# Millennium Prize Problems

The Millennium Prize Problems are seven well-known complex mathematical problems selected by the Clay Mathematics Institute in 2000

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# Announcement

- The millennium problems were first announced at Millennium Meeting on May 24, 2000 at the Collège de France.
- Timothy Gowers first presented a lecture titled The Importance of Mathematics as an introduction.
- After this, the British mathematician Michael Atiyah and the American John Tate announced the prize.
- One million dollars to anyone who could solve one of the seven most difficult open problems at the time.
- A small committee of mathematicians, selected by the scientific advisory board (SAB) of the Clay Mathematical Institute (which also had organized the meeting).
- This committee included such luminaries as Andrew Wiles.
- He aforementioned Atiyah and Tate, the American Edward Twitten, and the French Alaine Connes.

# Introduction to motivation

- Partly, the motive of the CMI and its founder (see "Rules and Financing") was the founder's support of mathematical research. However, specifically, the inspiration was a similar prize exactly a hundred years earlier.
- Paris had seen a similar event then, at the second International Congress of Mathematicians. The famous German mathematician David Hilbert drew up a list of 23 "Hilbert Problems" on August 8, "setting the agenda for the twentieth century".
- Some of these problems were either shown to be unsolvable, indefinite, or trivial. However, many were difficult problems, and enormous prestige was given to a mathematician who solved one of them as soon as the mathematical community had pronounced his solution correct.

# Solving of the Poincaré Conjecture

- On April 7–11, 2003, Russian mathematician Grigori Perelman, a member of the Steklov Institute of Mathematics, a division of the Russian Academy of Sciences in St.
- Petersburg, presented his proof of the Poincaré Conjecture during the Simons Lecture Series at the MIT.
- He gave three lectures, titled "Ricci Flow and Geometrization of Three-Manifolds.
- These were his first public presentation of the important results he had published earlier, in November 2002 and March 2003.
- Perelman's paper proved not only the Poincaré Conjecture, but a generalization known as Thurston's Geometrization Conjecture.
- The former merely stated that every closed, simply-connected three manifold is homeomorphic to the three sphere.

# Birch and Swinnerton-Dyer Conjecture

- The Birch and Swinnerton-Dyer conjecture relates the rank of the abelian group of points over a number field of an elliptic curve $E$ to the order of the zero of the associated $L$-function $L(E, s)$ at $s = 1$.
- As of 2005, it has been proved only in special cases, such as over certain quadratic fields.
- It has been an open problem for around 40 years.
- and has stimulated much research
- its status as one of the most challenging mathematical questions has become widely recognized.

# Hodge Conjecture

- The Hodge conjecture asserts that structures known as Hodge classes;
- which can be elementarily described as geometric representations of a given manifold's topological properties;
- are composed of algebraic cycles;
- More rigorously, the common phrasing for the conjecture is "Given a projective complex manifold;
- every Hodge class on it is a linear combination with rational coefficients of the cohomology classes of complex subvarieties of it;

# Navier-Stokes Equations

- The Navier-Stokes equations describe the motion of fluids.
- These equations establish that the acceleration of fluid particles is simply the product of changes in pressure and dissipative viscous forces (similar to friction) acting inside the fluid.
- These viscous forces originate in molecular interactions and dictate how sticky (viscous) a fluid is.
- Thus, the Navier-Stokes equations are a dynamical statement of the balance of forces acting at any given region of the fluid.

# P versus NP

- The problem of P versus NP is an important problem in computability and complexity theory relating to whether decision problems (problems admitting a yes or no answer) whose solutions can be verified in polynomial time (as a function of the input, often expressed using big-O notation) can also be solved in polynomial time.
- The set $P$ consists of decision problems such that there exists a deterministic computer program (or Turing machine) that decides $P$ in polynomial time.
- The set $NP$, informally, consists of decision problems whose "yes" instances can be verified by a deterministic program in polynomial time, given a certificate.
- Whether $P = NP$ unknown, though many problems can be shown to be NP-complete - that is, if a problem $L$ is NP-complete, then any NP problem can be reduced to $L$ in polynomial time. This implies that if any NP-complete

# Poincaré Conjecture

- In elementary terms, the Poincaré conjecture states that the only three-manifold with no "holes" is the three-sphere.
- This would also show that the only $n$-manifold with no "holes" is the $n$-sphere; the case $n = 1$ is trivial, the case $n = 2$ is a classic problem, and the truth of the statement for $n \geq 4$ was verified by Stephen Smale in 1961.
- More rigorously, the conjecture is expressed as "Every simply connected, compact three-manifold (without boundary) is homeomorphic to the three-sphere.

# Riemann Hypothesis

- The Riemann hypothesis is a well-known conjecture in analytic number theory that states that all nontrivial (the trivial roots are when $s = -2, -4, -6, \ldots$) zeros of the Riemann zeta function have real part $\frac{1}{2}$.
- The Riemann Hypothesis is an important problem in the study of prime numbers. Let $\pi(x)$ denote the number of primes less than or equal to x, and let $\mathrm{Li}(x) = \int_2^x \frac{1}{\ln t}\, dt$
- Then an equivalent statement of the Riemann hypothesis is that $\pi(x) = \mathrm{Li}(x) + O(x^{1/2}\ln(x))$.

# Yang-Mills Theory

- The quantum Yang-Mills theory (no quarks) with a non-abelian gauge group is an exception to the general rule that nontrivial (i.e. interacting) quantum field theories that we know of in 4D are effective field theories with a cutoff scale.
- It has a property known as asymptotic freedom, meaning that it has a trivial UV fixed point. Because of this, this is the simplest model to pin our hopes on for a nontrivial constructive QFT model in 4D. (QCD, with its fermionic quarks is obviously more complicated).
- It has already been well proven at the standards of theoretical physics, but not mathematical physics, that the quantum Yang–Mills theory for a non-abelian Lie group exhibits a property known as confinement.

# Grigori Perelman's solution

**The Ricci flow**

$$R(X,Y)Z = \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

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# Evolution of curvature

## Lemma 1

$$\frac{\partial R}{\partial t} = \delta R + |Ric|^2$$

## Corollary 2

On a compact manifold M , $min_M$ R is non-decreasing with time.

## Corollary 3

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$$

# Acknowledgments

*Thanks for attention!*

begin