Lecture-16

On

INTRODUCTION TO ELECTRICAL ENGINEERING (ESO203)

- The Source-Free Series RLC Circuit.
- Step Response of RLC Circuit.

The Source-Free Series RLC Circuit (Cont..)

- Since $i = Ae^{\sigma t}$ is the assumed solution, therefore, only expression in parentheses can be zero -

$$\sigma^2 + \frac{R}{L}\sigma + \frac{1}{LC} = 0 \qquad (8)$$

- This quadratic equation is known as the ***characteristic equation*** of the differential Eq. (4), since the roots of the equation dictate the character of i .

The two roots of Eq. (8) are

$$\sigma_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad (9a)$$

$$\sigma_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad (9b)$$

The Source-Free Series *RLC* Circuit (Cont..)

- A more compact way of expressing the roots is -

$$\sigma_1 = -\alpha + \sqrt{\alpha^2 - {\omega_0}^2}, \sigma_2 = -\alpha - \sqrt{\alpha^2 - {\omega_0}^2} \qquad (10)$$

where,

$$\alpha = \frac{R}{2L}, \qquad (11)$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

- The roots σ_1 and σ_2 are called *natural frequencies*, measured in nepers per second (Np/s). ω_0 is known as the *resonant frequency* or the *undamped natural frequency*, expressed in radians per second (rad/s); and α is the *damping factor*, expressed in nepers per second.

The Source-Free Series RLC Circuit (Cont..)

- In terms of α and ω_0, Eq. (8) can be written as

$$\sigma^2 + 2\alpha\sigma + {\omega_0}^2 = 0 \qquad (8a)$$

- The two values of σ in Eq. (10) indicate that there are two possible solutions for i, so,

$$i_1 = A_1 e^{\sigma_1 t}, i_2 = A_2 e^{\sigma_2 t} \qquad (12)$$

- Since Equation $\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$ is a linear equation, any linear combination of the two distinct solutions i_1 and i_2 is also a solution of this equation.

- A complete or total solution of above equation would therefore require a linear combination of i_1 and i_2. Thus, the natural response of the series RLC circuit can be written as:-

$$i(t) = A_1 e^{\sigma_1 t} + A_2 e^{\sigma_2 t} \qquad (13)$$

where the constants A_1 and A_2 can be determined from the initial values $i(0)$ and $di(0)/dt$.

The Source-Free Series *RLC* Circuit (Cont..)

From the values of σ_1 and σ_2, we can say that there are three types of solutions:

1. If $\alpha > \omega_0$, we have the *overdamped* case.
2. If $\alpha = \omega_0$, we have the *critically damped* case.
3. If $\alpha < \omega_0$, we have the *underdamped* case.

$$\sigma_1 = -\alpha + \sqrt{\alpha^2 - {\omega_0}^2}$$

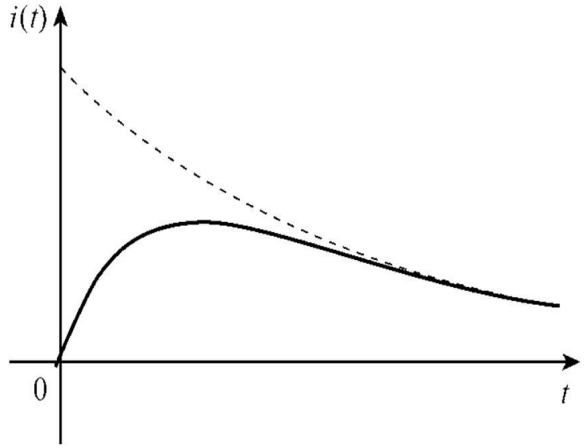
$$\sigma_2 = -\alpha - \sqrt{\alpha^2 - {\omega_0}^2}$$

Where, $\alpha = \frac{R}{2L}$, and $\omega_0 = \frac{1}{\sqrt{LC}}$

❑ Overdamped Case ($\alpha > \omega_0$)

$\alpha > \omega_0$ occurs when $C > 4L/R^2$. When this happens, both roots σ_1 and σ_2 are negative and real. The response is

$$i(t) = A_1 e^{\sigma_1 t} + A_2 e^{\sigma_2 t} \qquad (14)$$



which decays and approaches zero as t increases as shown in Figure

The Source-Free Series RLC Circuit (Cont..)

❑ Critically Damped Case ($\alpha = \omega_0$)

When $\alpha = \omega_0, C = 4L/R^2$ *and* $\sigma_1 = \sigma_2 = -\frac{R}{2L}$ (15)

For this case the solution yields -

$$i(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = A_3 e^{-\alpha t}$$

Where $A_3 = A_1 + A_2$.

This cannot be the solution, because the two initial conditions cannot be satisfied with the single constant A_3. It is so, because, our assumption of an exponential solution is incorrect for this special case of critical damping.

So, let us rewrite the equation $\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$ when $\alpha = \omega_0 = \frac{R}{2L}$,

$$\frac{d^2 i}{dt^2} + 2\alpha \frac{di}{dt} + \alpha^2 i = 0 \qquad \textit{or}$$

$$\frac{d}{dt}\left(\frac{di}{dt} + \alpha i\right) + \alpha\left(\frac{di}{dt} + \alpha i\right) = 0 \qquad (16)$$

The Source-Free Series RLC Circuit (Cont..)

If we let

$$f = \left(\frac{di}{dt} + \alpha i\right) \qquad (17)$$

then Eq. (16) becomes

$$\frac{df}{dt} + \alpha f = 0$$

which is a first-order differential equation with solution $f = A_1 e^{-\alpha t}$, where A_1 is a constant. Equation (17), then, becomes -

$$\frac{di}{dt} + \alpha i = A_1 e^{-\alpha t} \qquad or$$

$$e^{\alpha t}\frac{di}{dt} + e^{\alpha t}\alpha i = A_1 \qquad (18)$$

This can be written as

$$\frac{d}{dt}(e^{\alpha t} i) = A_1 \qquad (19)$$

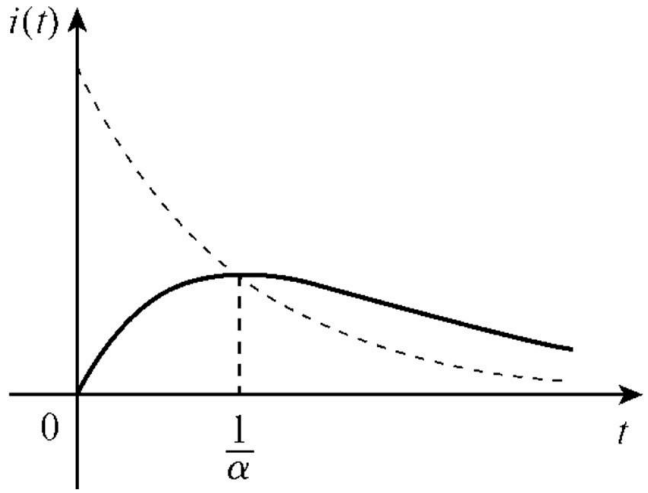
The Source-Free Series RLC Circuit (Cont..)

Integrating both sides yields

$$i = (A_1 t + A_2)e^{-\alpha t} \qquad (20)$$

The natural response of the critically damped circuit is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term.

Critically damped response is shown in Figure. It is a sketch of $i(t) = te^{-\alpha t}$, which reaches a maximum value of e^{-1}/α at $t = 1/\alpha$, i.e. one time constant, and then decays to zero.



The Source-Free Series RLC Circuit (Cont..)

❑ Underdamped Case ($\alpha < \omega_0$)

For $\alpha < \omega_0$ and $C < 4L/R^2$, the roots may be written as -

$$\sigma_1 = -\alpha + \sqrt{-(\omega_0{}^2 - \alpha^2)} = -\alpha + j\omega_d \qquad (21a)$$

$$\sigma_2 = -\alpha - \sqrt{-(\omega_0{}^2 - \alpha^2)} = -\alpha - j\omega_d \qquad (21b)$$

$\omega_d = \sqrt{\omega_0{}^2 - \alpha^2}$, which is called the *damping frequency*.

ω_0 is often called the *undamped natural frequency*, ω_d is called the *damped natural frequency*.

The Source-Free Series RLC Circuit (Cont..)

So, $i(t) = A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t}$

$$= e^{-\alpha t}\,(A_1 e^{j\omega_d t} + A_2 e^{j\omega_d t}) \qquad (22)$$

Using Euler's Identities,

$$e^{j\theta} = cos\theta + jsin\theta, \qquad e^{-j\theta} = cos\theta - jsin\theta \qquad (23)$$

We get,

$$i(t) = e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t e^{j\omega_d t} + j(A_1 - A_2)\sin\omega_d t e^{j\omega_d t}] \qquad (24)$$

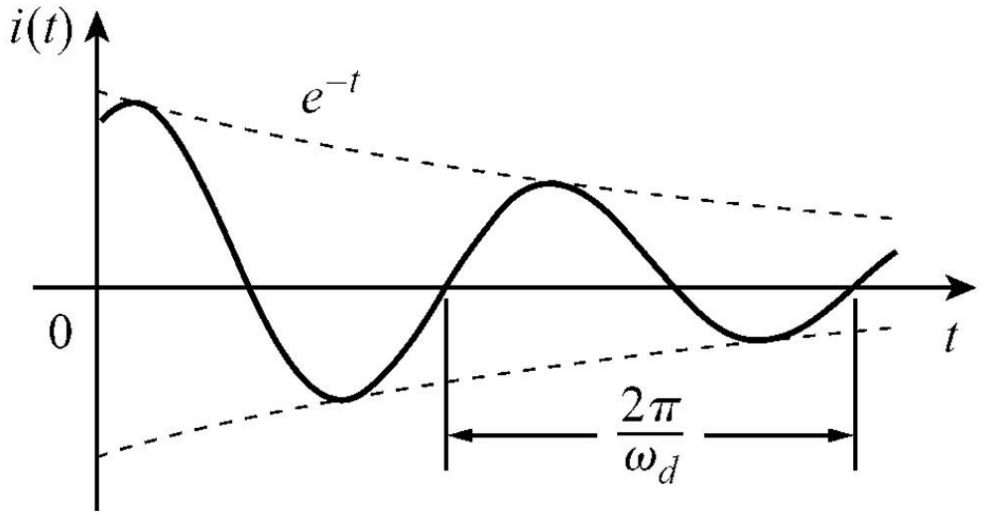
Replacing constants $(A_1 + A_2)$ and $j(A_1 - A_2)$ with constants B_1 and B_2, we write

$$i(t) = e^{-\alpha t}(B_1\cos\omega_d t e^{j\omega_d t} + B_2\sin\omega_d t e^{j\omega_d t}) \qquad (25)$$

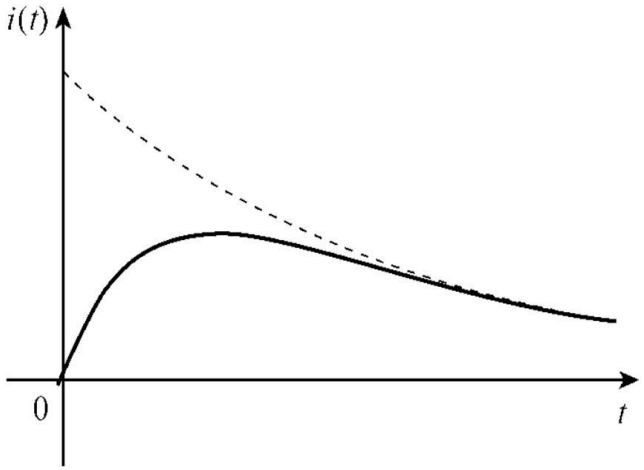
The Source-Free Series RLC Circuit (Cont..)

- With the presence of sine and cosine functions, the natural response for this case is exponentially damped and oscillatory in nature.

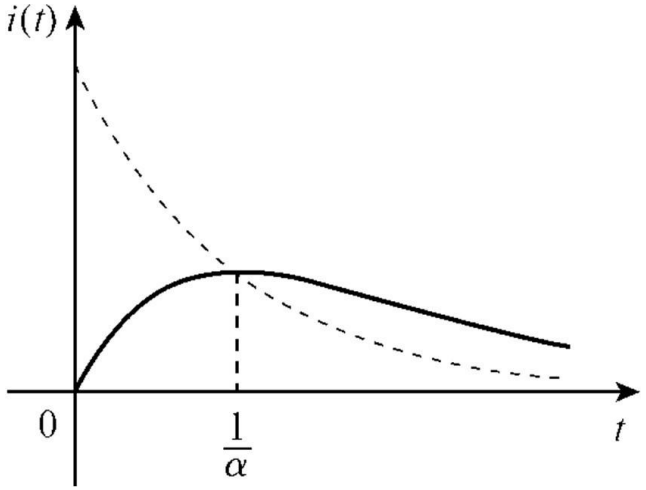
- The response has a time constant of $1/\alpha$ and a period of $T = 2\pi/\omega_d$. Figure below depicts a typical underdamped response.



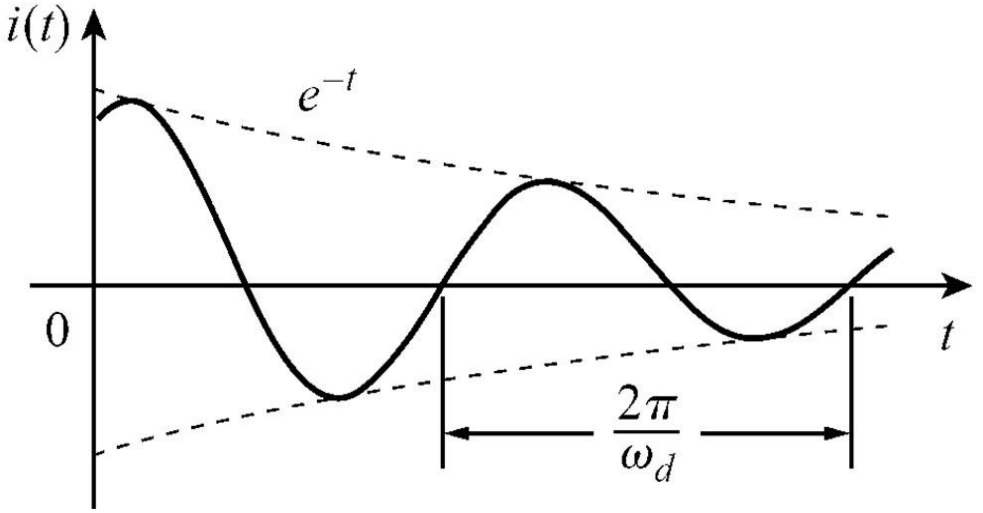
The Source-Free Series RLC Circuit (Cont..)



Overdamped response



Critically damped response



Underdamped response

Step Response of RLC Circuit

❑ Few Properties of a series RLC network:

- The damping effect is due to the presence of resistance R. The damping factor α determines the rate at which the response is damped.
- If $R = 0$, then $\alpha = 0$, and we have an LC circuit with $\frac{1}{\sqrt{LC}}$ as the undamped natural frequency.

Since $\alpha = \omega_0$ in this case, the response is not only undamped but also oscillatory. The circuit is said to be *lossless*, because the dissipating or damping element (R) is absent.

- By adjusting the value of R, the response may be made undamped, overdamped, critically damped, or underdamped.

- Oscillatory response is possible due to the presence of the two types of storage elements. Having both L and C allows the flow of energy back and forth between the two elements.

- The damped oscillation shown by the underdamped response of the circuit is known as *ringing*.

Step Response of RLC Circuit (Cont…)

- The critically damped case is the borderline between the underdamped and overdamped cases and decays the fastest.

- With the same initial conditions, the overdamped case has the longest settling time, because it takes the longest time to dissipate the initial stored energy.

- If we want the fastest response without oscillation or ringing, the critically damped circuit is the right choice.

Step Response of *RLC* Circuit (Cont…)

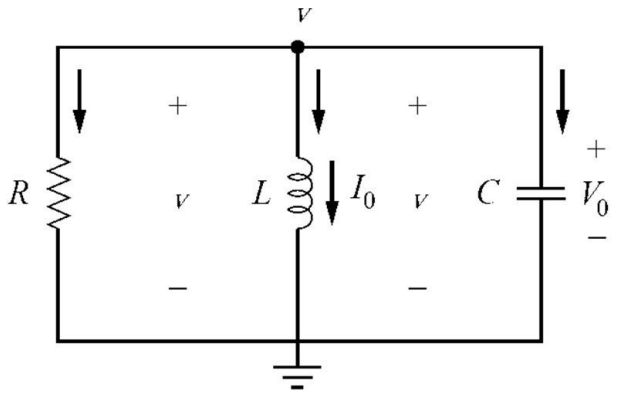
❑ The Source-free Parallel *RLC* circuit

Consider a parallel *RLC* circuit shown in Figure.

Assume initial inductor current I_0 and initial capacitor voltage V_0, then,

$$i(0) = I_0 = \frac{1}{L}\int_{-\infty}^{0} v(t)dt \qquad (1a)$$

$$v(0) = V_0 \qquad (1b)$$



Applying KCL at the top node gives-

$$\frac{v}{R} + \frac{1}{L}\int_{0}^{t} v(t)dt + C\frac{dv}{dt} = 0 \qquad (2)$$

Step Response of RLC Circuit (Cont…)

❑ The Source-free Parallel RLC circuit

Taking the derivative with respect to t and dividing by C results in

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0 \qquad (3)$$

We can obtain the characteristic equation by replacing the first derivative by σ and the second derivative by σ^2. The characteristic equation is obtained as :

$$\sigma^2 + \frac{1}{RC}\sigma + \frac{1}{LC} = 0 \qquad (4)$$

The roots of the characteristic equation are

$$\sigma_{1,2} = -\frac{1}{2R} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \qquad or$$

$$\sigma_{1,2} = -\alpha \pm \sqrt{\alpha^2 - {\omega_0}^2} \qquad (5)$$

Step Response of RLC Circuit (Cont…)

❑ The Source-free Parallel RLC circuit

Where

$$\alpha = \frac{1}{2RC}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \tag{6}$$

Again, there are three possible solutions, depending on whether $\alpha > \omega_0$, $\alpha = \omega_0$, or $\alpha < \omega_0$.

Overdamped Case ($\alpha > \omega_0$)

From above equation, $\alpha > \omega_0$ when $L > 4R^2C$. The roots of the characteristic equation are real and negative. The response is

$$v(t) = A_1 e^{\sigma_1 t} + A_2 e^{\sigma_2 t} \tag{7}$$

Critically Damped Case ($\alpha = \omega_0$)

For $\alpha = \omega_0$, $L = 4R^2C$. The roots are real and equal so the response is-

$$v(t) = (A_1 + A_2 t)e^{-\alpha t} \tag{8}$$

Step Response of RLC Circuit (Cont…)

❑ The Source-free Parallel RLC circuit

Underdamped Case ($\alpha < \omega_0$)

When $\alpha < \omega_0$, $L < 4R^2C$ .In this case the roots are complex and can be expressed as

$$\sigma_{1,2} = -\alpha \pm j\omega_d \qquad (9)$$

where

$$\omega_d = \sqrt{{\omega_0}^2 - \alpha^2} \qquad (10)$$

The response is

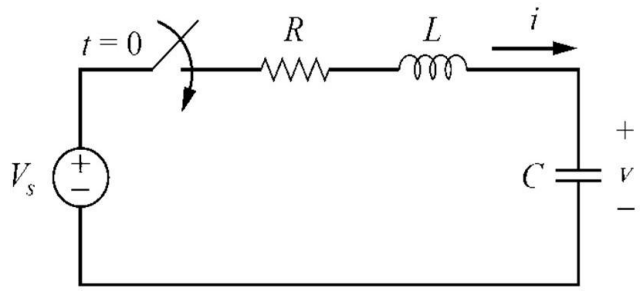
$$v(t) = e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t) \qquad (11)$$

Step Response of *RLC* Circuit (Cont…)

❑ Step Response of a Series *RLC* circuit

Consider the series *RLC* circuit shown in the Figure. Applying KVL around the loop for $t > 0$,

$$L\frac{di}{dt} + Ri + v = V_s \qquad (1)$$



But

$$i = C\frac{dv}{dt} \qquad (2)$$

Substituting for i in Eq. (1) and rearranging terms,

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC} \qquad (3)$$

Step Response of RLC Circuit (Cont…)

❑ Step Response of a Series RLC circuit

The characteristic equation for the series RLC circuit is not affected by the presence of the DC source.

The solution to $\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC}$ has two components: the natural response $v_n(t)$ and the forced response $v_f(t)$; that is,

$$v(t) = v_f(t) + v_n(t) \qquad (4)$$

The natural response is the solution when we set $V_s = 0$ in above differential equation.

The natural response v_n for the overdamped, underdamped, and critically damped cases are:

$$v(t) = A_1 e^{\sigma_1 t} + A_2 e^{\sigma_2 t} \quad \text{(Overdamped)} \qquad (5a)$$

$$v(t) = (A_1 + A_2 t)e^{-\alpha t} \quad \text{(Critically damped)} \qquad (5b)$$

$$v(t) = e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t) \quad \text{(Underdamped)} \qquad (5c)$$

Step Response of RLC Circuit (Cont…)

❑ Step Response of a Series RLC circuit

The forced response is the steady state or final value of $v(t)$.

The final value of the capacitor voltage is the same as the source voltage V_s. Hence

$$v_f(t) = v(\infty) = V_s \qquad (6)$$

Thus, the complete solutions for the overdamped, underdamped, and critically damped cases are:

$$v(t) = V_s + A_1 e^{\sigma_1 t} + A_2 e^{\sigma_2 t} \quad \text{(Overdamped)} \qquad (7a)$$

$$v(t) = V_s + (A_1 + A_2 t) e^{-\alpha t} \quad \text{(Critically damped)} \qquad (7b)$$

$$v(t) = V_s + e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t) \quad \text{(Underdamped)} \qquad (7c)$$

Circuit Analysis using Laplace Transform

The Laplace transform can be used to analyze a circuit.

This usually involves three steps.

- Transform the circuit from the time domain to the *s* domain.

- Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique.

- Take the inverse Laplace transform of the solution and thus obtain the solution in the time domain.

Circuit Analysis using Laplace Transform (Cont…)

In the *s* domain, the circuit elements are replaced as follows:

For a **resistor**, the voltage-current relationship in the time domain is

$$v(t) = i(t)R$$

Taking the Laplace transform, we get

$$V(s) = RI(s)$$

For an **inductor**,

$$v(t) = L\frac{di(t)}{dt}$$

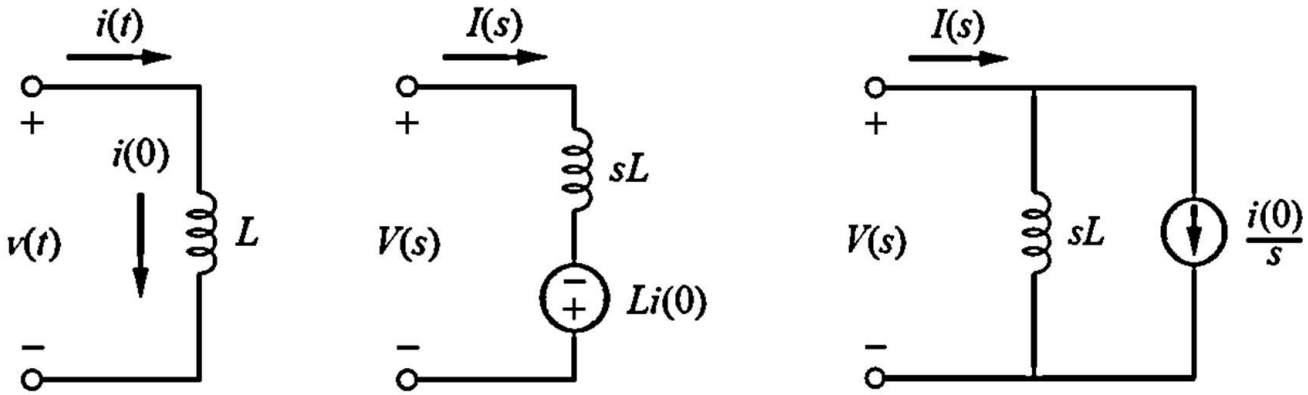
Taking the Laplace transform of both sides gives

$$V(s) = L[sI(s) - i(0^-)] = sLI(s) - Li(0^-)$$

Circuit Analysis using Laplace Transform (Cont…)

$$I(s) = \frac{1}{sL}V(s) + \frac{i(0^-)}{s}$$

The *s*- domain equivalents are shown in Figure below, where the initial condition is modeled as a voltage or current source.



Circuit Analysis using Laplace Transform (Cont…)

For a **capacitor**,

$$i(t) = C\frac{dv}{dt}$$

which transforms into the *s*-domain as

$$I(s) = C[sV(s) - v(0^-)] = sCV(s) - Cv(0^-)$$

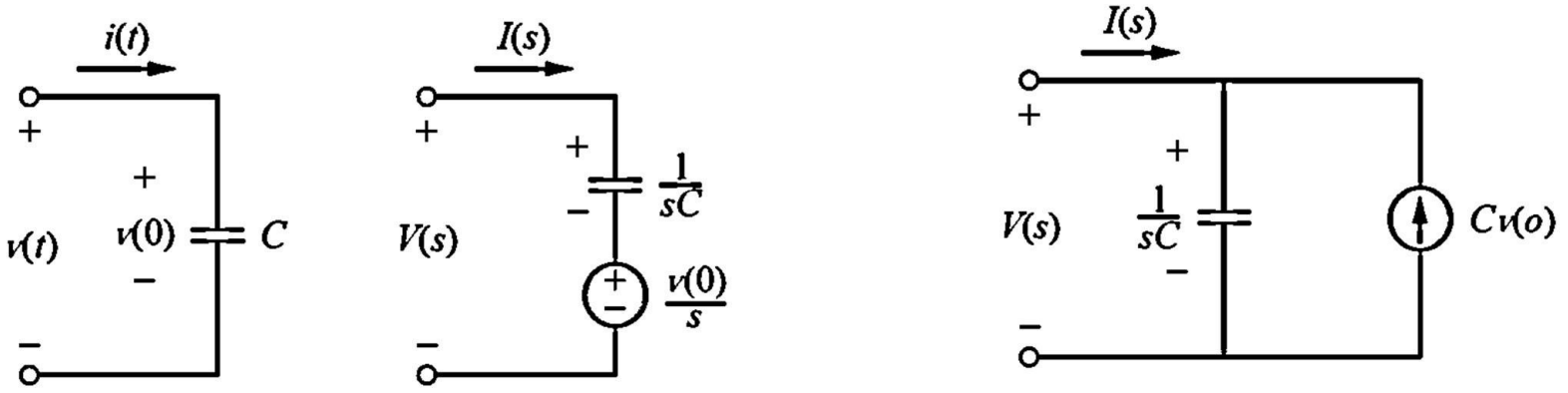
Or

$$V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}$$

Circuit Analysis using Laplace Transform (Cont…)

The *s*-domain equivalents for capacitor are shown in Figure below -

$$V(s) = \frac{1}{sC} I(s) + \frac{v(0^-)}{s} \qquad I(s) = C[sV(s) - v(0^-)] = sCV(s) - Cv(0^-)$$



Circuit Analysis using Laplace Transform (Cont…)

- It is observed from *Equations* that the initial conditions are part of the transformation.
- This is one advantage of using the Laplace transform in circuit analysis.
- Another advantage is that a complete response, i.e., transient and steady state response of a network is obtained.

- From the equations

$$V(s) = sLI(s) - Li(0^-) \text{ and } I(s) = sCV(s) - Cv(0^-)$$

we can observe the duality confirming that L and C, $I(s)$ and $V(s)$, and $v(0)$ and $i(0)$ are dual pairs.

Circuit Analysis using Laplace Transform (Cont…)

The impedance in the *s-domain* is the ratio of the voltage transform to the current transform under zero initial conditions, that is,

$$Z(s) = \frac{V(s)}{I(s)}$$

Thus, the impedances of the three circuit elements are

$$Resistor: \ Z(s) = R$$

$$Inductor: \ Z(S) = sL$$

$$Capacitor: \ Z(s) = 1/sC$$

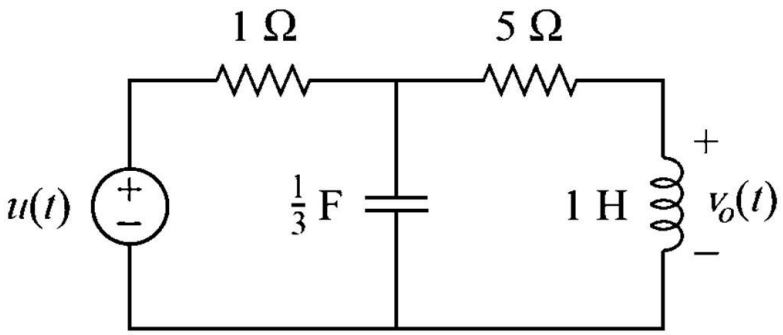
The admittance in the *s* domain is the reciprocal of the impedance,

The use of the Laplace transform in circuit analysis facilitates the use of various signal sources such as impulse, step, ramp, exponential, and sinusoidal.

Circuit Analysis using Laplace Transform (Cont…)

- Example:

Find $v_o(t)$ in the circuit, assuming zero initial conditions.



Solution:

We first transform the circuit from the time domain to the s-*domain*

$$u(t) \Rightarrow 1/s$$

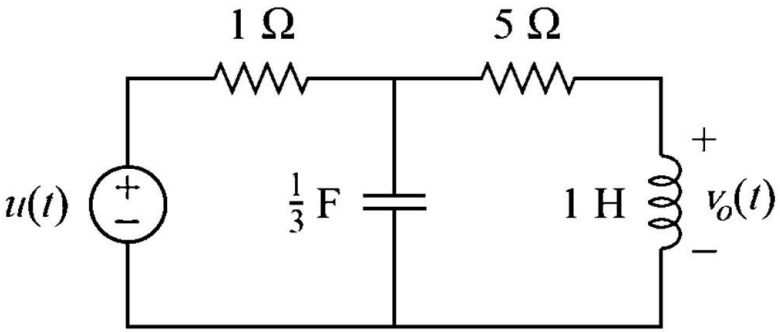
$$1\,H \Rightarrow sL = s$$

$$1/3\,F \Rightarrow 1/sC = 3/s$$

Circuit Analysis using Laplace Transform (Cont…)

- The resulting *s*-domain circuit is in the Figure given below. We now apply mesh analysis. For mesh 1,

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right) I_1 - \frac{3}{s} I_2$$

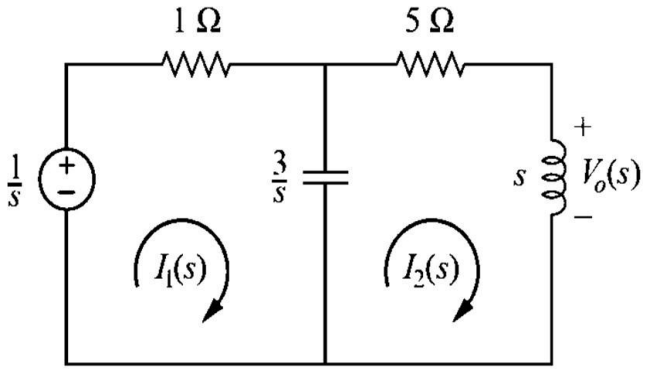


For mesh 2,

$$0 = -\frac{3}{s} I_1 + \left(s + 5 + \frac{3}{s}\right) I_1$$

Or

$$I_1 = \frac{1}{3}(s^2 + 5s + 3) I_2$$



Substituting this into first Equation,

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right) \frac{1}{3}(s^2 + 5s + 3) I_2 - \frac{3}{s} I_2$$

Circuit Analysis using Laplace Transform (Cont…)

- Multiplying through by $3s$ gives

$$3 = (s^3+8s^2+18s)I_2 \qquad \Rightarrow \qquad I_2 = \frac{3}{(s^3+8s^2+18s)}$$

$$V_0(s) = sI_2 = \frac{3}{s^2+8s+18}$$

$$= \frac{3}{\sqrt{2}}\frac{\sqrt{2}}{(s+4)^2+(\sqrt{2})^2}$$

Taking the inverse transform yields

$$v_0(t) = \frac{3}{\sqrt{2}}e^{-4t}\sin\sqrt{2}\,t\ V, \qquad t \geq 0\,.$$

THANKS
FOR YOUR
ATTENTION