Lecture-13

On

INTRODUCTION TO ELECTRICAL ENGINEERING (ESO203)

- RC circuit.
- RL circuit.

First Order Circuits

- In first order circuits, we will discuss about two types of simple circuits: a circuit comprising a resistor and a capacitor; and a circuit comprising a resistor and an inductor.
- These are called R-C and R-L circuits, respectively.
- Analysis of R-C and R-L circuits is done by applying Kirchhoff's laws, as is done for resistive circuits.
- Applying Kirchhoff's laws to purely resistive circuits results in algebraic equations, while applying the laws to R-C and R-L circuits produces differential equations, more difficult to solve than algebraic equations.
- The differential equations resulting from analyzing R-C and R-L circuits are of the first order. Hence, the circuits are collectively known as first-order circuits.
- A first-order circuit is characterized by a first-order differential equation.

First Order Circuits (Cont…)

Two ways to excite the first-order circuits:

- First way:
  - To excite the circuit by using initial conditions of the storage elements in the circuits.
  - These types of circuits are called as source-free circuits.
  - Here, it is assumed that energy is initially stored in the capacitive or inductive element.
  - The energy causes current to flow in the circuit and is gradually dissipated in the resistors.
  - Although source free circuits are free of independent sources, they may have dependent sources.
- Second way:
  - Exciting first-order circuits by independent sources.

The Source-Free R-C Circuit

- A source-free R-C circuit occurs when its dc source is suddenly disconnected.
- The energy already stored in the capacitor is released to the resistors.

- Consider a series combination of a resistor and an initially charged capacitor, as shown in Fig. 1 (The resistor and capacitor may be the equivalent resistance and equivalent capacitance of combinations of resistors and capacitors). The objective is to determine the circuit response.

A circuit response is the manner in which the circuit reacts

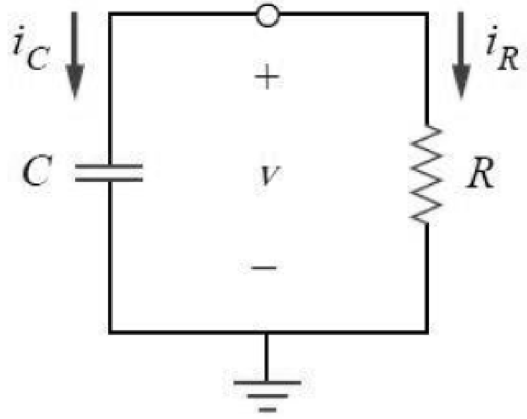


Fig. 1

The Source-Free R-C Circuit (Cont…)

- Assume a voltage v(t) across capacitor. Since the capacitor is initially charged, we can assume that at time t = 0, the initial voltage is

$$v(0) = V_0 \qquad (1)$$

with the corresponding value of the energy stored as

$$w(0) = \frac{1}{2}CV^2 \qquad (2)$$

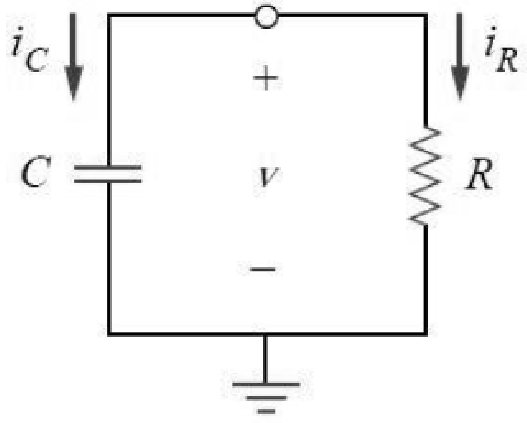


Fig. 1

- Applying KCL at the top node of the circuit in Fig. 1,

$$i_C + i_R = 0 \qquad (3)$$

- By definition, $i_C = C\frac{dv}{dt}$ and $i_R = \frac{v}{R}$, Thus,

$$C\frac{dv}{dt} + \frac{v}{R} = 0 \qquad (4)$$

The Source-Free R-C Circuit (Cont…)

- This is a first-order differential equation, since only the first derivative of v is involved. To solve it, we rearrange the terms as

$$\frac{dv}{v} = -\frac{1}{RC}dt \tag{5}$$

- Integrating both sides, we get

$$lnv = -\frac{t}{RC} + lnA$$

- where $ln\ A$ is the integration constant. Thus

$$ln\frac{v}{A} = -\frac{t}{RC} \tag{6}$$

The Source-Free R-C Circuit (Cont…)

- Taking powers of *e* on both sides produces

$$v(t) = Ae^{-t/RC}$$

- But from the initial conditions, $v(0) = A = V_0$. Hence,

$$v(t) = V_0 e^{-t/RC} \qquad (7)$$

- This shows that the voltage response of the R-C circuit is an exponential decay of the initial voltage.
- Since the response is due to the initial energy stored and the physical characteristics of the circuit; and not due to some external voltage or current source, it is called the natural response of the circuit.

The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.

The Source-Free R-C Circuit (Cont…)

- The natural response is illustrated graphically in the figure below. Note that, value at $t = 0$ is the initial condition.
- As t increases, the voltage decreases toward zero. The rapidity with which the voltage decreases is expressed in terms of the time constant, denoted by the lower-case Greek letter tau, τ .
- The value of τ is RC for the R-C circuit

The natural response depends on the nature of the circuit alone, with no external sources. In fact, the circuit has a response only because of the energy initially stored in the capacitor.

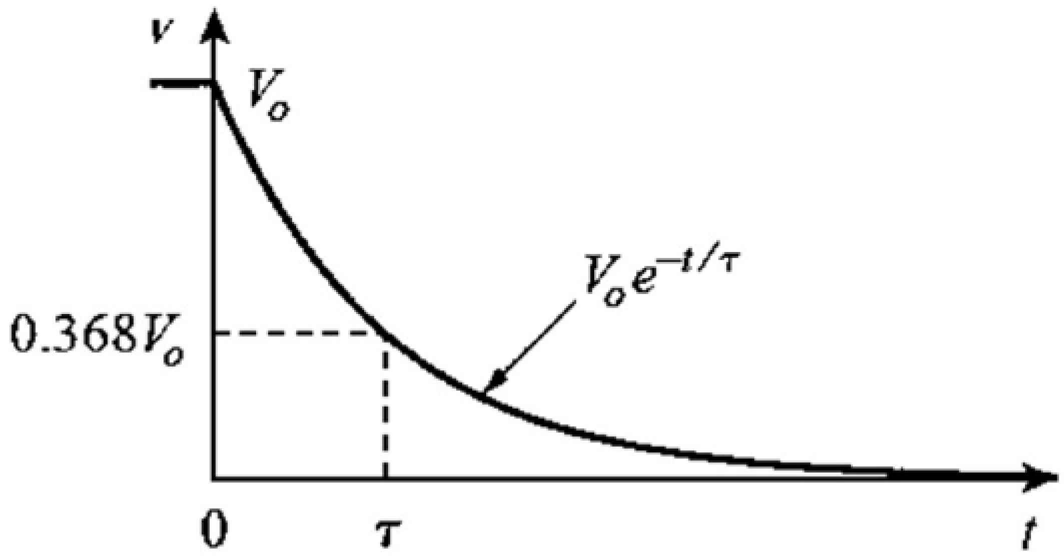


Fig. 2

The Source-Free R-C Circuit (Cont…)

- We can also say that the time constant of a circuit is the time required for the response to decay by a factor of $1/\boldsymbol{e}$ or 36.8 percent of its initial value.

  As,

$$v(t) = V_0 e^{-t/RC}$$

- This implies that at $t = \tau$,

$$V_0 e^{-t/RC} = V_0 e^{-1} = 0.368 V_0$$

- Here, $\tau = RC$
- In terms of the time constant, above equation can be written as

$$v(t) = V_0 e^{-t/\tau}$$

The Source-Free R-C Circuit (Cont…)

- The value of $v(t)/V_0$ is as shown in the Table. From Table, it can be verified that the voltage v(t) is less than 1 percent of V_0 after 5τ (five-time constants).
- Thus, it is customary to assume that the capacitor is fully discharged (or charged) after five-time constants.
- It takes 5τ for the circuit to reach its final state or steady state when no changes take place with time.
- For every time interval of τ , the voltage is reduced by 36.8 percent of its previous value - $v(t + \tau) = v(t)/e = 0.368v(t)$, regardless of the value of t.

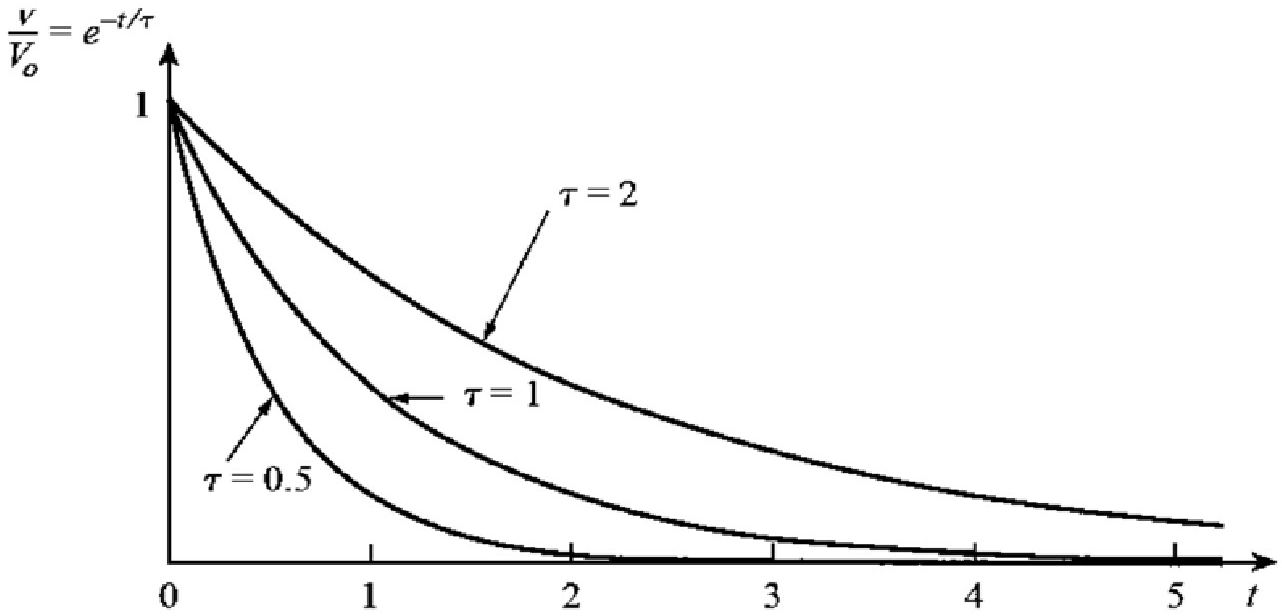
Table 1

$v(t)/V_0 = e^{-t/\tau}$.

t	$v(t)/V_0$
τ	0.36788
2τ	0.13534
3τ	0.04979
4τ	0.01832
5τ	0.00674

The Source-Free R-C Circuit (Cont…)

- Smaller the time constant, more rapidly the voltage decreases; that means, faster the response.
- A circuit with a small-time constant gives a fast response and reaches the steady state (or final state) quickly due to quick dissipation of energy stored. Whereas a circuit with a large time constant gives a slow response because it takes longer time to reach the steady state.



The Source-Free R-C Circuit (Cont…)

- At any rate, whether the time constant is small or large, the circuit reaches at steady state in five-time constants.

  Using voltage $v(t)$, we can find the current $i_R(t)$,

$$i_R(t) = \frac{V_0}{R} e^{-t/\tau}$$

- The power dissipated in the resistor is

$$p(t) = v * i_R = \frac{V_0^2}{R} e^{-2t/\tau}$$

- So, the energy absorbed by the resistor up to time t is

$$w_R(t) = \int_0^t p dt = \frac{1}{2} C V_0^2 \left(1 - e^{-2/\tau}\right)$$

The Source-Free R-C Circuit (Cont…)

- As $t \to \infty$, $w_R(\infty) \to \frac{1}{2}CV_0^2$, which is the same as $w_C(0)$, i.e. the energy initially stored in the capacitor. The energy that was initially stored in the capacitor is eventually dissipated in the resistor.

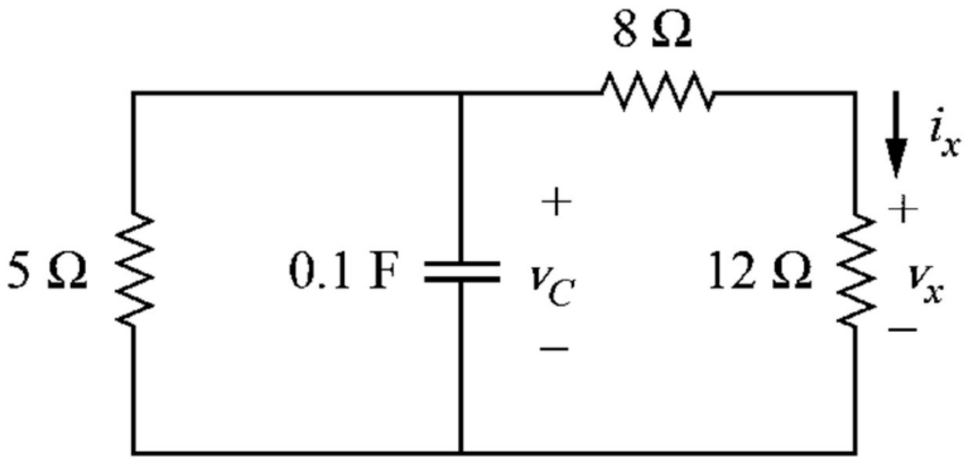
- The Key to Working with a Source - free RC Circuit -
  1. Find the initial voltage $v(0) = V_0$ across the capacitor.
  2. The time constant τ .

The time constant is the same regardless of what the output is defined to be.

In finding the time constant $\tau = RC$, R is often the Thevenin equivalent resistance at the terminals of the capacitor; that means, take out the capacitor C and find $R = R_{Th}$ at its terminals.

Example:

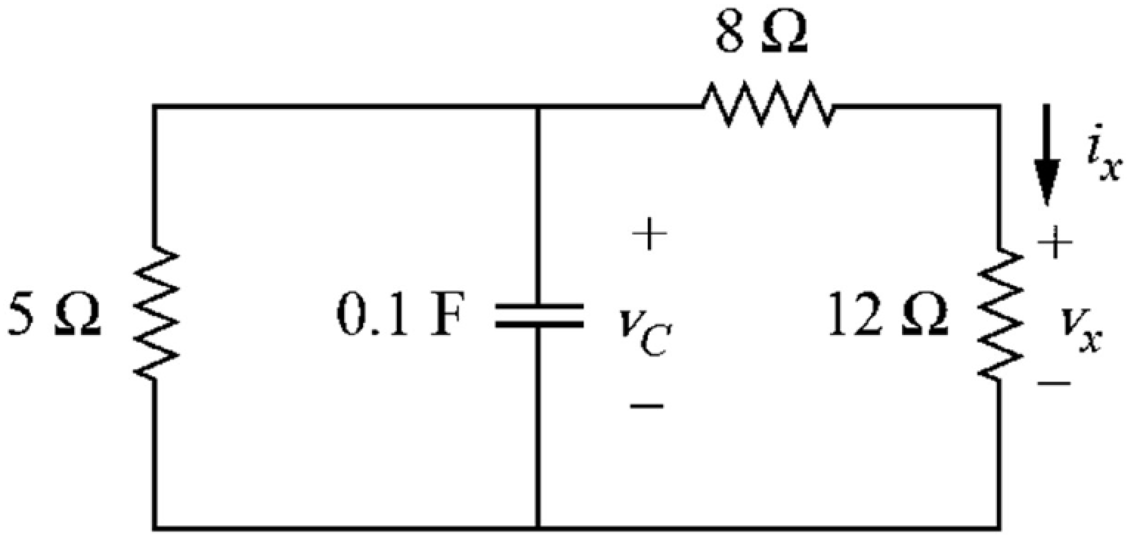
In the Figure shown below, let $v_C(0) = 15$ V. Find the values of v_C, v_x, and i_x for $t > 0$?



- Find the equivalent resistance or the Thevenin resistance at the capacitor terminals.
- First obtain capacitor voltage v_C. From this, determine v_x and i_x .

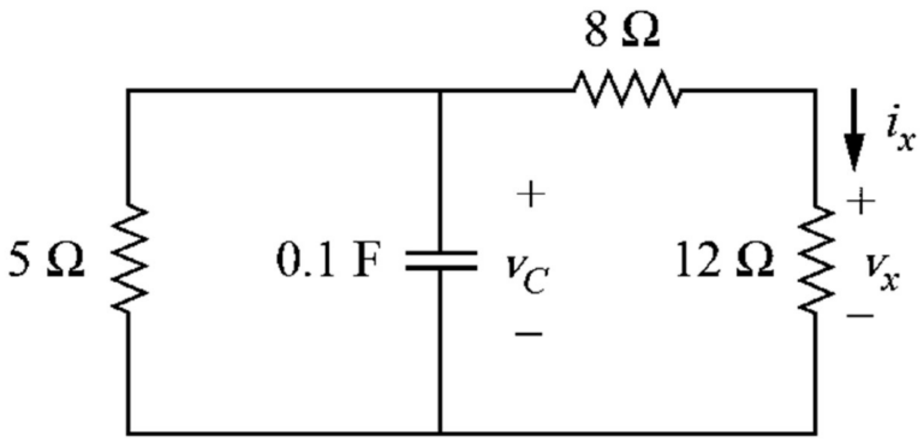
The 8Ω and 12 Ω resistors in series can be combined to give a 20 Ω resistor. This 20 Ω resistor in parallel with the 5 Ω resistor can be combined so that the equivalent resistance is-

$$R_{eq} = \frac{20 * 5}{20 + 5} = 4\Omega$$



Hence, the equivalent circuit is as shown in the Figure. The time constant is -

$$\tau = R_{\text{eq}}C = 4(0.1) = 0.4\ s$$



Thus,

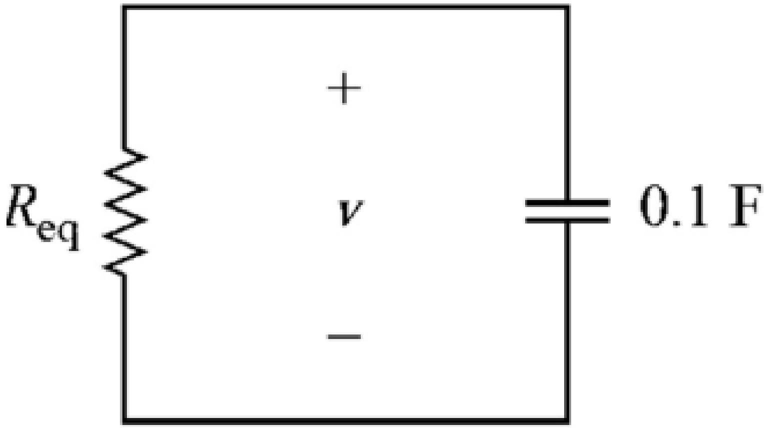
$$v = v(0)e^{-t/\tau} = 15e^{-t/0.4}\ V$$

So,

$$v_C = v\ = 15e^{-2.5t}\ V$$

use voltage division to get v_x; so -

$$v_x = \frac{12}{12+8}v\ = 0.6(15e^{-2.5t})=9e^{-2.5t}\ V$$



Finally,

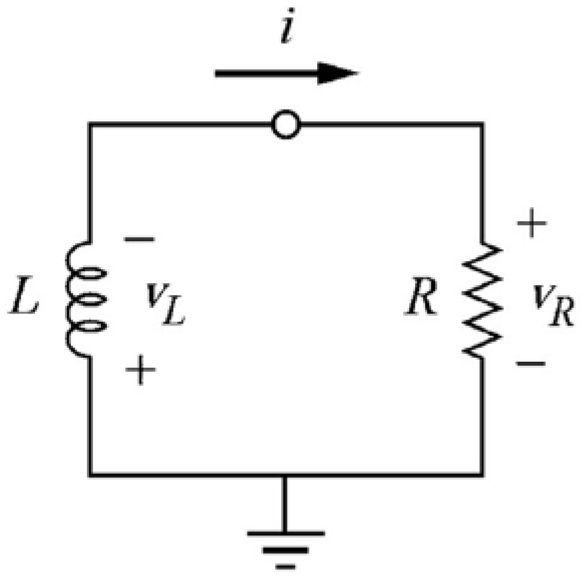
$$i_x = \frac{v_x}{12} = 0.75e^{-2.5t}\ \text{A}$$

The Source-Free R-L Circuit

- A series connection of a resistor and an inductor, as shown in the Figure below.
- The goal is to determine the circuit response, which is the current $i(t)$ through the inductor.
- Select the inductor current as the response in order to take advantage of the idea that the inductor current cannot change instantaneously.

At $t = 0$, the inductor has an initial current I_0 , or

$$i\,(0) = I_0$$



The Source-Free R-L Circuit (Cont…)

- with the corresponding energy stored in the inductor as

$$w(0) = \frac{1}{2} L I_0^2$$

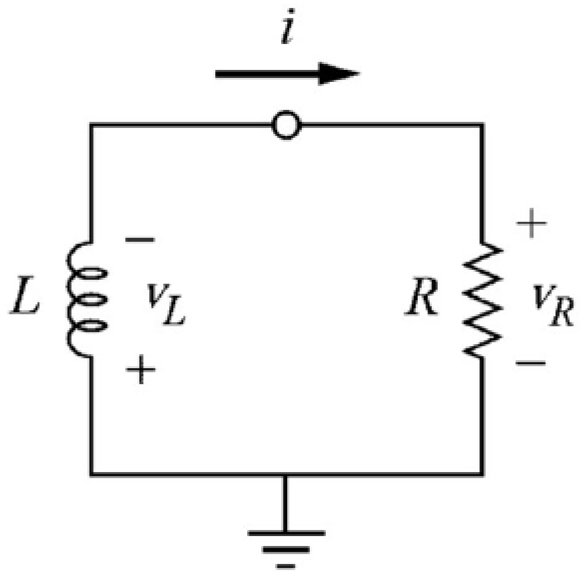
- Applying KVL around the loop in Figure,

$$v_L + v_R = 0$$

- But, $v_L = L\frac{di}{dt}$ and $v_R = \text{iR}$, Thus,

$$\text{L}\frac{di}{dt} + \text{Ri} = 0$$

$$\frac{di}{dt} + \frac{R}{L}\text{i} = 0$$



The Source-Free R-L Circuit (Cont…)

Rearranging terms and integrating gives

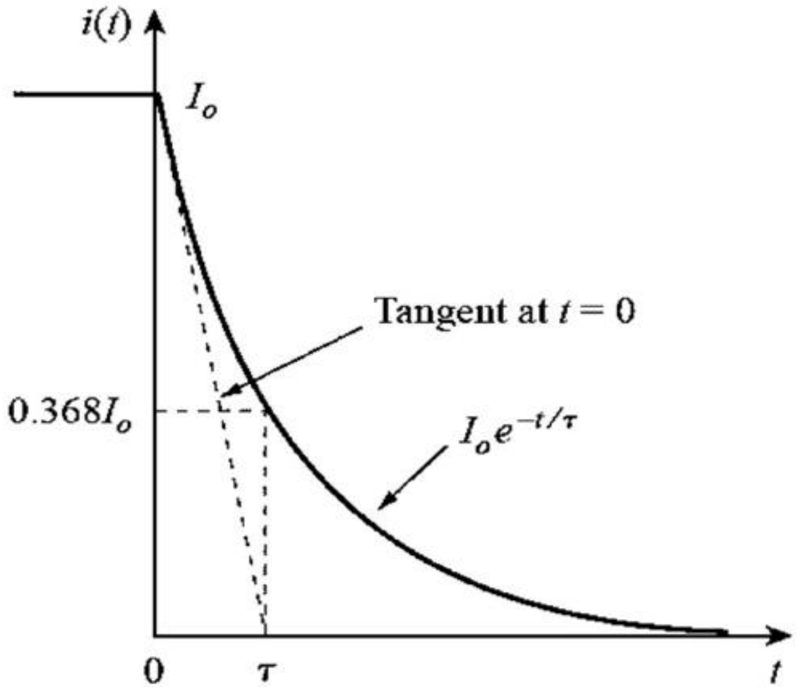
$$ln\frac{i(t)}{\mathrm{I}_0} = -\frac{R}{L}t$$

Taking the powers of e, we have

$$i(t) = I_0 e^{\frac{-Rt}{L}}$$

This shows that the natural response of the R-L circuit is an exponential decay of the initial current.

The current response is shown in the Figure.



The Source-Free R-L Circuit (Cont…)

So, from Equation $i(t) = I_0 e^{\frac{-Rt}{L}}$, we can say that the time constant for the R-L circuit is-

$$\tau = \frac{L}{R}$$

The time constant τ has the unit of seconds. Thus, above equation may be written as

$$i(t) = I_0 e^{\frac{-t}{\tau}}$$

The voltage across the resistor can be given as -

$$v_R(t) = iR = I_0 R e^{-t/\tau}$$

The Source-Free R-L Circuit (Cont…)

The power dissipated in the resistor is

$$p = iv_R = {I_0}^2 R e^{-2t/\tau}$$

The energy dissipated by the resistor is

$$w_R(t) = \int_0^t p\,dt = \frac{1}{2} L {I_0}^2 \left(1 - e^{-2t/\tau}\right)$$

$$\text{At } t \to \infty,\ \ w_R(\infty) \to \frac{1}{2} L {I_0}^2$$

Which is the same as $w_L(0)$, i.e. the initial energy stored in the inductor

So, the energy initially stored in the inductor is eventually dissipated in the resistor.

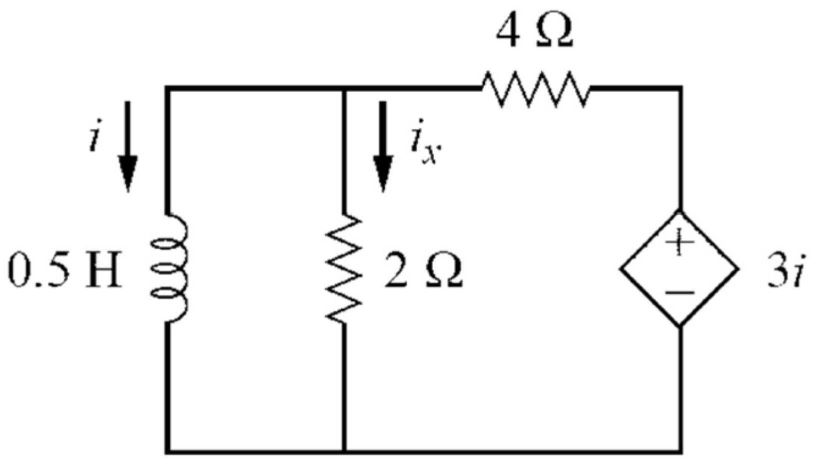
The Source-Free R-L Circuit (Cont…)

- Smaller the time constant, τ, of a circuit, faster the rate of decay of the response.
- Larger the time constant, slower the rate of decay of the response.
- At any rate, the response decays to less than 1 percent of its initial value (i.e., reaches at steady state) after 5τ .

- While Working with a Source - free R-L Circuit:
  1. Find the initial current $i(0) = I_0$ through the inductor.
  2. The time constant τ of the circuit.
  3. When a circuit has a single inductor and several resistors and dependent sources, the Thevenin equivalent can be found at the terminals of the inductor to form a simple R-L circuit. Also, one can use Thevenin's theorem when several inductors can be combined to form a single equivalent inductor.

Example:

Assuming that $i(0) = 10\ A$, calculate $i(t)$ and $i_x(t)$ in the circuit shown below.



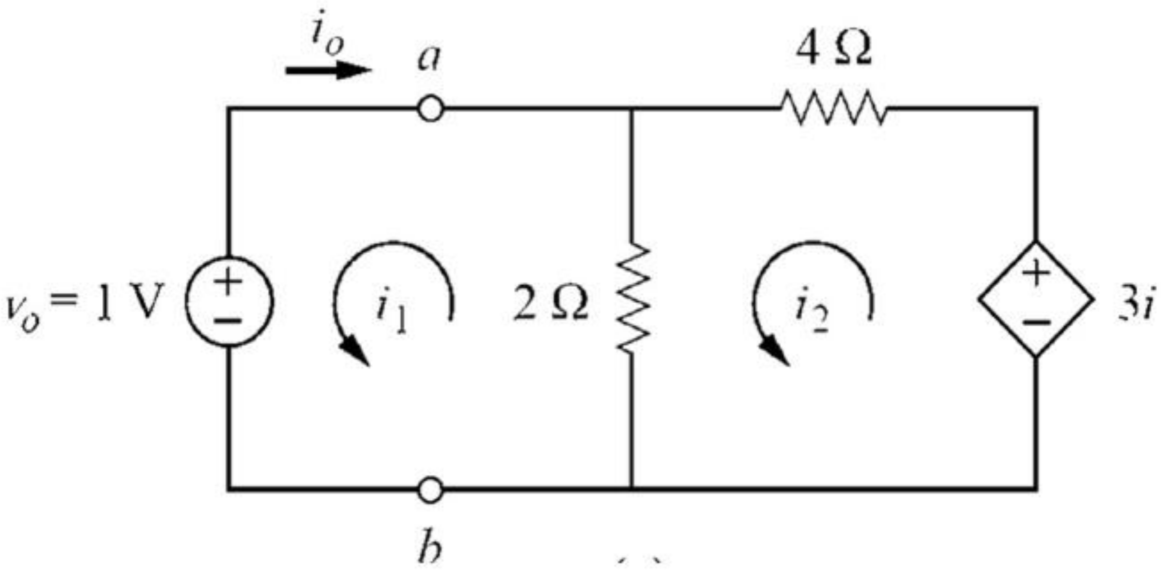
There are two ways to solve this problem.

- One way is to obtain the equivalent resistance at the inductor terminals.
- The other way is to start from scratch by using Kirchhoff's voltage law.

Whichever approach is taken, it is always better to first obtain the inductor current.

Method 1 : The equivalent resistance is the same as the Thevenin resistance at the inductor terminals. Because of the dependent source, a voltage source is inserted with $v_0 = 1\ V$ at the inductor terminals a-b, as shown in the Figure in next slide.

(We can also insert a 1-A current source at the terminals.)



$$2i_1 - 2i_2 + 1 = 0$$

$$i_1 - i_2 = -\frac{1}{2} \qquad (a)$$

$$6i_2 - 2i_1 - 3i_1 = 0$$

$$i_2 = \frac{5}{6} i_1 \qquad (b)$$

Substituting Eq. (b) into Eq. (a) gives-

$$i_1 = -3\ A, \qquad i_0 = -i_1 = 3A$$

Hence,

$$R_{eq} = R_{Th} = \frac{v_0}{i_0} = \frac{1}{3}\,\Omega$$

The time constant is-

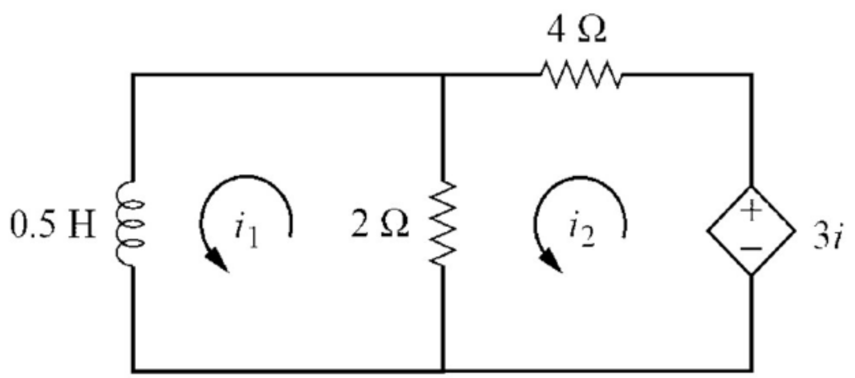
$$\tau = \frac{L}{R_{eq}} = \frac{\frac{1}{2}}{\frac{1}{3}}$$

$$\tau = \frac{L}{R_{eq}} = \frac{3}{2} S$$

Thus, the current through the inductor is-

$$i(t) = i(0)e^{\frac{-t}{\tau}} = 10e^{\frac{-2t}{3}} A, \qquad t > 0$$

Method 2: Apply KVL to the circuit as shown in the Figure below. For loop 1,



$$\frac{1}{2}\frac{di_1}{dt} + 2(i_1 - i_2) = 0$$

$$\frac{di_1}{dt} + 4(i_1 - i_2) = 0 \qquad (a)$$

For Loop 2, the current flowing through inductor, i.e. i_1, decides the value of dependent sources:

$$6i_2 - 2i_1 - 3i_1 = 0$$

$$i_2 = \frac{5}{6} i_1 \qquad\qquad (b)$$

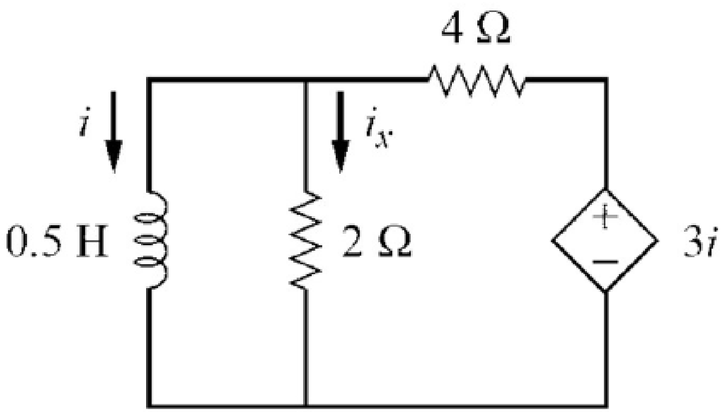
Substituting Eq. (b) into Eq. (a) gives

$$\frac{di_1}{dt} + \frac{2}{3} i_1 = 0$$

Rearranging terms,

$$\frac{di_1}{i_1} = -\frac{2}{3} dt$$

Since $i_1 = i$ in figure, replace i_1 with i and integrate:



$$ln\frac{i(t)}{i(0)} = -\frac{2t}{3}$$

Taking the powers of e, we finally obtain

$$i(t) = i(0)e^{\frac{-t}{\tau}} = 10e^{\frac{-2t}{3}}A, \qquad t > 0$$

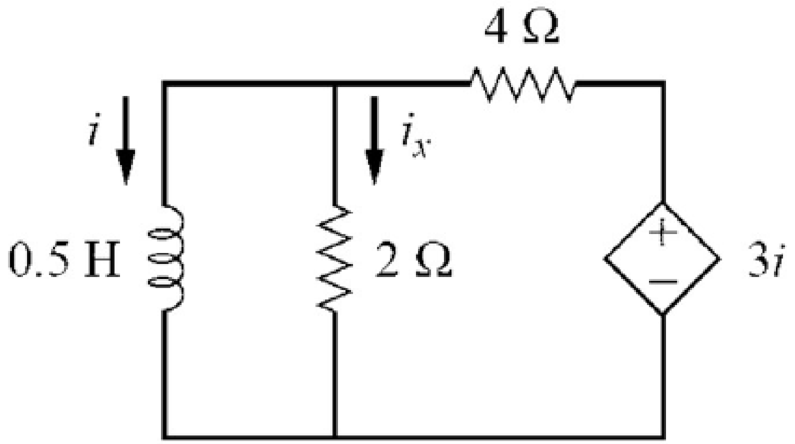
Which is the same as by Method 1.

The voltage across the inductor is-

$$v = L\frac{di}{dt} = 0.5(10)\left(\frac{-2}{3}\right)e^{-2t/3} = \left(\frac{-10}{3}\right)e^{-2t/3}\ V$$

Since the inductor and the 2 Ω resistor are in parallel,

$$i_x(t) = \frac{v}{2} = -1.667e^{\frac{-2t}{3}}A, \qquad t > 0$$

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