Lecture Notes 5: Probability and Statistics

So far we have talked about the discrete, continous and mixture distributions. In this course whenever we talk about the continuous distribution we actually mean absolutely continuous distribution. Although, we do not make it explicit. Remember, an absolutely continuous distribution function is a continuous distribution function $F(x)$, for which the probability density function exists, i.e.

$$F(x) = \int_{-\infty}^{x} f(u)du, \quad \text{for all } -\infty < x < \infty,$$

where for all x,

$$f(x) \geq 0, \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)dx = 1.$$

Please keep in mind that there are continuous distribution function $F(x)$, for which the density function does not exist, but we do not discuss any further in this course. The main reason we will avoind here because it may not have much practical value. Hence, from now on in this course whenever we talk about a continious distribution function $F(x)$ it is assumed that $F(x)$ has a probability density functio (PDF) and it can be obtained as

$$f(x) = \frac{d}{dx}F(x).$$

Therefore, a discrete distribution function is characterized by its probability mass function (PMF) and a continuous distribution function is characterized by the corresponding PDF. It has also observed that any cumulative distribution function (CDF) can be expressed uniquely as

$$F(x) = \alpha F_c(x) + (1-\alpha)F_d(x); \quad \text{for all} \quad -\infty < x < \infty.$$

Here $F_c(x)$ a continuous CDF, $F_d(x)$ is a discerete distribution function and $0 \leq \alpha \leq 1$. Therefore, any CDF can be written uniquely as a convex combination of a continuous CDF and a discrere CDF. If $\alpha = 0$, then $F(x) = F_d(x)$, i.e. a discrete CDF and if $\alpha = 1$, then $F(x) = F_c(x)$, i.e. a continuous CDF. If $0 < \alpha < 1$, then $F(x)$ has a mixture CDF. We have already discussed how to recover $F_c(x)$ and $F_d(x)$ from a given $F(x)$.

Now we will discuss about the transformation of random variables. First we will discuss about the discrete random variable and then we will discuss about the continous random variables. Suppose X is a discrete random variable with the following PMF

$$P(X = a_i) = p_i; \qquad i = 1, 2, \ldots,$$

where

$$-\infty < a_1, a_2, \ldots < \infty \quad \text{and} \quad p_1, p_2 \ldots \geq 0, \ \sum_{i=1}^{\infty} p_i = 1.$$

Let us define $\mathcal{D}_X = \{a_1, a_2, \ldots\}$ as the range of the random variable. If $g(\cdot)$ is a one-to-one function defined on $\mathcal{D}_X$, and Y is a new random variable such that $Y = g(X)$, the range of Y becomes $\mathcal{D}_Y = \{g(1), g(2), \ldots, \}$. Then the PMF of Y can be obtained as

$$P(Y = g(i)) = P(X = i) = p_i; \quad i = 1, 2, \ldots$$

Let us look at the following example;

Example: Suppose X is a Poisson random variable with the following PMF, for $\lambda > 0$

$$P(X = i) = \frac{e^{-\lambda}\lambda^i}{i!}; \qquad i = 0, 1, 2, \ldots.$$

First observe that it is a proper PMF as the expansion of e^{λ} is

$$e^{\lambda} = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}.$$

Here $\mathcal{D}_X = \{0, 1, 2, \ldots\}$. Suppose $g(i) = i^2 + i + 1$ defined on $\mathcal{D}_X$. Clearly, it is a one-to-one function. Therefore, the PMF of $Y = g(X)$ becomes

$$P(Y = i^2 + i + 1) = P(X = i) = \frac{e^{-\lambda}\lambda^i}{i!}; \qquad i = 0, 1, 2, \ldots.$$

Now we will discuss when $g(\cdot)$ is not a one-to-one function. It means it is a many-to-one function, i.e. there exists $i_1 \neq i_2 \neq i_3 \ldots$ such that $g(i_1) = g(i_2) = \ldots$. Suppose X is a discrete random variable as defined above and

$g(\cdot)$ is a many-to-one function, and $Y = g(X)$. Let the range of Y becomes $\mathcal{D}_Y = \{b_1, b_2, \ldots\}$. Then the PMF of $Y = g(X)$ can be written as

$$P(Y = b_i) = \sum_{j; g(j) = b_i} p_j; \qquad i = 1, 2,$$

Let us look at the following example

Example Suppose X is a discrete random variable with the following PMF, for $\lambda > 0$,

$$P(X = 0) = e^{-\lambda}, \quad P(X = \mp i) = \frac{e^{-\lambda}\lambda^i}{i!}; \quad i = 1, 2, 3, \ldots.$$

In this case $\mathcal{D}_X = \{0, \mp 1, \mp 2, \ldots\}$. Suppose $g(x) = |x|$ defined on $\mathcal{D}_X$. Clearly, $g(\cdot)$ is a many-to-one function, and the PMF of $Y = g(X) = |X|$ becomes

$$P(Y = i) = \frac{e^{-\lambda}\lambda^i}{i!}; \quad i = 0, 1, 2, 3, \ldots.$$

Now let us discuss abot the continuous random variable. Suppose X is a continuous random variable with the PDF $f_X(x)$ and range of X as $\mathcal{D}_X$. In this case we assume a slightly stronger assumption on the function $g(\cdot)$. It is assumed that $g(\cdot)$ is a one-to-one and differentiable function defined on $\mathcal{D}_X$. It will be clear why we need this condition. We want to obtain the CDF and PDF of $Y = g(X)$. First observe that since $g(\cdot)$ is a one-to-one function, therefore it has to be either an increasing or a decreasing function. Let us assume for the time being that it is an increasing function. Therefore, the CDF of Y can be written as

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(g(X) \le y) = P(X \le g^{-1}(y)) \\ &= F_X(g^{-1}(y)) = \int_{-\infty}^{g^{-1}(y)} f(u)du = F_X(g^{-1}(y)). \end{aligned}$$

Hence, the PDF of Y can be obtained as

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(g^{-1}(y)).$$

Let us denote $x = g^{-1}(y)$. Then the PDF of Y can be expressed as

$$f_Y(y) = \left(\frac{d}{dx}F_X(x)\right)\frac{dx}{dy} = f_X(x)\frac{dx}{dy}.$$

Similarly, if $g(\cdot)$ is a decreasing function, then

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(g(X) \leq y) = P(X \geq g^{-1}(y)) \\ &= 1 - F_X(g^{-1}(y)) = 1 - \int_{-\infty}^{g^{-1}(y)} f(u)du = 1 - F_X(g^{-1}(y)). \end{aligned}$$

Hence, the PDF of Y can be obtained as

$$f_Y(y) = \frac{d}{dy}F_Y(y) = 1 - \frac{d}{dy}F_X(g^{-1}(y)) = f_X(x)\left|\frac{dx}{dy}\right|.$$

Let us look at the following example.

Example: Suppose X has the following PDF for $\alpha > 0$

$$f_X(x) = \alpha x^{\alpha-1}e^{-x^{\alpha}} \qquad x > 0,$$

and 0, otherwise. Consider $Y = X^{\alpha}$. Then the CDF of Y for $y > 0$ becomes

$$F_Y(y) = P(Y \leq y) = P(X^{\alpha} \leq y) = P(X \leq y^{1/\alpha}) = \int_0^{y^{1/\alpha}} f_X(u)du = 1 - e^{-y}.$$

Hence, the PDF of Y becomes

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}(1 - e^{-y}) = e^{-y}; \quad y > 0,$$

and 0, otherwise.

Exercise: In the same example above if $Y = 1/X$, find the CDF and PDF of Y

Exercise: In the same example above if $Y = \ln X$, find the CDF and PDF of Y