Lecture Notes 7: Probability and Statistics

We have talked about the transformation of random variables when the function may or may not be monotone. We have mainly used two different approaches for this purpose. We have provided several examples of discrete and continuous random variables. Now we will discuss about the expectation of a random variables or a function of random variable. Suppose X is a discrete random variable, with PMF

$$P(X = a_i) = p_i; \qquad p_i \geq 0, \quad \text{and} \quad \sum_{i=1}^{\infty} p_i = 1.$$

Then $E(X)$ is defined as

$$E(X) = \sum_{i=1}^{\infty} a_i p_i; \qquad \text{provided} \qquad \sum_{i=1}^{\infty} |a_i| p_i < \infty.$$

Similarly, if X is a continuous random variable with the PDF $f_X(x)$, then $E(X)$ is defined as

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx \qquad \text{provided} \qquad \int_{-\infty}^{\infty} |x| f_X(x) dx < \infty.$$

Similarly, if $g(x)$ is a ny function on the range of X, and X is a discrete random variable as defined above, then $E(g(X))$ is defined as

$$E(g(X)) = \sum_{i=1}^{\infty} g(a_i) p_i; \qquad \text{provided} \qquad \sum_{i=1}^{\infty} |g(a_i)| p_i < \infty.$$

Similarly, if $g(x)$ is a continuous function, then $E(g(X))$ is defined as

$$E(g(X) = \int_{-\infty}^{\infty} g(x) f_X(x) dx \qquad \text{provided} \qquad \int_{-\infty}^{\infty} |g(x)| f_X(x) dx < \infty.$$

For example if you take $g(x) = x$,and if the $E(X)$ exists, then it gives the mean or the expected value of X. Similarly, if you take $g(x) = x^m$, for some integer $m \geq 1$, and if $E(X^m)$ exists, then we will call this as the m-th moment of X. Now let us look at some of the example of different $g(x)$. Suppose X is a Poisson random variable, with parameter $\lambda > 0$, i.e.

$$P(X=n) = \frac{e^{-\lambda}\lambda^n}{n!}; \qquad n = 0, 1, 2, \ldots$$

Then let us look at $E(X)$

$$E(X) = \sum_{n=0}^{\infty} \frac{ne^{-\lambda}\lambda^n}{n!} = \sum_{n=1}^{\infty} \frac{e^{-\lambda}\lambda^n}{(n-1)!} = \sum_{n=1}^{\infty} \frac{e^{-\lambda}\lambda^n}{(n-1)!} = \lambda \sum_{m=0}^{\infty} \frac{e^{-\lambda}\lambda^m}{m!} = \lambda$$

Let us take $g(x) = e^{tx}$, for some fixed t.

$$E(e^{tX}) = \sum_{n=0}^{\infty} \frac{e^{tn}e^{-\lambda}\lambda^n}{n!} = \sum_{n=0}^{\infty} \frac{e^{-\lambda}(e^t\lambda)^n}{n!} = e^{-\lambda(1-e^t)}.$$

Now let us another continuous distribution with PDF

$$f_X(x) = \frac{1}{\pi(1+x^2)}; \qquad -\infty < x < \infty.$$

Note that this a proper PDF,i.e $f_X(x) \geq 0$, for all $-\infty < x < \infty$, and further

$$\int_{-\infty}^{\infty} f_X(x)dx = 1.$$

It is a symmetric PDF,and it is symmetric about zero. It is known as Cauchy distribution. Therefore, the natural instinct will be that $E(X) = 0$. But before come to that conclusion, let us look at the following quantity:

$$I = \int_{-\infty}^{\infty} \frac{|x|}{\pi(1+x^2)}dx.$$

If $I < \infty$, then we can say that $E(X) = 0$. Now

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \frac{|x|}{\pi(1+x^2)}dx = 2\int_0^{\infty} \frac{x}{\pi(1+x^2)} = 2\int_0^1 \frac{x}{\pi(1+x^2)} + 2\int_1^{\infty} \frac{x}{\pi(1+x^2)} \\ &\geq 2\int_0^1 \frac{x}{\pi(1+x^2)} + \frac{1}{\pi}\int_1^{\infty} \frac{x}{x^2} = 2\int_0^1 \frac{x}{\pi(1+x^2)} + \frac{1}{\pi}\int_1^{\infty} \frac{1}{x} = I_1 + I_2. \end{aligned}$$

Note that $I_1 \geq 0$ and I_2 is not finite, hence I is not finite. Hence, $E(X)$ does not exist.

From now on if X is a random variable and the expectation $E(X)$ exists, we will denote it by $E(X) = \mu$. If we take $g(x) = (x - \mu)^2$, then if $E(g(X))$ exists then we will call this as the variance of X. Similarly, if $g(x) = e^{tx}$, for some fixed t and if $E(g(X))$ for $-a < t < a$, for some $a > 0$, then we will call this as the moment generating function (MGF) of X.

Before, progressing further, we will prove one simple result which will be useful in calculating many expectations. Suppose X is a random variable, and $g_1(x)$ and $g_2(x)$ are two functions such that $E(g_1(X))$ and $E(g_2(X))$ exist. Then for any two constants c_1 and c_2, consider a new function $g(x) = c_1 g_1(x) + c_2 g_2(x)$. In this case $E(g(X))$ exists and

$$E(g(X)) = c_1 E(g_1(X)) + c_2 E(g_2(X)).$$

The above equality indicates that the expectation is a linear function.

We will show the result for the continuous case, similar proof holds for the discrete case also. Suppose X has the PDF $f_X(x)$. Then

$$\int_{-\infty}^{\infty} |g_1(x)| f_X(x) dx < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |g_2(x)| f_X(x) dx < \infty$$

Hence

$$\begin{aligned}
\int_{-\infty}^{\infty} |g(x)| f_X(x) dx &= \int_{-\infty}^{\infty} |c_1 g_1(x) + c_2 g_2(x)| f_X(x) dx \\
&\le \int_{-\infty}^{\infty} |c_1 g_1(x)| f_X(x) dx + \int_{-\infty}^{\infty} |c_2 g_2(x)| f_X(x) dx \\
&\le |c_1| \int_{-\infty}^{\infty} |g_1(x)| f_X(x) dx + |c_2| \int_{-\infty}^{\infty} |g_2(x)| f_X(x) dx < \infty.
\end{aligned}$$

Hence, $E(g(X))$ exists. Now

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x) f_X(x) dx &= \int_{-\infty}^{\infty} c_1 g_1(x) + c_2 g_2(x) f_X(x) dx \\
&= c_1 \int_{-\infty}^{\infty} g_1(x) f_X(x) dx + c_2 \int_{-\infty}^{\infty} g_2(x) f_X(x) dx \\
&= c_1 E(g_1(X)) + c_2 E(g_2(X)).
\end{aligned}$$

The above result becomes very helpful in many calculations. For example if X is a random variable and its variance exists, i.e. $E(X-\mu)^2 < \infty$. Now we can compute it as follows:

$$\begin{aligned} E(X-\mu)^2 &= E(X^2 - 2\mu X + \mu^2) = E(X^2) - 2\mu E(X) + E(\mu^2) \\ &= E(X^2) - 2\mu^2 + \mu^2 = E(X^2) - \mu^2. \end{aligned}$$

Now let us look at another example where the above formula can be used quite effectively. Suppose X is a Poisson random variable with parameter $\lambda > 0$, i.e. the PMF of X is

$$P(X = n) = \frac{e^{-\lambda}\lambda^n}{n!}; \qquad n = 0, 1, 2, \ldots$$

Suppose we want to compute the Variance of X. Then we need to compute $E(X^2)$ or $E(X-\lambda)^2$. Please check both these quantities are not very easy to calculate directly. But, it can be done very simply as follows: Let us take $g(x) = x(x-1)$, therefore

$$E(X(X-1)) = \sum_{n=0}^{\infty} \frac{n(n-1)e^{-\lambda}\lambda^n}{n!} = \sum_{n=2}^{\infty} \frac{e^{-\lambda}\lambda^n}{(n-2)!} = \lambda^2 \sum_{m=0}^{\infty} \frac{e^{-\lambda}\lambda^m}{m!} = \lambda^2.$$

Hence you can see

$$E(X(X-1)) = E(X^2) - E(X) = \lambda^2 \qquad \Rightarrow \qquad E(X^2) = \lambda + \lambda^2.$$

Hence the variance of X becomes

$$V(X) = E(X^2) - (E(X))^2 = \lambda + \lambda^2 - \lambda^2 = \lambda.$$

Exercise: If X is a binomial random variable with the PMF

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}; \quad k = 0, 1, \ldots, n.$$

Find $V(X)$.

Exercise: If X is a geometric random variable with the PMF

$$P(X = k) = (1-p)^{k-1} p; \quad k = 1, 2, \ldots,$$

Find $E(X)$ and $V(X)$.

Note that the following form of the geometric random variable is also available in the literature.

Exercise: If X is a geometric random variable with the PMF

$$P(X = k) = (1 - p)^k p; \quad k = 0, 1, 2, \ldots,$$

Find $E(X)$ and $V(X)$.

We have already mentioned that if $g(x) = e^{tx}$ then $M(t) = E(e^{tX})$ is known as the moment generating function (MGF) of X if there exists an $a > 0$, such that $M(t) = E(e^{tX})$ exists for all $-a < t < a$. From now on the MGF of a random variable X will be denoted by $M_X(t)$. First note that if X and Y are two random variables such that they have the same distribution functions, i.e. $F_X(t) = F_Y(t)$, for all $-\infty < t < \infty$ and the MGF exists, then it is clear that

$$M_X(t) = M_Y(t).$$

Now we will provide an important result with out proof that if X and Y are two random variables and there exists an a,

$$M_X(t) = M_Y(t) \quad \text{for all } -a < t < a,$$

then $F_X(z) = F_Y(z)$, for all $-\infty < z < \infty$. Note that this is a very strong statement, and we will explore this more later.