Practice Problems for Mid-semester Exam of EE250, Spring 2025*

DEPARTMENT OF ELECTRICAL ENGINEERING, IIT KANPUR

1. Prove or disprove the pairs in the following table of Laplace transforms.

Table of Laplace Transforms		
$f(t)$, $t \geq 0$	$\mathcal{L}\{f\}$	
1	$\frac{1}{s}$	Here, $H_a(t) = \begin{cases} 0 & t \leq a \\ 1 & t > a \end{cases}$
e^{at}	$\frac{1}{s-a}$	**First shift theorem:** If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$.
t^n	$\frac{n!}{s^{n+1}}$ $(n = 0, 1, \ldots)$	**Third shift theorem:** If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\left\{H_a(t)f(t-a)\right\} = e^{-as}F(s)$
$\sin at$	$\frac{a}{s^2+a^2}$	
$\cos at$	$\frac{s}{s^2+a^2}$	
$\sinh at$	$\frac{a}{s^2-a^2}$	
$\cosh at$	$\frac{s}{s^2-a^2}$	
$H_a(t)$	$\frac{e^{-as}}{s}$	
$\delta(t-a)$	e^{-as}	
$f'(t)$	$s\mathcal{L}\{f\} - f(0)$	
$f''(t)$	$s^2\mathcal{L}\{f\} - sf(0) - f'(0)$	

check

2. Evaluate $\mathcal{L}\left\{\int_0^t f(t)\mathrm{d}t\right\}$ and write your final answer in the box. No marks will be given if you do not show your evaluation. $\boxed{}$

3. Open-loop transfer function of a certain unit-feedback system is as follows:

$$G(s) = \frac{K(s+1)}{s(s-1)}$$

Determine:

3.1. the range of values of $K (K > 0)$ for which the system is stable.
3.2. the value of K that will result in the system being marginally stable.
3.3. the location of the roots of the characteristic equation for the value of K found above.

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You may use whatever methods you know, including Nyquist stability theory.

4. NST problems

4.1. Sketch the Nyquist plot of the open-loop transfer function $G(s) = K/s$.
Your plot should also show (in a slightly magnified manner) the section that corresponds to $Re^{j\phi}(R \to \infty)$ section of the Nyquist contour.

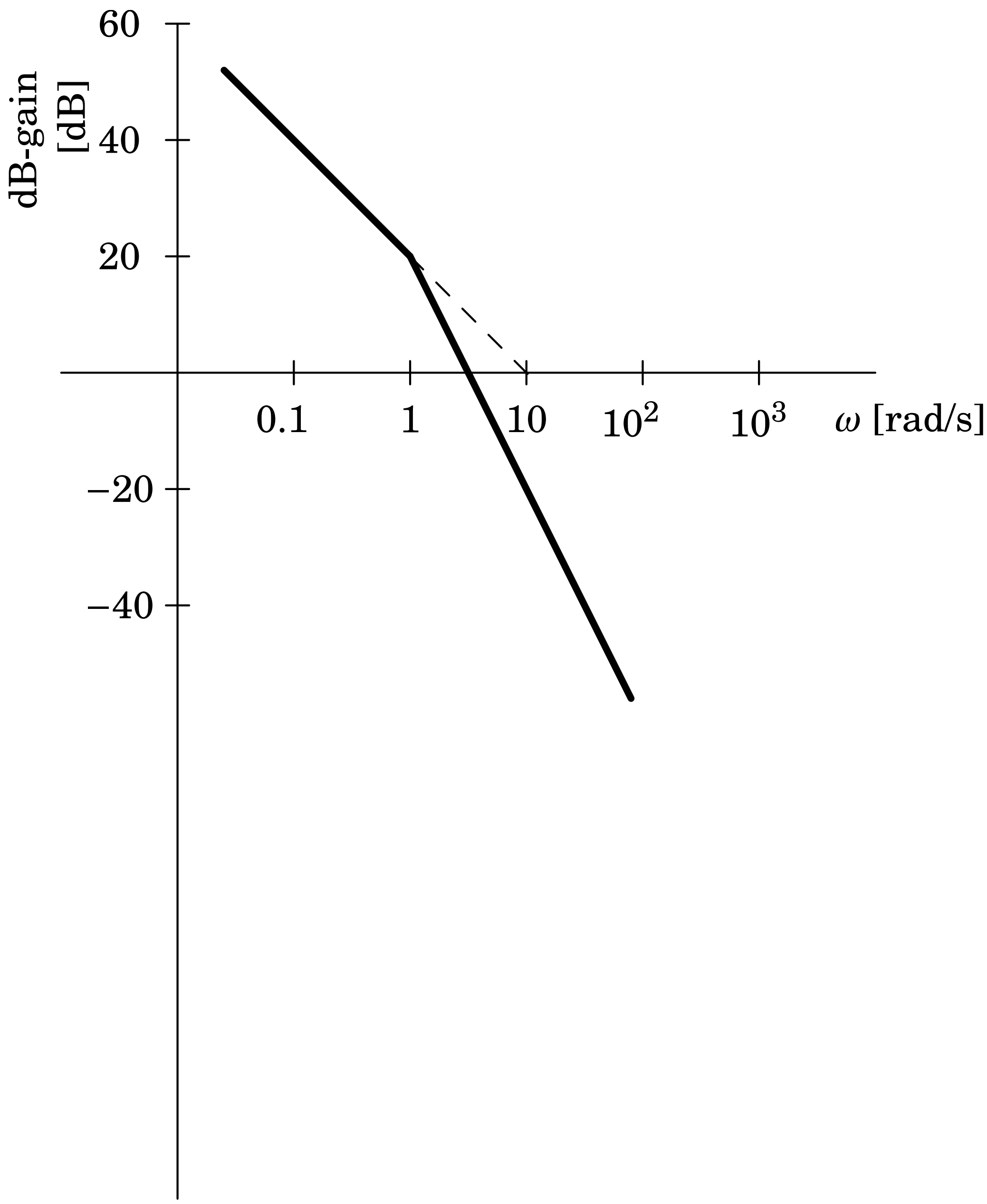
4.2. Identify in some unambiguous and easy-to-understand manner the polar plot part of this Nyquist plot.

4.3. Based on your Nyquist plot, determine the values of K for which corresponding unity-feedback control system is stable.

5. The asymptotic Bode magnitude plot of the open-loop transfer function

$$G(s) = \frac{{\omega_n}^2}{s(s + 2\zeta\omega_n)}$$

is shown in the figure below:



5.1. Label the slopes of the magnitude plot.

5.2. In the space below the magnitude plot, draw the Bode phase plot.

5.3. Determine ω_n and ζ.

6. Given the open-loop transfer function, $G_{\mathrm{OL}}(s) = \frac{K}{s^2+{\omega_0}^2}$, we wish to study the stability of the corresponding unity-feedback closed-loop system for $K \in [0, \infty)$.

7. NST problem.

7.1. Use "Bode plot to polar plot" method to sketch the Nyquist plot of $G_{\mathrm{OL}}(s)$. Your answer will show

7.1.1. a sketch of the Bode plot,

7.1.2. a sketch of the Nyquist plot,

7.1.3. explanation for how you constructed the various sections of the Nyquist plot.

7.2. Use the "tabulation method" to construct the Nyquist plot. Clearly show the s-plane contour, and which point on it maps to which point on the Nyquist plot.

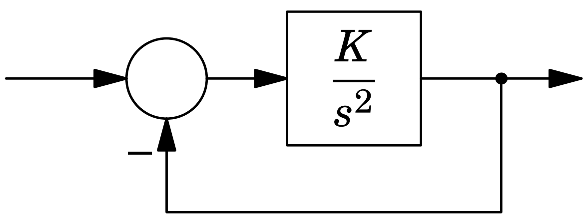
7.3. Use NSC to find the values of $K \in [0, \infty)$ for which the closed-loop system is stable, for which it is marginally stable, and for which it is unstable.

8. For the transfer function $\frac{K}{s+\omega_1}$, determine the dB error between the magnitude Bode plot and the asymptotic magnitude Bode plot at $\omega = \omega_2$ which is less than ω_1 but not significantly so.

9. Find the Laplace Transform of $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}$. To earn full points, you will need to write all the assumptions that you will make en route to the final answer.

10. For the transfer function $G(s) = \frac{K}{s^2+2\xi\omega_n s+{\omega_n}^2}$ determine the frequency ω_{max} at which the dB-gain of the frequency response attains the maximum value. What is the value of the dB-gain at ω_{max}?

11. Use NSC to determine the values of K for which the following system will be stable.



Your solution will contain the following items:

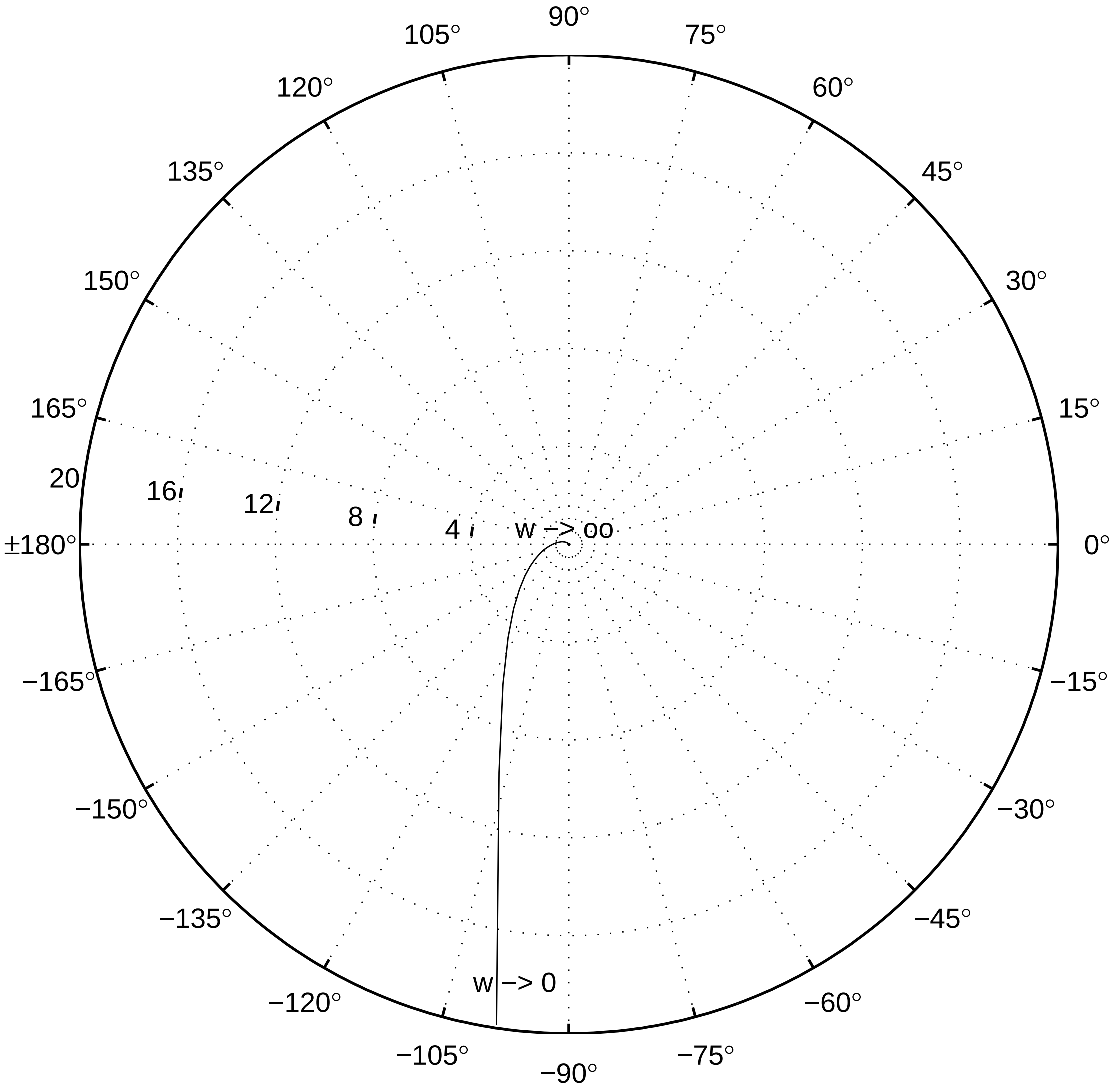
11.1. a neat sketch of the Nyquist contour (a.k.a. s-plane contour),

11.2. a table of values that will give you the Nyquist plot,

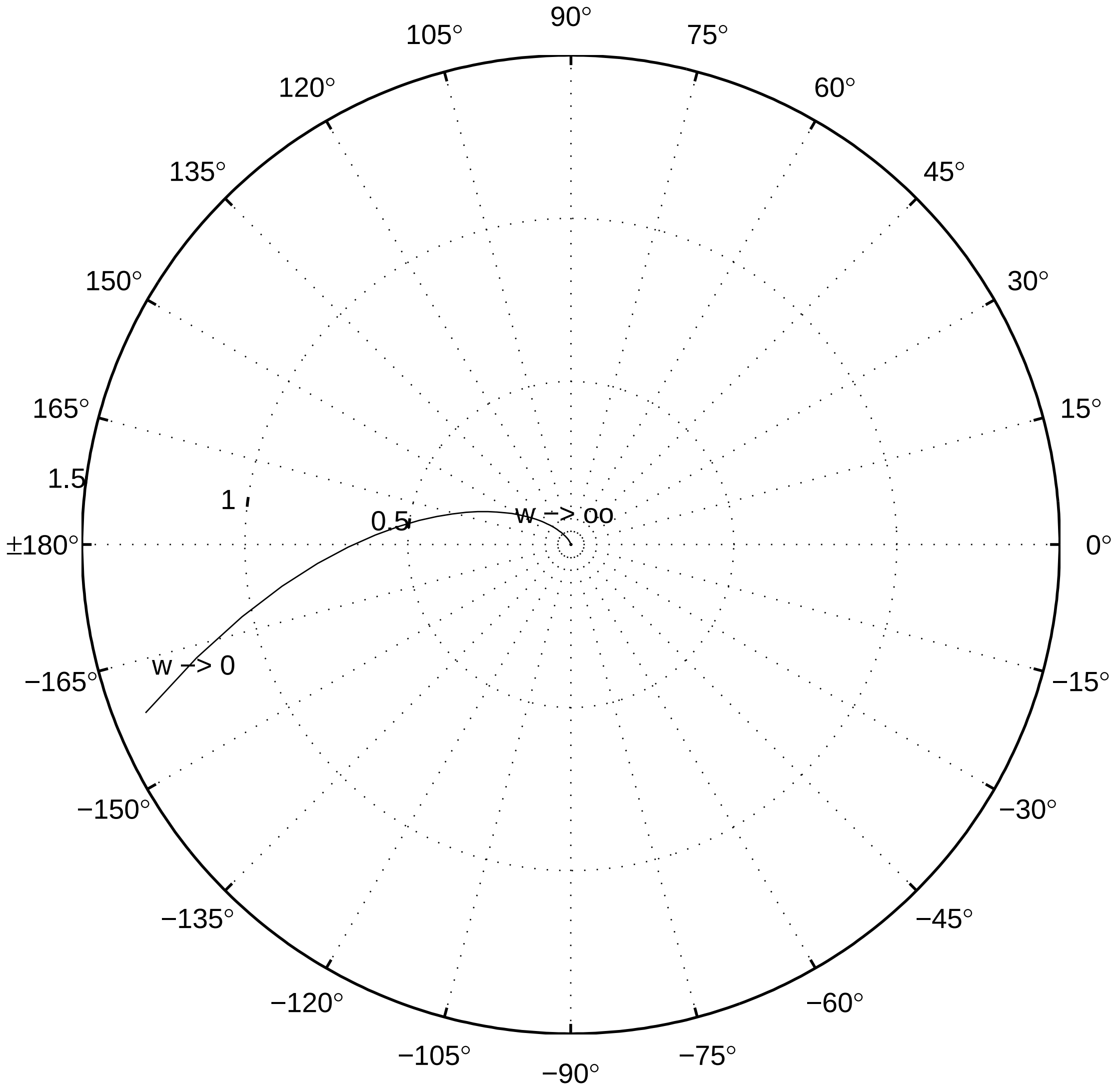
11.3. **[1 point]** a neat sketch of the Nyquist plot, with points appropriately labeled to show which points from the Nyquist contour are mapped to which points on the Nyquist plot,

11.4. inference about stability for positive values of K.

12. **[Your answers must be numerically as accurate as possible]** Consider the following Polar plot of a certain minimum-phase transfer function



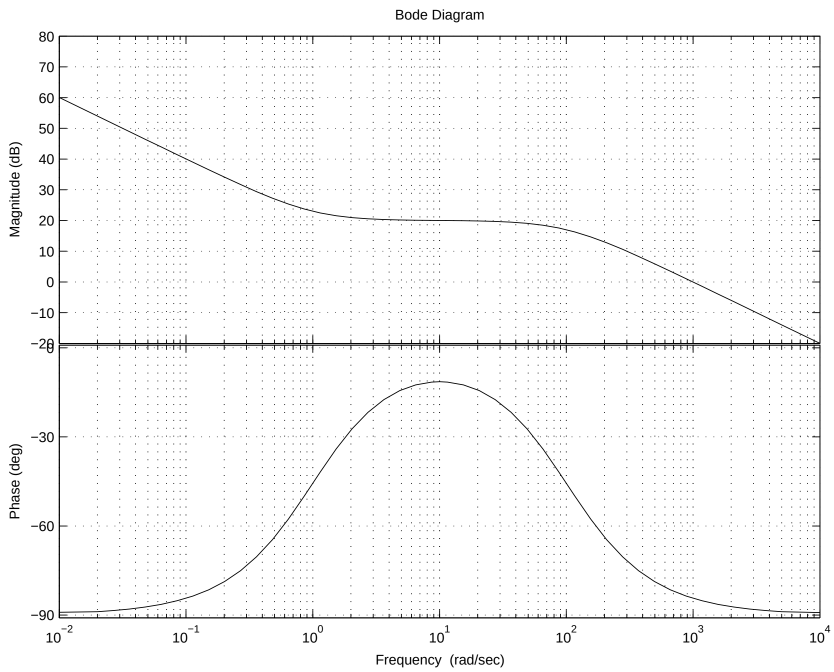
and its zoomed-in version



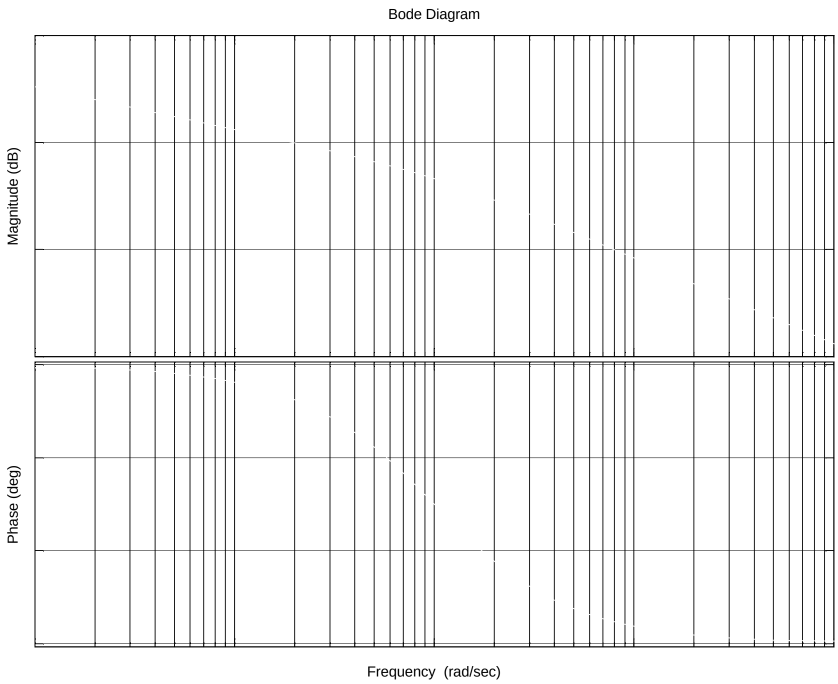
12.1. What is the transfer function corresponding to this polar plot?

12.2. What would be the slopes (in correct sequence) of the various sections of this Bode magnitude plot starting from low-frequencies?

13. Use ruler & pen to construct the asymptotic Bode magnitude plot on the following figure. Determine the TF. Is this a minimum phase TF or a non-minimum phase one? Write the value(s) of whichever of the following error constants you can determine: K_p, K_v, K_a.



14. On the semilog graph provided, draw the Bode plot of the transfer function $G(s) = \frac{20e^{-0.1s}}{s+1}$. Choose a frequency range that will best show the features of this BP. Fill the table for this purpose. Label the x and y ticks appropriately.



ω (rad/s)										
dB-gain										
Phase (degrees)										

15. Given the OL TF, $G(s) = \frac{K(s+1)}{s(\frac{s}{10}-1)}$, we wish to apply Nyquist Stability Theory (NST) to study the stability of the corresponding unity-feedback CL system for $K \in [0, \infty)$. We will use the Bode Plot (BP) method to sketch the NP of $G(s)$.

15.1. Sketch the BP.

15.2. Sketch the polar plot (PP) section of the NP. Show this section by a thick solid line.

15.3. Work out a few points on the s-plane contour and the $G(s)$-plane contour that will help complete the NP.

15.4. Sketch the NP. Label the sections of the s-plane contour $C1, C2, \cdots$ and the corresponding sections of the NP $C1', C2', \cdots$. Label a few points on the s-plane contour $1, 2, \cdots$, and the corresponding points on the NP $1', 2', \cdots$.

15.5. Use NST to determine the values of K for which the CL system is stable, for which it is marginally stable, and for which it is unstable.

16. A linear time-invariant system initially at rest, when subjected to unit ramp input ($r(t) = t \cdot 1(t)$), gives the response $y(t) = te^{-t} \cdot 1(t)$. Determine the TF of the system.

17. For the TF

$$KG(s) = \frac{K(s+2)}{s+10} e^{-t_d s}$$

we will use the Bode Plot (BP) method to sketch the NP of $G(s)$.

17.1. Sketch the BP.
17.2. Sketch the polar plot (PP) section of the NP. Show this section by a thick solid line.
17.3. Work out as many points as you find necessary on the s-plane contour and the $G(s)$-plane contour that will help complete the NP.
17.4. Sketch the NP. Label the sections of the s-plane contour $C1, C2, \cdots$ and the corresponding sections of the NP $C1', C2', \cdots$. Label a few points on the s-plane contour $1, 2, \cdots$, and the corresponding points on the NP $1', 2', \cdots$.
17.5. Use NST to determine the values of K and t_d for which the CL system is stable, for which it is marginally stable, and for which it is unstable.

18. For the CL TF $\frac{{\omega_n}^2}{s^2+2\zeta\omega_n s+{\omega_n}^2}$, derive the expression for the bandwidth ω_{BW} in terms of ω_n and ζ. (The bandwidth of a low-pass filter is the frequency at which its gain falls to $\frac{1}{\sqrt{2}}$ of its dc-gain. The dc-gain is the gain at $\omega = 0$.)

19. We wish to use Nyquist Stability Theory (NST) to evaluate the stability of the unity-feedback CL system built around a plant with the TF

$$G(s) = \frac{K(1-s)}{s+1}, \quad K > 0.$$

19.1. Sketch the BP.
19.2. Sketch the polar plot (PP) section of the NP. Show this section by a thick solid line.
19.3. Work out as many points as you find necessary on the s-plane contour and the $G(s)$-plane contour that will help complete the NP.
19.4. Sketch the NP. Label the sections of the s-plane contour $C1, C2, \cdots$ and the corresponding sections of the NP $C1', C2', \cdots$. Label a few points on the s-plane contour $1, 2, \cdots$, and the corresponding points on the NP $1', 2', \cdots$.
19.5. Use NST to determine the values of $K \geq 0$ for which the CL system is stable, for which it is marginally stable, and for which it is unstable.

20. For each of the following TFs, in the space provided after the TF,

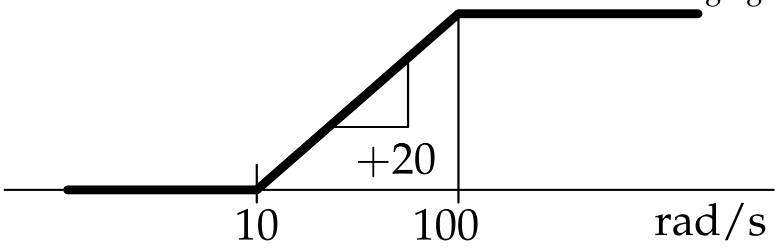
20.1. write the Bode form of the TF,
20.2. describe the form of the ABMP in terms of the slopes, written in the correct sequence, of its straight line segments, and
20.3. calculate one point that you would need on the ABMP to locate the ABMP on a semilog grid.

An example is worked out to show the format in which I want your answers.

20.1. $G(s) = 10\dfrac{s+10}{s+100}$.

20.1.1. Bode form: $G(s) = \left(\frac{s}{10}+1\right)\Big/\left(\frac{s}{100}+1\right)$.

20.1.2. The form of the ABMP is as shown in the following figure:



20.1.3. One point: $\omega = 1$ rad/s, dB-gain $= 0$.

20.2. $G(s) = 10\dfrac{(s+10)(s+1)}{(s+0.1)(s+100)}$.

20.3. $G(s) = 10\dfrac{(s+10)(s+1)}{(s+0.1)(s^2+100)}$.

20.4. $G(s) = 10\dfrac{(s+10)(s-1)}{(s+0.1)(s^2+100)}$.

20.5. $G(s) = 10\dfrac{(s-1)}{(s+0.1)(s^2+100)}$.

20.6. $G(s) = 100\dfrac{(s-1)}{(s+0.1)}$.

20.7. $G(s) = \dfrac{s+1}{s(s+0.1)(s+100)}$.

20.8. $G(s) = \dfrac{s+1}{s^2(s+0.1)(s+100)}$.

20.9. $G(s) = \dfrac{s+1}{s^2(s+100)}$.

20.10. $G(s) = \dfrac{s+1}{s(s+100)}$.

20.11. $G(s) = 100\dfrac{(s+1)^2}{s^2+1}$.

20.12. $G(s) = 10\dfrac{s+1}{(s+0.1)(s+100)}$.

20.13. $G(s) = 10\dfrac{(s+10)(s+1)}{s(s+0.1)}$.

20.14. $G(s) = \dfrac{(s+10)(s-1)}{(s+0.1)(s+100)}$.

20.15. $G(s) = 10\dfrac{s}{(s+0.1)(s+100)}$.

20.16. $G(s) = \dfrac{(s+1)}{(s+0.1)s}$.

21.

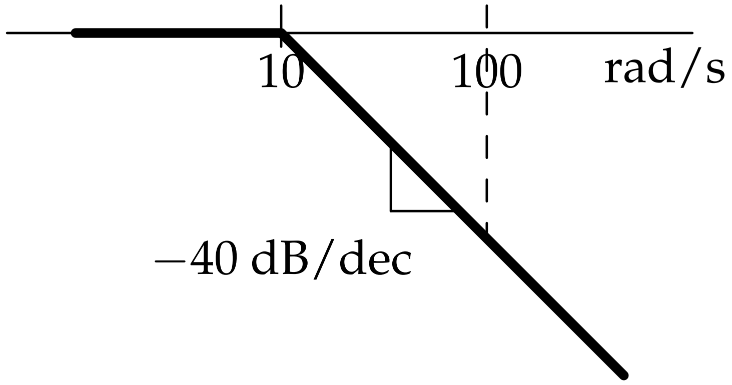
22. For each of the following TFs
    (a) write the Bode form of the TF,
    (b) describe the form of the asymptotic Bode magnitude plot (ABMP) in terms of the slopes, written in the correct sequence, of its straight line segments, and
    (c) calculate one point that you would need on the ABMP to locate the ABMP on a semilog grid.

    An example is worked out to show the format in which I want your answers.

    $G(s) = \dfrac{10}{s^2 + 100}$.

    (a) Bode form: $G(s) = \dfrac{1}{10} \dfrac{1}{s^2/100 + 1}$.

    (b) The form of the ABMP is as follows:

    (c) One point: $\omega = 1$ rad/s, dB-gain $= -20$.

    22.1. $H_1(s) = 10\dfrac{(s+1)}{(s^2+s+100)(s+100)}$.

    22.2. $H_2(s) = \dfrac{s^2+10s+10^4}{1000(s^2+1)}$.

23. The asymptotic Bode phase plot (ABPP) is a straight line approximation of the Bode phase plot. It is characterized by the slopes of its straight line segments and the frequencies at which these slopes change.

    Sketch the ABPP of the following TFs as <u>sums of elemental ABPPs</u>:

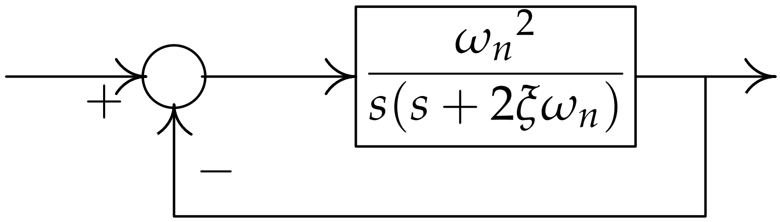
    23.1. $G_1(s) = 100\dfrac{(s-1)}{(s+0.1)}$.

    23.2. $G_2(s) = 100\dfrac{(s-1)}{(s+10)}$.

24. Assume that an s-plane contour Γ_s encloses 3 zeros and 4 poles of a certain transfer function $G(s)$. How many times does the $G(s)$-contour encircle the origin in the $G(s)$-plane in the clockwise direction?

25. Consider a series $R-L$ circuit. Assume that we turn on the voltage $V_{\text{in}} = 1$ V to this circuit at $t=0$. Determine the form of the current $i(t)$ through the $R-L$ series combination <u>after</u> the switch is turned on, given that $i(0^-) = 1$ A. Use the Laplace transform.

26. For the following standard second order system

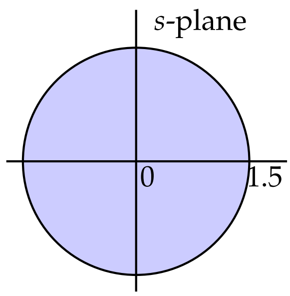


26.1. Derive the expression for the bandwidth ω_b (same as 3-dB frequency) in terms of ω_n and ζ.

26.2. Derive the expression for the OL gain crossover frequency ω_g.

27. For one clockwise traversal of the following s-plane contour, determine the number of times the $G(s)$-plane contour encircles the origin of the $G(s)$-plane in the clockwise direction, given that

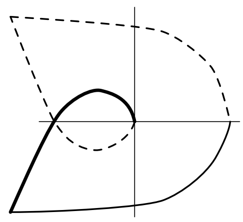
$$G(s) = \frac{(s+1)(s+2)}{(s^2+s+1)}.$$



28. Circle the <u>one</u> transfer function that generates the following Nyquist plot.

Write a brief reason for your choice.

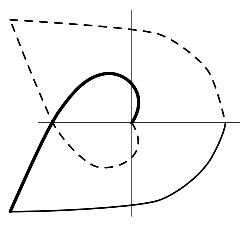
(1) $\frac{K}{s^2(s+1)}$, (2) $\frac{K}{(s+1)^3}$, (3) $\frac{K}{(s+1)^4}$,
(4) $\frac{K}{s(s+1)^2}$, (5) $\frac{K}{s^2(s+1)}$, (6) $\frac{K}{s(s+1)^3}$,
(7) $\frac{K}{s^2(s+1)^2}$, (8) $\frac{1}{s^2(s+10)^2}$,
(9) $\frac{1}{s^3(s+1)}$, (10) $\frac{1}{s(s+1)}$.



29. Circle the <u>one</u> transfer function that generates the following Nyquist plot.

Write a brief reason for your choice.

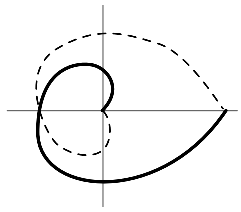
(1) $\frac{K}{s^2(s+1)}$, (2) $\frac{K}{(s+1)^3}$, (3) $\frac{K}{(s+1)^4}$,
(4) $\frac{K}{s(s+1)^2}$, (5) $\frac{K}{s^2(s+1)}$, (6) $\frac{K}{s(s+1)^3}$,
(7) $\frac{K}{s^2(s+1)^2}$, (8) $\frac{1}{s^2(s+10)^2}$,
(9) $\frac{1}{s^3(s+1)}$, (10) $\frac{1}{s(s+1)}$.



30. Circle the <u>one</u> transfer function that generates the following Nyquist plot.

Write a brief reason for your choice.

(1) $\dfrac{K}{s^2(s+1)}$, (2) $\dfrac{K}{(s+1)^3}$, (3) $\dfrac{K}{(s+1)^4}$,
(4) $\dfrac{K}{s(s+1)^2}$, (5) $\dfrac{K}{s^2(s+1)}$, (6) $\dfrac{K}{s(s+1)^3}$,
(7) $\dfrac{K}{s^2(s+1)^2}$, (8) $\dfrac{1}{s^2(s+10)^2}$,
(9) $\dfrac{1}{s^3(s+1)}$, (10) $\dfrac{1}{s(s+1)}$.



31. Given the OL TF, $G(s) = Ke^{-s}$, we wish to apply Nyquist Stability Theory (NST) to study the stability of the corresponding unity-feedback CL system for $K \in [0, \infty)$. Each problem is of 1 point.

31.1. Sketch the BP.

31.2. Sketch the polar plot (PP) section of the NP. Show this section by a thick solid line. You may mark $-\frac{1}{K} + j0$ as the critical point, if you wish.

31.3. If necessary, work out a few points on the s-plane contour and the $G(s)$-plane contour that will help complete the NP.

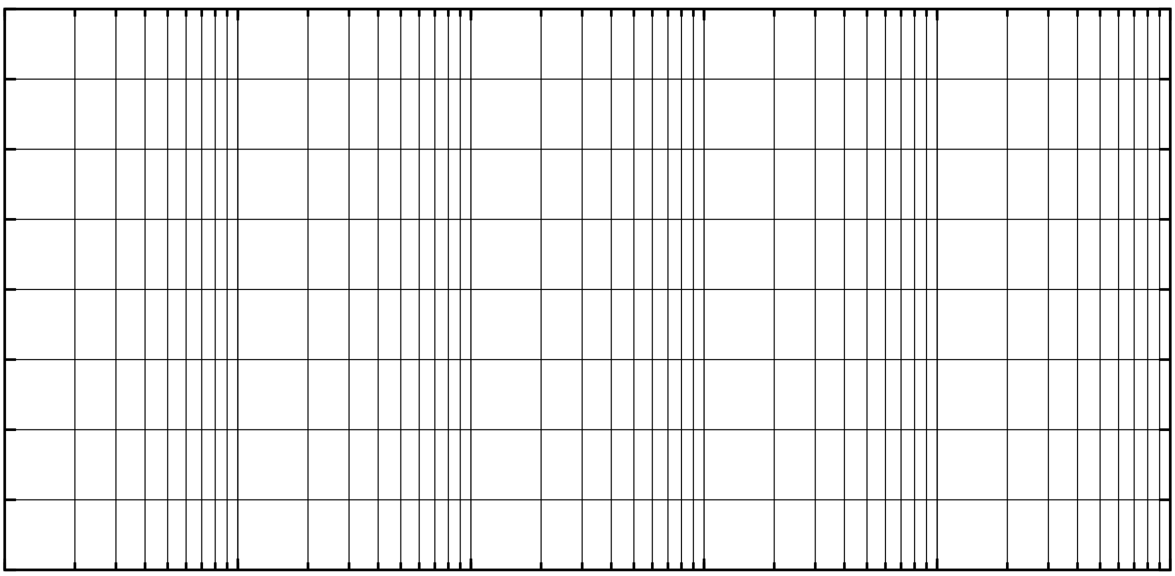
31.4. Sketch the NP. Label the sections of the s-plane contour $C1, C2, \cdots$ and the corresonding sections of the NP $C1', C2', \cdots$. If necessary, label a few points on the s-plane contour $1, 2, \cdots$, and the corresponding points on the NP $1', 2', \cdots$.

31.5. Use NST to determine the values of K for which the CL system is stable.

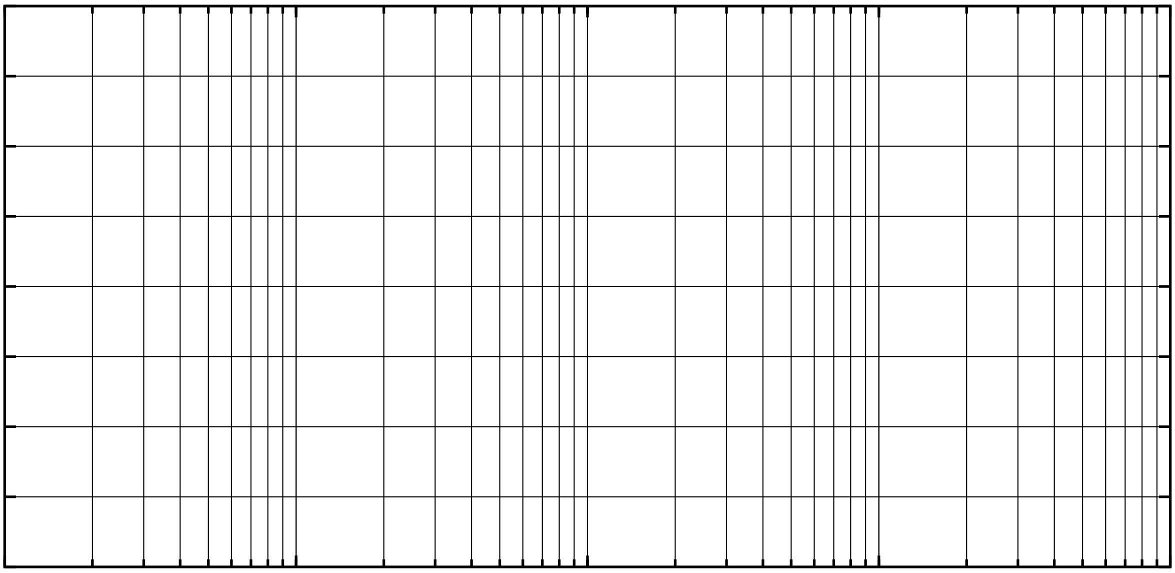
31.6. Based on the experience gained from this problem, what do you think will happen if you model a plant as $K/(s+1)$ neglecting a dead-time such as $e^{-t_d s}$ that may actually be present in the plant?

32. Draw the asymptotic Bode magnitude plot (ABMP) of each of the following transfer functions (TFs) on the semilog grids provided. Label the axes and slopes with units.

32.1. $H_1(s) = 10\dfrac{(s+1)}{(s^2+s+100)(s+100)}$.

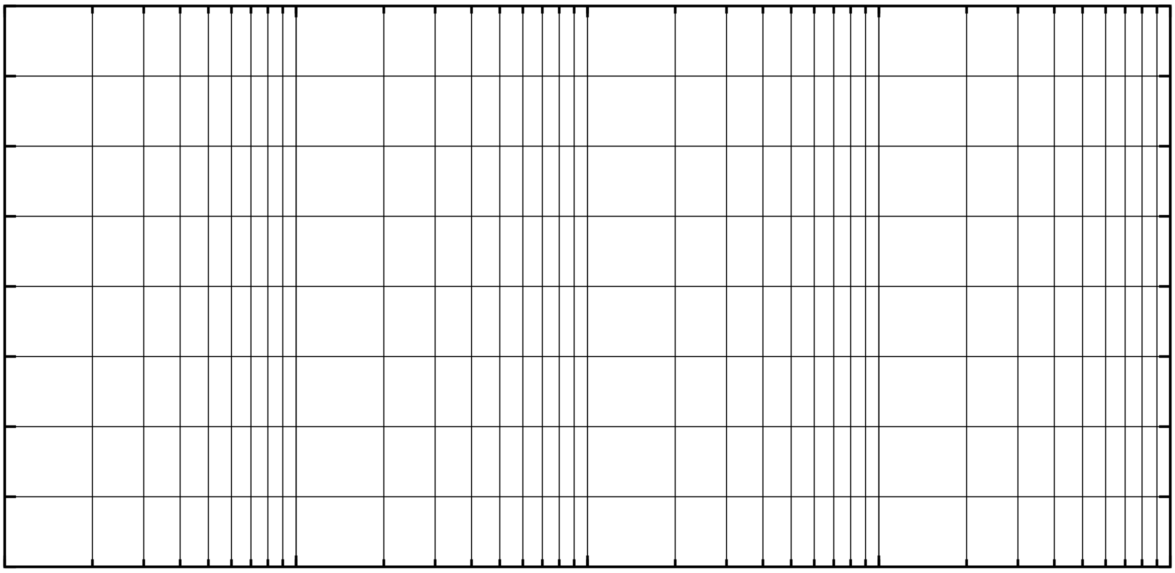


32.2. $H_2(s) = \dfrac{s^2+10s+100}{1000(s^2+1)}$.

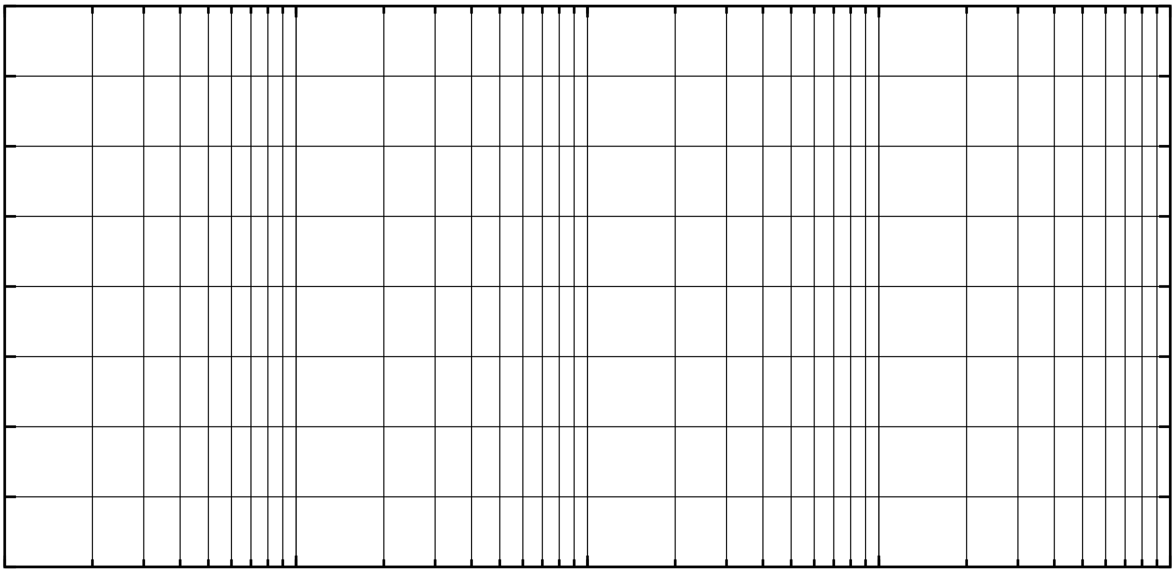


33. Sketch the asympotic Bode phase plots (ABPPs) of the following TFs as <u>sums of elemental ABPPs</u> in the space provided:

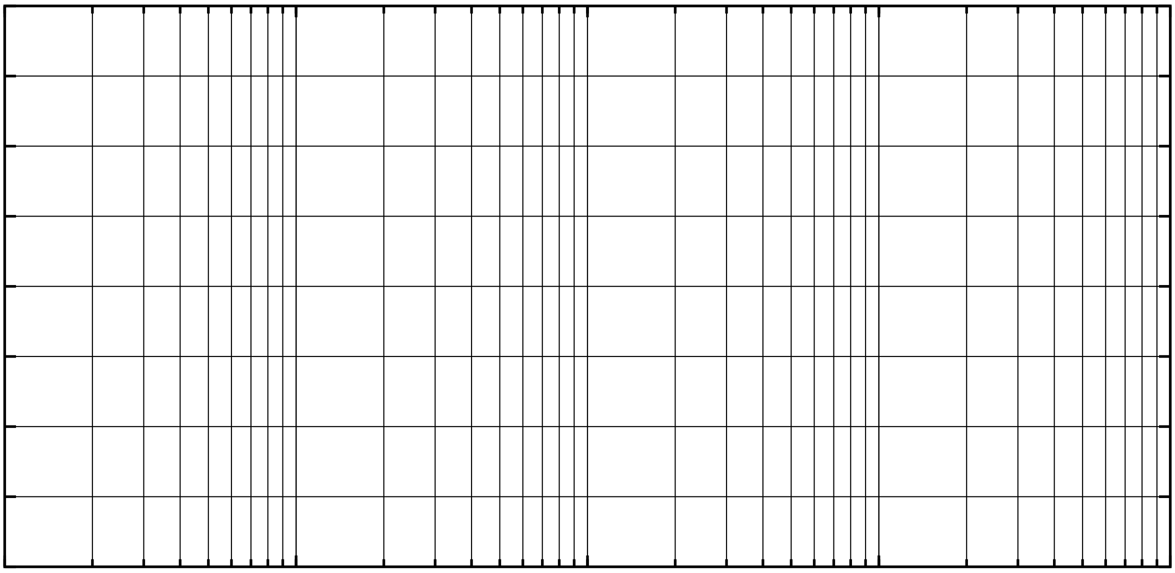
33.1. $G_1(s) = 100(s-1)\,/\,(s+10)$.



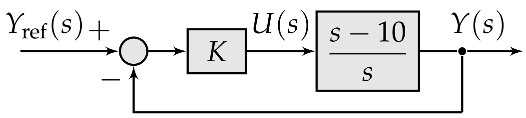
33.2. $G_2(s) = 100\,/\left(s^3+s^2+s+1\right)$.



33.3. $G_3(s) = 100 \, / \left((s+1)(s^2+s+1) \right).$



34. Determine, using Nyquist stability theory, the values of $K \geq 0$ for which the following control system is stable. Each problem is of 1 point.



34.1. Sketch the BP.

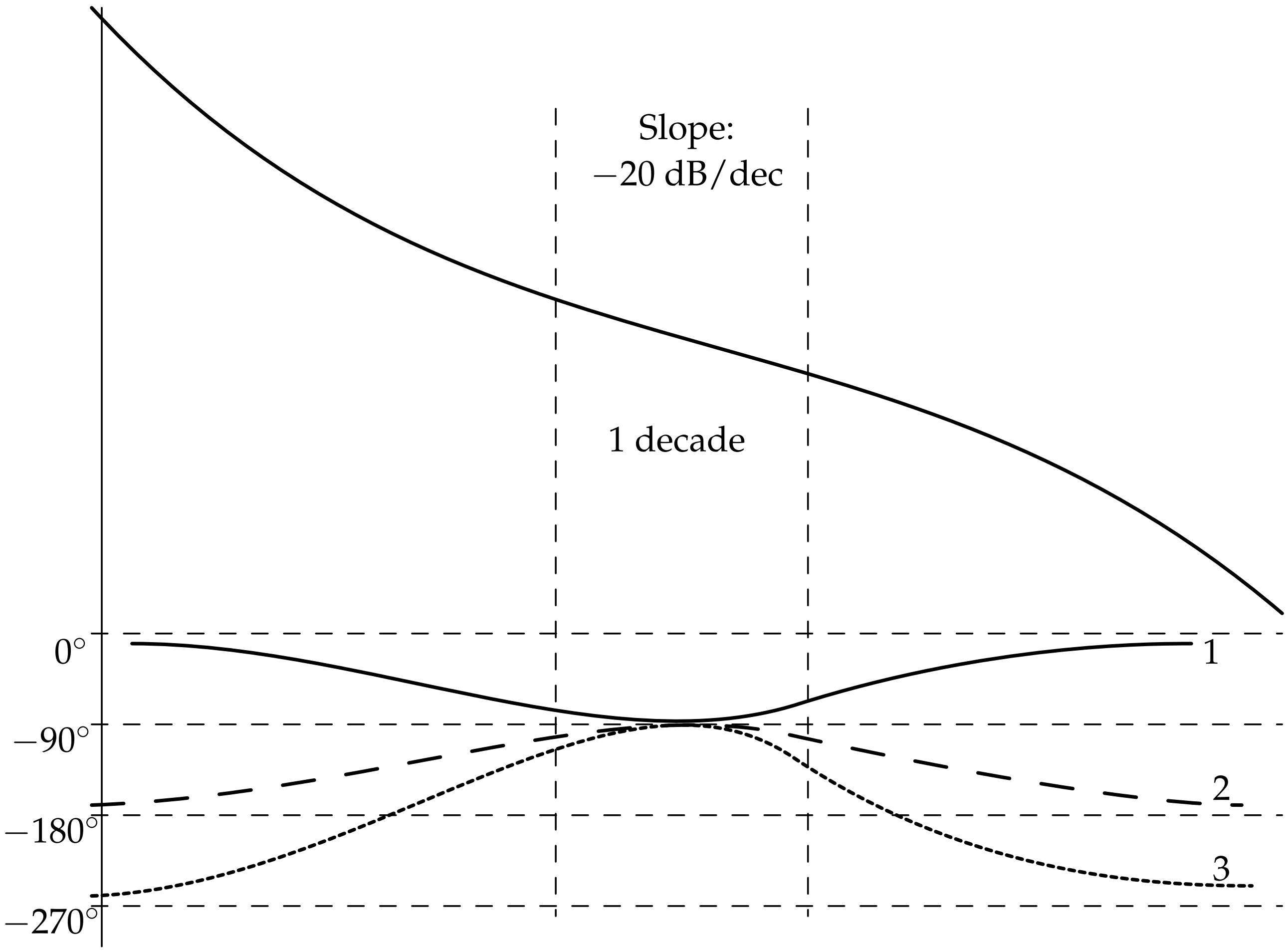
34.2. Sketch the polar plot (PP) section of the NP. Show this section by a thick solid line. Mark the critical point.

34.3. If necessary, work out a few points on the s-plane contour and the $G(s)$-plane contour that will help complete the NP.

34.4. Sketch the NP. Label the sections of the s-plane contour $C1, C2, \cdots$ and the corresponding sections of the NP $C1', C2', \cdots$.

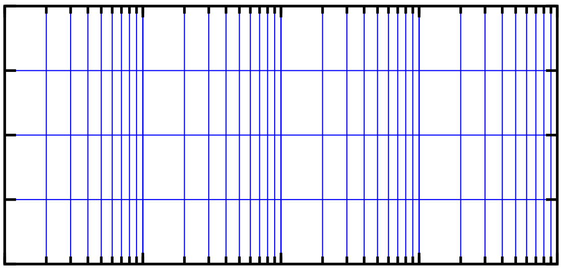
34.5. Use NST to determine the values of K for which the CL system is stable.

35. Consider the BMP and BPPs shown below for a certain minimum-phase TF. Which of the three BPPs are valid candidates? Why?

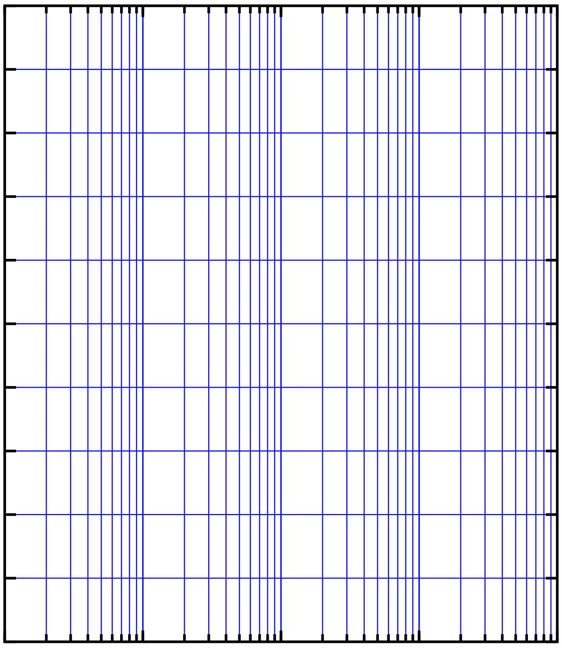


36. For the transfer function $\frac{(s-1)(s-10)}{(s+1)(s+10)}$

36.1. Sketch the asymptotic Bode magnitude plot on the given grid. Label the axes.



36.2. Sketch the asymptotic Bode phase plot on the given grid. Label the axes.

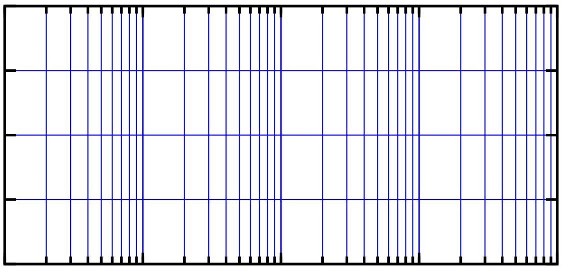


36.3. Calculate the phase lag at $\omega = 0$, at $\omega = \infty$, and the increase in phase lag from $\omega = 0$ to $\omega = \infty$.

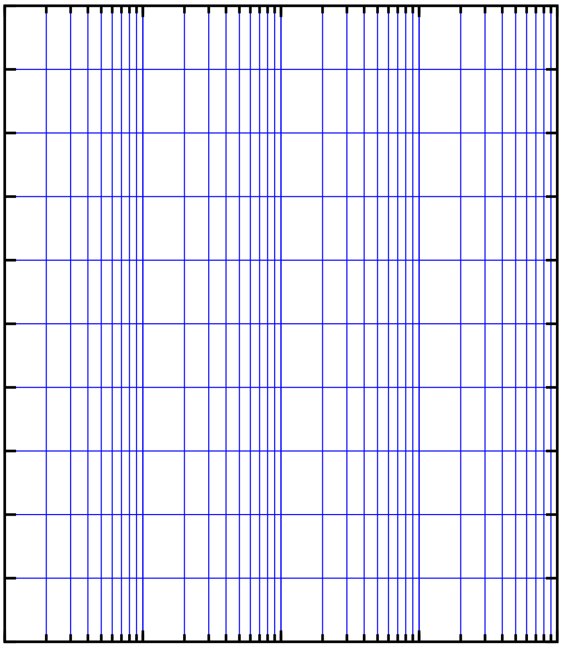
36.4. Sketch a polar plot of the sinusoidal version of this transfer function.

37. For the transfer function $\frac{(s-1)}{(s+1)(s+10)}$

37.1. Sketch the asymptotic Bode magnitude plot on the given grid. Label the axes.



37.2. Sketch the asymptotic Bode phase plot on the given grid. Label the axes.

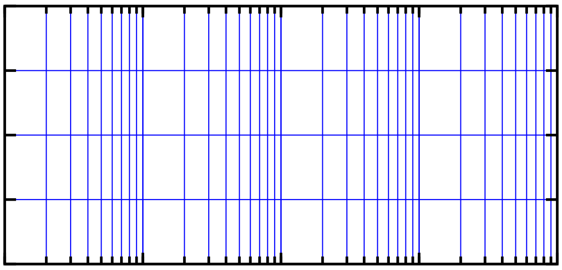


37.3. Calculate the phase lag at $\omega = 0$, at $\omega = \infty$, and the increase in phase lag from $\omega = 0$ to $\omega = \infty$.

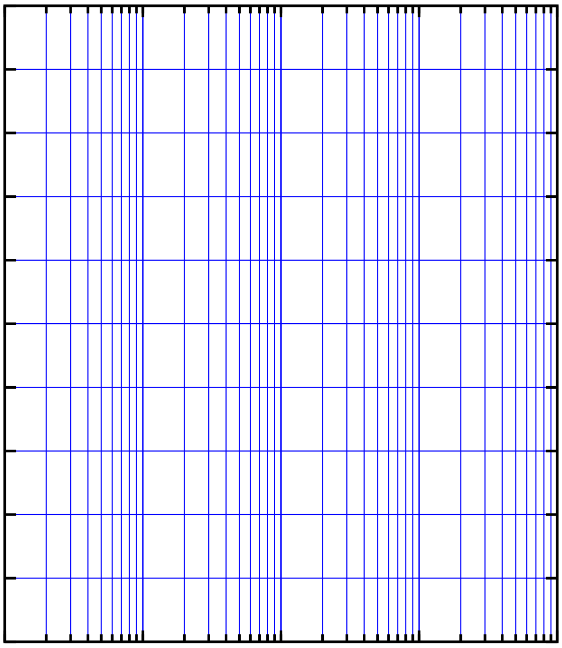
37.4. Sketch a polar plot of the sinusoidal version of this transfer function.

38. For the transfer function $\frac{(s-10)}{(s+1)(s+10)}$

38.1. Sketch the asymptotic Bode magnitude plot on the given grid. Label the axes.



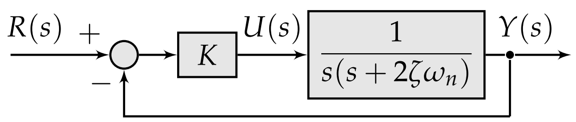
38.2. Sketch the asymptotic Bode phase plot on the given grid. Label the axes.



38.3. Calculate the phase lag at $\omega = 0$, at $\omega = \infty$, and the increase in phase lag from $\omega = 0$ to $\omega = \infty$.

38.4. Sketch a polar plot of the sinusoidal version of this transfer function.

39. Use Nyquist stability test to determine the values of $K \geq 0$ for which the following closed-loop system is stable. Here $0 < \zeta < 1$ and $\omega_n > 0$.



40. Prove or disprove the following Laplace transform pairs:

40.1. $\frac{m!}{s^{m+1}} \leftrightarrow t^m 1(t)$

40.2. $\frac{1}{(s+a)^m} \leftrightarrow \frac{1}{(m-1)!} t^{m-1} e^{-at}$

40.3. $1 \leftrightarrow \delta(t)$

40.4. $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$

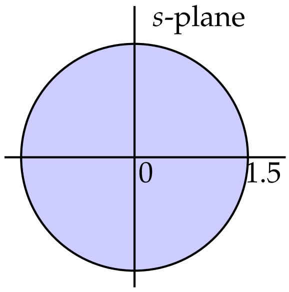
40.5. $\lim_{t\to 0+} y(t) = \lim_{s\to\infty} sY(s)$

41. For one clockwise traversal of the following s-plane contour, determine the number of times the $G(s)$-plane contour encircles the origin of the $G(s)$-plane in the clockwise direction, given that

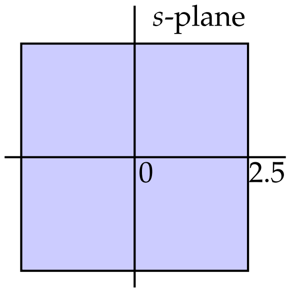
$$G(s) = \frac{(s+1)(s+2)}{(s^2+s+1)}.$$

[Each contour is worth 1 point]

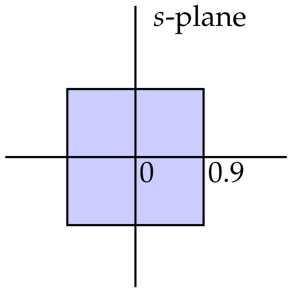
41.1.



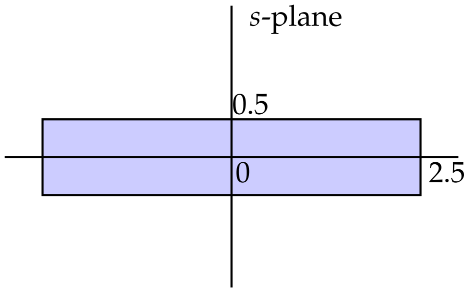
41.2.



41.3.



41.4.



42. Using Nyquist stability theory determine the values of $K > 0$ for the stability of each of the following control systems.