Lecture Notes 6: Probability and Statistics

In the previous lecture note we have discussed about the transformation of random variable when the function is one-to-one. We have seen that if X is a discrete random variable, with a given PMF, then how to obtain the PMF of $Y = g(X)$, when $g(\cdot)$ is a one-to-one function. We have also seen if X is a continuous random variable with a PDF $f_X(x)$, and if $Y = g(X)$, where $g(\cdot)$ is a one-to-one function, then how to obtain the PDF of Y. We have seen that we can obtain it two different ways, one is the direct way by using the CDF, and the second one by using the formula

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-y} \right|.$$

It should be mentioned that the distribution function approach can be used even if the function $g(\cdot)$ is not a one-to-one function, where as Jacobian approach can be used always.

Note that the transformation of a random variable approach is very important in all areas. First we will give an example how it can be used to generate sample from a given distribution from an uniform random variable. Suppose U is an uniform random variable in $(0, 1)$, i.e. the PDF of U can be written as

$$f_U(u) = \begin{cases} 1 & \text{if} \quad 0 < u < 1 \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

Now consider the following new random variable $X = -\ln(1-U)$. Note that here the function $g(u) = -\ln(1-u)$ and in this case $g^{-1}(x) = 1 - e^{-x}$. Then the PDF of X can be obtained as

$$f_X(x) = f_U(g^{-1}(x)) \left| \frac{d}{dx} g^{-1}(x) \right|.$$

Hence, the PDF of X can be written as

$$f_X(x) = \begin{cases} e^{-x} & \text{if} \quad 0 < x < \infty \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$

Therefore, it can be seen that X has exponential distribution. In fact if we want X to follow a specific distribution function, i.e. $F_X(x)$, then it can be obtained by making the transformation $X = F_X^{-1}(U)$. It can be easily seen as

$$P(X \leq x) = P(F_X^{-1}(U) \leq x) = P(U \leq F_X(x)) = F_X(x).$$

Note that the last equality follows as U has uniform distribution, therefore

$$P(U \leq u) = \begin{cases} 0 & \text{if} \quad u < 0 \\ u & \text{if} \quad 0 < u < 1 \\ 1 & \text{if} \quad u \geq 1. \end{cases}$$

Hence, if F_X^{-i} has an explicit form, then we can generate random sample from X by using the transformation F_X^{-U}.

Exercise: How how you can generate sample from X which has the following PDF

$$f_X(x) = \begin{cases} 2xe^{-2x^2} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0. \end{cases}$$

Now we will be discussing few specific distribution functions which are being used quite extensively in different areas of applications in various fields.

Exponential: We have already talked about this specific distribution. Suppose X is a random variable which has the PDF for $\lambda > 0$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0 \end{cases}$$

Here λ is known as the parameter of the random variable or the distribution. The corresponding CDF of X is

$$P(X \leq x) = F_X(x) = \int_0^x \lambda e^{-\lambda u} du = 1 - e^{-\lambda x}; \quad \text{for } x > 0,$$

and zero otherwise. It can be easily verified that for any $0 < a < b < \infty$,

$$P(a < X < b) = \int_a^b \lambda e^{-\lambda u} du = e^{-\lambda a} - e^{-\lambda b}.$$

In this case observe that

$$F_X^{-1}(u) = -\frac{1}{\lambda} \ln(1-u).$$

Hence, if U is an uniform (0,1) random variable $X = -\frac{1}{\lambda}\ln(1-U)$ has an exponential distribution. The exponential distribution is one of the most used distributions in different areas. Further, observe that the expected value of X can be obtained as

$$E(X) = \int_0^\infty x\lambda e^{-\lambda x}dx = -\frac{1}{\lambda}xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x} = \frac{1}{\lambda}.$$

Normal Distribution:

Now let us look at another random variable X. Suppose X has the following PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}; \qquad -\infty < x < \infty.$$

Before, moving further, we need to show that $f_X(x)$ is a proper PDF, i.e. $f(x) \geq 0$, for all $-\infty < x < \infty$ and

$$\int_{-\infty}^{\infty} f(x)dx = 1. \tag{1}$$

To prove (1) let us denote

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx.$$

Therefore

$$\begin{aligned} I^2 &= \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx\right]\left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}dxdy. \end{aligned}$$

Now make the following transformation $x = R\cos(\theta)$ and $y = R\sin(\theta)$, hence

$$\begin{aligned} I^2 &= \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} Re^{-\frac{R^2}{2}}d\theta dR \\ &= \int_0^\infty Re^{-\frac{R^2}{2}}dR = \int_0^\infty e^{-u}du = 1. \end{aligned}$$

Hence $I = 1$. Therefore the above $f_X(x)$ is a proper PDF.

If X is a random variable with the following PDF

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}; \qquad -\infty < x < \infty,$$

then it is called the standard normal random variable. Note that it is a symmetric PDF,i.e. $\phi(x) = \phi(-x)$, for all $-\infty < x < \infty$. The corresponding CDF is denoted by

$$\Phi(x) = P(X \leq x) = \int_{-\infty}^{x} \phi(x) dx.$$

It is very clear that if $x > 0$, then $P(X < -x) = P(X > x)$, hence

$$\Phi(-x) = 1 - \Phi(x) \Rightarrow \Phi(x) + \Phi(-x) = 1.$$

The CDF of a normal distribution $\Phi(x)$ cannot be obtained analytically. It has to be computed numerically. Due to this reason many statistical text books have tabulated values of $\Phi(x)$ for different values of x. The expected value of X can be obtained as

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx = 0,$$

and $xe^{-\frac{x^2}{2}}$ is a odd function of x.

Gamma Distribution

Another important continuous distribution is gamma distribution. It can be defined as follows. First let us consider the following notation:

$$\Gamma(\alpha) = \int_{0}^{\infty} x^{\alpha-1} e^{-x} dx; \qquad \alpha > 0.$$

The notation $\Gamma(\alpha)$ is called 'gamma-alpha'. Verify

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha).$$

It can be easily checked using induction hypothesis that

$$\cancel{\Gamma(n) = n!} \qquad \text{for } n = 1, 2, \ldots.$$

$\Gamma(n+1) = n! \;\; \forall n \in \mathbb{N}$

A random variable X is said to have a gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$, if it has the PDF

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}; \qquad x > 0,$$

and it is zero otherwise. Here $\alpha > 0$ and $\lambda > 0$ are the two parameters of the distribution. When $\alpha = 1$, the PDF becomes

$$f_X(x) = \lambda e^{-\lambda x}; \qquad x > 0,$$

and zero otherwise. Therefore, it is an extension of the exponential distribution. It is being used quite extensively in analyzing different lifetime data set in practice. The shape of the PDF of a gamma distribution is either an unimodal function, when $\alpha > 1$, and it is a decreasing function when $\alpha \leq 1$. The mode of the PDF, i.e. the maximum value of the PDF can be easily obtained by taking derivative of $f_X(x)$ with respect to x. Note that, since the $\ln(x)$ is an increasing function, it implies that the mode can be obtained by taking the derivative of $\ln f_X(x)$. In this case

$$\frac{d}{dx} \ln f_X(x) = \frac{d}{dx} \left[\alpha \ln \lambda - \ln \Gamma(\alpha) + (\alpha - 1) \ln x - \lambda x\right]$$

Hence, for $\alpha > 1$, the mode occurs at

$$\text{mode} = \frac{\alpha - 1}{\lambda}.$$

In this case the expected value of X is

$$\begin{aligned} E(X) &= \int_0^\infty x \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx \\ &= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha} e^{-\lambda x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} = \frac{\alpha}{\lambda}. \end{aligned}$$

Now we will be discussing some of the discrete random variable.

Binomial Distribution: Suppose we toss a coin n times, and let us assume that $P(H) = p$. We observe the number of heads out of n tosses. We denote this the random variable X. The possible values of X are $\{0, 1, \ldots, n\}$. In this case

$$P(X = i) = \binom{n}{i} p^i (1-p)^{n-i}; \qquad i = 0, 1, \ldots, n.$$

Verify

$$\sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = 1.$$

Find the value of i, such that $P(X = i)$ is maximum, and it is called the mode of the binomial distribution. In this case the expected value of X is

$$E(X) = \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i} = np \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i} = np.$$

$r\,{}^{n}C_{r} = n\,{}^{n-1}C_{r-1}$

Geometric Distribution

Suppose we toss a coin until we get the head. Let us consider the random variable X as follows from this experiment. Let X denote the number trials needed to get the first head. Therefore, the possible values of X are $\{1, 2, \ldots, \}$. Hence, if $P(H) = p$, then,

$$P(X = i) = (1-p)^{i-1} p; \qquad i = 1, 2, \ldots.$$

The expected value of X is

$$E(X) = \sum_{i=1}^{\infty} i(1-p)^{i-1} p = p \sum_{i=1}^{\infty} i(1-p)^{i-1}.$$

Since

$$\sum_{i=0}^{\infty} (1-p)^i = \frac{1}{p},$$

therefore, taking derivative with respect to p on both sides gives

$$\sum_{i=1}^{\infty} i(1-p)^{i-1} = \frac{1}{p^2}.$$

Hence,

$$E(X) = \frac{1}{p}.$$