TA # 0, EE 250 (Control System Analysis) - Spring 2025*

DEPARTMENT OF ELECTRICAL ENGINEERING, IIT KANPUR

1. Given the equation $\dot{x} + 5x = \delta(t)$, $t \geq 0-$, $x(0-) = 2$, determine $x(t)$ for $t > 0$.

   Here, δ is the unit impulse function, and has the properties $\delta(t) = 0, \forall t \neq 0$, and $\int_{-\infty}^{\infty} \delta(t)dt = 1$.

   Which definition of the Laplace transform (LT) is it convenient to use in this problem — the $\mathfrak{L}_-$ or the $\mathfrak{L}_+$?

$$\mathfrak{L}_-\{x(t)\} = \int_{0-}^{\infty} x(t)e^{-st}dt$$
$$\mathfrak{L}_+\{x(t)\} = \int_{0+}^{\infty} x(t)e^{-st}dt.$$

   If the system is instead described by the equation $\dot{x} + 5x = \delta(t), t \geq 0-, x(0+) = 2$, determine $x(t)$ for $t > 0$, which definition of the LT is it convenient to use?

2. Determine the Laplace transform of $f(at)$ given that the Laplace transform of $f(t)$ is $F(s)$.

   For simplicity, we don't have to bother in this problem about the distinction between $\mathfrak{L}_-, \mathfrak{L}, \mathfrak{L}_+$.

3. Determine the unit step response of the transfer function

$$\frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}, \quad 0 < \zeta < 1,$$

   given the information

$$\mathfrak{L}\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2 + \omega^2}$$

   and

$$\mathfrak{L}\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2 + \omega^2}$$

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Solutions

Ex: $\dot{x} + 5x = \delta(t)$, $t \geq 0$, $x(0-) = 2$.

[Kailath p.11]

Determine $x(t)$ for $t > 0$.

Ans: For $t > 0$, we have

Classical solution

$$\dot{x} + 5x = 0$$ [redacted] starting from $x(0+)$.

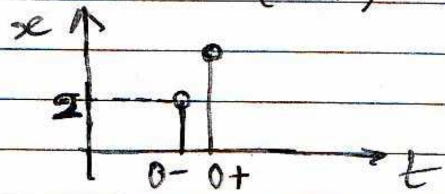
$$\Rightarrow x(t) = \text{[redacted]} + x(0+) \cdot e^{-5t}.$$

Check: $\dot{x} = -5x(0+) \cdot e^{-5t}$

$$\Rightarrow \dot{x} = -5x$$

$$\Rightarrow \dot{x} + 5x = 0 \checkmark.$$

What is $x(0+)$?



@ $t = 0$, $\boxed{\cancel{\dot{x}(0) = \delta(0)}}$ $\cdot\ \dot{x}(0) + 5x(0) = \delta(0)$ (*)

[redacted] As $\delta(0) = \infty$, one way this equality will be true is if $x(0) =$ some finite value and $x(0+) = 2 + 1 = 3$. This means that there is a step change in $x(t)$ from $x(0-)$ to $x(0+)$, so that $\dot{x}(0) = \delta(0)$. ($\because$ $5x(0)$ is negligible compared to $\dot{x}(0)$) $\therefore x(0+) = 3$.

Is there any other possibility?

E.g., $x(0+) = 1$. $\Rightarrow \dot{x}(0) = -\delta(0)$. Doesn't satisfy (*).

Eg., $x(0) = \delta(0) \Rightarrow \dot{x}(0) = \left.\frac{d}{dt}\{\delta(t)\}\right|_{t=0}$

Dimensionally inconsistent in (*).

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So, seems like $x(0+) = 3$ is the only possibility.

Did a lot of work to figure out $x(0+)$

$\mathcal{L}_+$ solution:

$$\mathcal{L}_+\{\dot{x}\} + 5\mathcal{L}_+\{x\} = \mathcal{L}_+\{\delta(t)\}$$

$$\Rightarrow sX_+(s) - x(0+) + 5X_+(s) = 0.$$

$$\Rightarrow X_+(s) = \frac{x(0+)}{s+5}$$

$$\Rightarrow x(t) = x(0+) \cdot e^{-5t} = 3e^{-5t}$$

$x(0+)$ is found $\underbrace{}_{\text{from } x(0-)}$ as in the classical method.

$\mathcal{L}_-$ method:

$$sX_-(s) - x(0-) + 5X_-(s) = 1$$

$$\Rightarrow X_-(s) = \frac{1 + x(0-)}{s+5}$$

$$x(t) = [1 + x(0-)]\, e^{-5t} = 3e^{-5t}$$

$\mathcal{L}_-$ method gave answer without much work when ICs @ ~~x(0)~~ $t = 0-$.

Conclusion: when ICs are @ $t = 0+$, $\mathcal{L}_+$ way is fine; only $\mathcal{L}_+\{\delta(t)\} = 0$.

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"Time scaling" property of LT:

$\mathcal{L}\{f(at)\} = ?$ given $\mathcal{L}\{f(t)\} = F(s)$.

$$= \int_0^\infty f(at) \cdot e^{-st} dt$$

Let $at = \theta$, Then $dt = \frac{d\theta}{a}$. $t = \frac{\theta}{a}$

$t = 0 \Rightarrow \theta = 0$

$t = \infty \Rightarrow \theta = \infty \Rightarrow$ for $a > 0$.

$$\Rightarrow \mathcal{L}\{f(at)\} = \frac{1}{a}\int_0^\infty f(\theta) \cdot e^{-\frac{s\theta}{a}} \cdot d\theta \quad \text{for } a>0$$

$$\Rightarrow \mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right).$$

Works for $a > 0$ only, as pointed out by Abhishek (09/Jan/2013). (11021).

For $a < 0$, the time is scaled negatively, and ∞ is negative. The one-sided LT does not work for $-$ve time.

$$y(t) = \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot\frac{1}{s}\right\} = ?$$

$$\Rightarrow \frac{As+B}{s^2+2\zeta\omega_n s+\omega_n^2} + \frac{C}{s} \equiv \frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)s}$$

$$\Rightarrow (A+C)s^2 + (B+2\zeta\omega_n C)s + C\omega_n^2 \equiv \omega_n^2$$

$$\Rightarrow A+C=0,\ B+2\zeta\omega_n C=0,\ C=1$$

$$\Rightarrow A=-1,\ B=-2\zeta\omega_n.$$

$$\therefore \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot\frac{1}{s} = \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2} \quad (*)$$

$$\mathcal{L}\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2+\omega^2}$$

We have $\mathcal{L}\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2+\omega^2}$

$\therefore$ (*) can be written as

$$\frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$= \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\omega_n\zeta)^2+\omega_d^2} - \frac{\omega_d}{(s+\omega_n\zeta)^2+\omega_d^2}\cdot\frac{\zeta\omega_n}{\omega_d}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Taking inverse L.T, we have:

$$y(t) = 1(t) - \left[e^{-\zeta\omega_n t}\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\sin\omega_d t\right]1(t)$$

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$$= 1(t) - e^{-\zeta\omega_n t}\, 1(t) \cdot \left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \cdot \sin\omega_d t\right]$$

Let $r\cos\theta = \frac{\zeta}{\sqrt{1-\zeta^2}}$, $r\sin\theta = 1$.

$$\Rightarrow \quad r = \sqrt{\frac{\zeta^2}{1-\zeta^2} + 1} = \frac{1}{\sqrt{1-\zeta^2}}.$$

$$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}.$$

Then,

$$y(t) = 1(t) - e^{-\zeta\omega_n t}\, 1(t) \cdot r\sin(\omega_d t + \theta)$$

$$= \left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cdot \sin\left(\omega_d t + \arctan\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right] 1(t).$$