# Part 4 – Assignment

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This assignment is worth 50 marks. Solutions must be written in **R** markdown and submitted *both* as Rmd-file and compiled pdf-file. Feel free to copy code from the lecture material if it helps solve a problem. Use equation environments (`$...$` or `$$...$$`) to show analytical calculations, and use embedded **R**-code for numerical calculations. *For full marks numerical results must be correct to 5 significant digits.*

## Hernquist halo [10 marks]

The so-called Hernquist profile is an analytical model for spherical mass distributions. In this model, the volume density as a function of radius $r$ is given by

$$\rho(r) = \frac{M}{2\pi a^3}\frac{1}{(r/a)(1+r/a)^3},$$

where $M$ is the total mass (to $r=\infty$) and $a$ is a scale radius. The file `hernquist.csv` (in the `data` directory in the Dropbox) contains the Cartesian coordinates of 100 mass points sampled from such a density distribution, centred at $x=y=z=0$. [Hint: For the following questions, bear in mind that $\rho(r)$ is a *volume* density, not a probability density as a function of $r$.]

a) Plot the log-likelihood of $a$ around its maximum. [5 marks]
b) Compute the MAP solution of $a$ and its uncertainty in the Laplace approximation, assuming a logarithmic prior. [5 marks]

## Extreme value analysis [20 marks]

You draw five numbers from a normal distribution with unknown mean $\mu$ and standard deviation $\sigma$. The lowest random number is $x_{\min}=-0.253$, the highest is $x_{\max}=0.704$. [Hint: use *cubintegrate* from the **cubature** package to accurately evaluate 2D integrals.]

a) Write down the likelihood function $\mathcal{L}(\mu,\sigma)$. You can use the functions $\rho(x;\mu,\sigma)$ and $C(x;\mu,\sigma)$ as short-hands for the probability density and cumulative probability, respectively. [5 marks]
b) Compute the MAP solution for $(\mu,\sigma)$, assuming a flat prior for $\mu$ and a prior proportional to $\sigma^{-2}$ for the standard deviation. [5 marks]
c) Compute the expectation of $\sigma$. [5 marks]
d) Why does the expectation of $\sigma$ (in c) differ from the MAP solution (in b)? Answer concisely in no more than two short sentences and without using equations or numbers. [5 marks]

## Cosmic rays [10 marks]

Cosmic rays are high-energy particles that travel through space at nearly the speed of light. Each ray can be characterized by its energy $E$ or, equivalently, its dimensionless energy value $x=E/\text{GeV}$. The average number of cosmic rays in the interval $x\pm\delta x/2$, hitting the Earth per square metre and per second, can be written as $\lambda(x)\delta x$. The number density function $\lambda(x)$ is well approximated by a power law $\lambda(x)=kx^{\alpha}$ with empirical parameters $k$ and $\alpha$. A sensor that detects all and only cosmic rays with energies between 10 GeV and 1.5 TeV is collecting rays in one square metre for exactly one hour, registering 1412 events with energies (in GeV) listed in the file `cosmicrays.txt` (in the `data` directory in the Dropbox).

a) Determine the MLE solution for the parameters $k$ and $\alpha$. [5 marks]

b) Produce a plot of the $(k,\alpha)$-plane showing the MLE solution, surrounded by two elliptical contours containing 68% and 95% of the posterior probability in the Laplace approximation. [5 marks]

## Which distribution? [10 marks]

A fair die is rolled. If it shows six, a number is generated using `x=rnorm(1,mean=0,sd=1)`; otherwise, a number is generated using `x=rnorm(1,mean=5,sd=3)`.

a) If the random number is $x = 0.431$, what is the posterior probability that the die showed six? [5 marks]

b) Determine the values of $x$, where the odds ratio is one. [5 marks]