STAT 443

Assignment 3

October, 2017

1.Solution:

$$E(X_t) = E[2w_{t+1} - w_t] = 2E[2w_{t+1}] - E[w_t] = 0 - 0 = 0$$
$$Var(X_t) = Var(2w_{t+1} - w_t) = 4Var(w_{t+1}) + Var(w_t) = 4\sigma^2 + \sigma^2 = 5\sigma^2$$

For $h = 1$,

$$Cov(X_t, X_{t+1}) = E[(2w_{t+1} - w_t)(2w_{t+2} - w_{t+1})] = -2E[w_{t+1}^2] = -2\sigma^2$$

For $h \geq 2$,

$$Cov(X_t, X_{t+h}) = 0$$

This means that,

$$Cov(X_t, X_{t+h}) = \begin{cases} -2\sigma^2 & \text{if } h = 1 \\ 0 & \text{if } h \geq 2 \end{cases}$$

In conclusion, $\{X_t\}$ is stationary.

2.Solution:
(projection method)
Observe

$$\begin{aligned} &E[(X_3 - \phi_1 X_2 - \phi_2 X_1)X_2] = E[w_3 X_2] = 0 \quad \textit{by assumption} \\ &E[(X_3 - \phi_1 X_2 - \phi_2 X_1)X_1] = E[w_3 X_1] = 0 \quad \textit{by assumption} \\ &E[(X_3 - \phi_1 X_2 - \phi_2 X_1)1] = E[X_3] - \phi_1 E[X_2] - \phi_2 E[X_1] = 0 \quad \textit{by assumption} \end{aligned} \tag{2.1}$$

Therefore, by uniqueness of projection,

$$\phi_1 X_2 + \phi_2 X_1$$

is the BLP of X_3 in terms of X_1 and X_2.

3.Solution:From the question, we know that

$$\begin{aligned} &E[X_1] = E[X_2] = ... = E[X_n] = E[X_n + 1] = \mu \\ &Corr(X_n, X_n + 1) = \rho(1) \end{aligned} \tag{3.1}$$

Since the best linear predictor of $\{X_{n+1}\}$ is

$$X_{n+1} = aX_n + b \tag{3.2}$$

So, from the formula (3.2)

$$\begin{aligned} a &= \frac{Cov(X_n, X_{n+1})}{Var(X_n)} = Corr(X_n, X_{n+1}) = \rho(1) \\ b &= E(X_{n+1}) - \frac{Cov(X_n, X_{n+1})}{Var(X_n)} E(X_n) = \mu - \rho(1)\mu = \mu(1 - \rho(1)) \end{aligned} \tag{3.3}$$

4.Solution:

$$\begin{aligned} &Z_t = a_t + \delta a_{t-1} \\ &Y_t = Z_t + v_t = a_t + \delta a_{t-1} + v_t \\ &E(a_t v_{t-j}) = 0 \end{aligned} \tag{4.1}$$

We can get the mean and covariance of the process

$$E[Y_t] = E[a_t + \delta a_{t-1} + v_t] = 0 \tag{4.2}$$

For $h = 0$,

$$Cov(Y_t, Y_t) = E[(a_t + \delta a_{t-1} + v_t)(a_t + \delta a_{t-1} + v_t)] = (1 + \sigma^2)\sigma_a^2 + \sigma_v^2$$

For $h = 1$,

$$Cov(Y_t, Y_{t+1}) = E[(a_t + \delta a_{t-1} + v_t)(a_{t+1} + \delta a_t + v_{t+1})] = \sigma\sigma_a^2$$

For $h \geq 2$,

$$Cov(X_t, X_{t+h}) = 0$$

This means that,

$$Cov(X_t, X_{t+h}) = \begin{cases} (1 + \sigma^2)\sigma_a^2 + \sigma_v^2 & \text{if } h = 0 \\ \sigma\sigma_a^2 & \text{if } h = 1 \\ 0 & \text{if } h \geq 2 \end{cases} \tag{4.3}$$

We can calculate $\rho(h)$ directly,

$$\rho(h) = \begin{cases} 1 & \text{if } h = 0 \\ \frac{\sigma\sigma_a^2}{(1+\sigma^2)\sigma_a^2 + \sigma_v^2} & \text{if } h = 1 \\ 0 & \text{if } h \geq 2 \end{cases} \tag{4.3}$$

The ACF of $\{Y_t\}$ cuts of after lag 1, which means it is an MA(1) process.