1. In this problem you are asked to generate and plot 5 paths for each of the three processes described below (the notation is the same as in class). You may assume that the number of time steps for each process is $N = 200$.

   (i) Random walk with $Z_1 \sim N(0, \sigma^2 = 0.2)$ and $X_0 = 0$.

   (ii) MA(1) process with $Z_1 \sim N(0, \sigma^2 = 0.2)$ and $\theta = 0.7$. To start your iteration, for each path you may use

$$X_1 = Z_1, \quad \text{that is } Z_0 = 0.$$

   (iii) Process $Y_1, Y_2, \ldots, Y_{200}$ defined as

$$Y_t = 10\sqrt{t} + X_t, \; t = 1, 2, \ldots, 200,$$

   where X_t, $t = 1, 2, \ldots,$ follow an AR(1) process with $\phi = -0.5$, $\sigma^2 = 0.2$ and $X_0 = 0$.

   For each process present:

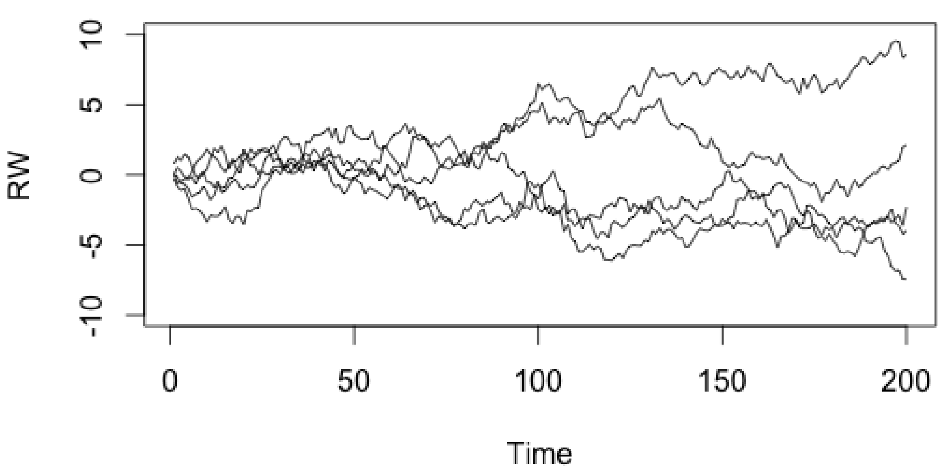
   (1) -a single graph that depicts all five simulated paths,

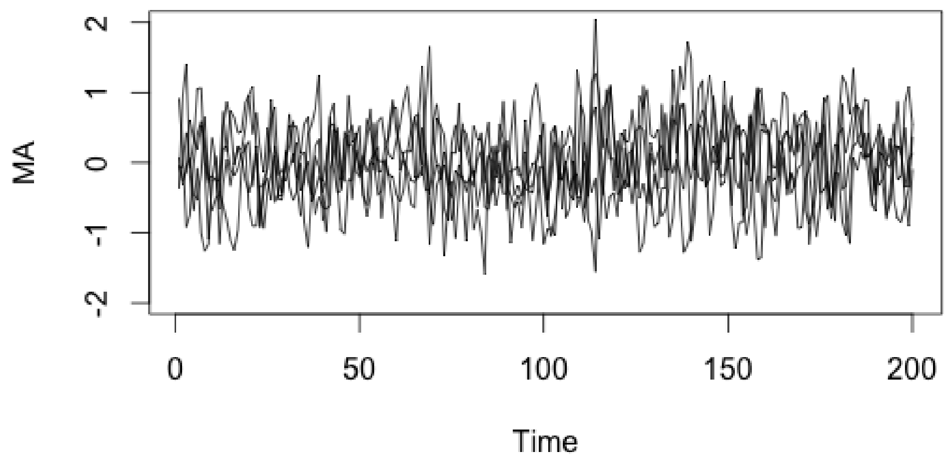
   (2) -the R code that used to simulate the paths,

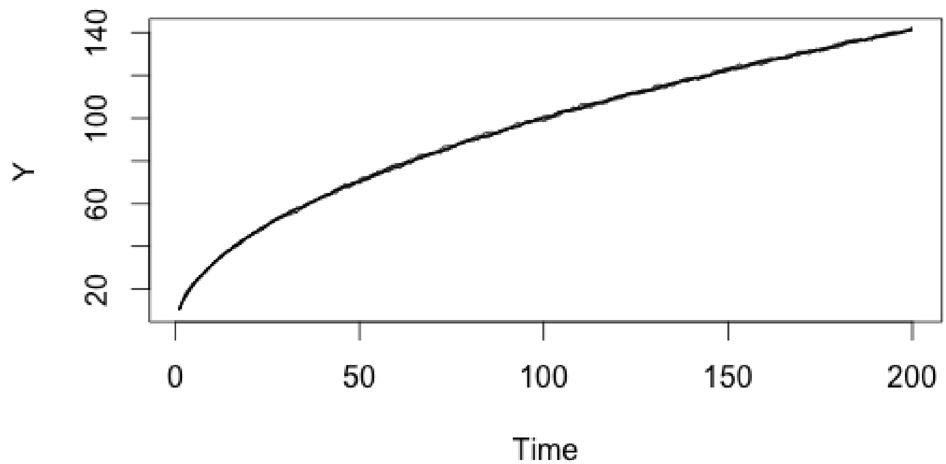
   (3) -a brief comment (2-3 sentences) as to whether or not the paths show "stationary" behavior over time (when plotting a path use lines connecting the generated points; this will help you see better the shape of the path).

   Solution:

   1 :







2 : R code is attached in the appendix.

3 : The conclusions are

(i) non-stationary as the variability increases with time;

(ii) stationary;

(iii) non-stationary as there is an upward trend

2. : Suppose that random variables X and Y are defind in the following way:

$$X = Z + \epsilon_1$$

$$Y = Z + \epsilon_2$$

where $Z \sim N(0, \sigma^2)$, $\sigma > 0$, $\epsilon_1 \sim N(0,1)$, and $\epsilon_2 \sim N(0,1)$. Assume also that Z is independent of ϵ_1 and ϵ_2, and

$$\mathrm{Corr}(\epsilon_1, \epsilon_2) = \rho,$$

where $\rho \in [-1, 1]$.

(i) Find correlation coefficient between X and Y.

(ii) Can you find values of ρ and σ for which X and Y are perfectly correlated? Briefly explain your answer.

(iii) Can you find values of ρ and σ for which X and Y are independent ? Briefly explain your answer.

Solution:

1 : By the property of covariance, we have

$$\mathrm{Cov}(X,Y) = \mathrm{Cov}(Z + \epsilon_1, Z + \epsilon_2) = \mathrm{Var}(Z) + \mathrm{Cov}(\epsilon_1, \epsilon_2) = \sigma^2 + \rho,$$

and $\mathrm{Var}(X) = \sigma^2 + 1$, $\mathrm{Var}(Y) = \sigma^2 + 1$. Hence, $\mathrm{Corr}(X,Y) = \frac{\sigma^2+\rho}{\sigma^2+1}$.

2 : If X and Y are perfectly correlated, then $\mathrm{Corr}(X,Y) = 1$ or -1, which gives $\rho = 1$. From the definition of X and Y, it is easy to see that if ϵ_1 and ϵ_2 are perfect positive correlated, which means $\epsilon_1 = \epsilon_2$, then X and Y are perfectly correlated, .

3 : If X and Y are independent, then $\mathrm{Corr}(X,Y) = 0$, which means $\rho = -\sigma^2$. Take $\rho = -1$, $\sigma = 1$, then ϵ_1 and ϵ_2 are perfect negative correlated, that is $\epsilon_1 = -\epsilon_2$. Then $X = Z + \epsilon_1$ and $Y = Z - \epsilon_1$, where $Z \sim N(0,1)$, $\epsilon_1 \sim N(0,1)$, Z and ϵ_1 are independent, so X and Y are independent.

3. The density of a random variable Y conditioned on $X = x$ and the marginal density f_X of the variable X are given respectively by

$$f_{Y|X}(y|x) = xe^{-xy}, y > 0, \quad \text{and } f_X(x) = 1 \text{ if } x \in (0,1), \text{ and 0 otherwise.}$$

Find the best non linear predictor of Y in terms of X, and calculate the corresponding minimum mean-square error for your predictor.

Solution:

$$\mathbb{E}[Y|X=x] = \int_0^\infty y f_{Y|X}(y|x)\mathrm{d}y = \int_0^\infty xye^{-xy}\mathrm{d}y = \frac{1}{x},$$

then the best non linear predictor of Y in terms of X is $\hat{Y} = \mathbb{E}[Y|X] = \frac{1}{X}$. The corresponding minimum mean-square error is

$$\mathbb{E}[\hat{Y} - Y]^2 = \int_0^1 \int_0^\infty \left(y - \frac{1}{x}\right)^2 f_{Y|X}(y|x) f_X(x)\mathrm{d}y\mathrm{d}x = \int_0^1 \frac{1}{x^2}\mathrm{d}x = \infty.$$

4. Obtain the historical data **hist_y.txt** which consists of 200 consecutive observations $Y_1, \ldots, Y_{200}$ from a constant mean model

$$Y_t = \beta_0 + \epsilon_t, \quad \text{where } \epsilon_t \sim \text{OLSA}.$$

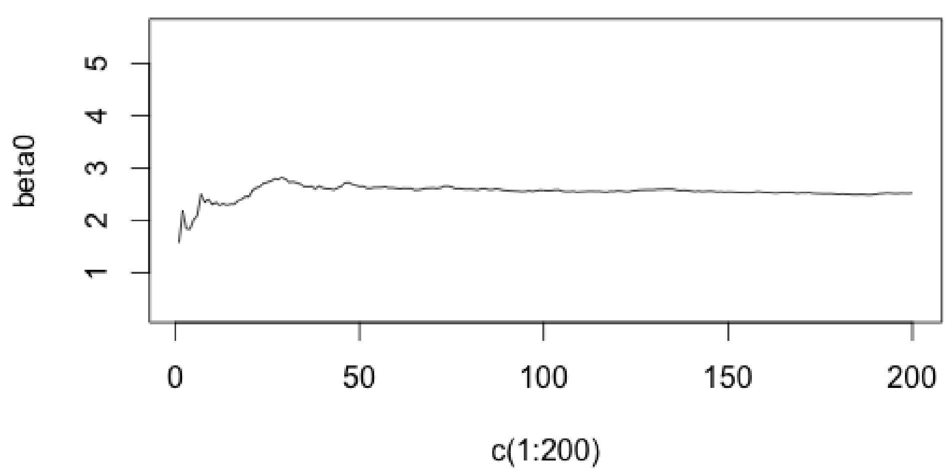
Let $D_n = \{Y_1, \ldots, Y_n\}$, where $n = 1, 2, \ldots, 200$. We know that if D_n is the only data available to us then the least squares estimate of β_0 is given by $\beta_0^*(n) = \bar{Y}_n$, the sample mean.

(i) Plot $\beta_0^*(n)$ vs. n, where $n = 1, 2, \ldots, 200$. To what value does $\beta_0^*(n)$ appear to be converging? What value does this represent in the constant mean model ?

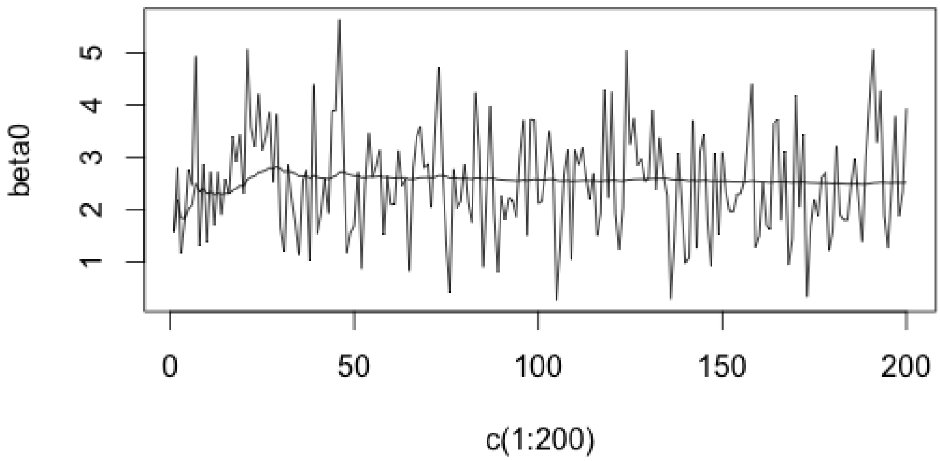
(ii) Provide a plot of $\beta_0^*(n)$ vs. n superimposed on a plot of Y_n vs. n. Compare the variability of the curves. At what point does the updating of $\beta_0^*(n)$ appear to become almost negligible here ?

Solution:

1 : β_0^* converges to 2.5. This value is the true value of β_0 in this model.



2 : Y_n vs. n is more variable than $\beta_0^*(n)$ vs. n. When n goes to 100, the updating of β_0^* becomes almost negligible.



```
# Simulation of a path of a RW process
N <- 200 # the number of simulated observations
RW.vect <- 0*c(1:N) # here you will store your path
sigma <- sqrt(0.2)
# Simulate one path of length N
noise.vect <- rnorm(N) # generate N iid N(0,1) variates
RW.vect[1] <- sigma*noise.vect[1]
for ( i in 2:N) {
  # use the definition of the model:
  RW.vect[i] <- RW.vect[i-1] + sigma*noise.vect[i]
} # end of the loop
par(mfrow=c(1,1))
plot(c(1:N), RW.vect,type="l",xlab="Time",ylab="RW")
# run it 4 additional times and plot the paths

# Simulation of a path of a MA(1) process
N <- 200 # the number of simulated observations
# here you will store your path
MA.vect <- 0*c(1:N)
theta <- 0.7
sigma <- sqrt(0.2)
# Simulate one path of length N
noise.vect <- rnorm(N)
# generate N iid N(0,1) variates
MA.vect[1] <- sigma*noise.vect[1]
for ( i in 2:N) {
  # use the definition of the model:
  MA.vect[i] <- sigma*noise.vect[i] + theta*sigma*noise.vect[i-1]
} # end of the loop
par(mfrow=c(1,1))
plot(c(1:N), MA.vect,type="l",xlab="Time",ylab="MA")
# run it 4 additional times and plot the paths

# Simulation of a path of a Y process
N <- 200 # the number of simulated observations
```

```
# here you will store your path
AR.vect <- 0*c(1:N)
Y.vect <- 0*c(1:N)
sigma <- sqrt(0.2)
phi <-0.5
# Simulate one path of length N
noise.vect <- rnorm(N)
# generate N iid N(0,1) variates
AR.vect[1] <-sigma*noise.vect[1]
for ( i in 2:N) {
  # use the definition of the model:
  AR.vect[i] <- phi*AR.vect[i-1] + sigma*noise.vect[i]
} # end of the loop
for (i in 1:N){
  Y.vect[i] <- 10*sqrt(i)+AR.vect[i]
}
par(mfrow=c(1,1))
plot(c(1:N), Y.vect,type="l",xlab="Time",ylab="Y")
# run it 4 additional times and plot the paths

# plot beta*0(n) vs. n and Yn vs. n
mypath <-"~/Desktop/"
data.set <- "hist_y.txt"
hist.y <- scan(paste(mypath,data.set,sep=""))
hist.ts <- ts(scan(paste(mypath,data.set,sep="")))
beta0 <- 0*c(1:200)
for (i in c(1:200)) {
  beta0[i] <- mean(hist.y[c(1:i)])
}
par(mfrow=c(1,1))
plot(c(1:200), beta0,type="l",ylim=c(min(hist.y),max(hist.y)))
lines(c(1:200),hist.y,type="l")
```