Lecture Notes 9: Probability and Statistics

So far we have talked about univariate or single random variable. Now we will be talking about two random variables together. We usually call it as the bivariate random vector. Hence, we will be discussing about bivariate random vector (X, Y), when both X and Y are univariate random variables. We can easily see that there are four possibilities namely when (a) X is discrete and Y is also discrete random variables, (b) X is continuous and Y is continuous random variables, (c) X is discrete and Y is continuous and (d) X is continuous and Y is discrete. We will be mainly discussing about (a) and (b), i.e. when either both are discrete or both are continuous.

Suppose both X and Y are discrete random variables. Then the following quantity

$$p_{ij} = P(X = i, Y = j), \quad i, j = 0, 1, 2, \ldots,$$

where $p_{ij} \geq 0$, for all $i, j = 0, 1, 2, \ldots$ and $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_{ij} = 1$, is known as the joint probability mass function (PMF) and X and Y. The joint probability mass function provides everything about the bivariate random vector (X, Y). For example, we want to the marginal PMF of X or the marginal PMF of Y, they can be easily obtain from the joint PMF of X and Y as follows;

$$\begin{aligned} P(X = i) &= \sum_{j=0}^{\infty} P(X = i, Y = j) = \sum_{j=0}^{\infty} p_{ij} = p_{i\bullet} \quad \text{and} \\ P(Y = j) &= \sum_{i=0}^{\infty} P(X = i, Y = j) = \sum_{i=0}^{\infty} p_{ij} = p_{\bullet j}. \end{aligned}$$

Similarly, suppose we want to find $P(X \geq Y)$, it will be

$$P(X \geq Y) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} p_{ij}.$$

The joint cumulative distribution function (CDF) of X and Y is defined as

$$F_{X,Y}(x, y) = \sum_{i \leq x} \sum_{j \leq y} p_{ij}.$$

Exercise: Suppose X and Y has the following joint PMF

$$P(X=i,Y=j) = p(1-p)^i q(1-q)^j; \quad i,j = 0,1,2,\ldots.$$

Find the marginal PMF of X and the marginal PMF of Y. Find $P(X > Y)$ and $P(X+Y > 5)$.

Two random variables X and Y are called independent random variables if

$$P(X=i)P(Y=j) = P(X=i,Y=j); \quad \text{for all } i,j = 0,1,2,\ldots.$$

Hence, the two random variables X and Y are independent if and only if

$$p_{ij} = p_{i\bullet}p_{\bullet j}; \quad \text{for all } i,j = 0,1,2,\ldots.$$

Further, the conditional PMF of X given $Y=j$ is given by

$$P(X=i|Y=j) = \frac{P(X=i,Y=j)}{P(Y=j)} = \frac{p_{ij}}{p_{\bullet j}}; \quad i = 0,1,2,\ldots,$$

given that $P(Y=j) > 0$. Observe that for a given j, $P(X=i|Y=j)$ is a proper PMF, i.e. $P(X=i|Y=j) \ge 0$ and $\sum_{i=0}^{\infty} P(X=i|Y=j) = 1$. Hence, what ever we have done in case of a univariate random variable, we will be able to in case of conditional random variable. For example

$$\begin{aligned} E(X|Y=j) &= \sum_{i=0}^{\infty} iP(X=i|Y=j) \\ V(X|Y=j) &= E(X^2|Y=i) - (E(X|Y=i))^2. \end{aligned}$$

From the random vector (X,Y), suppose we have a transformation T such that $T(X,Y) = (U,V)$, then it is easy to obtain the joint PMF of (U,V). It is as follows

$$P(U=m,V=n) = \sum_{(i,j):T(i,j)=(m,n)} P(X=i,Y=j).$$

Exercise: Suppose X and Y has the following joint PMF

$$P(X=i,Y=j) = p(1-p)^i q(1-q)^j; \quad i,j = 0,1,2,\ldots.$$

Let $U = 2X + 3Y$ and $V = X + Y$, find the joint PMF of (U, V).

Now we will discuss about the random vector (X, Y), when both are continuous. Suppose both X and Y are continuous random variables. The joint probability density function of X and Y is defined as $f_{X,Y}(x, y)$, where $f_{X,Y}(x, y) \geq 0$, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1$ and

$$P((X, Y) \in A) = \int \int_A f_{X,Y}(x, y) dx dy, \quad \text{for} \quad A \subset \mathbb{R}.$$

If we have the joint PDF, we can obtain the joint cumulative distribution function (CDF) as

$$P(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u, v) du dv, \quad x, y \in \mathbb{R}.$$

If we have the joint PDF, we can easily obtain the joint CDF and similarly, if we have the joint CDF, we can obtain the joint PDF as

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y).$$

From the joint CDF we can obtain the marginal PDFs as follows:

$$F_X(x) = \lim_{y \to \infty} P(X \leq x, Y \leq y) = \lim_{y \to \infty} F_{X,Y}(x, y),$$

and

$$F_Y(y) = \lim_{x \to \infty} P(X \leq x, Y \leq y) = \lim_{x \to \infty} F_{X,Y}(x, y),$$

Similarly, if we have the joint PDF of (X, Y), we can obtain the marginal PDFs as follows:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx.$$

Two random variables are said to be independent if and only if

$$P(X \leq x, Y \leq y) = P(X \leq x) P(Y \leq y), \quad \text{for all } x, y \in \mathbb{R}.$$

Equivalently, it can be written as

$$f_{X,Y}(x,y) = f_X(x)f_Y(y), \quad \text{for all } x, y \in \mathbb{R}.$$

Further, the conditional PDF of X given $Y = y$ is defined as

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \text{provided} \quad f_Y(y) \neq 0.$$

Similarly, the conditional PDF of Y given $X = x$ is defined as

$$f_{Y|X=x}(x) = \frac{f_{X,Y}(x,y)}{f_X(x)} \quad \text{provided} \quad f_Y(y) \neq 0.$$

Example: Consider the following example. Suppose the random vector (X, Y) has the following joint PDF

$$f_{X,Y}(x,y) = x + y; \qquad 0 < x, y < 1,$$

and zero, otherwise. First observe that $f_{X,Y}(x,y)$ is a proper joint PDF, as $f_{X,Y}(x,y) \geq 0$ for all $x, y \in \mathbb{R}$ and $\int_0^1 \int_0^1 (x+y)dxdy = 1$. The joint CDF for $0 \leq x, y \leq 1$ becomes

$$F_{X,Y}(x,y) = \int_0^x \int_0^y (u+v)dvdu = \int_0^x (uy + \frac{1}{2}y^2)du = \frac{1}{2}(yx^2 + xy^2).$$

If $x < 0$ or $y < 0$, $F_{X,Y}(x,y) = 0$. If $x \geq 1$ and $0 < y < 1$, then $F_{X,Y}(x,y) = F_{X,Y}(1,y) = \frac{1}{2}(y + y^2)$. Similarly, if $0 < x < 1$ and $y \geq 1$, then $F_{X,Y}(x,y) = F_{X,Y}(x,1) = \frac{1}{2}(x + x^2)$. If $x \geq 1$ and $y \geq 1$, then $F_{X,Y}(x,y) = 1$. The marginal CDF of X and Y become

$$F_X(x) = F_{X,Y}(x,1) = \frac{1}{2}(x + x^2) \qquad \text{and} \qquad F_Y(y) = F_{X,Y}(1,y) = \frac{1}{2}(y + y^2),$$

respectively. Hence, the PDF of X and Y become

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{1}{2}(1 + 2x) \qquad \text{and} \qquad f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{1}{2}(1 + 2y).$$

respectively. Clearly X and Y are not independent as

$$F_{X,Y}(x,y) \neq F_X(x)F_Y(y); \qquad \text{for} \quad x = y = \frac{1}{2}$$

Let us consider the conditional PDF of $\{X|Y = y\}$, for $0 < y < 1$ and it is

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2(x+y)}{1+2y}; \quad 0 < x < 1,$$

and zero, otherwise. The conditional CDF of $\{X|Y = y\}$ becomes

$$F_{X|Y=y}(x) = \int_0^x \frac{2(u+y)}{1+2y} du = \frac{1}{1+2y}(x^2 + 2xy); \quad 0 \le x \le 1,$$

zero if $x < 0$ and it is one, for $x \ge 1$. Similarly, we can get

$$\begin{aligned} E(X|Y = y) &= \int_0^1 x f_{X|Y=y}(x) dx = \int_0^1 \frac{2(x^2 + xy)}{1+2y} dx = \frac{2}{1+2y}\left(\frac{1}{3} + \frac{1}{2}y\right) \\ &= \frac{2+3y}{3(1+2y)}. \end{aligned}$$

$$\begin{aligned} E(X^2|Y = y) &= \int_0^1 x^2 f_{X|Y=y}(x) dx = \int_0^1 \frac{2(x^3 + x^2 y)}{1+2y} dx = \frac{2}{1+2y}\left(\frac{1}{4} + \frac{1}{3}y\right) \\ &= \frac{3+4y}{6(1+2y)}. \end{aligned}$$

Hence,

$$V(X|Y = y) = \frac{3+4y}{6(1+2y)} - \left(\frac{2+3y}{3(1+2y)}\right)^2$$

Exercise Suppose X and Y has the following joint PDF

$$f_{X,Y}(x,y) = c(x+y); \quad 0 < x < y < 1,$$

and zero, otherwise. (a) Find c. (b) Find joint CDF of (X,Y), (c) Find the marginals CDF and PDF. (d) Find the conditional PDF of $\{X|Y = y\}$, (e) Find the conditional PDF of $\{Y|X = x\}$. (f) Find $P(X < Y)$, (g) Find $P(X + Y < 1)$.