Notes of Real Analysis

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Contents

0.1 Introduction

This note is to provide an easy understanding of Real Analysis, and to continue to understand and work on more specific areas. And the content are almost come from bilibili

Chapter 1: Abstract Measures

1.1 Building Block

Definition 1.1.1. (closed rectangle) A **closed rectangle** R in $\mathbb{R}^d$ is defined as the Cartesian product of closed intervals. Specifically, R can be written as:

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d] \,,$$

where $[a_i, b_i]$ are closed intervals on the real line, and $a_i \leq b_i$ for all i from 1 to d. In other words,

$$R = \{(X_1, X_2, \ldots, X_d) \in \mathbb{R}^d : a_j \leq b_j, j = 1, \ldots, d\},$$

The volume of R is

$$|R| = (b_1 - a_1) \times (b_2 - a_2) \times \cdots \times (b_d - a_d) \,,$$

An **open rectangle** is the product of open intervals, and the interior of the rectangle R is

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d) \,.$$

Example 1.1.1. (Closed Rectangles) There are some examples of **closed rectangle**:

1. In $\mathbb{R}^2$ (the plane), a building block might be a rectangle defined by $R = [1, 3] \times [2, 4]$. This rectangle includes all points (x, y) where $1 \leq x \leq 3$ and $2 \leq y \leq 4$.

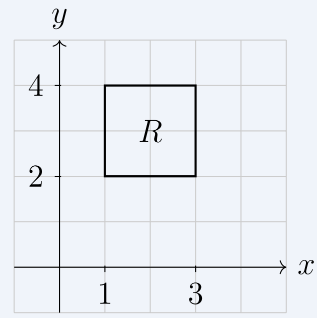
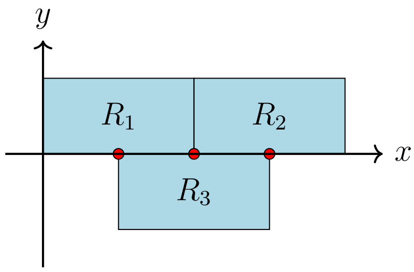


Figure 1.1: The rectangle $R = [1,3] \times [2,4]$ in $\mathbb{R}^2$

2. In $\mathbb{R}^3$ (three-dimensional space), a typical building block could be a rectangular prism (or box) defined by $R = [0,1] \times [0,1] \times [0,1]$. This includes all points (x, y, z) where $0 \le x \le 1, 0 \le y \le 1$, and $0 \le z \le 1$.

Definition 1.1.2. (Almost Disjoint) A union of rectangles is said to be almost disjoint if the interiors of them are disjoint.



The rectangle R_1 and R_2 share a common boundary along the line $x = 2$ but do not overlap. R_3 is positioned such that it touches the bottom edges of R_1 and R_2 at points along the line $y = 0$. The points where the rectangles touch are highlighted with red dots to emphasize the boundary interactions but no interior overlap.

Lemma 1. If a rectangles is the almost disjoint union of finitely many rectangles: $R = \bigcup_{N}^{k=1} R_k$, then $|R| = \sum_{k=1}^{N} |R_k|$.

这个引理描述了一种特殊情况，其中一个矩形 R 是有限个几乎不相交的矩形的并集，即 $R = \bigcup_{k=1}^{N} R_k$，并且这些矩形的内部不相交. 在这种情况下，$R$ 的体积等于所有这些子矩形体积的总和.

Proof. 定义每个 R_k 为闭矩形 $[a_{k1}, b_{k1}] \times [a_{k2}, b_{k2}] \times \cdots \times [a_{kd}, b_{kd}]$, 对于任何 $i \neq j$，
Almost Disjoint $\operatorname{int}(R_i) \cap \operatorname{int}(R_j) = \emptyset$. 其中每个 R_k 的体积计算为 $|R_k| = \prod_{j=1}^{d} (b_{kj} - a_{kj})$.
最小包含矩形 R 设 $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$，其中 $a_j = \min_k a_{kj}$，$b_j = \max_k b_{kj}$.
由于 $\operatorname{int}(R_i) \cap \operatorname{int}(R_j) = \emptyset$，可以断定每个 R_k 的体积贡献是独立的，即它们的体积之和给出了 R 中被覆盖的全部体积.

R_k 的边界可能与其他 R_k 的边界重合，但由于边界的测度在整体测度中不起主导作用（在高维中测度为零），因此不影响总体积计算.

因此，可以通过各 R_k 的体积独立累加，无需减去重叠部分，从而得到 R 的总体积，即 $|R| = \sum_{k=1}^{N} |R_k|$. Q.E.D.

Lemma 2. If $R, R_1, \ldots, R_N$ are rectangles, and $R \subset \bigcup$