Lecture-14

On

INTRODUCTION TO ELECTRICAL ENGINEERING (ESO203)

- Singularity Functions.
- Step Response of an RC Circuit.

Singularity Functions

- **Singularity functions are functions that either are discontinuous or have discontinuous derivatives.**

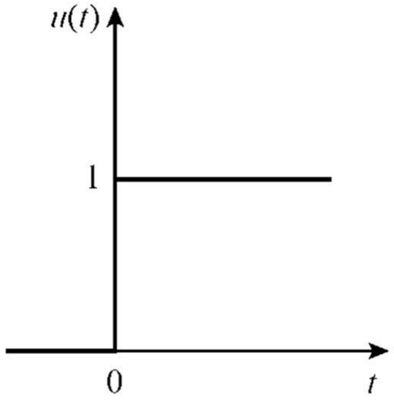
- A basic understanding of singularity functions will help to understand the response of first- order circuits to a sudden application of an independent DC voltage or current source.
- It serves as good approximations to the switching signals that arise in circuits with switching operations.
- It is helpful in the neat, compact description of some circuit phenomena, especially the step response of *R-C* or *R-L* circuits.
- The three most widely used singularity functions in circuit analysis are the *unit step*, the *unit impulse*, and the *unit ramp* functions.

Singularity Functions (Cont..)

- The unit step function $u(t)$ is 0 for negative values of t and 1 for positive values of t as shown in Figure below

In mathematical terms,

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



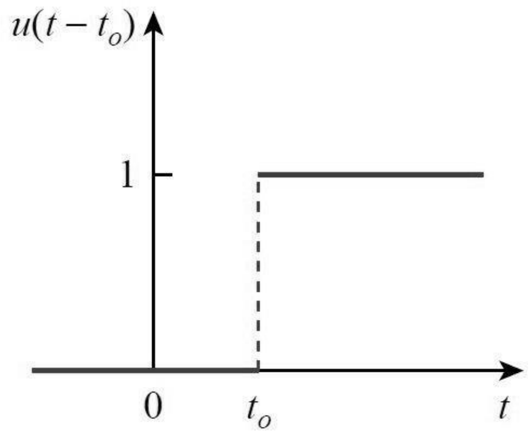
- The unit step function is undefined at $t = 0$, where it changes abruptly from 0 to 1.

- It is dimensionless, like other mathematical functions such as sine and cosine.

Singularity Functions (Cont..)

- If the abrupt change occurs at $t = t_0$ $(where\ t_0 > 0)$ instead of $t = 0$, the unit step function becomes

$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$

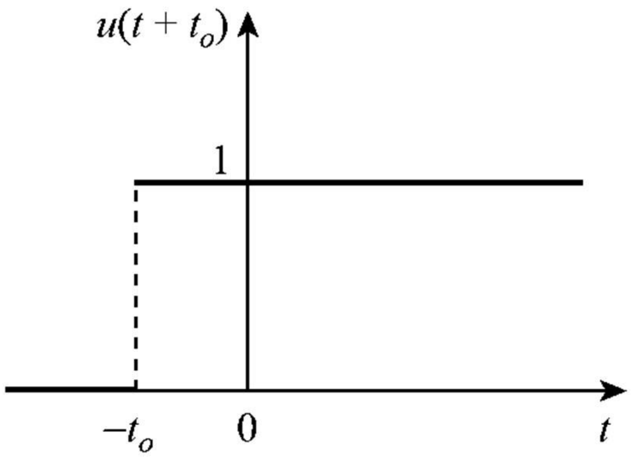


- which is the same as saying that $u(t)$ is delayed by t_0 seconds, as shown in the Figure

Singularity Functions (Cont..)

- If the change is at $t = -t_0$, the unit step function becomes

$$u\,(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t \geq -t_0 \end{cases}$$



- meaning that $u(t)$ is advanced by t_0 seconds, as shown in Figure.

Singularity Functions (Cont..)

- Step function is used to represent an abrupt change in voltage or current, like the changes that occur in the circuits of control systems and digital computers. For example, the voltage

$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0\,, & t \geq t_0 \end{cases}$$

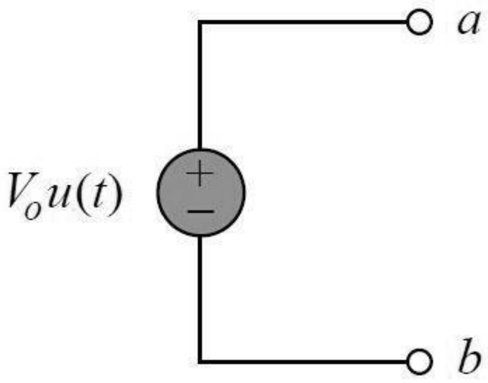
- may be expressed in terms of the unit step function as

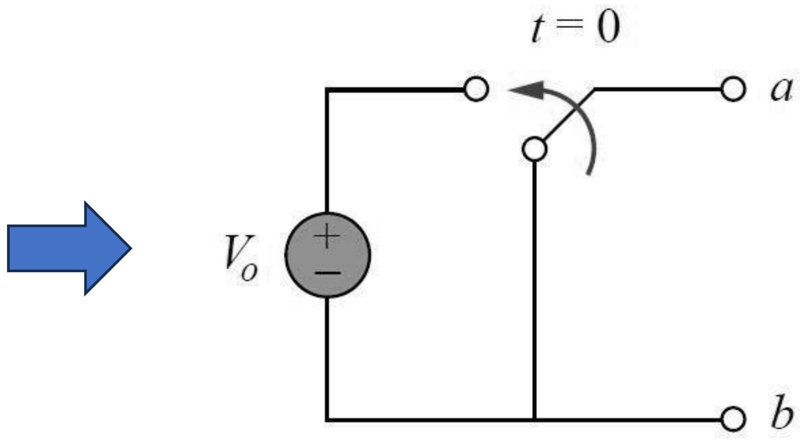
$$v(t) = V_0 u(t - t_0)$$

- If $t_0 = 0$,then $v(t)$ is simply the step voltage $V_0 u(t)$

Singularity Functions (Cont..)

- A voltage source of $V_0 u(t)$ is shown in left Figure below; its equivalent circuit is shown in right figure
- It is evident that terminals $a - b$ are short circuited $(v = 0)$ for $t < 0$ and that $v = V_0$ appears at the terminals for $t > 0$.

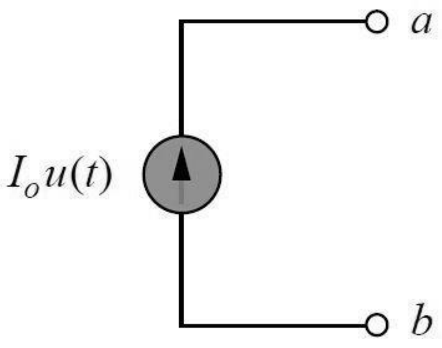


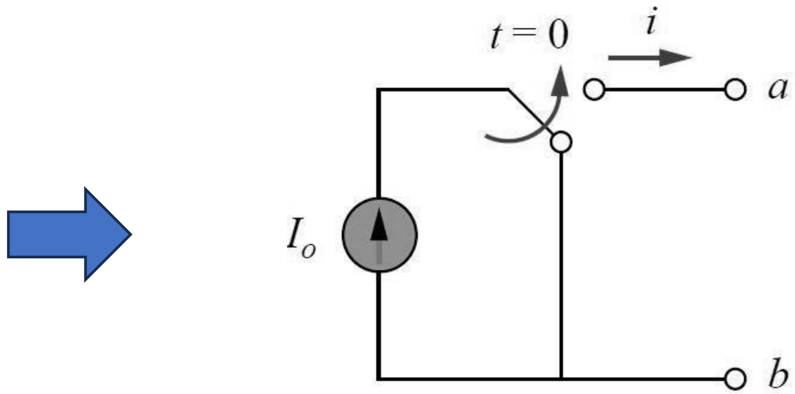


Singularity Functions (Cont..)

- Similarly, a current source of $I_0 u(t)$ is shown in left figure below, while its equivalent circuit is shown in the right figure.

- For $t < 0$, there is an open circuit ($i = 0$), and that $i = i_0$ flows for $t > 0$

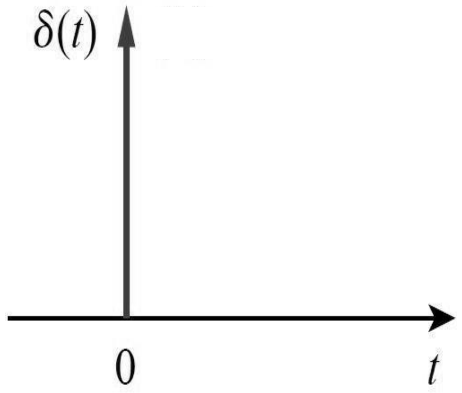




Singularity Functions (Cont..)

- The derivative of the unit step function $u(t)$ is the *unit impulse function* $\delta(t)$, which can be written as :

$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0; & t < 0 \\ undefined\,; & t = 0 \\ 0; & t > 0 \end{cases}$$



- The unit impulse function—also known as the *dirac-delta* function is shown in the Figure.

- The unit impulse function $\delta(t)$ is zero everywhere except at $t = 0$, where it is undefined.

Singularity Functions (Cont..)

- Impulsive currents and voltages occur in electric circuits as a result of switching operations or impulsive sources.

- Although the unit impulse function is not physically realizable (just like ideal sources, ideal resistors, etc.), it is a very useful mathematical tool.

- The unit impulse may be regarded as an applied or resulting shock.

- It may be visualized as a very short duration pulse of unit area.

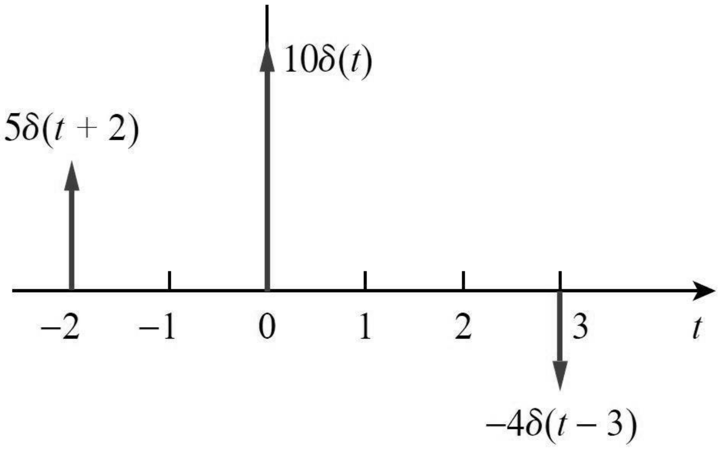
It can be expressed mathematically as :

$$\int_{0^-}^{0^+} \delta(t)dt = 1$$

- where $t = 0^-$ denotes the time just before $t = 0$ and $t = 0^+$ is the time just after $t = 0$.

Singularity Functions (Cont..)

- The unit area is known as the *strength* of the impulse function.
- When an impulse function has a strength other than unity, the area of the impulse is equal to its strength.



- For example, an impulse function $10\delta(t)$ has an area of 10.

- Figure in right shows the impulse functions:

$$5\delta(t+2), 10\delta(t) \ \ and \ \ -4\delta(t-3).$$

Singularity Functions (Cont..)

❑ How the impulse function affects other functions ?

Let us evaluate the integral

$$\int_a^b f(t)\delta(t - t_0)\, dt$$

where $a < t_0 <$ b. Since $\delta(t - t_0) = 0$ except at $t = t_0$, the integrand is zero except at t_0. Thus,

$$\int_a^b f(t)\delta(t - t_0)dt = \int_a^b f(t_0)\delta(t - t_0)\, dt = f(t_0)\int_a^b \delta(t - t_0)\, dt = f(t_0)$$

$$\int_a^b f(t)\delta(t - t_0) = f(t_0)$$

Singularity Functions (Cont..)

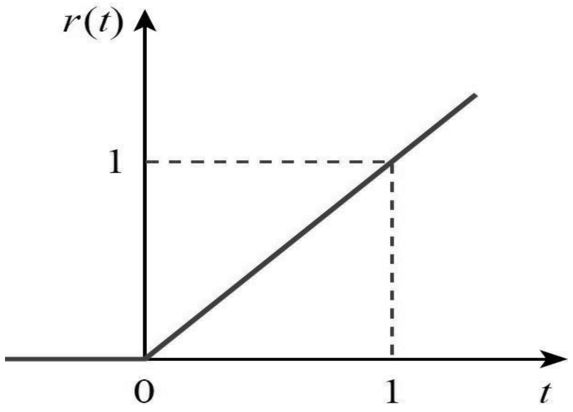
- This shows that when a function is integrated with the impulse function, we obtain the value of the function at the point where the impulse occurs.

- This is a highly useful property of the impulse function known as the ***sampling*** or ***shifting*** property.

- Integrating the unit step function *u(t)* results in the *unit ramp function r(t)*, *as shown below*;

$$r(t) = \int_{-\infty}^{t} u(t)\,dt = tu(t) \;\; or,$$

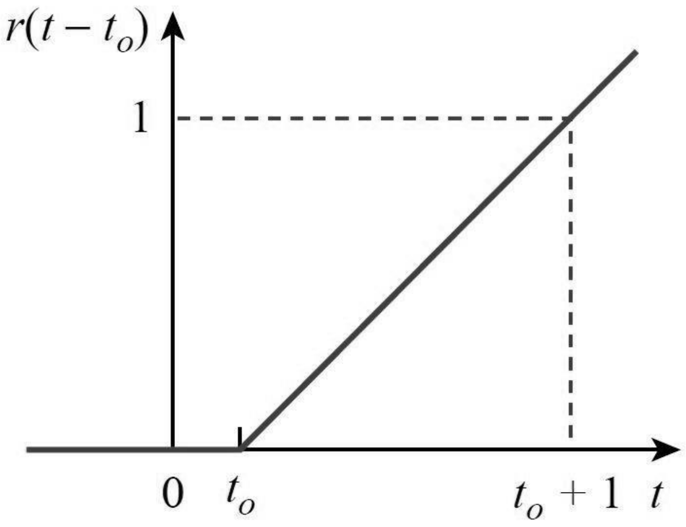
$$r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases}$$

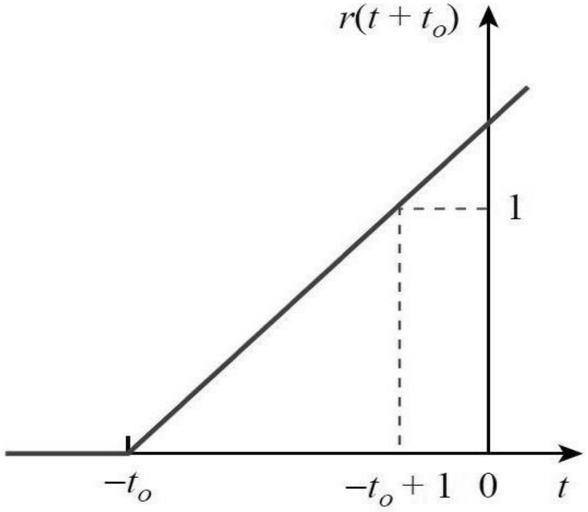


- The unit ramp function is zero for negative values of *t* and has a unit slope for positive values of *t*.

Singularity Functions (Cont..)

- In general, a ramp is a function that changes at a constant rate. The unit ramp function may be delayed or advanced as shown in Figures below.





Singularity Functions (Cont..)

- For the delayed unit ramp function

$$r(t - t_0) = \begin{cases} 0, & t \le t_0 \\ t - t_0, & t \ge t_0 \end{cases}$$

- For the advanced unit ramp function,

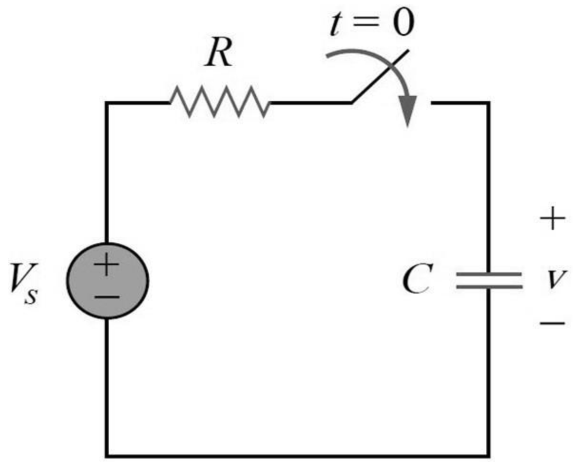
$$r(t + t_0) = \begin{cases} 0, & t \le -t_0 \\ t + t_0, & t \ge -t_0 \end{cases}$$

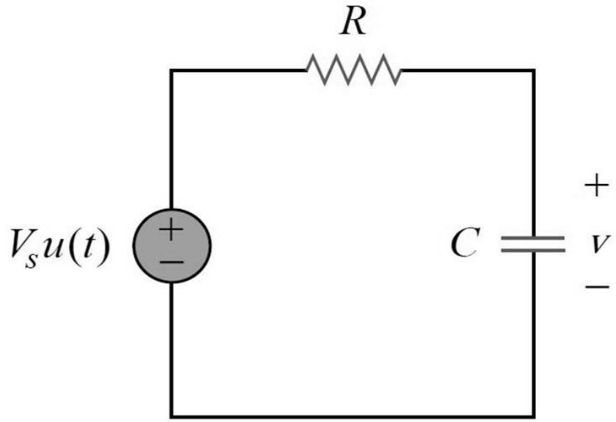
$$\boldsymbol{\delta(t) = \frac{d}{dt} u(t)} \qquad \boldsymbol{u(t) = \frac{d}{dt} u(t)}$$

$$\boldsymbol{r(t) = \int_{-\infty}^{t} u(t)\, dt} \qquad \boldsymbol{u(t) = \int_{-\infty}^{t} \delta(t)\, dt}$$

Step Response of an RC Circuit

- The step response is the response of the circuit due to a sudden application of a DC voltage or current source.
- Consider the *R*-*C* circuit, shown in left Figure, is replaced by the circuit, shown in right Figure, V_s is a constant, DC voltage source. Select the capacitor voltage as the circuit response to be determined.
- Assume an initial voltage V_0 on the capacitor.





Step Response of an RC Circuit (Cont..)

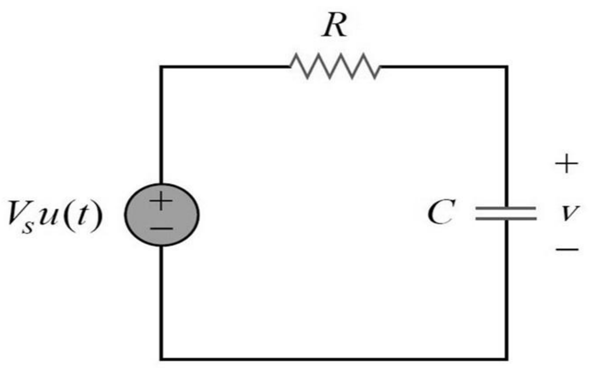
- Since the voltage of a capacitor cannot change instantaneously,

$$v(0^-) = v(0^+) = V_0 \qquad (1)$$

- Where $v(0^-)$ is the voltage across the capacitor just before switching and $v(0^+)$ is its voltage immediately after switching, Applying KCL, we have

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0 \qquad \text{or}$$

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t) \qquad (2)$$



- where v is the voltage across the capacitor. For $t > 0, Eq. 2$ becomes

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} \qquad (3)$$

Step Response of an RC Circuit (Cont..)

- Rearranging terms gives

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} \qquad (3)$$

$$\frac{dv}{dt} = -\frac{v - V_s}{RC} \qquad OR$$

$$\frac{dv}{v - V_s} = -\frac{dt}{RC} \qquad (4)$$

- Integrating both sides and introducing the initial conditions,

$$ln(v - V_s)\,\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_{0}^{t}$$

$$ln\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC} \qquad (5)$$

Step Response of an RC Circuit (Cont..)

- Taking the exponential of both sides

$$\frac{v - V_s}{V_0 - V_s} = e^{-\frac{t}{RC}} = e^{-\frac{t}{\tau}}, \qquad \tau = RC$$

$$v - V_s = (V_0 - V_s)\, e^{-\frac{t}{\tau}} \qquad \qquad or$$

$$v(t) = V_s + (V_0 - V_s)\, e^{-\frac{t}{\tau}}\ ;\ \ t > 0 \qquad \qquad (6)$$

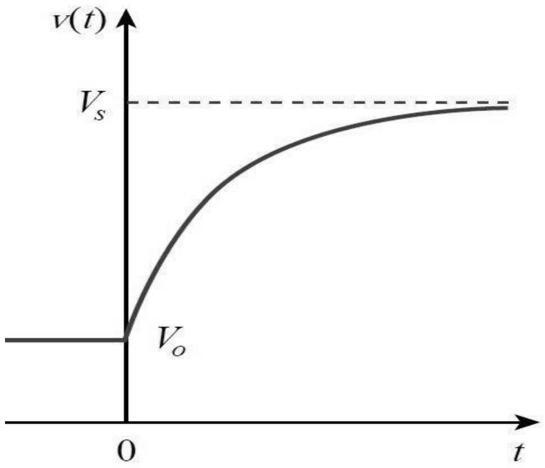
Thus,

$$V(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)\, e^{-\frac{t}{\tau}}\ ; & t > 0 \end{cases}$$

This is known as the *complete response* of the *RC* circuit to a sudden application of a DC voltage source, assuming the capacitor is initially charged.

Step Response of an RC Circuit (Cont..)

- Assuming that $V_s > V_0$, a plot of $v(t)$ is shown in Figure.



$$V(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)\, e^{-\frac{t}{\tau}} \;; & t > 0 \end{cases}$$

- If we assume that the capacitor is uncharged initially, set $V_0 = 0$, So :

$$V(t) = \begin{cases} 0, & t < 0 \\ V_s\left(1 - e^{-\frac{t}{\tau}}\right), & t > 0 \end{cases} \qquad (7)$$

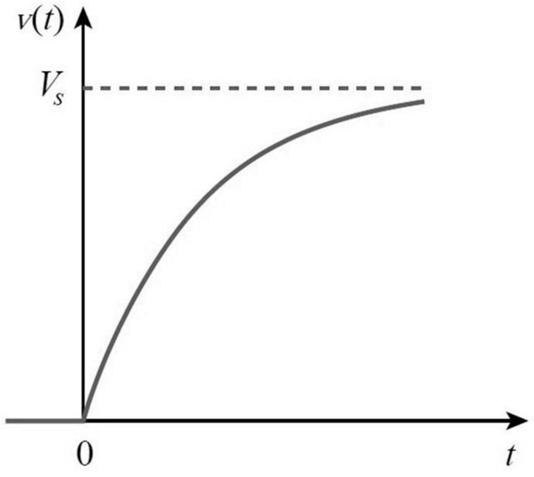
- The current through the capacitor is obtained from Above equation using -

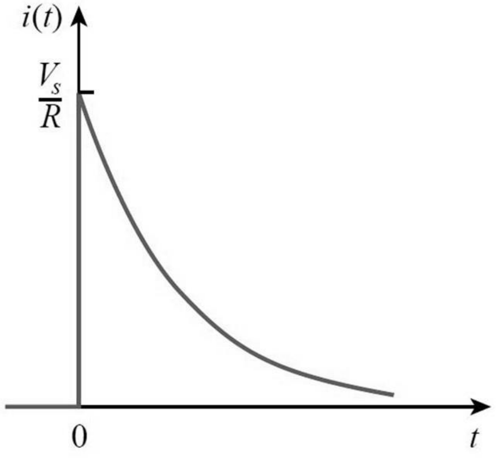
$$i(t) = C\frac{dv}{dt} = \frac{C}{\tau} V_S e^{-\frac{t}{\tau}} \qquad t > 0 \qquad or$$

$$i(t) = \frac{V_S}{R} e^{-\frac{t}{\tau}} u(t) \qquad (8)$$

Step Response of an RC Circuit (Cont..)

- Figure below shows the plots of capacitor voltage $v(t)$ and capacitor current $i(t)$.





$$i(t) = \frac{V_S}{R} e^{-\frac{t}{\tau}} u(t) \qquad (8)$$

Step Response of an RC Circuit (Cont..)

❑ An alternate method for finding the step response of an *R-C* or *R-L* circuit.

- Re-examine Equation- $v(t) = V_s + (V_0 - V_s)\, e^{-\frac{t}{\tau}}$; $t > 0$

- It is evident that $v(t)$ has two components. Thus, we may write

$$v = v_f + v_n \qquad (9)$$

where

$$v_f = V_s \qquad (10)$$

$$v_n = (V_0 - V_s)\, e^{-\frac{t}{\tau}} \qquad (11)$$

Step Response of an RC Circuit (Cont..)

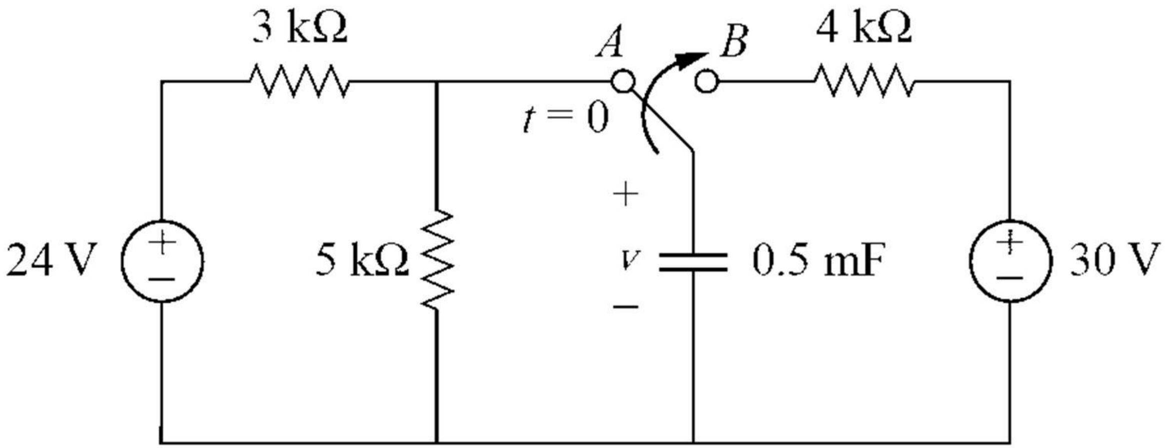
- v_n is called the natural response of the circuit.
- As v_n part of the response decays to almost zero after five time constants, it can be termed as *transient* response or natural response.
- v_f is known as the *forced* response because it is produced by the circuit when an external "force" is applied (In this case voltage source is the external force).
- It is also known as the *steady-state response*, because it remains for a long time period after the circuit is excited.
- Complete response of the circuit is, therefore, the sum of natural response and forced response.

$$\boldsymbol{v(t) = v(\infty) + [v(0) - v(\infty)]e^{-\frac{t}{\tau}}}$$

- Where, $v(0)$ is the initial voltage at $t = 0^+$ and $v(\infty)$ is the final or steady state value.

❑ EXAMPLE:

- The switch in Fig. below has been in position A for a long time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1s$ and $4s$.




THANKS
FOR YOUR
ATTENTION