Lecture-17

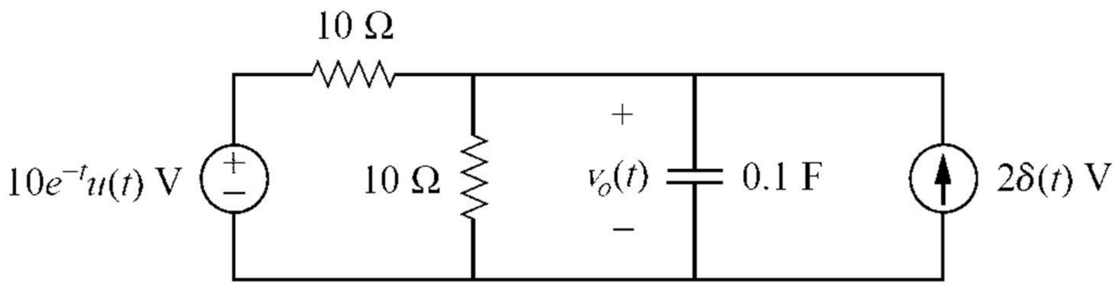
On

INTRODUCTION TO ELECTRICAL ENGINEERING (ESO203)

- Circuit Analysis using Laplace Transform.
- Transfer Function.
- Network Stability.

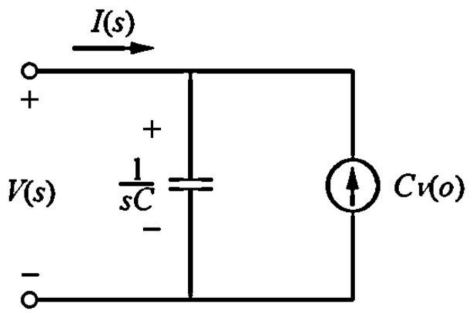
❑ Example:

Find $v_o(t)$ in the circuit. Assume $v_o(0) = 5\ V$.



Solution:

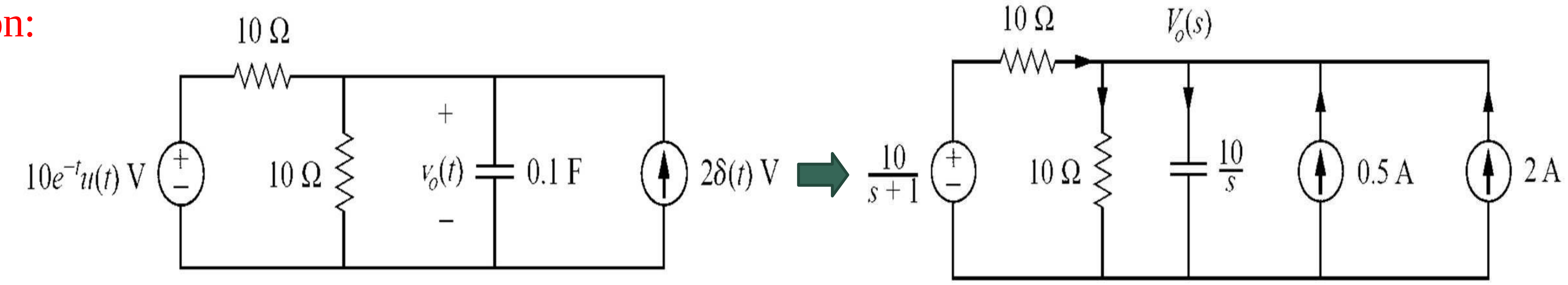
We transform the circuit to the *s* domain as shown in the Figure in next slide. The initial condition is included in the form of the current source



$$Cv_0(0) = 0.1(5) = 0.5A$$

Circuit Analysis using Laplace Transform (Cont…)

Solution:



Apply nodal analysis. At the top node,

$$\frac{10/(s+1) - V_0}{10} + 2 + 0.5 = \frac{V_0}{10} + \frac{V_0}{10/s}$$

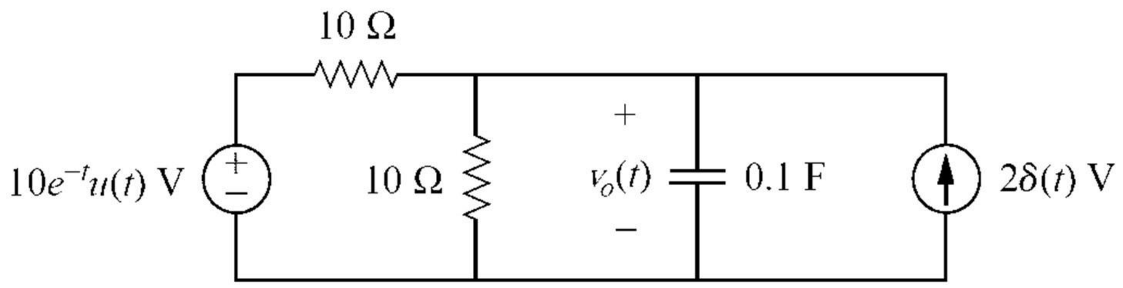
Multiplying through by 10,

$$\frac{10}{s+1} + 25 = V_0(s+2) \qquad or$$

$$V_0 = \frac{25s+35}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

Solution:

On solving, we get $A = 10$, $B = 15$



Thus,

$$V_0(s) = \frac{10}{s+1} + \frac{15}{s+2}$$

Taking the inverse Laplace transform, we obtain

$$v_0(t) = (10e^{-t} + 15e^{-2t})u(t)$$

Transfer Function

- The *transfer function* is a key concept in signal processing because it indicates how a signal is processed as it passes through a network.

- It is a fitting tool for finding the network response, determining (or designing for) network stability, and network synthesis.

- The transfer function of a network describes how the output behaves in respect to the input.

- It specifies the transfer from the input to the output in the *s* domain, assuming no initial energy.

Transfer Function (Cont…)

The transfer function $H(s)$ is the ratio of the output response $Y(s)$ to the input excitation $X(s)$, assuming all initial conditions are zero. Thus,

$$H(s) = \frac{Y(s)}{X(s)}$$

- The transfer function depends on what we define as input and output.
- Since the input and output can be either current or voltage at any place in the circuit, there are four possible transfer functions:

$$H(S) = Voltage\ Gain = \frac{V_0(s)}{V_i(s)} \qquad H(S) = Current\ Gain = \frac{I_0(s)}{I_i(s)}$$

$$H(s) = Impedance = \frac{V(s)}{I(s)} \qquad H(s) = Admittance = \frac{I(s)}{V(s)}$$

Transfer Function (Cont…)

- A circuit can have many transfer functions.
- Note that $H(s)$ is dimensionless for first two transfer functions.
- In the transfer function equations, we assume that $X(s)$ and $Y(s)$ are known.
- Sometimes we know only $X(s)$ and $\mathrm{H}(s)$.
- In that case $Y(s)$ can be found as:- $Y(S) = H(s)X(s)$.

Transfer Function (Cont…)

- We take inverse Laplace transform of $Y(s)$ to get the $y(t)$

- As a special case, $when\ x(t) = \delta(t)$, then $X(s) = 1$

- In that case, $Y(s) = H(s)$, and hence, $y(t) = h(t)$

- Therefore,

$$h(t) = \mathcal{L}^{-1}[H(s)],$$

represents unit impulse response of the circuit.

Transfer Function (Cont…)

The set of transfer functions shown in previous slide can be found in two ways.

- One way is to assume any convenient input $X(s)$, use any circuit analysis technique (such as current or voltage division, nodal or mesh analysis) to find the output $Y(s)$, and then obtain the ratio of the two.

- The other approach is to apply the *ladder method*, which involves walking through the circuit.

- By this approach, we assume that the output is $1\ V$, or $1\ A$ as appropriate, and use the basic laws of Ohm and Kirchhoff (KCL only) to obtain the input.

- The transfer function becomes unity divided by the input.

Transfer Function (Cont…)

- This approach may be more convenient to use when the circuit has many loops or nodes and applying nodal or mesh analysis becomes cumbersome.

- In the first method, assume an input and find the output; in the second method, assume the output and find the input.

- In both methods, calculate $H(s)$ as the ratio of output to input transforms. The two methods rely on the linearity property.

❑ Example:

The output of a linear system is $\boldsymbol{y(t) = 10e^{-t}\cos 4t\, u(t)}$ when the input is $\boldsymbol{x(t) = e^{-t}\, u(t)}$. Find the transfer function of the system and its impulse response.

Solution:

$$y(t) = 10e^{-t}\cos 4t\, u(t) \qquad \text{and} \qquad x(t) = e^{-t}\, u(t), \qquad \text{then} -$$

$$X(s) = \frac{1}{s+1} \qquad \text{and} \qquad Y(s) = \frac{10(s+1)}{(s+1)^2+4^2}$$

$$\text{Hence, } H(s) = \frac{Y(s)}{X(s)} = \frac{10(s+1)^2}{(s+1)^2+1} = \frac{10(s^2+2s+1)}{s^2+2s+17}$$

Transfer Function (Cont…)

- **Solution:**

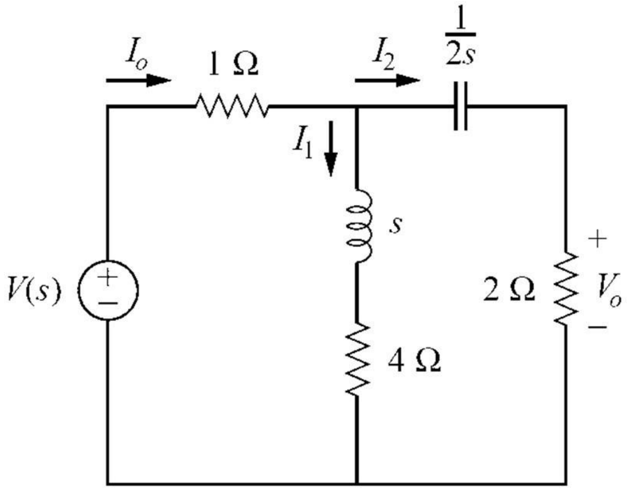
To find $h(t)$ we write $H(s)$ as –

$$H(S) = 10 - 40\frac{4}{(s+1)^2 + 4^2}$$

$$So, \qquad h(t) = 10\delta(t) - 40e^{-t}\sin 4t\, u(t)$$

- Example:

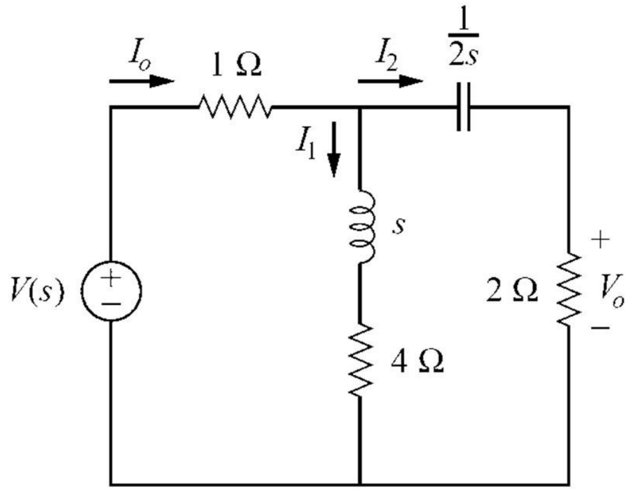
Determine the transfer function $H(S) = V_0(s)/I_0(s)$ of the circuit



- Solution:

METHOD 1: **By Current Division**

$$I_2 = \frac{(s+4)I_0}{s+4+2+1/2s}$$



$$But, \qquad V_0 = 2I_2 = \frac{2(s+4)I_0}{s+6+1/2s}$$

$$Hence, \qquad H(s) = \frac{V_0(s)}{I_0(s)} = \frac{4s(s+4)}{2s^2+12s+1}$$

Transfer Function (Cont…)

❑ Solution:

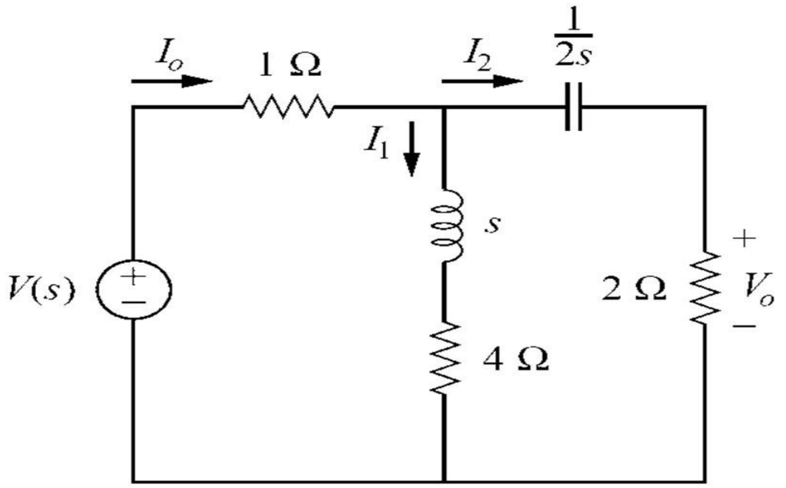
Method 2: We can apply the ladder method. We let $V_o = 1\ V$.

By Ohm's law, $I_2 = Vo/2 = 1/2\ A$. The voltage across the $(2 + 1/2s)$ impedance is

$$V_1 = I_2\left(2 + \frac{1}{2s}\right) = 1 + \frac{1}{4s} = \frac{4s+1}{4s}$$

This is the same as the voltage across the $(s + 4)$ impedance. Hence,

$$I_1 = \frac{V_1}{s+4} = \frac{4s+1}{4s(s+4)}$$



Applying KCL at the top node yields

$$I_0 = I_1 + I_2 = \frac{4s+1}{4s(s+4)} + \frac{1}{2} = \frac{2s^2+12s+1}{4s(s+4)}$$

Hence

$$H(s) = \frac{V_0}{I_0} = \frac{4s(s+4)}{2s^2+12s+1}$$

As before.

Network Stability

A circuit is *stable* if its impulse response $h(t)$ is bounded (i.e., $h(t)$ converges to a finite value) as $t \longrightarrow \infty$; it is *unstable* if $h(t)$ grows without bound as $t \longrightarrow \infty$. In mathematical terms, a circuit is stable when

$$\lim_{t\rightarrow\infty} |h(t)| = finite$$

Since the transfer function $H(s)$ is the Laplace transform of the impulse response $h(t)$,

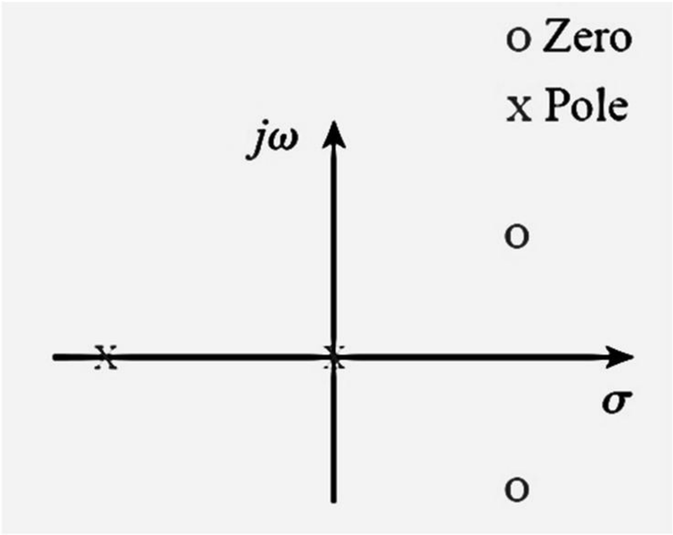
$H(s)$ must meet certain requirements in order for above Equation to hold. As we know that

$H(s)$ may be written as:-

$$H(s) = \frac{N(s)}{D(s)}$$

Network Stability (Cont…)

- where the roots of $N(s\ ) = 0$ are called the *zeros* of $H(s)$ because they make $H(s) = 0$,
- while the roots of $D(s) = 0$ are called the *poles* of $H(s)$ since they cause $H(s) \rightarrow \infty$.
- The zeros and poles of $H(s)$ are often located in the s plane as shown in Figure.



$H(s)$ can also be written in terms of its poles as

$$H(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s + p_1)(s + p_2) \dots \dots (s + p_n)}$$

Network Stability (Cont…)

***$H(s)$* must meet two requirements for the circuit to be stable.**

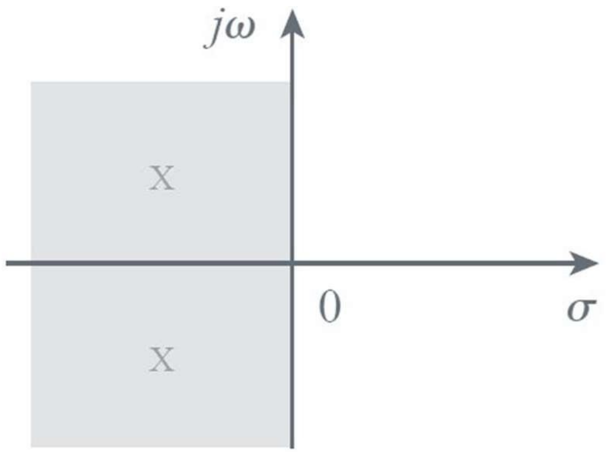
1. The degree of $N(s)$ must be less than the degree of $D(s)$; otherwise, long division would produce

$$H(s) = k_n s^n + k_{n-1} s^{n-1} + \cdots + k_1 s + k_0 + \frac{R(s)}{D(s)}$$

where $R(s)$ is remainder of the long division. The degree of $R(s)$ is less than the degree of $D(s)$.

The inverse of $H(s)$ in this case does not meet the stability condition .

2. All the poles of $H(s)$ (i.e., all the roots of $D(s) = 0$) must have negative real parts, i.e. all the poles must lie in the left half of the s plane, as shown in Figure.



Network Stability (Cont…)

The inverse of $H(s)$ is given as :

$$h(t) = k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \cdots \ldots \ldots \ldots \ldots + k_n e^{-p_n t}$$

It is observed from this equation that

- each pole $-p_i$ must be positive (i.e., pole $s = -p_i$ in the left-half plane) in order for $e^{-p_i t}$ to decrease with increasing t.
- An unstable circuit never reaches steady state because the transient response does not decay to zero.
- Consequently, steady-state analysis is only applicable to stable circuits.

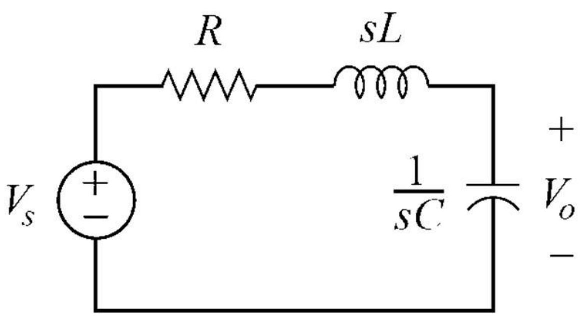
Network Stability (Cont…)

- A circuit made up of passive elements (R, L, *and* C) and *independent sources* cannot be unstable, because that would imply that some branch currents or voltages would grow indefinitely with sources set to zero.

- Passive elements cannot generate such indefinite growth. Passive circuits either are stable or have poles with zero real parts.

- To verify the above aspect, consider the series RLC circuit in Figure. The transfer function is given by

$$H(s) = \frac{V_0}{V_S} = \frac{1/sC}{R + sL + 1/sC} = \frac{1/LC}{s^2 + \frac{sR}{L} + 1/LC}$$



Network Stability (Cont…)

Here, $D(s) = s^2 + \frac{sR}{L} + 1/LC = 0$ is the same as the characteristic equation obtained for the series RLC circuit. The circuit has poles at

$$p_{1,2} = -\alpha \pm \sqrt{\alpha^2 - {\omega_0}^2}$$

where

$$\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

For $R, L, C > 0$, the two poles always lie in the left half of the s plane, implying that the circuit is always stable.

However, when $R = 0$, $\alpha = 0$ and the circuit becomes oscillatory. Although ideally this is possible, it does not really happen practically, because R is never zero.

Network Stability (Cont…)

- Active circuits or passive circuits with controlled sources can supply energy, and they can be unstable.

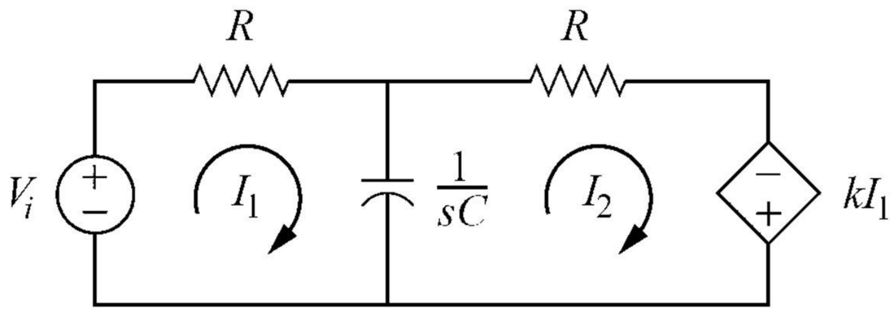
- An oscillator is an example of a circuit designed to be unstable. An oscillator is designed such that its transfer function is of the form

$$H(s) = \frac{N(s)}{s^2 + {\omega_0}^2} = \frac{N(s)}{(s + j\omega_0)(s - j\omega_0)}$$

So that its output is sinusoidal.

- Example:

Determine the values of k for which the circuit in $Figure$ is stable.



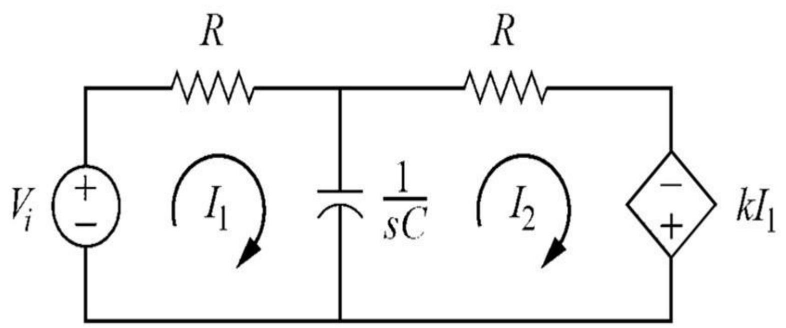
Network Stability (Cont…)

- Solution:

Applying mesh analysis to the first-order circuit in *Figure* gives

Mesh-1:

$$V_i = \left(R + \frac{1}{sC}\right) I_1 - \frac{I_2}{sC}$$



Mesh-2:

$$0 = -kI_1 + \left(R + \frac{1}{sC}\right) I_2 - \frac{I_1}{sC}$$

or

$$0 = -\left(k + \frac{1}{sC}\right) I_1 - \left(R + \frac{1}{sC}\right) I_2$$

Network Stability (Cont…)

Writing $Eqs.$ in matrix form as

$$\begin{bmatrix} V_i \\ 0 \end{bmatrix} = \begin{bmatrix} \left(R + \frac{1}{sC}\right) & -\frac{1}{sC} \\ \left(k + \frac{1}{sC}\right) & \left(R + \frac{1}{sC}\right) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

The determinant is

$$\Delta = \left(R + \frac{1}{sC}\right)^2 - \frac{k}{sC} - \frac{1}{s^2C^2} = \frac{sR^2C + 2R - k}{sC}$$

The characteristic equation ($\Delta = 0$) gives the single pole as

$$p = \frac{k - 2R}{R^2C}$$

which is negative when $k < 2R$. Thus, we conclude the circuit is stable when $k < 2R$ and unstable for $k > 2R$.

Network Synthesis

- Network synthesis is the process of obtaining an appropriate network for a given transfer function.

- Network synthesis is easier to carry out in the s domain than in the time domain.

In network analysis, we find the transfer function of a given network. While in network synthesis, we reverse the approach: i.e. we are required to find a suitable network for a given transfer function

- In synthesis, there may be many different answers or possibly no answers, because there may be many circuits that can be used to represent the same transfer function.

- While in network analysis, there is only one answer.

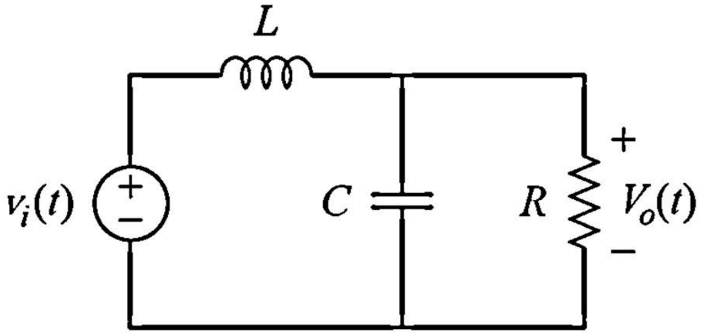
- Example: **Given the transfer function**

$$H(s) = \frac{V_0(s)}{V_i(s)} = \frac{10}{s^2 + 3s + 10}$$

Realize the function using the circuit shown in Figure.

(a) Select $R = 5\Omega$, and find L and C.

(b) Select $R = 1\Omega$, and find L and C.



- Solution:

The *s*-domain equivalent of the circuit shown in figure is shown in next slide.

- Solution:

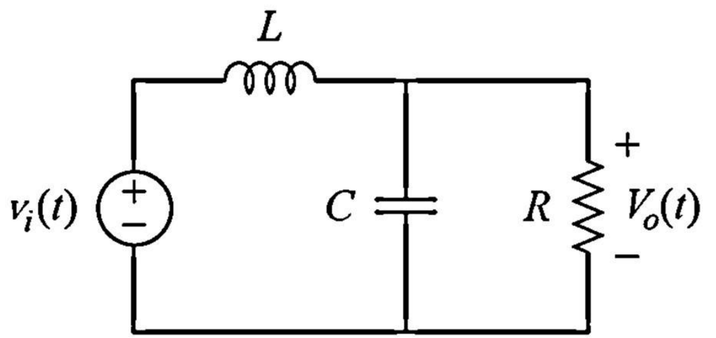
The parallel combination of R and C gives

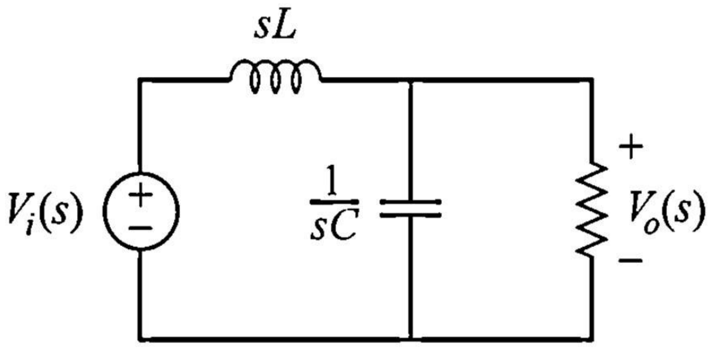
$$R\,||\,\frac{1}{sC} = \frac{R/_{sC}}{R + \frac{1}{sC}} = \frac{R}{1 + sRc}$$

Using the voltage division principle,

$$V_0 = \frac{R/(1 + sRc)}{sL + R/(1 + sRc)} V_i$$

$$\frac{V_0}{V_i} = \frac{R}{s^2RLC + sL + R} = \frac{1/_{LC}}{s^2 + \frac{s}{RC} + 1/_{LC}}$$





- Solution:

$$\frac{V_0}{V_i} = \frac{R}{s^2RLC + sL + R} = \frac{1/LC}{s^2 + \frac{s}{RC} + 1/LC}$$

$$\boldsymbol{H(s) = \frac{V_0(s)}{V_i(s)} = \frac{10}{s^2 + 3s + 10}}$$

Comparing this with the given transfer function $H(s)$ reveals that

$$\frac{1}{LC} = 10, \qquad \frac{1}{RC} = 3$$

There are several values of R, L, and C that satisfy these requirements. So, specifying one element value, can help in determining others.

(a) If we select $R = 5\Omega$, then

$$C = \frac{1}{3R} = 66.67\ mF,$$

$$L = \frac{1}{10C} = 1.5\ H$$

Solution:

(b) If we select $R = 1\Omega$, then

$$C = \frac{1}{3R} = 0.333\ F,$$

$$L = \frac{1}{10C} = 0.3\ H$$

Making $R = 1\Omega$ can be regarded as ***normalizing*** the design.

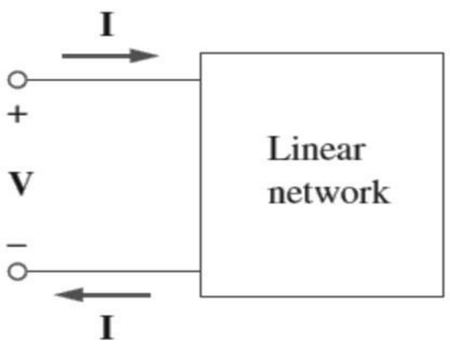
Impedance Parameters

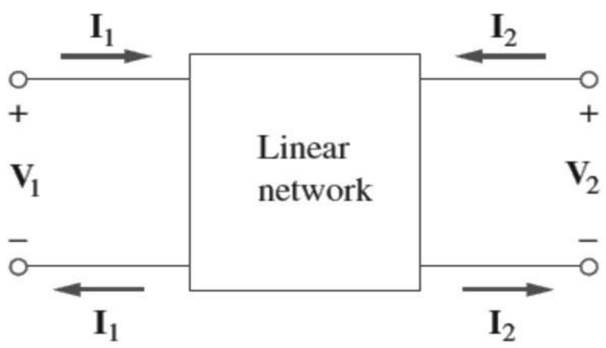
❑ Introduction

- A pair of terminals through which a current may enter or leave a network is known as a port.
- A port is an access to the network and consists of a pair of terminals; the current entering one terminal leaves through the other terminal so that the net current entering the port equals zero.
- Two-terminal devices or elements (such as resistors, capacitors, and inductors) result in one-port networks.
- Most of the circuits we have dealt with so far are two-terminal or one-port circuits.
- In general, a network may have n ports.
- In this module, the discussion is mainly focused on two-port networks.

❑ Introduction

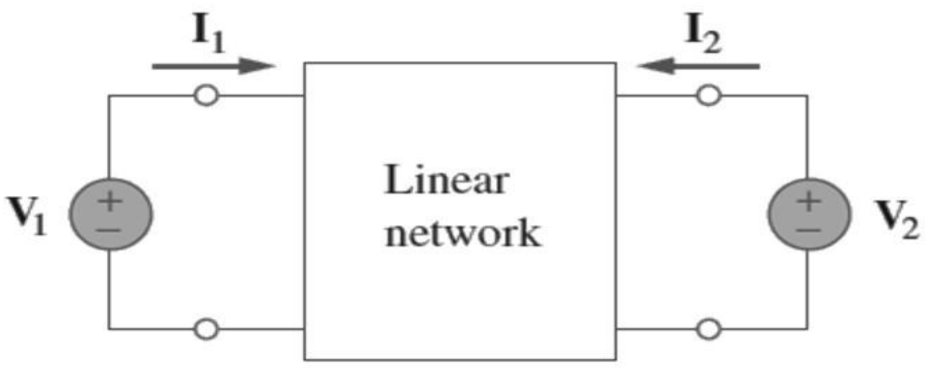
- A two-port network is an electrical network with two separate ports for input and output.
- Thus, a two-port network has two terminal pairs acting as access points unlike the one port network.
- As shown in the figure below, the current entering one terminal of a pair leaves the other terminal in the pair.
- To characterize a two-port network, it is required that we relate the terminal quantities $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{I}_1$, and $\mathbf{I}_2$, out of which two are independent.
- The various terms that relate these voltages and currents are called parameters.

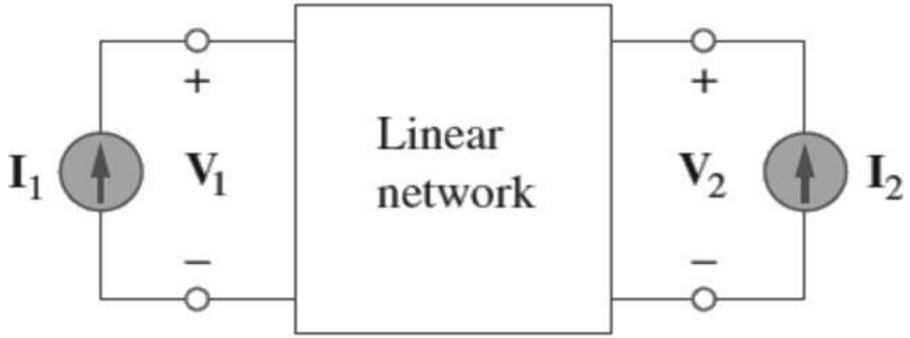




Impedance Parameters (Cont…)

- A two-port network may be voltage driven or current driven as illustrated in the figure below.
- To determine the impedance parameters the terminal voltages are expressed in terms of the terminal currents.



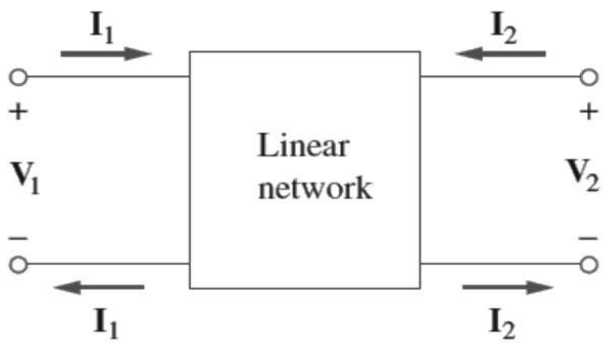


Impedance Parameters (Cont…)

- The relation between the terminal voltage and the terminal current can be established using the following equations:

$$\mathbf{V}_1 = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2$$

$$\mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{z}_{22}\mathbf{I}_2$$



- This can alternatively be expressed in matrix form as,

$$\begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\end{bmatrix} = \begin{bmatrix}\mathbf{z}_{11} & \mathbf{z}_{12}\\ \mathbf{z}_{21} & \mathbf{z}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{I}_1\\ \mathbf{I}_2\end{bmatrix} = [\mathbf{z}]\begin{bmatrix}\mathbf{I}_1\\ \mathbf{I}_2\end{bmatrix}$$

- Here the **z** terms are known as the impedance parameters or the z parameters.

- Impedance parameters are expressed in ohms.

Impedance Parameters (Cont…)

- The values of the parameters can be evaluated by setting $\mathbf{I_1 = 0}$ (input port open-circuited) or $\mathbf{I_2 = 0}$ (output port open-circuited).

- Since the *z* parameters are obtained by open-circuiting the input or output port, they are also called the open-circuit impedance parameters.

- The individual parameters are evaluated as follows:

$$\mathbf{V}_1 = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2$$

$$\mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{z}_{22}\mathbf{I}_2$$

- If $\mathbf{I}_2 = 0$, then

$$\mathbf{z}_{11} = \frac{\mathbf{V}_1}{\mathbf{I}_1}, \mathbf{z}_{21} = \frac{\mathbf{V}_2}{\mathbf{I}_1}$$

- If $\mathbf{I}_1 = 0$, then

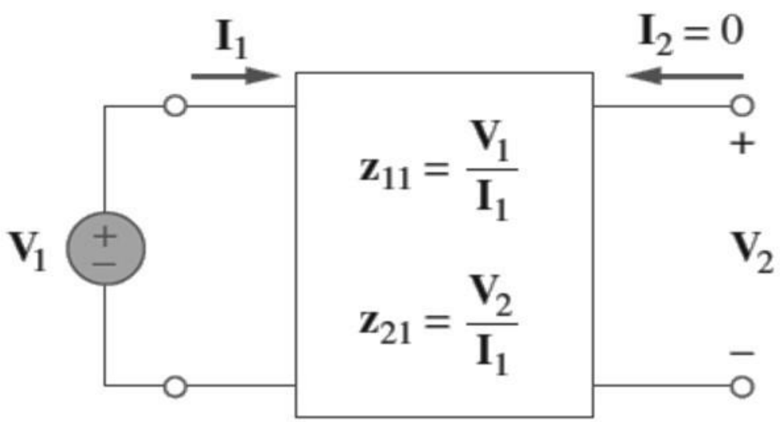
$$\mathbf{z}_{12} = \frac{\mathbf{V}_1}{\mathbf{I}_2}, \mathbf{z}_{22} = \frac{\mathbf{V}_2}{\mathbf{I}_2}$$

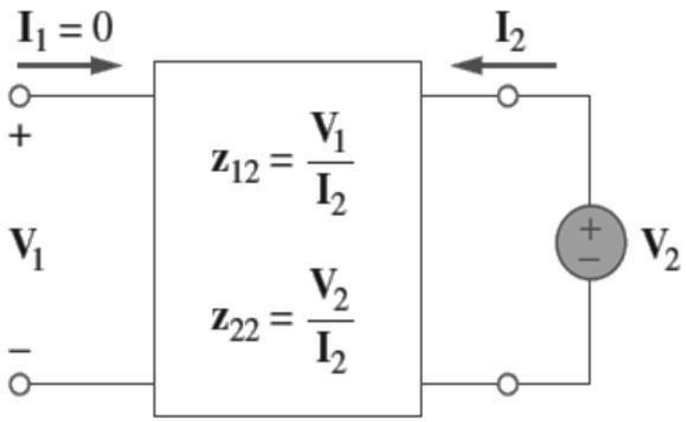
Impedance Parameters (Cont…)

- Specifically,
  - $\mathbf{z}_{11}$ = Open-circuit input impedance
  - $\mathbf{z}_{12}$ = Open-circuit transfer impedance from port 1 to port 2
  - $\mathbf{z}_{21}$ = Open-circuit transfer impedance from port 2 to port 1
  - $\mathbf{z}_{22}$ = Open-circuit output impedance
  - The values of $\mathbf{z}_{11}$ and $\mathbf{z}_{21}$ can be obtained by connecting a voltage $\mathbf{V}_1$ (or current $\mathbf{I}_1$) to port 1 with port 2 open-circuited ($\mathbf{I}_2 = 0$).
  - $\mathbf{I}_1, \mathbf{V}_1$ and $\mathbf{V}_2$ are then obtained which can be used to evaluate $\mathbf{z}_{11} = \frac{\mathbf{V}_1}{\mathbf{I}_1}$ and $\mathbf{z}_{21} = \frac{\mathbf{V}_2}{\mathbf{I}_1}$
  - Similarly, the values of $\mathbf{z}_{12}$ and $\mathbf{z}_{22}$ can be obtained by connecting a voltage $\mathbf{V}_2$ (or current $\mathbf{I}_2$) to port 2 with port 1 open-circuited ($\mathbf{I}_1 = 0$).
  - $\mathbf{I}_2, \mathbf{V}_2$ and $\mathbf{V}_1$ are then obtained which can be used to evaluate $\mathbf{z}_{12} = \frac{\mathbf{V}_1}{\mathbf{I}_2}$ and $\mathbf{z}_{22} = \frac{\mathbf{V}_2}{\mathbf{I}_2}$

Impedance Parameters (Cont…)

- The above procedure can be easily understood using the following Figure.






THANKS
FOR YOUR
ATTENTION