbf{p}') \\ \int d^3p |\mathbf{p}\rangle \langle \mathbf{p}| = \hat{1} \end{array} \right. \end{array} \right.$$

Energy-representation, Fork-state representation (occupation number representation)

$$\begin{aligned} |\psi\rangle &= \left(\int d^3x |\mathbf{x}\rangle \langle \mathbf{x}| \right) |\psi\rangle = \int d^3x \psi(\mathbf{x}) |\mathbf{x}\rangle \\ &= \left(\int d^3p |\mathbf{p}\rangle \langle \mathbf{p}| \right) |\psi\rangle = \int d^3p \tilde{\psi}(\mathbf{p}) |\mathbf{p}\rangle \end{aligned}$$

$$\psi(\mathbf{x}) = \langle \mathbf{x} | \psi \rangle \quad \tilde{\psi}(\mathbf{p}) = \langle \mathbf{p} | \psi \rangle$$

$$\begin{aligned} \tilde{\psi}(\mathbf{p}) &= \langle \mathbf{p} | \psi \rangle = \langle \mathbf{p} | \hat{1} | \psi \rangle = \langle \mathbf{p} | \left(\int d^3x |\mathbf{x}\rangle \langle \mathbf{x}| \right) | \psi \rangle = \int d^3x \langle \mathbf{p} | \mathbf{x} \rangle \psi(\mathbf{x}) \\ &= \int d^3x \psi(\mathbf{x}) \psi_{\mathbf{p}}^*(\mathbf{x}) \quad \text{where } \psi_{\mathbf{p}}(\mathbf{x}) = \langle \mathbf{x} | \mathbf{p} \rangle = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar} \\ &\implies \text{in } \mathbf{x}\text{-representation, } \hat{\mathbf{x}} = \mathbf{x}, \quad \hat{\mathbf{p}} = -i\hbar \frac{\partial}{\partial \mathbf{x}} \end{aligned}$$

$$\implies \begin{cases} i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle & \xrightarrow{\int d^3x |\mathbf{x}\rangle \langle \mathbf{x}| = \hat{1}} i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{x}, t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(\mathbf{x}, t) + V(\mathbf{x}) \Psi(\mathbf{x}, t) \\ \hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{x}}) \end{cases}$$

$$\text{where } \begin{cases} \Psi(\mathbf{x}, t) = \langle \mathbf{x} | \psi(t) \rangle \\ \hat{\mathbf{x}} = \mathbf{x}, \quad \hat{\mathbf{p}} = -i\hbar \nabla \end{cases}$$

$$\text{in } \mathbf{p}\text{-representation, } \hat{\mathbf{x}} = +i\hbar \frac{\partial}{\partial \mathbf{p}}, \quad \hat{\mathbf{p}} = \mathbf{p}$$

$$\text{Schrödinger eq. } \xrightarrow{\int d^3p |\mathbf{p}\rangle \langle \mathbf{p}| = \hat{1}} i\hbar \frac{\partial}{\partial t} \tilde{\psi}(\mathbf{p}, t) = \frac{\mathbf{p}^2}{2m} \tilde{\psi}(\mathbf{p}, t) + V \left(i\hbar \frac{\partial}{\partial \mathbf{p}} \right) \tilde{\psi}(\mathbf{p}, t)$$

$$\text{where } \begin{cases} \tilde{\psi}(\mathbf{p}) = \langle \mathbf{p} | \psi(t) \rangle \\ \hat{\mathbf{p}} = \mathbf{p}, \quad \hat{\mathbf{x}} = +i\hbar \frac{\partial}{\partial \mathbf{p}} \end{cases}$$

9.2 Matrix Mechanics: State Vector

$$|\psi\rangle = \hat{1} |\psi\rangle = \sum_n c_n |u_n\rangle = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ \vdots \end{pmatrix} = (c_1 \quad c_2 \quad \cdots \quad c_n \quad \cdots)^T$$

$$\hat{A} = \mathbf{I} \hat{A} \mathbf{I} = \left(\sum_n |u_n\rangle \langle u_n| \right) \hat{A} \left(\sum_m |u_m\rangle \langle u_m| \right) = \sum_{n,m} \underbrace{\langle u_n | \hat{A} | u_m \rangle}_{=A_{nm}} (|u_n\rangle \langle u_m|) = \begin{pmatrix} & A_{nm} \end{pmatrix}$$

9.3 Matrix Mechanics: Hermitian Conjugate

if the operator \hat{A} can be written as a matrix,

$$\langle \psi | \hat{A}^\dagger | \phi \rangle = \left(\langle \phi | \hat{A} | \psi \rangle \right)^* \implies \hat{A}^\dagger = \left(\hat{A}^T \right)^*$$

$$\text{Pf. Let } \begin{cases} |\psi\rangle = \sum_n c_n |u_n\rangle \\ |\phi\rangle = \sum_n d_n |u_n\rangle \end{cases},$$

$$\begin{aligned} \langle\psi|\hat{A}^\dagger|\phi\rangle &= \left(\sum_n c_n^* \langle u_n|\right) \hat{A}^\dagger \left(\sum_m d_m |u_m\rangle\right) = \sum_{n,m} c_n^* d_m \langle u_n|\hat{A}^\dagger|u_m\rangle = \sum_{n,m} c_n^* d_m (\hat{A}^\dagger)_{nm} \\ (\langle\phi|\hat{A}|\psi\rangle)^* &= \left[\left(\sum_n d_n^* \langle u_n|\right) \hat{A} \left(\sum_m c_m |u_m\rangle\right)\right]^* = \left(\sum_{n,m} d_n^* c_m \langle u_n|\hat{A}|u_m\rangle\right)^* \\ &= \sum_{n,m} d_n c_m^* A_{nm}^* = \sum_{n,m} d_m c_n^* A_{mn}^* = \sum_{n,m} c_n^* d_m A_{mn}^* \implies (\hat{A}^\dagger)_{nm} = (\hat{A})_{mn}^* = (\hat{A}^T)_{nm}^* \end{aligned}$$

10 Commutation

10.0.1 Definition

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \quad (1)$$

10.0.2 Compare

$$\begin{cases} \text{q-number : } \hat{A}\hat{B} \neq \hat{B}\hat{A} & (\text{QM}) \\ \text{c-number : } AB = BA & (\text{CM}) \end{cases}$$

$$[A, B]_{\text{PB}} = \sum_{\alpha} \left(\frac{\partial A}{\partial q_{\alpha}} \frac{\partial B}{\partial p_{\alpha}} - \frac{\partial A}{\partial p_{\alpha}} \frac{\partial B}{\partial q_{\alpha}} \right) \quad (2)$$

Classical Canonical Commutator

10.0.3 Relation

$$\boxed{[A, B]_{\text{PB}} \longleftrightarrow \frac{1}{i\hbar} [\hat{A}, \hat{B}]} \quad (3)$$

11 Properties of Commutations

$$[\hat{A}, \hat{B}] \neq 0 \implies \text{Uncertainty Relation}$$

$$\Delta\hat{A} \cdot \Delta\hat{B} \geq \frac{|\langle [\hat{A}, \hat{B}] \rangle|}{2} \quad (4)$$

$$[\hat{A}, \hat{B}] = 0 \implies \begin{matrix} \text{Common} \\ \text{Simultaneous} \end{matrix} \text{ Eigenstates}$$

1. **Linearity**

$$[a\hat{A}, b\hat{B} + c\hat{C}] = ab[\hat{A}, \hat{B}] + ac[\hat{A}, \hat{C}] \quad (5)$$

2. **Leibniz rule**

$$\begin{cases} [\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}] \\ [\hat{A}\hat{B}, \hat{C}] = [\hat{A}, \hat{C}]\hat{B} + \hat{A}[\hat{B}, \hat{C}] \end{cases} \quad (6)$$

3. **Power related**

$$[\hat{A}, \hat{B}^n] = n[\hat{A}, \hat{B}]\hat{B}^{n-1} \quad (7)$$

provided that $[\hat{B}, [\hat{A}, \hat{B}]] = 0$

Pf. $[\hat{A}, \hat{B}] = 1[\hat{A}, \hat{B}]\hat{B}^{1-1}$ ($n = 1$). Suppose $[\hat{A}, \hat{B}^k] = k[\hat{A}, \hat{B}]\hat{B}^{k-1}$, then when $n = k + 1$, one has $[\hat{A}, \hat{B}^{k+1}] = [\hat{A}, \hat{B}^k]\hat{B} + \hat{B}^k[\hat{A}, \hat{B}] = (k + 1)[\hat{A}, \hat{B}]\hat{B}^k \Rightarrow [\hat{A}, \hat{B}^n] = n[\hat{A}, \hat{B}]\hat{B}^{n-1}$

4.

$$[\hat{A}, f(\hat{B})] = [\hat{A}, \hat{B}] \frac{\partial f}{\partial \hat{B}} \quad (8)$$

provided that $[\hat{B}, [\hat{A}, \hat{B}]] = 0$

$$\begin{aligned} \text{Pf. } [\hat{A}, f(\hat{B})] &= \left[\hat{A}, \sum_n \frac{f^{(n)}(0)}{n!} \hat{B}^n \right] = \sum_n \frac{f^{(n)}(0)}{n!} [\hat{A}, \hat{B}^n] = \sum_n \frac{f^{(n)}(0)}{(n-1)!} \hat{B}^{n-1} [\hat{A}, \hat{B}] \\ &= \sum_n \frac{f^{(n)}(0)}{n!} [\hat{A}, \hat{B}] \hat{B}^n = [\hat{A}, \hat{B}] \frac{\partial f}{\partial \hat{B}} \end{aligned}$$

5. **Jacobi identity** \rightarrow Lie Algebra

$$[A, [B, C]_{\text{PB}}]_{\text{PB}} + [B, [C, A]_{\text{PB}}]_{\text{PB}} + [C, [A, B]_{\text{PB}}]_{\text{PB}} = 0 \quad (\text{CM}) \quad (9)$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0 \quad (\text{VA}) \quad (10)$$

$$\mathbf{F}_{\mu\nu, \lambda} + \mathbf{F}_{\nu\lambda, \mu} + \mathbf{F}_{\lambda\mu, \nu} = 0 \quad (\text{CED}) \quad (11)$$

$$[\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] = 0 \quad (\text{QM}) \quad (12)$$

12 Fundamental Commutation Relations

$$\text{CM} \begin{cases} [x_i, p_j]_{\text{PB}} = \delta_{ij} \\ [x_i, x_j]_{\text{PB}} = [p_i, p_j]_{\text{PB}} = 0 \end{cases} \longleftrightarrow \text{QM} \begin{cases} [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \\ [\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0 \end{cases} \quad (13)$$

Pf. in \mathbf{x} -representation, $\hat{\mathbf{x}} = \mathbf{x}$, $\hat{\mathbf{p}} = -i\hbar\nabla \implies \hat{x} = x$, $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$. $\forall\psi(x)$,

$$\begin{aligned} [\hat{x}, \hat{p}_x]\psi(x) &= (\hat{x}\hat{p}_x - \hat{p}_x\hat{x})\psi(x) = -i\hbar\left(x\frac{\partial}{\partial x} - \frac{\partial}{\partial x}x\right)\psi(x) = -i\hbar\left[x\frac{\partial\psi}{\partial x} - \frac{\partial}{\partial x}(x\psi)\right] \\ &= i\hbar\psi(x) \implies [\hat{x}, \hat{p}_x] = i\hbar. \end{aligned}$$

Similarly, $[\hat{y}, \hat{p}_y] = [\hat{z}, \hat{p}_z] = i\hbar$. $[\hat{x}_i, \hat{p}_j] = 0$, ($i \neq j$).

13 Commutation Relations of Orbital Angular Momentum

$$\hat{\mathbf{L}} = \hat{\mathbf{x}} \times \hat{\mathbf{p}} \iff \hat{L}_i = \varepsilon_{ijk}\hat{x}_j\hat{p}_k \quad (14)$$

$$\implies \boxed{[\hat{L}_i, \hat{L}_j] = i\hbar\varepsilon_{ijk}\hat{L}_k \iff \hat{\mathbf{L}} \times \hat{\mathbf{L}} = i\hbar\hat{\mathbf{L}}} \quad (15)$$

$$\implies [\hat{L}_i, \hat{\mathbf{L}}^2] = 0 \implies [\hat{L}_z, \hat{\mathbf{L}}^2] = 0, z: \text{quantum axis} \quad (16)$$

$$\begin{cases} [\hat{L}_i, \hat{x}_j] = i\hbar\varepsilon_{ijk}\hat{x}_k \iff \hat{\mathbf{L}} \times \hat{\mathbf{x}} + \hat{\mathbf{x}} \times \hat{\mathbf{L}} = 2i\hbar\hat{\mathbf{x}} \\ [\hat{L}_i, \hat{p}_j] = i\hbar\varepsilon_{ijk}\hat{p}_k \iff \hat{\mathbf{L}} \times \hat{\mathbf{p}} + \hat{\mathbf{p}} \times \hat{\mathbf{L}} = 2i\hbar\hat{\mathbf{p}} \end{cases} \quad (17)$$

$$\forall \text{ vector } \hat{\mathbf{v}}, [\hat{L}_i, \hat{v}_j] = i\hbar\varepsilon_{ijk}\hat{v}_k \iff \hat{\mathbf{L}} \times \hat{\mathbf{v}} + \hat{\mathbf{v}} \times \hat{\mathbf{L}} = 2i\hbar\hat{\mathbf{v}} \quad (18)$$

$$\text{for } V(\mathbf{x}) = V(r), [\hat{L}_i, V(r)] = 0 \quad (19)$$

$$\text{for } \hat{T} = \frac{\hat{\mathbf{p}}^2}{2m}, [\hat{L}_i, \hat{T}] = 0 \quad (20)$$

$$\text{for } \hat{H} = \hat{T} + V(r), [\hat{L}_i, \hat{H}] = [\hat{\mathbf{L}}^2, \hat{H}] = 0 \quad (21)$$

$$\text{Pf. } (\hat{\mathbf{L}} \times \hat{\mathbf{L}})_i = \varepsilon_{ijk}\hat{L}_j\hat{L}_k = \frac{1}{2}\varepsilon_{ijk}\hat{L}_j\hat{L}_k - \frac{1}{2}\varepsilon_{ijk}\hat{L}_k\hat{L}_j = \frac{1}{2}\varepsilon_{ijk}[\hat{L}_j, \hat{L}_k]$$

$$= \frac{1}{2}\varepsilon_{ijk}(i\hbar)\varepsilon_{jkl}\hat{L}_l = i\hbar\delta_{il}\hat{L}_l = i\hbar\hat{L}_i \implies \hat{\mathbf{L}} \times \hat{\mathbf{L}} = i\hbar\hat{\mathbf{L}}$$

$$[\hat{L}_i, \hat{L}_j] = i\hbar\varepsilon_{ijk}\hat{L}_k \implies [\hat{L}_i, \hat{\mathbf{L}}^2] = [\hat{L}_i, \hat{L}_j\hat{L}_j] = [\hat{L}_i, \hat{L}_j]\hat{L}_j + \hat{L}_j[\hat{L}_i, \hat{L}_j]$$

$$= i\hbar\varepsilon_{ijk}(\hat{L}_k\hat{L}_j + \hat{L}_j\hat{L}_k) = 0$$

$$[\hat{L}_i, \hat{x}_j] = [\varepsilon_{iab}\hat{x}_a\hat{p}_b, \hat{x}_j]$$

$$[\hat{L}_i, V(r)] = [\hat{L}_i, \hat{x}_j]\frac{\partial V}{\partial x_j} = i\hbar\varepsilon_{ijk}\hat{x}_k\frac{dV}{dr}\frac{\hat{x}_j}{r} = \left(\frac{i\hbar}{r}\frac{dV}{dr}\right)\varepsilon_{ijk}\hat{x}_k\hat{x}_j = 0$$

$$[\hat{L}_i, \hat{\mathbf{p}}^2] = [\hat{L}_i, \hat{p}_j\hat{p}_j] = [\hat{L}_i, \hat{p}_j]\hat{p}_j + \hat{p}_j[\hat{L}_i, \hat{p}_j] = i\hbar\varepsilon_{ijk}(\hat{p}_k\hat{p}_j + \hat{p}_j\hat{p}_k) = 0$$

14 Eigenvalue Problem

15 Observables

1. Position (canonical)

$$\hat{\mathbf{x}} = \begin{cases} \mathbf{x} & \mathbf{x}\text{-representation} \\ +i\hbar \frac{\partial}{\partial \mathbf{p}} & \mathbf{p}\text{-representation} \end{cases} \quad (22)$$

2. Momentum (canonical)

$$\hat{\mathbf{p}} = \begin{cases} -i\hbar \frac{\partial}{\partial \mathbf{x}} & \mathbf{x}\text{-representation} \\ \mathbf{p} & \mathbf{p}\text{-representation} \end{cases} \quad (23)$$

$$\text{Fundamental Commutation Relation } \begin{cases} [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij} \\ [\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0 \end{cases}$$

3. Orbital Angular Momentum

$$\hat{\mathbf{L}} = \hat{\mathbf{x}} \times \hat{\mathbf{p}} \text{ or } \hat{L}_i = \varepsilon_{ijk} \hat{x}_j \hat{p}_k$$

$$\text{Commutation Relations } [\hat{L}_i, \hat{L}_j] = i\hbar \varepsilon_{ijk} \hat{L}_k \implies [\hat{L}_i, \hat{\mathbf{L}}^2]$$

Remark $\hat{\mathbf{L}} \times \hat{\mathbf{L}} \neq 0$, but $\hat{\mathbf{x}} \times \hat{\mathbf{p}} = -\hat{\mathbf{p}} \times \hat{\mathbf{x}}$

4. Intrinsic Angular Momentum (No classical correspondence) - SPIN

$$\begin{aligned} [\hat{S}_i, \hat{S}_j] &= i\hbar \varepsilon_{ijk} \hat{S}_k \iff \hat{\mathbf{S}} \times \hat{\mathbf{S}} = i\hbar \hat{\mathbf{S}} \\ [\hat{L}_i, \hat{S}_j] &= 0 \end{aligned}$$

5. Hamiltonian

$$\hat{H} = \hat{T} + \hat{V} = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}(\mathbf{x}) \quad \begin{cases} \hat{T} : & \mathbf{p}\text{-dependent only} \\ \hat{V} : & \mathbf{x}\text{-dependent only} \end{cases}$$

6. Energy

$$\begin{cases} \hat{E} = i\hbar \frac{\partial}{\partial t} & \text{wavefunctions} \\ \hat{E} = i\hbar \frac{d}{dt} & \text{state vector} \end{cases}$$

Remark In non-relativistic QM, time t is NOT observable.

16 Eigenvalue & Eigenstates

1. Position

$$\begin{aligned}\hat{\mathbf{x}} |\mathbf{x}'\rangle &= \mathbf{x}' |\mathbf{x}'\rangle \\ \text{eigenvalue eq.} \\ \begin{cases} \langle \mathbf{x}' | \mathbf{x}'' \rangle = \delta(\mathbf{x}' - \mathbf{x}'') & (\text{ON}) \\ \int d^3x |\mathbf{x}\rangle \langle \mathbf{x}| = \hat{1} & (\text{RI}) \end{cases} \\ \langle \mathbf{x} | \hat{\mathbf{x}} | \mathbf{x}' \rangle &= \mathbf{x}' \delta(\mathbf{x} - \mathbf{x}')\end{aligned}$$

In $\hat{\mathbf{x}}$ -representation, wavefunction of $|\mathbf{x}\rangle$ is $\langle \mathbf{x}' | \mathbf{x} \rangle = \delta(\mathbf{x} - \mathbf{x}')$, and eigenvalue eq. is

$$\hat{\mathbf{x}} \delta(\mathbf{x} - \mathbf{x}') = \mathbf{x}' \delta(\mathbf{x} - \mathbf{x}')$$

2. Momentum

$$\begin{aligned}\hat{\mathbf{p}} |\mathbf{p}'\rangle &= \mathbf{p}' |\mathbf{p}'\rangle \\ \begin{cases} \langle \mathbf{p}' | \mathbf{p}'' \rangle = \delta(\mathbf{p}' - \mathbf{p}'') & (\text{ON}) \\ \int d\mathbf{p} |\mathbf{p}\rangle \langle \mathbf{p}| = \mathbf{I} & (\text{RI}) \end{cases}\end{aligned}$$

In $\hat{\mathbf{p}}$ -representation, wavefunction of $|\mathbf{p}\rangle$ is $\langle \mathbf{p}' | \mathbf{p} \rangle = \delta(\mathbf{p} - \mathbf{p}')$, and eigenvalue eq. is

$$\hat{\mathbf{p}} \delta(\mathbf{p} - \mathbf{p}') = \mathbf{p}' \delta(\mathbf{p} - \mathbf{p}')$$

Remark eigenspectrum: set of eigenvalues of observable

3. Relation

$$\psi(\mathbf{x}) = \psi_{\mathbf{p}}(\mathbf{x}) \tilde{\psi}(\mathbf{p}), \quad \psi_{\mathbf{p}}(\mathbf{x}) = \langle \mathbf{x} | \mathbf{p} \rangle$$

$$|\psi\rangle = \hat{I} |\psi\rangle = \begin{cases} \int d\mathbf{x} |\mathbf{x}\rangle \langle \mathbf{x} | \psi \rangle = \int d\mathbf{x} |\mathbf{x}\rangle \psi(\mathbf{x}) \\ \int d\mathbf{p} |\mathbf{p}\rangle \langle \mathbf{p} | \psi \rangle = \int d\mathbf{p} |\mathbf{p}\rangle \tilde{\psi}(\mathbf{p}) \end{cases} = \begin{cases} \psi(\mathbf{x}) & \hat{\mathbf{x}}\text{-representation} \\ \tilde{\psi}(\mathbf{p}) & \hat{\mathbf{p}}\text{-representation} \end{cases}$$

$$\begin{aligned}|\psi\rangle &= \int d\mathbf{p} |\mathbf{p}\rangle \tilde{\psi}(\mathbf{p}) = \int d\mathbf{x} |\mathbf{x}\rangle \left(\langle \mathbf{x} | \mathbf{p} \rangle \tilde{\psi}(\mathbf{p}) \right) = \int d\mathbf{x} |\mathbf{x}\rangle \psi(\mathbf{x}) \\ \implies \psi(\mathbf{x}) &= \langle \mathbf{x} | \mathbf{p} \rangle \tilde{\psi}(\mathbf{p}) = \psi_{\mathbf{p}}(\mathbf{x}) \tilde{\psi}(\mathbf{p})\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{p}} |\mathbf{p}\rangle &= \mathbf{p} |\mathbf{p}\rangle \implies \int d\mathbf{x} |\mathbf{x}\rangle \langle \mathbf{x} | \hat{\mathbf{p}} | \mathbf{p} \rangle = \mathbf{p} \int d\mathbf{x} |\mathbf{x}\rangle \langle \mathbf{x} | \mathbf{p} \rangle \quad (\hat{\mathbf{p}} = -i\hbar \frac{\partial}{\partial \mathbf{x}}) \\ \implies -i\hbar \frac{\partial}{\partial \mathbf{x}} \psi_{\mathbf{p}}(\mathbf{x}) &= \mathbf{p} \psi_{\mathbf{p}}(\mathbf{x})\end{aligned}$$

$$-i\hbar \frac{d}{dx} \psi_p(x) = p \psi_p(x) \implies \psi_p(x) = N \exp\left(\frac{i}{\hbar} p x\right) \implies \psi_{\mathbf{p}}(\mathbf{x}) = N \exp\left(\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{x}\right)$$

17 Spin and Pauli Matrices

17.1 Stern-Gerlach Experiment

17.1.1 Meanings

1. Quantization of Spatial Orientation of Angular Momentum
2. Intrinsic Angular Momentum (Spin)
3. S-G experiment \longrightarrow Rabi oscillation \longrightarrow Ramsey (atom clock, Hydrogen maser)
 \longrightarrow Quantum measurement

17.1.2 Analysis

1. (Silver) neutral atom beam \implies no Lorentz force
2. Magnetic field is inhomogeneous

$$\mathbf{F} = -\nabla W_m = \boldsymbol{\mu} \cdot \nabla \mathbf{B} \quad (24)$$

$$W_m = -\boldsymbol{\mu} \cdot \mathbf{B} \quad (25)$$

$\boldsymbol{\mu}$: magnetic dipole moment

3. Deflection: two values $\begin{cases} \text{up} & 50\% \\ \text{down} & 50\% \end{cases}$

4. furnace: low-temperature $\longrightarrow \mathbf{v}$ is small $\longrightarrow \mathbf{p}$ is small $\longrightarrow \mathbf{L}$ is small $\xrightarrow{\boldsymbol{\mu}_L \propto \mathbf{L}} \boldsymbol{\mu}_L$ is small \implies Intrinsic freedom?

17.1.3 History

1921-1922 Stern, Gerlach - S-G experiment

1924 Pauli - Two-valuedness not described classically \longrightarrow Pauli exclusion principle

1925 Kronig - Self-rotation of electron (unpublished)

1925 Uhlenbeck, Goudsmit - Self-rotation

1927 Pauli - Pauli matrices, Pauli equation (wavefunction is a spinor with 2-component)

1928 Dirac - Relativistic QM, Dirac equation (4-component spinor)

17.1.4 Cascaded S-G Experiment

$$\text{Ag} \xrightarrow{\uparrow \mathbf{B}} \mu_z \rightarrow \begin{cases} +\mu_B & \text{up } 50\% \\ -\mu_B & \text{down } 50\% \end{cases} \quad \mu_B = \frac{e\hbar}{2mc} \quad (\text{Bohr magneton})$$

$$\text{Ag} \longrightarrow \mu_x \rightarrow \begin{cases} |\mu_{x,+}\rangle \rightarrow \mu_z \rightarrow \begin{cases} +\mu_B & 50\% \\ -\mu_B & 50\% \end{cases} \\ |\mu_{x,-}\rangle \rightarrow \mu_z \rightarrow \begin{cases} +\mu_B & 50\% \\ -\mu_B & 50\% \end{cases} \end{cases}$$

18 Spin Angular Momentum

18.0.1 Significance

1. Spin is an Intrinsic Angular Momentum
2. Spin is a signature to distinguish two families

$$\begin{aligned}
 (a) \quad & \begin{cases} s = 0, 1, 2, \dots \text{ (integer)} & \text{Bosons} \\ s = \frac{1}{2}, \frac{3}{2}, \dots \text{ (half-integer)} & \text{Fermion} \end{cases} \\
 (b) \quad \text{Boson} & \begin{cases} \text{Statistics: Bose-Einstein statistics} \\ \text{Permutation: } \psi_S(\dots, x_i, \dots, x_j, \dots) = \psi_S(\dots, x_j, \dots, x_i, \dots) \\ & \text{unchanged} \end{cases} \\
 (c) \quad \text{Fermion} & \begin{cases} \text{Statistics: Fermi-Dirac statistics} \\ \text{Permutation: } \psi_A(\dots, x_i, \dots, x_j, \dots) = -\psi_A(\dots, x_j, \dots, x_i, \dots) \\ & \text{opposite sign} \end{cases} \\
 (d) \quad \text{Quantum Statistics} & \begin{cases} \text{Bose-Einstein} \\ \text{Fermi-Dirac} \end{cases} \xrightarrow{\hbar \rightarrow 0} \text{Maxwell-Boltzmann statistics}
 \end{aligned}$$

$$[\hat{S}_i, \hat{S}_j] = i\hbar \varepsilon_{ijk} \hat{S}_k \iff \hat{\mathbf{S}} \times \hat{\mathbf{S}} = i\hbar \hat{\mathbf{S}} \quad (26)$$

$$\implies [\hat{\mathbf{S}}^2, \hat{S}_i] = 0 \implies \begin{cases} \hat{\mathbf{S}}^2 |s, m_s\rangle = \hbar^2 s(s+1) |s, m_s\rangle \\ \hat{S}_z |s, m_s\rangle = \hbar m_s |s, m_s\rangle \end{cases} \quad \text{common eigenstates: } |s, m_s\rangle$$

$$\text{Dirac eq.} \longleftrightarrow s = \frac{1}{2} \quad (\text{electron \& positron, proton, neutron, } \dots) \quad (27)$$

$$\left| s = \frac{1}{2}, m_s = +\frac{1}{2} \right\rangle = |+\rangle = |\uparrow\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (28)$$

$$\left| s = \frac{1}{2}, m_s = -\frac{1}{2} \right\rangle = |-\rangle = |\downarrow\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (29)$$

$$\begin{cases} |0\rangle \langle 0| + |1\rangle \langle 1| = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \langle 0|0\rangle = \langle 1|1\rangle = 1; \langle 1|0\rangle = \langle 0|1\rangle = 0 \end{cases}$$

$$\implies \text{In } \hat{S}_z\text{-representation} \begin{cases} \hat{S}_z |0\rangle = +\frac{\hbar}{2} |0\rangle \\ \hat{S}_z |1\rangle = -\frac{\hbar}{2} |1\rangle \end{cases} \implies \hat{\mathbf{S}} = \frac{\hbar}{2} \hat{\boldsymbol{\sigma}} \implies \begin{cases} \hat{\boldsymbol{\sigma}}_z |0\rangle = |0\rangle \\ \hat{\boldsymbol{\sigma}}_z |1\rangle = -|1\rangle \end{cases}$$

$$\implies \hat{\boldsymbol{\sigma}}_z = |0\rangle \langle 0| - |1\rangle \langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{\mathbf{S}}^2 = \frac{3}{4} \hbar^2 \mathbf{I} \implies \hat{\boldsymbol{\sigma}}^2 = 3\mathbf{I} \implies [\hat{\boldsymbol{\sigma}}_i, \hat{\boldsymbol{\sigma}}_j] = 2i\varepsilon_{ijk} \hat{\boldsymbol{\sigma}}_k$$

19 Algebraic Method to Eigenvalue Problem of Angular Momentum

20 Review

Fundamental Commutations

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \quad [\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0 \quad (30)$$

20.1 Orbital Angular Momentum

20.1.1 Definition of Orbital AM

$$\hat{\mathbf{L}} = \hat{\mathbf{x}} \times \hat{\mathbf{p}} \quad \text{or} \quad \hat{L}_i = \varepsilon_{ijk} \hat{x}_j \hat{p}_k \quad (31)$$

20.1.2 Commutation Relations of Orbital AM

$$[\hat{L}_i, \hat{L}_j] = i\hbar\varepsilon_{ijk} \hat{L}_k \iff \hat{\mathbf{L}} \times \hat{\mathbf{L}} = i\hbar\hat{\mathbf{L}} \quad (32)$$

$$[\hat{L}^2, \hat{L}_i] = 0 \implies \text{Common Eigenstates} \quad (33)$$

20.1.3 Eigenvalue Equation

$$\begin{cases} \hat{\mathbf{L}}^2 |l, m_l\rangle = \hbar^2 l(l+1) |l, m_l\rangle \\ \hat{L}_z |l, m_l\rangle = \hbar m_l |l, m_l\rangle \end{cases} \quad m_l = \underbrace{-l, -l+1, \dots, l-1, l}_{2l+1} \quad (34)$$

20.2 Intrinsic Angular Momentum (Spin)

Generalize to Intrinsic AM (Spin):

$$[\hat{S}_i, \hat{S}_j] = i\hbar\varepsilon_{ijk} \hat{S}_k \iff \hat{\mathbf{S}} \times \hat{\mathbf{S}} = i\hbar\hat{\mathbf{S}} \quad (35)$$

$$[\hat{S}^2, \hat{S}_i] = 0 \implies \text{Common Eigenstates} \quad (36)$$

20.2.1 Eigenvalue Equation

$$\begin{cases} \hat{\mathbf{S}}^2 |s, m_s\rangle = \hbar^2 s(s+1) |s, m_s\rangle \\ \hat{S}_z |s, m_s\rangle = \hbar m_s |s, m_s\rangle \end{cases} \quad m_s = \underbrace{-s, -s+1, \dots, s-1, s}_{2s+1} \quad (37)$$

$$s = \frac{1}{2} \implies |s, m_s\rangle = \begin{cases} |+\rangle = |0\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}^T \\ |-\rangle = |1\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix}^T \end{cases} \quad (38)$$

20.3 Pauli Matrix

$$\hat{\mathbf{S}} = \frac{\hbar}{2} \hat{\boldsymbol{\sigma}} \quad (39)$$

$$\begin{cases} \hat{\mathbf{S}}^2 = \frac{3}{4} \hbar^2 \hat{1}_2 \implies \hat{\boldsymbol{\sigma}}^2 = 3 \hat{1}_2 \\ \hat{S}_z^2 = \frac{1}{4} \hbar^2 \hat{1}_2 \implies \hat{\sigma}_z^2 = \hat{1}_2 \end{cases} \quad (40)$$

$$\text{Cascaded SGE} \implies \begin{cases} |\mu_x, +\rangle = c_1 |0\rangle + c_2 |1\rangle \\ |c_1|^2 = |c_2|^2 = \frac{1}{2} \end{cases} \implies \hat{\sigma}_x^2 = \hat{\sigma}_y^2 = \hat{1}_2$$

$$\hat{\sigma}_i = \hat{1}_2 \quad (41)$$

$$\{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{ij} \hat{1}_2 \quad (42)$$

$$[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk} \hat{\sigma}_k \quad (43)$$

$$\hat{\sigma}_i \hat{\sigma}_j = \delta_{ij} \hat{1}_2 + i\epsilon_{ijk} \hat{\sigma}_k \quad (44)$$

$$\text{tr}(\hat{\sigma}_i) = 0 \quad \text{tr}(\hat{\sigma}_i \hat{\sigma}_j) = 2\delta_{ij} \quad (45)$$

$$\hat{S}_{\pm} = \hat{S}_x \pm i\hat{S}_y = \frac{\hbar}{2} (\hat{\sigma}_x \pm i\hat{\sigma}_y) = \hbar \hat{\sigma}_{\pm} \quad (46)$$

$$\begin{cases} \hat{\sigma}_x = \hat{\sigma}_+ + \hat{\sigma}_- \\ \hat{\sigma}_y = -i(\hat{\sigma}_+ - \hat{\sigma}_-) \end{cases} \quad (47)$$

20.3.1 Ladder / Raising & Lowering / Transition Operator

$$\hat{\sigma}_+ = |0\rangle \langle 1| \quad \& \quad \hat{\sigma}_- = |1\rangle \langle 0| \quad (48)$$

20.3.2 Pauli Matrices ($S_z(\sigma_z)$ -representation)

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (49)$$

21 Arbitrary Angular Momentum

$$\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}} \quad (50)$$

$$[\hat{J}_i, \hat{J}_j] = i\hbar \epsilon_{ijk} \hat{J}_k \quad (51)$$

OAM & Spin are in DIFFERENT State Vector Spaces:

$$[\hat{L}_i, \hat{S}_j] \equiv 0 \quad (52)$$

$$[\hat{J}_i, \hat{J}_j] = [\hat{L}_i + \hat{S}_i, \hat{L}_j + \hat{S}_j] = [\hat{L}_i, \hat{L}_j] + [\hat{S}_i, \hat{S}_j] = i\hbar \epsilon_{ijk} (\hat{L}_k + \hat{S}_k) = i\hbar \epsilon_{ijk} \hat{J}_k$$

21.1 Ladder Operator

21.1.1 Definition of Ladder Operators

$$\hat{J}_{\pm} = \hat{J}_x \pm i\hat{J}_y \quad (53)$$

$$\left[\hat{J}_i, \hat{\mathbf{J}}^2 \right] = 0 \implies \begin{cases} \hat{\mathbf{J}}^2 |j, m_j\rangle = \hbar j(j+1) |j, m_j\rangle \\ \hat{J}_z |j, m_j\rangle = \hbar m_j |j, m_j\rangle \end{cases} \quad j \geq 0, \quad m_j = \underbrace{-j, -j+1, \dots, j-1, j}_{(2j+1)\text{-fold (degeneracy)}} \quad (54)$$

$$\begin{cases} j: \text{azimuthal quantum number} \\ m_j: \text{magnetic quantum number} \end{cases}$$

21.1.2 Effect of Ladder Operators on Common Eigenstates

$$\hat{J}_{\pm} |j, m_j\rangle = \hbar \sqrt{j(j+1) - m_j(m_j \pm 1)} |j, m_j \pm 1\rangle \quad (55)$$

21.1.3 Restriction

$$\hat{J}_+ |j, j\rangle = \hat{J}_- |j, -j\rangle = 0 \quad (56)$$

21.1.4 Commutations

1. $\left[\hat{J}_i, \hat{J}_j \right] = i\hbar \varepsilon_{ijk} \hat{J}_k$
2. $\left[\hat{\mathbf{J}}^2, \hat{J}_i \right] = 0$
3. $\left[\hat{\mathbf{J}}^2, \hat{J}_{\pm} \right] = 0$
4. $\left[\hat{J}_z, \hat{J}_{\pm} \right] = \pm \hbar \hat{J}_{\pm}$

$$\left[\hat{J}_z, \hat{J}_{\pm} \right] = \left[\hat{J}_z, \hat{J}_x \pm i\hat{J}_y \right] = \left[\hat{J}_z, \hat{J}_x \right] \pm i \left[\hat{J}_z, \hat{J}_y \right] = i\hbar \hat{J}_y \pm i(-i\hbar) \hat{J}_x = \pm \hbar \hat{J}_{\pm}$$

$$5. \hat{J}_+ \hat{J}_- = \hat{\mathbf{J}}^2 - \hat{J}_z^2 + \hbar \hat{J}_z$$

$$\hat{J}_+ \hat{J}_- = (\hat{J}_x + i\hat{J}_y)(\hat{J}_x - i\hat{J}_y) = \hat{J}_x^2 + \hat{J}_y^2 - i \left[\hat{J}_x, \hat{J}_y \right] = \hat{\mathbf{J}}^2 - \hat{J}_z^2 + \hbar \hat{J}_z$$

$$6. (\hat{J}_{\pm})^{\dagger} = \hat{J}_{\mp}$$

\hat{J}_{\pm} are NOT Hermitian Operators, but \hat{J}_+ & \hat{J}_- are mutually Hermitian Conjugates.

21.2 Eigenvalue Equation of Angular Momentum Operator

$$\begin{cases} \hat{\mathbf{J}}^2 |\lambda, m\rangle = \hbar^2 \lambda |\lambda, m\rangle \\ \hat{J}_z |\lambda, m\rangle = \hbar m |\lambda, m\rangle \end{cases} \quad (57)$$

$$[\hat{\mathbf{J}}^2, \hat{J}_\pm] = 0 \implies \hat{J}_\pm \hat{\mathbf{J}}^2 |\lambda, m\rangle = \hat{\mathbf{J}}^2 \hat{J}_\pm |\lambda, m\rangle \implies \lambda \hbar^2 (\hat{J}_\pm |\lambda, m\rangle) = \hat{\mathbf{J}}^2 (\hat{J}_\pm |\lambda, m\rangle) \quad (58)$$

$$\begin{aligned} [\hat{J}_\pm, \hat{J}_z] &= \pm \hbar \hat{J}_\pm \implies \hat{J}_z \hat{J}_\pm = \hat{J}_\pm \hat{J}_z \pm \hbar \hat{J}_\pm \\ \implies \hat{J}_z (\hat{J}_\pm |\lambda, m\rangle) &= (\hat{J}_\pm \hat{J}_z \pm \hbar \hat{J}_\pm) |\lambda, m\rangle = m \hbar \hat{J}_\pm |\lambda, m\rangle \pm \hbar \hat{J}_\pm |\lambda, m\rangle \\ &= (m \pm 1) \hbar (\hat{J}_\pm |\lambda, m\rangle) \end{aligned} \quad (59)$$

$\implies \hat{J}_\pm |\lambda, m\rangle$ is an eigenstate of \hat{J}_z , with eigenvalue of $(m \pm 1)\hbar$.

$$\implies \hat{J}_\pm |\lambda, m\rangle = C_\pm |\lambda, m \pm 1\rangle \quad (60)$$

$$\implies \hat{J}_\pm^n |\lambda, m\rangle = D_\pm |\lambda, m \pm n\rangle \quad (61)$$

$$\begin{cases} \langle \hat{\mathbf{J}}^2 \rangle = \langle \lambda, m | \hat{\mathbf{J}}^2 | \lambda, m \rangle = \lambda \hbar^2 \\ \langle \hat{J}_z^2 \rangle = \langle \lambda, m | \hat{J}_z^2 | \lambda, m \rangle = m^2 \hbar^2 \end{cases} \quad (62)$$

$$\langle \hat{\mathbf{J}}^2 \rangle \geq \langle \hat{J}_z \rangle \implies \lambda \geq m^2 \geq 0 \quad (63)$$

$\exists m_0$ (minimal), $\exists N$ (integer), $m_0 + N$ is the maximal

$$\hat{J}_- |\lambda, m_0\rangle = 0 \quad \hat{J}_+ |\lambda, m_0 + N\rangle = 0 \quad (64)$$

$$0 = \hat{J}_+ \hat{J}_- |\lambda, m_0\rangle = (\hat{\mathbf{J}}^2 - \hat{J}_z^2 + \hbar \hat{J}_z) |\lambda, m_0\rangle = \hbar^2 (\lambda - m_0^2 + m_0) \quad (65)$$

$$0 = \hat{J}_- \hat{J}_+ |\lambda, m_0 + N\rangle = (\hat{\mathbf{J}}^2 - \hat{J}_z^2 - \hbar \hat{J}_z) |\lambda, m_0 + N\rangle = \hbar^2 [\lambda - (m_0 + N)^2 - (m_0 + N)] \quad (66)$$

$$\begin{cases} \lambda - m_0^2 + m_0 = 0 \\ \lambda - (m_0 + N)^2 - (m_0 + N) = 0 \end{cases} \implies \begin{cases} m_0 = -\frac{N}{2} \\ m_0 + N = \frac{N}{2} \\ \lambda = -\frac{N}{2} \left(\frac{N}{2} + 1 \right) \end{cases} \quad (67)$$

let $j = \frac{N}{2}$, $\lambda = j(j+1)$, $m_j = \underbrace{-j, -j+1, \dots, j-1, j}_{2j+1}$

$$\implies \boxed{\begin{cases} \hat{\mathbf{J}}^2 |j, m_j\rangle = j(j+1) \hbar^2 |j, m_j\rangle \\ \hat{J}_z |j, m_j\rangle = m_j \hbar |j, m_j\rangle \end{cases}} \quad (68)$$

$$\hat{J}_\pm |j, m_j\rangle = C_\pm |j, m_j \pm 1\rangle \quad (69)$$

$$\langle j, m_j | j', m_{j'} \rangle = \delta_{jj'} \delta_{m_j m_{j'}} \quad (\hat{J}_\pm)^\dagger = \hat{J}_\mp \quad (70)$$

$$\implies 1 = \langle j, m_j \pm 1 | j, m_j \pm 1 \rangle = \langle j, m_j | \hat{J}_\mp \hat{J}_\pm | j, m_j \rangle = |C_\pm|^2 \langle j, m_j | j, m_j \rangle \quad (71)$$

$$\hat{J}_\mp \hat{J}_\pm = (\hat{J}_x \mp i\hat{J}_y)(\hat{J}_x \pm i\hat{J}_y) = \hat{J}_x^2 + \hat{J}_y^2 \pm i[\hat{J}_x, \hat{J}_y] = \hat{\mathbf{J}}^2 - \hat{J}_z^2 \mp \hbar \hat{J}_z \quad (72)$$

$$\begin{aligned} \implies \langle j, m_j | \hat{J}_\mp \hat{J}_\pm | j, m_j \rangle &= \langle j, m_j | (\hat{\mathbf{J}}^2 - \hat{J}_z^2 \mp \hbar \hat{J}_z) | j, m_j \rangle \\ &= \hbar^2 [j(j+1) - m_j^2 \mp m_j] \langle j, m_j | j, m_j \rangle = |C_\pm|^2 \langle j, m_j | j, m_j \rangle \\ \implies |C_\pm|^2 &= [j(j+1) - m_j(m_j \pm 1)] \hbar^2 \end{aligned} \quad (73)$$

for simplicity, let $C_\pm \in \mathbb{R}$

$$\implies C_\pm = \hbar \sqrt{j(j+1) - m_j(m_j \pm 1)} \quad (74)$$

$$\implies \boxed{\hat{J}_\pm | j, m_j \rangle = \hbar \sqrt{j(j+1) - m_j(m_j \pm 1)} | j, m_j \pm 1 \rangle} \quad (75)$$

22 Commutation Relations between Angular Momentum Operators and Hamiltonian

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(r) + \xi(r) \hat{\mathbf{L}} \cdot \hat{\mathbf{S}} \quad (76)$$

$V(r)$: central potential, $\xi(r) \hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$: $\hat{\mathbf{L}}\text{-}\hat{\mathbf{S}}$ coupling \longleftrightarrow fine structure

$$[\hat{L}_i, \hat{p}_j] = i\hbar \epsilon_{ijk} \hat{p}_k \implies [\hat{L}_i, \hat{\mathbf{p}}^2] = [\hat{L}_i, \hat{p}_j] \hat{p}_j + \hat{p}_j [\hat{L}_i, \hat{p}_j] = i\hbar \epsilon_{ijk} (\hat{p}_k \hat{p}_j + \hat{p}_j \hat{p}_k) = 0 \quad (77)$$

$$[\hat{S}_i, \hat{\mathbf{p}}^2] = 0 \implies [\hat{J}_i, \hat{\mathbf{p}}^2] = 0 \quad (78)$$

$$[\hat{L}_i, V(r)] = 0 \quad \& \quad [\hat{S}_i, V(r)] = 0 \implies [\hat{J}_i, V(r)] = 0 \quad (79)$$

$$[\hat{L}_i, \hat{\mathbf{L}} \cdot \hat{\mathbf{S}}] = [\hat{L}_i, \hat{L}_j \hat{S}_j] = i\hbar \epsilon_{ijk} \hat{L}_k \hat{S}_j \neq 0 \quad (80)$$

$$[\hat{\mathbf{L}}^2, \hat{\mathbf{L}} \cdot \hat{\mathbf{S}}] = [\hat{L}_i \hat{L}_i, \hat{L}_j \hat{S}_j] = [\hat{L}_i, \hat{L}_j] \hat{L}_i \hat{S}_j + \hat{L}_i [\hat{L}_i, \hat{L}_j] \hat{S}_j = i\hbar \epsilon_{ijk} (\hat{L}_k \hat{L}_i + \hat{L}_i \hat{L}_k) \hat{S}_j = 0 \quad (81)$$

similarly, $[\hat{S}_i, \hat{\mathbf{L}} \cdot \hat{\mathbf{S}}] \neq 0$, but $[\hat{\mathbf{S}}^2, \hat{\mathbf{L}} \cdot \hat{\mathbf{S}}] = 0$. futher, $[\hat{J}_i, \hat{\mathbf{L}} \cdot \hat{\mathbf{S}}] = 0$, $[\hat{\mathbf{J}}^2, \hat{\mathbf{L}} \cdot \hat{\mathbf{S}}] = 0$.

$$\hat{\mathbf{L}}^2, \hat{\mathbf{S}}^2, \hat{\mathbf{J}}^2, \hat{J}_i \text{ commutes with } \hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(r) + \xi(r) \hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$$

23 Addition of Angular Momenta

assume $\hat{\mathbf{J}}_1, \hat{\mathbf{J}}_2, \hat{\mathbf{J}} = \hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2$, and Commutation Relations:

$$[\hat{J}_{\alpha i}, \hat{J}_{\beta j}] = i\hbar \delta_{\alpha\beta} \varepsilon_{ijk} \hat{J}_{\alpha k} \quad (\alpha, \beta = 1, 2) \quad (82)$$

$$[\hat{\mathbf{J}}_\alpha^2, \hat{J}_{\alpha i}] = 0 \quad (\alpha = 1, 2) \quad (83)$$

Eigenvalue Equations:

$$\begin{cases} \hat{\mathbf{J}}_1^2 |j_1, m_1\rangle = j_1(j_1 + 1)\hbar^2 |j_1, m_1\rangle \\ \hat{J}_{1z} |j_1, m_1\rangle = m_1\hbar |j_1, m_1\rangle \end{cases} \quad (84)$$

$$\begin{cases} \hat{\mathbf{J}}_2^2 |j_2, m_2\rangle = j_2(j_2 + 1)\hbar^2 |j_2, m_2\rangle \\ \hat{J}_{2z} |j_2, m_2\rangle = m_2\hbar |j_2, m_2\rangle \end{cases} \quad (85)$$

$$[\hat{J}_i, \hat{J}_j] = i\hbar \varepsilon_{ijk} \hat{J}_k \quad (86)$$

24 Uncertainty Principle

24.1 Heisenberg Uncertainty Principle

$\forall |\psi\rangle$,

$$\underbrace{\Delta \hat{x}^2 \cdot \Delta \hat{p}^2}_{\text{variance}} \geq \frac{\hbar^2}{4} \quad \text{or} \quad \underbrace{\Delta \hat{x} \cdot \Delta \hat{p}}_{\text{root-mean-square}} \geq \frac{\hbar}{2} \quad (87)$$

24.2 Generalized Uncertainty Principle

$\forall |\psi\rangle$, if $[\hat{A}, \hat{B}] \neq 0$, then

$$\Delta \hat{A}^2 \cdot \Delta \hat{B}^2 \geq \frac{|\langle [\hat{A}, \hat{B}] \rangle|^2}{4} \quad \text{or} \quad \Delta \hat{A} \cdot \Delta \hat{B} \geq \frac{|\langle [\hat{A}, \hat{B}] \rangle|}{2} \quad (88)$$

Variance:

$$\Delta \hat{A}^2 = \langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 \quad (89)$$

root-mean-square:

$$\Delta \hat{A} = \sqrt{\Delta \hat{A}^2} = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2} \quad (90)$$

24.2.1 Essential Mathematical Tools

1. If $\hat{F}^\dagger = \hat{F}$ (Hermitian), then eigenvalue of \hat{F} is REAL
2. If $\hat{F}^\dagger = -\hat{F}$ (skew-Hermitian), then \hat{F} is IMAGINARY
3. Cauchy-Schwarz Inequality

$$(a) \left(\sum_n a_n^2 \right) \left(\sum_n b_n^2 \right) \geq \left(\sum_n a_n b_n \right)^2, \text{ equality iff } a_n = k b_n$$

$$(b) |\mathbf{u}|^2 |\mathbf{v}|^2 \geq |\mathbf{u} \cdot \mathbf{v}|^2, \text{ equality iff } \mathbf{u} = k \mathbf{v}$$

$$(c) \boxed{\langle \psi | \psi \rangle \langle \phi | \phi \rangle \geq |\langle \psi | \phi \rangle|^2, \text{ equality iff } |\psi\rangle = \lambda |\phi\rangle}$$

24.2.2 Define Deviation Operator

$$\hat{\sigma}_A = \hat{A} - \langle \hat{A} \rangle \quad \& \quad \hat{\sigma}_B = \hat{B} - \langle \hat{B} \rangle \quad (91)$$

$$\implies \begin{cases} \Delta \hat{A}^2 = \langle \hat{\sigma}_A^2 \rangle \\ \Delta \hat{B}^2 = \langle \hat{\sigma}_B^2 \rangle \end{cases} \quad (92)$$

24.2.3 Properties of $\hat{\sigma}_A, \hat{\sigma}_B$

1. $\hat{\sigma}_A^\dagger = \hat{\sigma}_A, \hat{\sigma}_B^\dagger = \hat{\sigma}_B$ (Hermitian)
2. $\hat{\sigma}_A^\dagger \hat{\sigma}_A = \hat{\sigma}_A^2, \hat{\sigma}_B^\dagger \hat{\sigma}_B = \hat{\sigma}_B^2$
3. $[\hat{\sigma}_A, \hat{\sigma}_B] = [\hat{A}, \hat{B}]$
4. $\hat{\sigma}_A \hat{\sigma}_B = \frac{1}{2} [\hat{\sigma}_A, \hat{\sigma}_B] + \frac{1}{2} \{\hat{\sigma}_A, \hat{\sigma}_B\}$
5. $[\hat{\sigma}_A, \hat{\sigma}_B]^\dagger = -[\hat{\sigma}_A, \hat{\sigma}_B]$ (skew-Hermitian)
6. $\{\hat{\sigma}_A, \hat{\sigma}_B\}^\dagger = \{\hat{\sigma}_A, \hat{\sigma}_B\}$ (Hermitian)
7. $\Delta \hat{A}^2 = \langle \hat{\sigma}_A^2 \rangle, \Delta \hat{B}^2 = \langle \hat{\sigma}_B^2 \rangle$

24.2.4 Proof of GUP

$$\begin{aligned} \text{def: } \hat{\sigma}_A |\psi\rangle &= |\psi_A\rangle \quad \& \quad \hat{\sigma}_B |\psi\rangle = |\psi_B\rangle \\ \langle \hat{\sigma}_A^2 \rangle &= \langle \psi | \hat{\sigma}_A \hat{\sigma}_A | \psi \rangle = \langle \psi | \hat{\sigma}_A^\dagger \hat{\sigma}_A | \psi \rangle = \langle \psi_A | \psi_A \rangle \quad \& \quad \langle \hat{\sigma}_B^2 \rangle = \langle \psi_B | \psi_B \rangle \\ \implies \langle \hat{\sigma}_A^2 \rangle \langle \hat{\sigma}_B^2 \rangle &= \langle \psi_A | \psi_A \rangle \langle \psi_B | \psi_B \rangle \geq |\langle \psi_A | \psi_B \rangle|^2 = |\langle \psi | \hat{\sigma}_A \hat{\sigma}_B | \psi \rangle|^2 = |\langle \hat{\sigma}_A \hat{\sigma}_B \rangle|^2 \\ \langle \hat{\sigma}_A \hat{\sigma}_B \rangle &= \frac{1}{2} \langle [\hat{\sigma}_A, \hat{\sigma}_B] \rangle + \frac{1}{2} \langle \{\hat{\sigma}_A, \hat{\sigma}_B\} \rangle \\ \implies |\langle \hat{\sigma}_A \hat{\sigma}_B \rangle|^2 &\geq \frac{1}{4} |\langle [\hat{\sigma}_A, \hat{\sigma}_B] \rangle|^2 = \frac{1}{4} \left| \langle [\hat{A}, \hat{B}] \rangle \right|^2 \implies \text{GUP} \end{aligned}$$

24.3 Energy-Time Uncertainty Relation

$$\Delta E \cdot \Delta T \geq \frac{\hbar}{2} \quad (93)$$

24.3.1 Ehrenfest Theorem

$$\begin{aligned} [\hat{A}, \hat{H}] \neq 0, \quad \frac{\partial \hat{A}}{\partial t} = 0, \\ \frac{d}{dt} \langle \hat{A} \rangle = \frac{1}{i\hbar} \langle [\hat{A}, \hat{H}] \rangle \end{aligned} \quad (94)$$

24.3.2 Proof

$$\Rightarrow \Delta \hat{A} \cdot \Delta \hat{H} \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{H}] \rangle \right| = \frac{\hbar}{2} \left| \frac{d}{dt} \langle \hat{A} \rangle \right| \Rightarrow \Delta \hat{H} \frac{\Delta \hat{A}}{\left| \frac{d}{dt} \langle \hat{A} \rangle \right|} = \frac{\hbar}{2}$$

$$\text{define } \Delta \hat{H} = \Delta E, \Delta T = \frac{\Delta \hat{A}}{\left| \frac{d}{dt} \langle \hat{A} \rangle \right|}$$

$$\Rightarrow \Delta E \cdot \Delta T \geq \frac{\hbar}{2}$$

24.4 Minimum Uncertainty State

$\frac{\hbar}{2}$: quantum limit (Heisenberg limit) \Rightarrow Minimum Uncertainty State

$$\exists |\psi\rangle_{\min}, \Delta \hat{A} \cdot \Delta \hat{B} = \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle \right|$$

$|\psi\rangle_{\min}$: Minimum Uncertainty State

24.4.1 MUS of \hat{x} & \hat{p}

$$\begin{cases} \hat{\sigma}_x = \hat{x} - \langle \hat{x} \rangle \\ \hat{\sigma}_p = \hat{p} - \langle \hat{p} \rangle \end{cases}, \text{ let } |\phi\rangle = (\hat{\sigma}_x - i\lambda \hat{\sigma}_p) |\psi\rangle \text{ (testing state)}$$

$$\begin{aligned} \langle \phi | \phi \rangle &= \langle \psi | (\hat{\sigma}_x + i\lambda \hat{\sigma}_p)(\hat{\sigma}_x - i\lambda \hat{\sigma}_p) |\psi\rangle = \langle \psi | (\hat{\sigma}_x^2 - i\lambda \hat{\sigma}_x \hat{\sigma}_p + i\lambda \hat{\sigma}_p \hat{\sigma}_x + \lambda^2 \hat{\sigma}_p^2) |\psi\rangle \\ &= \langle \psi | \hat{\sigma}_x^2 |\psi\rangle - i\lambda \langle \psi | [\hat{\sigma}_x, \hat{\sigma}_p] |\psi\rangle + \lambda^2 \langle \psi | \hat{\sigma}_p^2 |\psi\rangle = \langle \hat{\sigma}_x^2 \rangle + \lambda \hbar + \lambda^2 \langle \hat{\sigma}_p^2 \rangle \\ |\psi\rangle = 0 &\Rightarrow \langle \phi | \phi \rangle = 0 \Rightarrow \langle \hat{\sigma}_p^2 \rangle \lambda^2 + \hbar \lambda + \langle \hat{\sigma}_x^2 \rangle = 0 \\ \Delta &= \hbar^2 - 4 \langle \hat{\sigma}_x^2 \rangle \langle \hat{\sigma}_p^2 \rangle \lambda^2 \leq 0 \Rightarrow \langle \hat{\sigma}_x^2 \rangle \langle \hat{\sigma}_p^2 \rangle \lambda^2 \geq \frac{\hbar^2}{4} \\ &\Rightarrow \Delta \hat{x}^2 \cdot \Delta \hat{p}^2 \geq \frac{\hbar^2}{4} \quad \text{or} \quad \Delta \hat{x} \cdot \Delta \hat{p} \geq \frac{\hbar}{2} \end{aligned} \quad (95)$$

$$\Delta = 0 \implies \begin{cases} \lambda = -\frac{2}{\hbar}\Delta\hat{x}^2 \\ |\phi\rangle = \left(\hat{\sigma}_x + \frac{2i}{\hbar}\Delta\hat{x}^2\hat{\sigma}_p\right)|\psi\rangle = 0 \end{cases} \quad (96)$$

$$\langle x|(\hat{x} - \langle\hat{x}\rangle) - i\lambda\left(-i\hbar\frac{d}{dx} - \langle\hat{p}\rangle\right)|\psi\rangle = (x - \langle\hat{x}\rangle)\psi(x) - \lambda\hbar\frac{d}{dx}\psi(x) + i\lambda\langle\hat{p}\rangle\psi(x)$$

25 Quantum Dynamics

25.1 Schrödinger Equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \quad (97)$$

where $\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{x}})$

25.1.1 Remark

1. Pure State
2. Closed System $\implies \langle\psi(t)|\psi(t)\rangle = 1$
3. Non-Relativistic QM
4. Initial Value Problem (IVP): $|\psi(t = t_0)\rangle = |\psi(t_0)\rangle$
5. Schrödinger Picture: $\frac{d\hat{A}}{dt} \equiv 0$

25.1.2 Representation of Position

In \mathbf{x} -representation: $\hat{\mathbf{p}} = -i\hbar\nabla$, $\langle\mathbf{x}|\psi(t)\rangle = \psi(\mathbf{x})$

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \implies i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x}, t) = \left(-\frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{x}})\right)\psi(\mathbf{x}, t) \quad (98)$$

1. In general, $\frac{\partial\hat{A}}{\partial t} \equiv 0$ & $\frac{\partial\hat{H}}{\partial t} \equiv 0$. Unless there is external influence.
2. $\langle\psi(t)|\psi(t)\rangle = 1 \iff \int d^3x |\psi(\mathbf{x}, t)|^2 = 1$
3. $\psi(\mathbf{x}, t = t_0) = \psi(\mathbf{x}, t_0)$

25.2 Time Evolution Operator

$$|\psi(t)\rangle = \hat{U}(t, t_0)|\psi(t_0)\rangle \quad (99)$$

$$\text{Schrödinger eq.} \implies i\hbar\frac{d}{dt}\left[\hat{U}(t, t_0)|\psi(t_0)\rangle\right] = \hat{H}\left[\hat{U}(t, t_0)|\psi(t_0)\rangle\right] \quad (100)$$

$$\implies i\hbar\frac{d}{dt}\hat{U}(t, t_0) = \hat{H}\hat{U}(t, t_0) \quad (101)$$

25.2.1 TEO for Conservative System

If the system is conservative, that is $\frac{\partial \hat{H}}{\partial t} \equiv 0$.

$$\Rightarrow \frac{d\hat{U}}{\hat{U}} = -\frac{i}{\hbar} \hat{H} dt \Rightarrow \boxed{\hat{U}(t, t_0) = e^{-i\hat{H}(t-t_0)/\hbar}}$$

If $\frac{\partial \hat{H}}{\partial t} \neq 0$, then

$$\begin{aligned} \hat{U}(t, t_0) &= \hat{1} - \frac{i}{\hbar} \int dt' \hat{U}(t', t_0) \\ &= \hat{1} - \frac{i}{\hbar} \dots (\text{Dyson Series}) \end{aligned} \quad (102)$$

25.2.2 Properties of $\hat{U}(t, t_0)$

1. $\hat{U}^\dagger(t, t_0) = \hat{U}^{-1}(t, t_0 = \hat{U}(t_0, t))$ or $\hat{U}^\dagger(t, t_0)\hat{U}(t, t_0) = \hat{1}$
2. $\hat{U}(t, t_1)\hat{U}(t_1, t_0) = \hat{U}(t, t_0) \xrightarrow{\text{expand}} \hat{U}(t, t_n)\hat{U}(t_n, t_{n-1}) \dots \hat{U}(t_2, t_1)\hat{U}(t_1, t_0) = \hat{U}(t, t_0)$
3. $\hat{U}(t, t_0) = \hat{U}(t - t_0) = \hat{U}(\tau)$ ($\tau \stackrel{\text{def}}{=} t - t_0$)

25.3 Continuity Equation

$$\begin{aligned} &\begin{cases} i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t) = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{x}, t) + V(\mathbf{x})\psi(\mathbf{x}, t) \\ -i\hbar \frac{\partial}{\partial t} \psi^*(\mathbf{x}, t) = -\frac{\hbar^2}{2m} \nabla^2 \psi^*(\mathbf{x}, t) + V(\mathbf{x})\psi^*(\mathbf{x}, t) \end{cases} \\ \Rightarrow &\begin{cases} \psi^*(\mathbf{x}, t) \frac{\partial}{\partial t} \psi(\mathbf{x}, t) = -\frac{\hbar}{2mi} \psi^*(\mathbf{x}, t) \nabla^2 \psi(\mathbf{x}, t) + \frac{1}{i\hbar} \psi^*(\mathbf{x}, t) V(\mathbf{x}) \psi(\mathbf{x}, t) \\ \psi(\mathbf{x}, t) \frac{\partial}{\partial t} \psi^*(\mathbf{x}, t) = \frac{\hbar}{2mi} \psi(\mathbf{x}, t) \nabla^2 \psi^*(\mathbf{x}, t) - \frac{1}{i\hbar} \psi(\mathbf{x}, t) V(\mathbf{x}) \psi^*(\mathbf{x}, t) \end{cases} \\ \Rightarrow &\psi^* \frac{\partial}{\partial t} \psi + \psi \frac{\partial}{\partial t} \psi^* = \frac{\partial}{\partial t} (\psi^* \psi) = -\frac{\hbar}{2mi} (\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*) = -\frac{\hbar}{2mi} \nabla \cdot (\psi^* \nabla \psi - \psi \nabla \psi^*) \\ \Rightarrow &\frac{\partial}{\partial t} (\psi^* \psi) + \nabla \cdot \left[\frac{\hbar}{2mi} (\psi^* \nabla \psi - \psi \nabla \psi^*) \right] = 0 \end{aligned}$$

Let:

1. Probability Density

$$\rho(\mathbf{x}, t) = \psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t) = |\psi(\mathbf{x}, t)|^2 \quad (103)$$

2. Probability Current Density

$$\mathbf{j} = \frac{\hbar}{2mi} (\psi^* \nabla \psi - \psi \nabla \psi^*) \quad (104)$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0} \quad (105)$$

25.4 Ehrenfest Theorem

$$\begin{cases} i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle & \implies \frac{d}{dt} |\psi(t)\rangle = \frac{\hat{H}}{i\hbar} |\psi(t)\rangle \\ -i\hbar \frac{d}{dt} \langle\psi(t)| = \langle\psi(t)| \hat{H} & \implies \frac{d}{dt} \langle\psi(t)| = -\frac{1}{i\hbar} \langle\psi(t)| \hat{H} \end{cases}$$

Consider an observable \hat{O} , with its expectation value $\langle\hat{O}\rangle = \langle\psi(t)| \hat{O} |\psi(t)\rangle$

$$\begin{aligned} \frac{d}{dt} \langle\hat{O}\rangle(t) &= \left(\frac{d}{dt} \langle\psi(t)| \right) \hat{O} |\psi(t)\rangle + \langle\psi(t)| \frac{d}{dt} \hat{O} |\psi(t)\rangle + \langle\psi(t)| \hat{O} \left(\frac{d}{dt} |\psi(t)\rangle \right) \\ &= -\frac{1}{i\hbar} \langle\psi(t)| \hat{H} \hat{O} |\psi(t)\rangle + \langle\psi(t)| \frac{d}{dt} \hat{O} |\psi(t)\rangle + \frac{1}{i\hbar} \langle\psi(t)| \hat{O} \hat{H} |\psi(t)\rangle \\ &\implies \boxed{\frac{d}{dt} \langle\hat{O}\rangle(t) = \left\langle \frac{\partial \hat{O}}{\partial t} \right\rangle + \frac{1}{i\hbar} \langle [\hat{O}, \hat{H}] \rangle} \end{aligned} \quad (106)$$

25.4.1 Conserved Quantity (Constant of Motion)

If $\frac{\partial \hat{O}}{\partial t} \equiv 0$ & $[\hat{O}, \hat{H}] = 0$, \hat{O} is a conserved quantity. \hat{O} & \hat{H} have common eigenstates.

$$\begin{cases} \frac{d}{dt} O = \frac{\partial O}{\partial t} + [O, H]_{\text{PB}} & \text{CM} \\ \frac{d}{dt} \hat{O}_{\text{H}} = \left(\frac{\partial \hat{O}_{\text{S}}}{\partial t} \right)_{\text{H}} + \frac{1}{i\hbar} [\hat{O}_{\text{H}}, \hat{H}] & \text{QM} \end{cases}$$

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