Lecture-23

On

# INTRODUCTION TO ELECTRICAL ENGINEERING (ESO203)

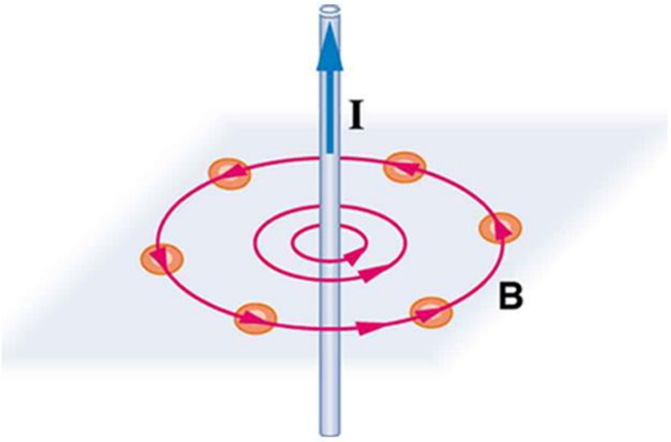
- Magnetically Coupled Circuits.
- Energy in a Coupled Circuit.
- Operating principle of transformer.

# Magnetically Coupled Circuits

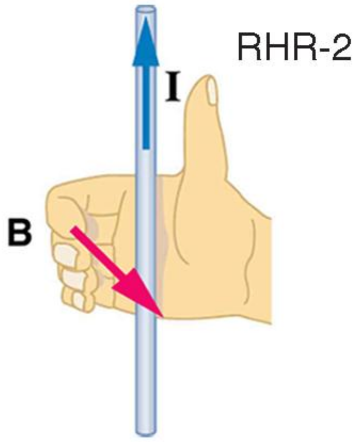
- The circuits, we have considered so far, are the conductively coupled, because one loop affects the neighboring loop through current conduction.
- When two loops with or without contacts between them affect each other through the magnetic field generated by one of them, they are said to be magnetically coupled.
- The transformer is an electrical device designed on the basis of the concept of magnetic coupling.
- It uses magnetically coupled coils to transfer energy from one circuit to another.
- Transformers are key circuit elements of the power system network.
- They are used in power systems for stepping up or stepping down AC voltages or currents.

# Magnetically Coupled Circuits (cont…)

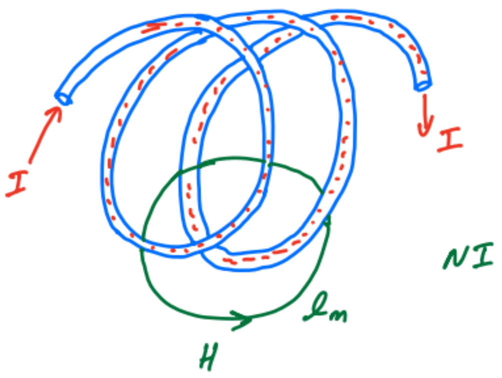
## ❑ Ampere's law:




(a)




(b)




Flux density:-

$$B = \frac{\phi}{A_c} = \text{Tesla} = \frac{\text{weber}}{m^2}$$

Ampere's law:-

$$\int_0^{l_m} H \cdot dl = NI = \text{mmf}$$

$$H \int_0^{l_m} dl = NI$$

$$H \cdot l_m = NI$$

$$\boxed{H = \frac{NI}{l_m}}$$

Amp/m

Where, H is magnetic field intensity and B is flux density.

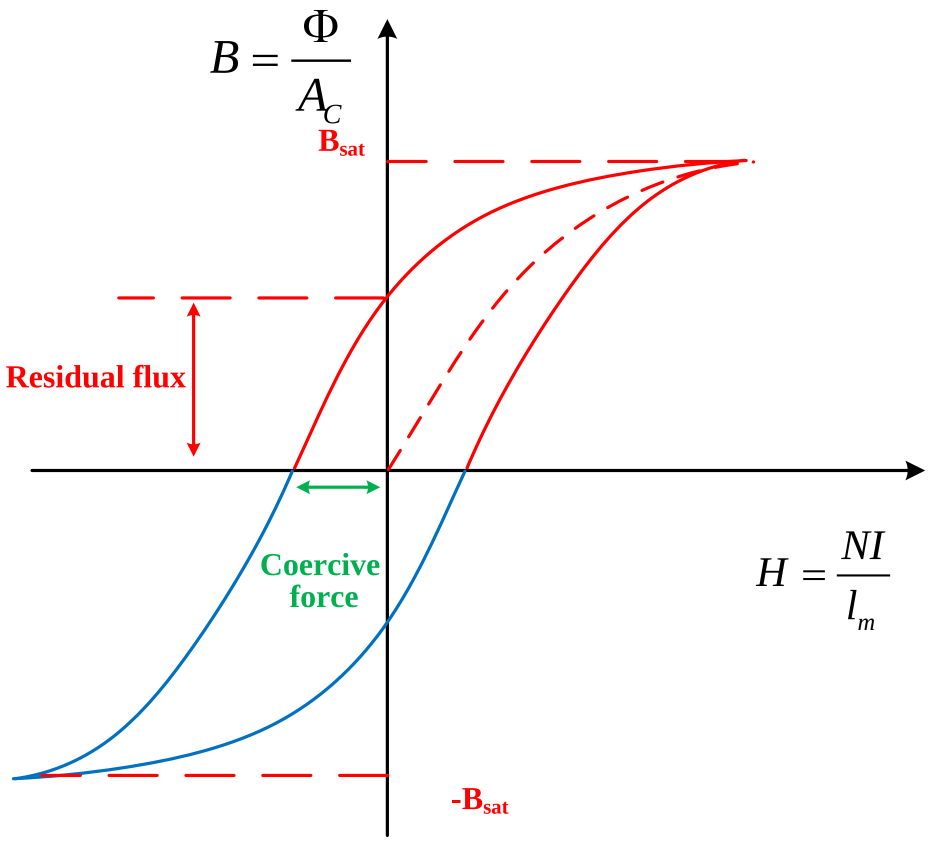



Fig. BH loop

$$H = \frac{NI}{l_m} = \frac{mmf}{l_m}$$

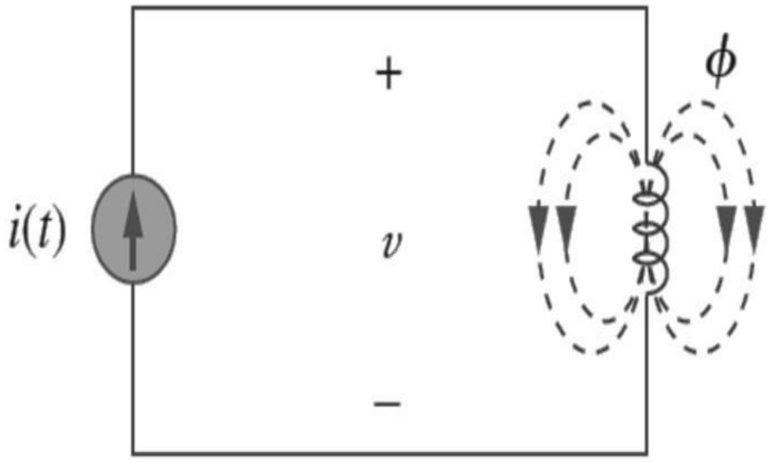
$$B = \frac{\phi}{A_c}$$

$$B = \mu H$$

Where μ Permeability

## ❑ Mutual Inductance:

- When two inductors (or coils) are in a close proximity to each other, the magnetic flux caused by current in one coil links with the other coil, thereby inducing voltage in the latter.
- This phenomenon is known as mutual inductance.
- Let us first consider a single inductor, a coil with $N$ turns.
- When current $i$ flows through the coil, a magnetic flux $\Phi$ is produced around it.



# Magnetically Coupled Circuits (cont…)

- According to Faraday's law of electromagnetism, the voltage $\boldsymbol{v}$ induced in the coil is proportional to the number of turns $\boldsymbol{N}$ and the rate of change of magnetic flux, $\boldsymbol{\Phi}$; and is given by,

$$v = N\frac{d\Phi}{dt}$$

- But the flux is produced by current $\boldsymbol{i}$ so that any change in $\boldsymbol{\Phi}$ is caused by a change in the current.
- The above equation can therefore be rewritten as,

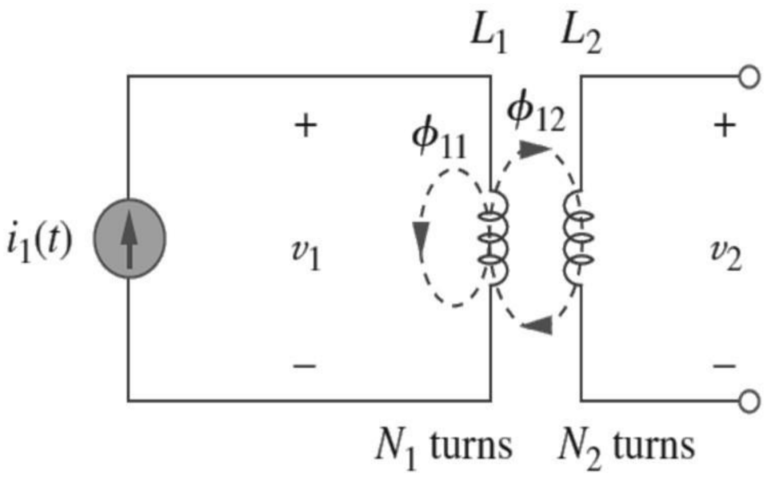
$$v = N\frac{d\Phi}{di}\frac{di}{dt} = L\frac{di}{dt}$$

which is the voltage current relationship for the inductor.

- This inductance is commonly called self-inductance, because it relates the voltage induced in a coil by a time-varying current in the same coil.

# Magnetically Coupled Circuits (cont…)

- Now consider two coils with self-inductances $L_1$ and $L_2$ that are in close proximity with each other.
- Coil 1 has $N_1$ turns, while coil 2 has $N_2$ turns.
- For the sake of simplicity, assume that the second inductor carries no current.
- The magnetic flux $\Phi_1$ emanating from coil 1 has two components: One component $\Phi_{11}$ that links only coil 1, and another component $\Phi_{12}$ links both coils.
- Hence, $\Phi_1 = \Phi_{11} + \Phi_{12}$.



# Magnetically Coupled Circuits (cont…)

- Although the two coils are physically separated, they are said to be magnetically coupled.

- Since the entire flux $\Phi_1$ links coil 1, the voltage induced in coil 1 is

$$v_1 = N_1 \frac{d\Phi_1}{dt} = N_1 \frac{d\Phi_1}{di_1}\frac{di_1}{dt} = L_1 \frac{di_1}{dt}$$

where $\boldsymbol{i_1}$ is the current in **coil 1** and $\boldsymbol{L_1}$ is the self-inductance of **coil 1**.

- Only flux $\Phi_{12}$ links coil 2, so the voltage induced in coil 2 is

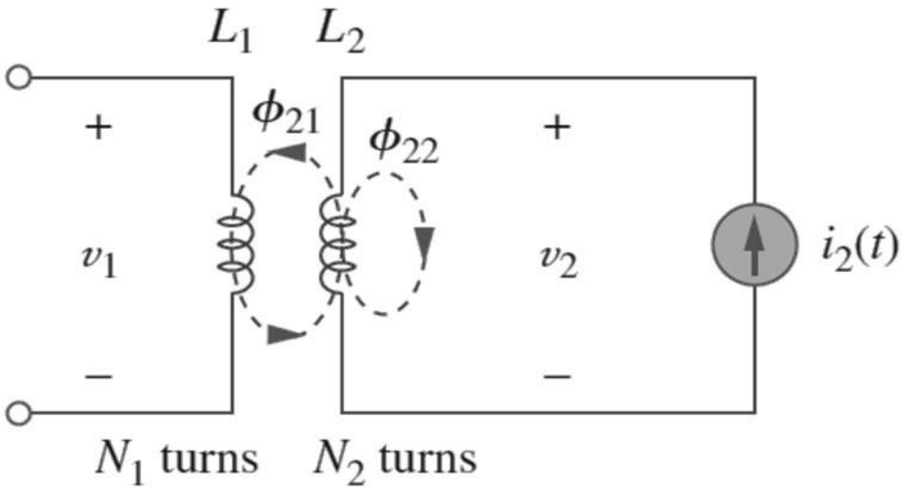
$$v_2 = N_2 \frac{d\Phi_{12}}{dt} = N_2 \frac{d\Phi_{12}}{di_1}\frac{di_1}{dt} = M_{21} \frac{di_1}{dt}$$

- $\boldsymbol{M_{21}}$ is the mutual inductance of **coil 2** with respect to **coil 1**.

- Subscript **21** indicates that the inductance relates the voltage induced in coil 2 due to the current in coil 1.

# Magnetically Coupled Circuits (cont…)

- Now consider current $i_2$ flowing through coil 2.
- The magnetic flux $\Phi_2$ emanating from coil 2 has two components:
- One component $\Phi_{22}$ links only coil 2, and another component $\Phi_{21}$ links both coils.
- Hence, $\Phi_2 = \Phi_{21} + \Phi_{22}$.



# Magnetically Coupled Circuits (cont…)

- Since the entire flux $\Phi_2$ links coil 2, the voltage induced in coil 2 is

$$v_2 = N_2 \frac{d\Phi_2}{dt} = N_2 \frac{d\Phi_2}{di_2} \frac{di_2}{dt} = L_2 \frac{di_2}{dt}$$

where $i_2$ is the current in coil 2 and $L_2$ is the self-inductance of coil 2.

- Only flux $\Phi_{21}$ links coil 1, so the voltage induced in coil 1 is

$$v_1 = N_1 \frac{d\Phi_{21}}{dt} = N_1 \frac{d\Phi_{21}}{di_2} \frac{di_2}{dt} = M_{12} \frac{di_2}{dt}$$

- $M_{12}$ is the mutual inductance of coil 1 with respect to coil 2.
- Mutual inductance is the ability of one inductor to induce a voltage across a neighboring inductor, measured in henrys (H).

# Magnetically Coupled Circuits (cont…)

- Although mutual inductance $M$ is always a positive quantity, the mutual voltage $Mdi/dt$ may be negative or positive, just like the self-induced voltage $Ldi/dt$.

- However, unlike the self-induced $Ldi/dt$, whose polarity is determined by the reference direction of the current and the reference polarity of the voltage (according to the passive sign convention), the polarity of mutual voltage $Mdi/dt$ is not easy to determine, because four terminals are involved.

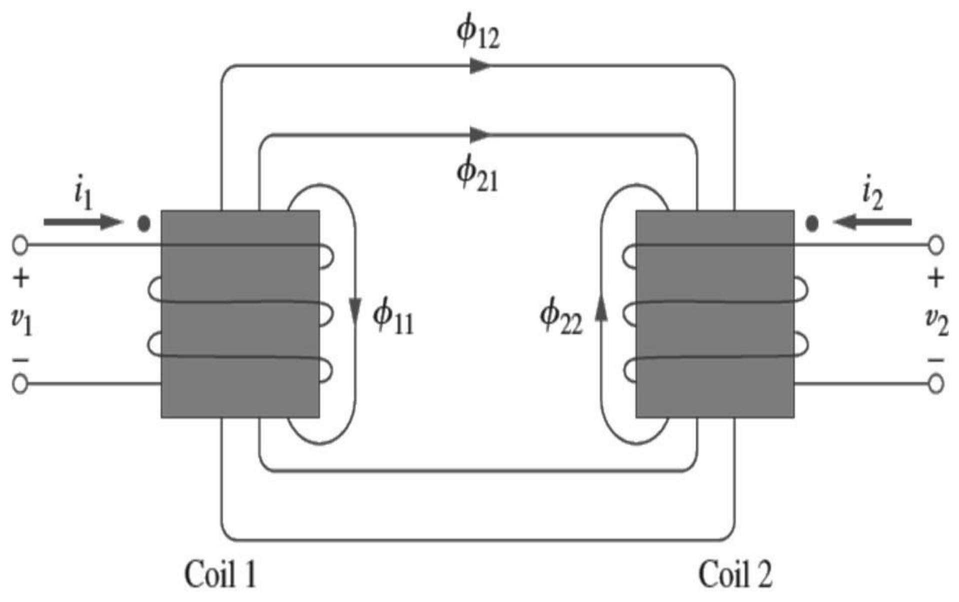
- The choice of the correct polarity for $Mdi/dt$ is made by examining the orientation (the particular way in which both coils are physically wound) and applying Lenz's law along with the right-hand rule.

- Since it is not easy to show the construction details of coils on a circuit schematic, we apply the *dot convention* in circuit analysis.

- By this convention, a dot is placed in the circuit at one end of each of the two magnetically coupled coils to indicate the direction of the magnetic flux if current enters that dotted terminal of the coil.

# Magnetically Coupled Circuits (cont…)

- This concept is illustrated in the following figure.




- Given a circuit, the dots are already placed beside the coils so that we need not bother about how to place them.

- The dots are used along with the dot convention to determine the polarity of the mutual voltage.

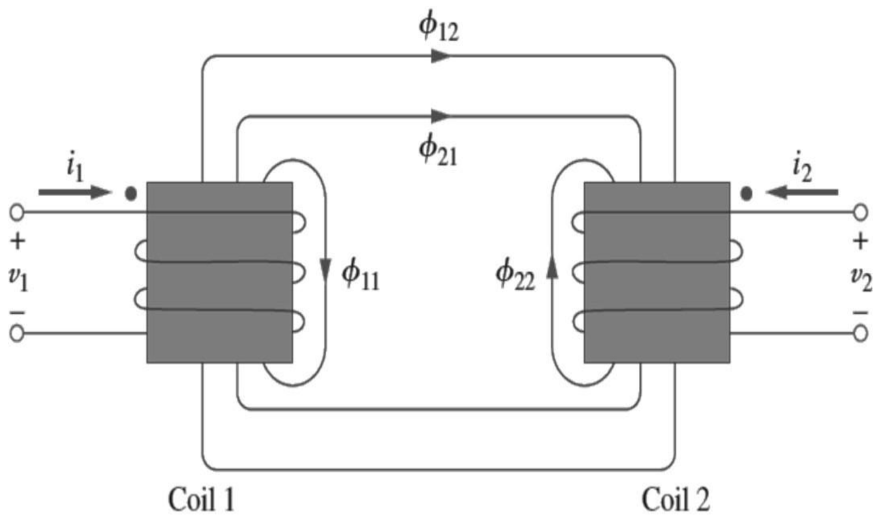
# Magnetically Coupled Circuits (cont…)

The dot convention can be explained as follows:

- If a current enters the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is positive at the dotted terminal of the second coil.

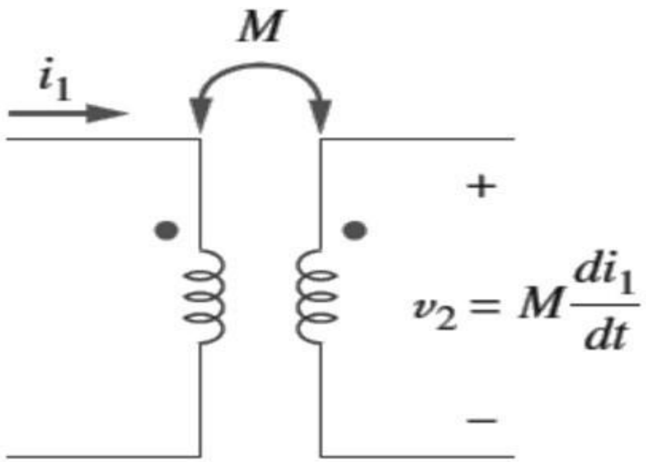
- Alternatively, if a current leaves the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is negative at the dotted terminal of the second coil.

- Thus, the reference polarity of the mutual voltage depends on the reference direction of the inducing current and the dots on the coupled coils.



# Magnetically Coupled Circuits (cont…)

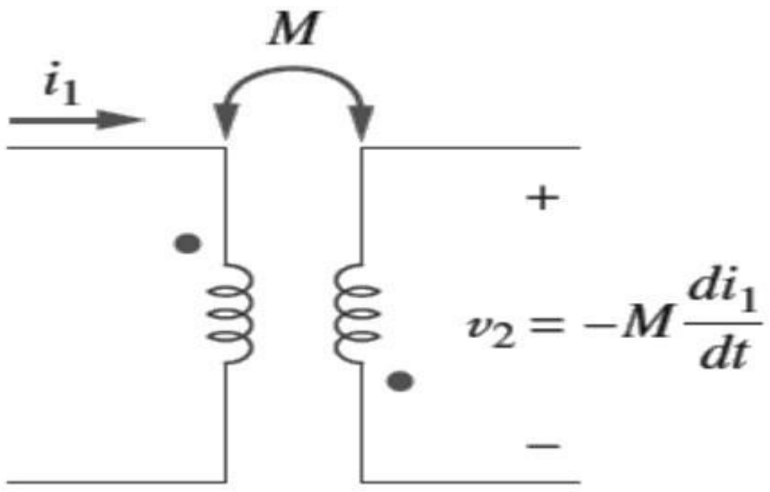
- For the figure given below,




- The sign of the mutual voltage $v_2$ is determined by the reference polarity for $v_2$ and the direction of $i_1$.
- Since $i_1$ enters the dotted terminal of coil 1 and $v_2$ is positive at the dotted terminal of coil 2, the mutual voltage is $Mdi_1/dt$.

# Magnetically Coupled Circuits (cont…)

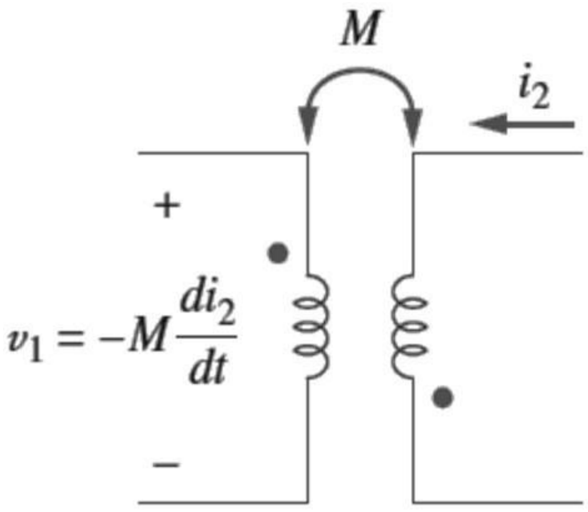
- For the figure given below,

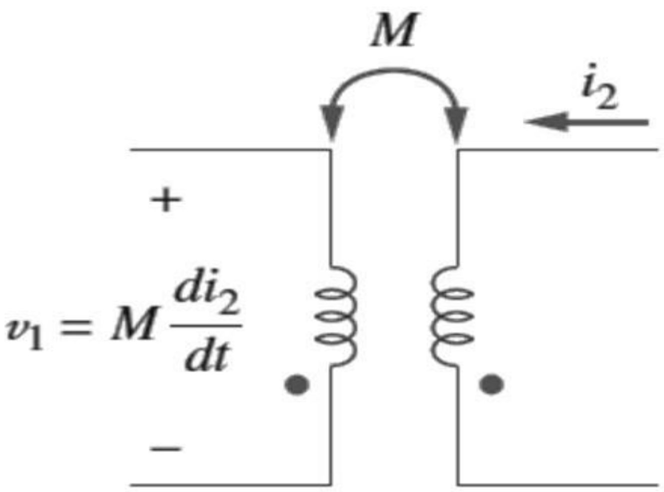



- Since $i_1$ enters the dotted terminal of coil 1 and $v_2$ is negative at the dotted terminal of coil 2, the mutual voltage is $-Mdi_1/dt$.

# Magnetically Coupled Circuits (cont…)

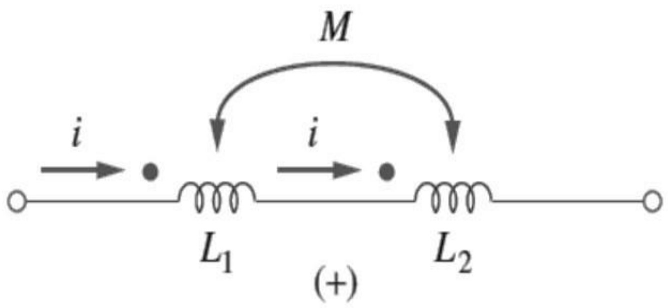
- The reasoning applied to the previous two circuits can be used to understand the following two circuits as well.



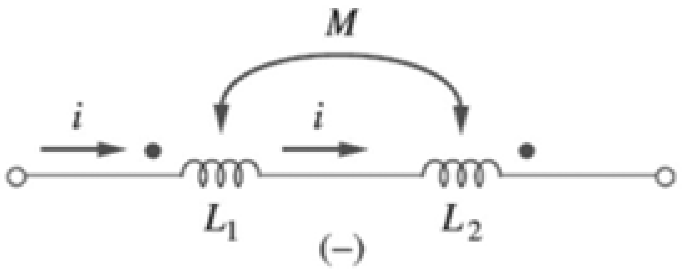
# Magnetically Coupled Circuits (cont…)

- Consider the series circuit shown below.




- The total inductance is calculated as,

$$L = L_1 + L_2 + 2M \Rightarrow Series - aiding\ connection$$

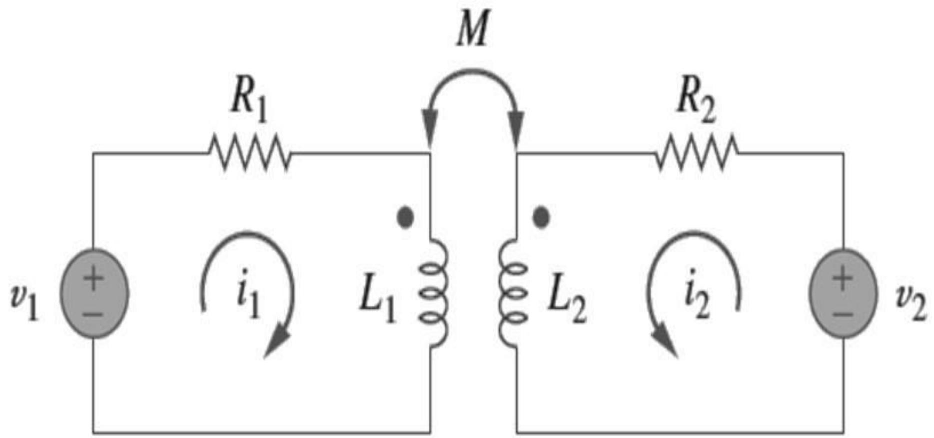



- For the above circuit, the total inductance is calculated as,

$$L = L_1 + L_2 - 2M \Rightarrow Series - opposing\ connection$$

# Magnetically Coupled Circuits (cont…)

- Now that we know how to determine the polarity of the mutual voltage, we are prepared to analyze circuits involving mutual inductance.
- Consider the following circuit:




- Applying KVL to coil 1 gives,

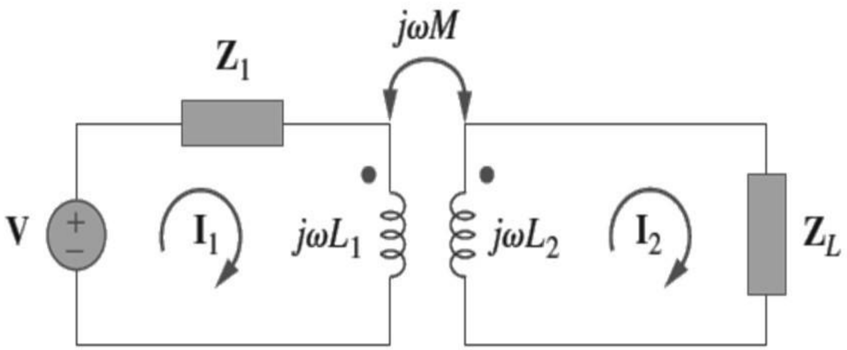
$$v_1 = i_1R_1 + L_1\frac{di_1}{dt} + M\frac{di_2}{dt} \Rightarrow \mathbf{V}_1 = (R_1 + j\omega L_1)\mathbf{I}_1 + j\omega M\mathbf{I}_2$$

- Applying KVL to coil 2 gives,

$$v_2 = i_2R_2 + L_2\frac{di_2}{dt} + M\frac{di_1}{dt} \Rightarrow \mathbf{V}_2 = (R_2 + j\omega L_2)\mathbf{I}_2 + j\omega M\mathbf{I}_1$$

# Magnetically Coupled Circuits (cont…)

- As second example, consider the following circuit:




- Applying KVL to coil 1, in frequency domain, gives,

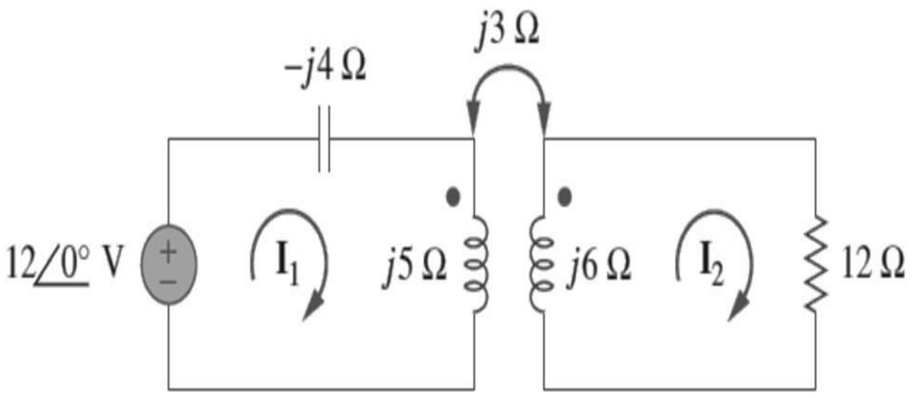
$$\mathbf{V} = (\mathbf{Z}_1 + j\omega L_1)\mathbf{I}_1 - j\omega M\mathbf{I}_2$$

- Applying KVL to coil 2 gives,

$$0 = (\mathbf{Z}_L + j\omega L_2)\mathbf{I}_2 - j\omega M\mathbf{I}_1$$

Example:

- Calculate $\mathbf{I}_1$ and $\mathbf{I}_2$ in the circuit?




Solution: Applying KVL to mesh 1,

$$-12 + (-j4 + j5)\mathbf{I}_1 - j3\mathbf{I}_2 = 0 \Rightarrow j\mathbf{I}_1 - j3\mathbf{I}_2 = 12$$

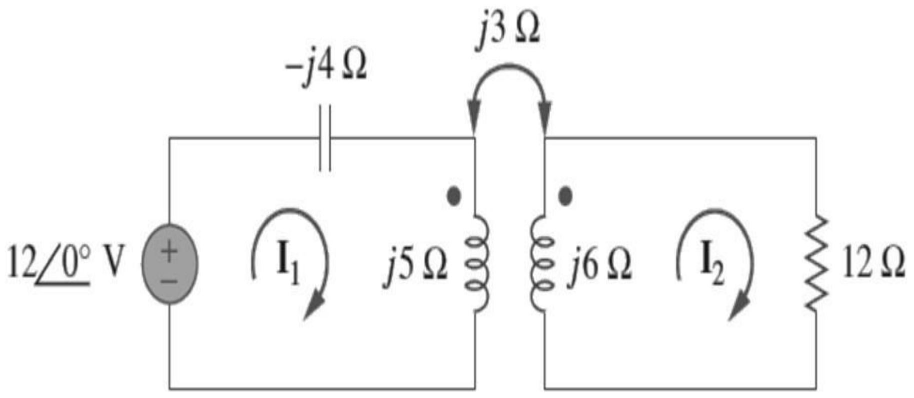
For mesh 2,

$$-j3\mathbf{I}_1 + (12 + j6)\mathbf{I}_2 = 0 \Rightarrow \mathbf{I}_1 = (2 - j4)\mathbf{I}_2$$

# Magnetically Coupled Circuits (cont…)

Using mesh 2 equation in mesh 1 equation ,

$$(j2 + 4 - j3)\mathbf{I}_2 = 12$$




Therefore,

$$\mathbf{I}_2 = \frac{12}{4 - j} = 2.91\angle 14.04° A$$

The current $\mathbf{I}_1$ is given by,

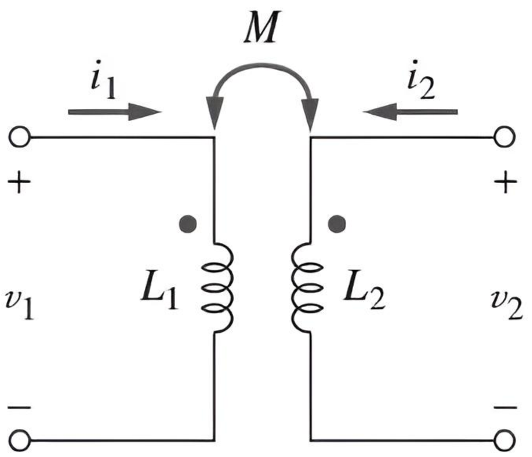
$$\mathbf{I}_1 = (2 - j4)\mathbf{I}_2 = 13.01\angle - 49.39°\ \text{A}$$

# Energy in a Coupled Circuit

- The energy stored in an inductor is given by,

$$w = \frac{1}{2}Li^2$$

- To determine the energy stored in a magnetically coupled coil, consider the following circuit.



# Energy in a Coupled Circuit (cont…)

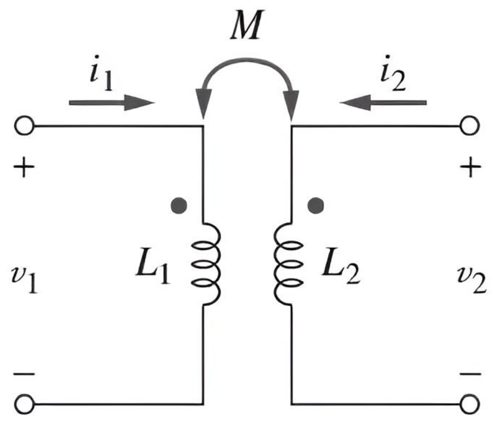
- We assume that currents $i_1$ and $i_2$ are zero initially, so that the stored energy is 0.

- If $i_1$ is allowed to increase from 0 to $I_1$ while maintaining $i_2 = 0$ , the power in coil 1 is given by,

$$p_1(t) = v_1 i_1 = i_1 L_1 \frac{di_1}{dt}$$

and the energy stored in the circuit is given by

$$w_1 = \int p_1 dt = L_1 \int_0^{I_1} i_1 di_1 = \frac{1}{2} L_1 I_1^2$$



# Energy in a Coupled Circuit (cont…)

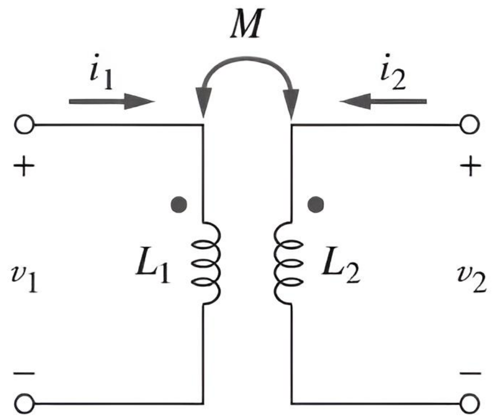
- If, now, $i_2$ is allowed to increase from 0 to $I_2$ while maintaining $i_1 = I_1$, the induced mutual voltage in coil 1 is $M_{12}\frac{di_2}{dt}$, while in coil 2 it is 0 as $i_1$ is not changing.

- The power in coils due to increase in $i_2$ is given by,

$$p_2(t) = i_1 M_{12}\frac{di_2}{dt} + i_2 v_2 = i_1 M_{12}\frac{di_2}{dt} + i_2 L_2\frac{di_2}{dt}$$

and the energy stored in the circuit is given by

$$w_2 = \int p_2 dt = M_{12} I_1 \int_0^{I_2} di_2 + L_2 \int_0^{I_2} i_2 di_2 = M_{12} I_1 I_2 + \frac{1}{2} L_2 I_2^2$$



# Energy in a Coupled Circuit (cont…)

- The total energy stored in the circuit when both $i_1$ and $i_2$ have reached constant values is

$$w = w_1 + w_2 = \frac{1}{2}L_1 I_1^2 + M_{12} I_1 I_2 + \frac{1}{2}L_2 I_2^2$$

- If we reverse the order by which the current reach their final values, that is, if we increase $i_2$ from $0$ to $I_2$ and then increase $i_1$ from $0$ to $I_1$ , the total storage in the circuit is,

$$w = \frac{1}{2}L_1 I_1^2 + M_{12} I_1 I_2 + \frac{1}{2}L_2 I_2^2$$

- Since the total energy stored should be the same regardless of how we reach the final conditions, comparing the above equations leads us to conclude that

$$M = M_{12} = M_{21}$$

$$w = \frac{1}{2}L_1 I_1^2 + M I_1 I_2 + \frac{1}{2}L_2 I_2^2$$

- This equation was derived based on the assumption that the currents in both coils entered the dotted terminals.

# Energy in a Coupled Circuit (cont…)

- If one current enters one dotted terminal while the other current leaves the other dotted terminal, the mutual voltage is negative, so that the mutual energy is also negative.

- In that case

$$w = \frac{1}{2}L_1I_1^2 - MI_1I_2 + \frac{1}{2}L_2I_2^2$$

- Also, since $I_1$ and $I_2$ are arbitrary values, they may be replaced by $i_1$ and $i_2$ which gives the instantaneous energy stored in the circuit using the general expression

$$w = \frac{1}{2}L_1i_1^2 \pm Mi_1i_2 + \frac{1}{2}L_2i_2^2$$

- The positive sign is selected for the mutual term if both currents enter or leave the dotted terminals of the coils; the negative sign is selected otherwise.

# Energy in a Coupled Circuit (cont…)

- We will now establish an upper limit for the mutual inductance $M$.

- Therefore,

$$\frac{1}{2}L_1 i_1^2 - M i_1 i_2 + \frac{1}{2}L_2 i_2^2 \geq 0$$

- To complete the square we both add and subtract $i_1 i_2\sqrt{L_1 L_2}$ on the right hand side and obtain,

$$\frac{1}{2}\left(i_1\sqrt{L_1} - i_2\sqrt{L_2}\right)^2 + i_1 i_2\left(\sqrt{L_1 L_2} - M\right) \geq 0$$

## Energy in a Coupled Circuit (cont…)

- The squared term is never negative; at least it can be zero.
- Therefore, the second term of the previous equation must be greater than zero.
- So,

$$\left(\sqrt{L_1 L_2} - M\right) \geq 0 \Rightarrow M \leq \sqrt{L_1 L_2}$$

- Thus, the mutual inductance cannot be greater than the geometric mean of the self-inductances of the coils.
- The extent to which the mutual inductance M approaches the upper limit is specified by the coefficient of coupling $k$, given by

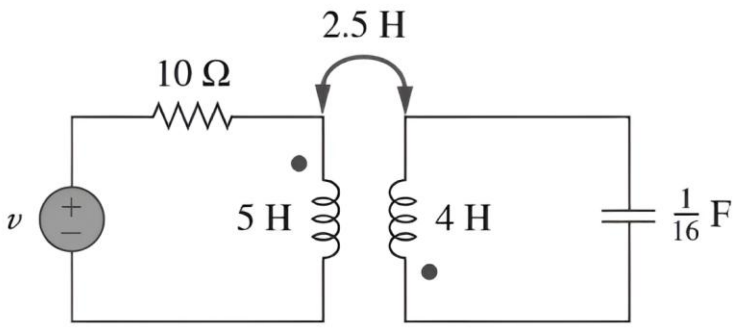
$$M = k\sqrt{L_1 L_2}$$

- The coupling coefficient $k$ is a measure of the magnetic coupling between two coils; $0 \leq k \leq 1$

# Energy in a Coupled Circuit (cont…)

## ❑ Example:

- Consider the circuit shown in the figure. Determine the coupling coefficient? Calculate the energy stored in the coupled inductors at time $t = 1s$ if $v = 60\cos(4t + 30)\,V$




- Solution: The coupling coefficient is

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{2.5}{\sqrt{20}} = 0.56$$

- Note: If the entire flux produced by one coil links another coil, then $k = 1$ and we have 100 percent coupling, or the coils are said to be perfectly coupled. For $k < 0.5$ coils are said to be loosely coupled; and for $k > 0.5$, they are said to be tightly coupled.

# Energy in a Coupled Circuit (cont…)

- To find the energy stored, we need to calculate the current.
- To find the current, we need to obtain the frequency-domain equivalent of the circuit.

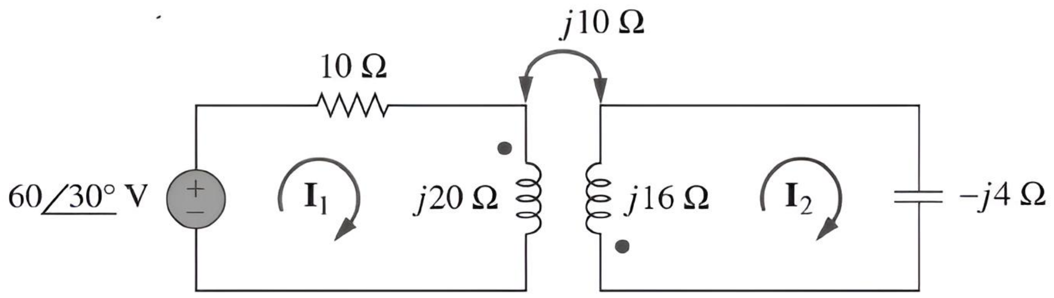
$$60\cos(4t+30) \Rightarrow 60\angle 30°, \omega = 4\, rad/s$$

$$5H \Rightarrow j\omega L_1 = j20$$

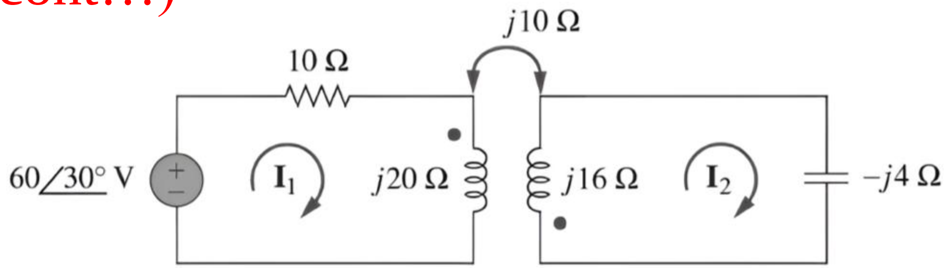
$$2.5H \Rightarrow j\omega M = j10$$

$$4H \Rightarrow j\omega M = j16$$

$$\frac{1}{16}F \Rightarrow \frac{1}{j\omega C} = -j4$$



## Energy in a Coupled Circuit (cont…)




- We now apply mesh analysis.

- For mesh 1,

$$(10 + j20)I_1 + j10I_2 = 60\angle 30°$$

- For mesh 2,

$$(-j4 + j16)I_2 + j10I_1 = 0 \Rightarrow I_1 = -1.2I_2$$

- Substituting the above in the first loop gives,

$$(-12 - j14)I_2 = 60\angle 30° \Rightarrow I_2 = 3.254\angle 160.6°$$

$$I_1 = -1.2I_2 = 3.905\angle - 19.4° A$$

# Energy in a Coupled Circuit (cont…)

- In time domain,

$$i_1 = 3.905\cos(4t - 19.4)\,, \qquad i_2 = 3.254\cos(4t + 160.6)$$

- At $t = 1s, 4t = 4\,rad = 229.2°$

$$i_1 = 3.905\cos(229.2° - 19.4) = -3.389A$$

$$i_2 = 3.254\cos(229.2° + 160.6) = 2.284A$$

- The total energy stored in a coupled inductor,

$$w = \frac{1}{2}L_1 i_1^2 + M i_1 i_2 + \frac{1}{2}L_2 i_2^2 = 20.73J$$

# Operating principle of transformer

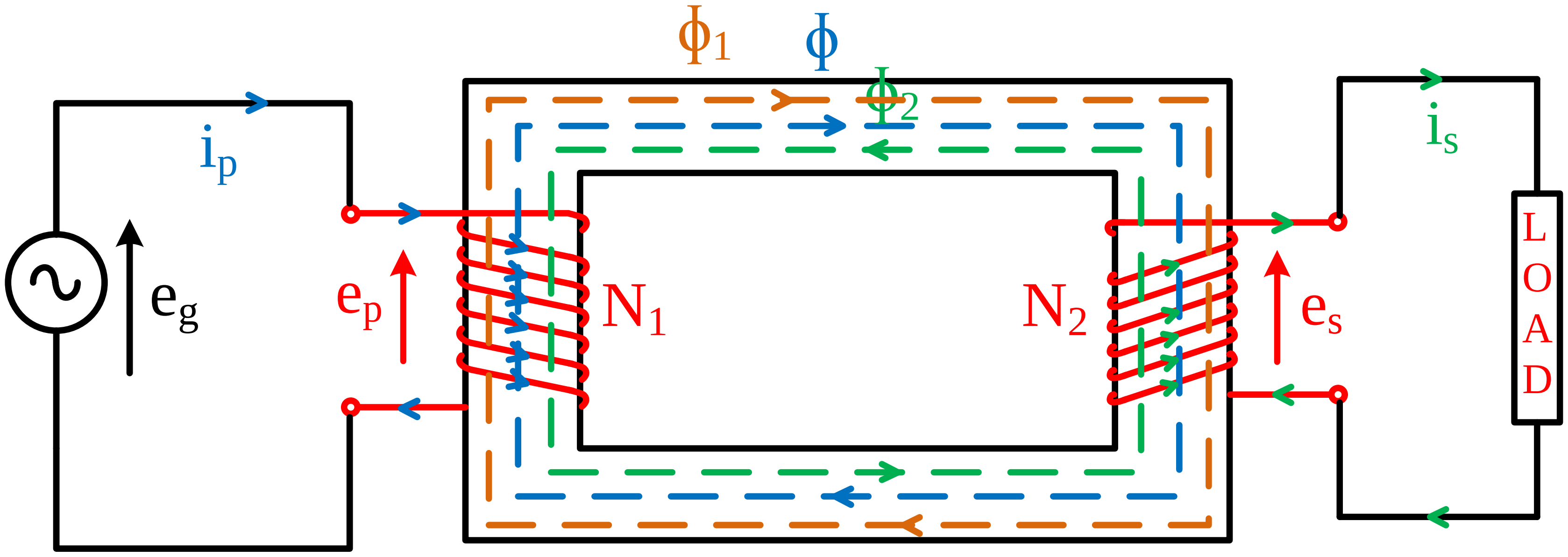



Fig. Operating principle of transformer.


THANKS
FOR YOUR
ATTENTION