Assignment 4

STAT 440/840 - CM 761

Due Tuesday July 30 at 9am - to be submitted through crowdmark

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Q1

1. The Weibull distribution with shape parameter $\alpha > 0$ and scale parameter $\theta > 0$ is given by

$$f(x|\theta,\alpha) = \begin{cases} \frac{\alpha}{\theta}\left(\frac{x}{\theta}\right)^{\alpha-1} e^{-(x/\theta)^{\alpha}} & x \geq 0, \\ 0 & x < 0, \end{cases}$$

Reparametrize the model using $\gamma = \theta^{\alpha}$. Suppose we obtain the following data;

```
set.seed(440)
x = rweibull(24, shape=1/2, scale=64)
```

$$f(x|\gamma,\alpha) = \begin{cases} \frac{\alpha x^{\alpha-1}}{\gamma} e^{-x^{\alpha}/\gamma} & x \geq 0, \\ 0 & x < 0, \end{cases}$$

- **Note:** Here we are interested in inference for $\gamma = \theta^{\alpha}$ conditional on α.

a) **[4 Marks]** Show that the conjugate prior for γ is a inverse-gamma distribution with hyperparameters (λ, β) denoted by $\text{inv}\Gamma(\lambda, \beta)$. The density is given by

$$g(x|\lambda,\beta) = \begin{cases} \frac{\beta^{\lambda}}{\Gamma(\lambda)}(1/x)^{\lambda+1}\exp\left(-\beta/x\right) & x \geq 0, \\ 0 & x < 0, \end{cases}$$

b) **[6 Marks]** Using each of the following priors plot the prior & posterior and then calculate a credible interval for γ,

   i) a inverse gamma prior with $\lambda = 1$ and $\beta = 10$,

   ii) a inverse gamma prior with $\lambda = 10$ and $\beta = 1$, and

   iii) an improper prior with $p(\gamma) = 1$.

- **Note** the R package `invgamma` might be helpful.

c) **[4 Marks]** Calculate a confidence interval using the log-likelihood ratio.

d) **[2 Marks]** Compare and discuss the intervals generated in b) and d).

e) In b) we can calculate the credible intervals exactly. Instead use MCMC to estimate the credible interval from b iii).

   i) **[1 Mark]** Construct a `R` function that generates a random walk **MCMC** algorithm to sample from the posterior. The input is the random walk length, the standard deviation for the random walk density σ, and the initial position.

   ii) **[8 Marks]** Run a random walk MCMC using $\sigma = 0.1, 1, 5, 20$, $T = 10^4$ iterations and starting state $\gamma^{(0)} = 8.9$. Then for each MCMC provide

   - traceplot, autocorrelation,

- summary table with the acceptance rate, estimate of the posterior mean, naive estimate of the credible interval, and a measure of mixing using

$$\frac{1}{T}\sum_{t=1}^{T}(x_t - x_{t-1})^2$$

- and then comment on the results.

Q2

Here we are interested in Bayesian inference for α conditional on the scale scale parameter being known and equal to $\theta = 64$. Use an improper of $p(\alpha) = 1$ for $\alpha \geq 0$.

```
set.seed(440)
x = rweibull(n=24, shape=1/2, scale=64)
```

a) **[2 Mark]** Write a `R` function that is proportional to the posterior on the log-scale.

b) Write `R` functions that generates a MCMC sample from the posterior using the following the proposals or candidate densities;

   i) **[2 Marks]** follows a gamma with shape equal to 1 and rate equal to 2

   ii) **[2 Marks]** follows a gamma with shape equal to 1 and rate equal to $1/X_t$

   iii) **[2 Marks]** follows a $N(X_t, 0.1)$, and

   iv) **[2 Marks]** follows a $N(\widehat{\alpha}, \mathcal{O}(\widehat{\alpha})^{-1})$. i.e. an Gaussian independence sampler with mean equal to the MLE and variance equal to the inverse fisher information.

   v) **[6 Marks]** Generate a sample path of length $10,000$ using each of the above MCMCs and the initial value equal to $1/2$.

   - Summarize and comment on the traceplots, autocorrelations, histogram of the posterior from each MCMC.
   - Use a table to summarize the acceptance ratio, the estimate of the posterior average & variance and the mixing criteria, $\frac{1}{T}\sum_{t=1}^{T}(x_t - x_{t-1})^2$.

Q3 Inference over both parameters.

Suppose we obtain the following data from a Weibull distribution with shape parameter $\alpha > 0$ and scale parameter $\theta > 0$.

```
set.seed(440)
x = rweibull(24, 1/2, 64)
```

- **Note:** Here we are interested in inference for (α, θ) even though in Q1 we did inference over $\gamma = \theta^\alpha$ conditional on α.

a) **[4 Marks]** Frequentist Inference; Plot the $90, 95, 99$ % confidence regions based on the Log-likelihood ratio using a contour plot and plot the MLE.

b) Bayesian Inference; Using an improper of $p(\alpha, \theta) = 1$ for $\alpha \geq 0$ and $\theta \geq 0$.

c) **[8 Marks]** Implement a bivariate random walk to sample from the posterior. Use an initial position of $(\alpha^{(0)}, \theta^{(0)}) = c(0.446, 72.439)$ and initial variance for the random walk of $(1, 1)$.

- Tune the random walk to so that the acceptance probability is roughly .2 .5.

- Include evidence of tunning such traceplot and acceptance probability.
- Overlay the final or resulting sample path (of length 1000) on to log-likelilhood ratio confidence regions given in a).

c) **[6 Marks]** Implement a conditional MCMC to sample from the posterior using a gibbs sampler for θ and using the independence sampler for α from Q2 b) iv).

- Include a traceplot and the acceptance probability for each variable.
- Overlay the final or resulting sample path (of length 1000) on to log-likelilhood ratio confidence regions given in a).