Example of Regime Switching State Space Model

Renhe W.

Contents

List of Figures

List of Tables

1 Kim(1994)-Regime Switching State Space Model

1.1 Regime Switching State Space Model

As an example of a regime switching state space model, Prof. Kim used the following generalized Hamilton model for the log of real GNP (Lam; 1990) in his paper and book.

$$\begin{aligned}
\ln(GNP_t) &= n_t + x_t \\
n_t &= n_{t-1} + \mu_0 + \mu_1 s_t \\
x_t &= \phi_1 x_{t-1} + \phi_2 x_{t-2} + u_t \\
u_t &\sim N\left(0, \sigma^2\right) \\
s_t &= 0, 1 \quad P_{tj} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}
\end{aligned}$$

where $\ln(GNP_t)$ is a real GNP level. n_t is a deterministic series with a regimeswitching growth rate and x_t is stationary AR(2) cycle process. Since $\ln(GNP_t)$ is the log level variable, the difference of it, $y_t = \ln(GNP_t) - \ln(GNP_{t-1})$, can be represented as a state space model in the following way.

$$\begin{aligned}
y_t &= \mu_0 + \mu_1 s_t + x_t - x_{t-1} \\
x_t &= \phi_1 x_{t-1} + \phi_2 x_{t-2} + u_t
\end{aligned}$$

It is a typical approach that a state space model is represented as a vector-matrix form for using Kalman filter as follows.

$$
\begin{aligned}
y_t &= \mu_{s_t} + \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} \\
\begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} &= \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_{t-2} \end{bmatrix} + \begin{bmatrix} u_t \\ 0 \end{bmatrix} \\
u_t &\sim N\left(0, \sigma^2\right) \\
\mu_{s_t} &= \mu_0 + \mu_1 s_t, \quad \mu_1 > 0 \\
s_t &= 0, 1 \quad P_{tj} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} \\
&\Downarrow \\
y_t &= \mu_{s_t} + F\mathbf{x}_t \\
\mathbf{x}_t &= A\mathbf{x}_{t-1} + v_t
\end{aligned}
$$

For the sake of notational simplicity, we use x_t instead of $\mathbf{x}_t$.

1.2 Kalman Filtering

Kalman filter with regime switching is used to get state estimates from a state space model taking regime transition into account and has the following recursion.

$$
\begin{aligned}
x_{t|t-1}^{ij} &= Ax_{t-1|t-1}^{i} \\
P_{t|t-1}^{ij} &= AP_{t-1|t-1}^{i}A^T + Q \\
\eta_{t|t-1}^{ij} &= y_t - \mu_j - Fx_{t|t-1}^{ij} \\
H_{t|t-1}^{ij} &= FP_{t|t-1}^{ij}F^T + R \\
K^{ij} &= P_{t|t-1}^{ij}F^T \left[H_{t|t-1}^{ij}\right]^{-1} \\
x_{t|t}^{ij} &= x_{t|t-1}^{ij} + K^{ij}\eta_{t|t-1}^{ij} \\
P_{t|t}^{ij} &= \left(I - K^{ij}F\right) P_{t|t-1}^{ij}
\end{aligned}
$$

In regime-dependent Kalman filter, all the notations are augmented with superscript $\{ij\}$ except $x_{t-1|t-1}^{i}$ and $P_{t-1|t-1}^{i}$ since these two estimates are in i-state (two-state) but other estimates must reflect state transitions from i to j (four-state). For example,

$x_{t|t-1}$ and $x^{ij}_{t|t-1}$ are different in terms of conditioning information.

$$\begin{aligned} x_{t|t-1} &= E\left[X_t \mid \psi_{t-1}\right] \\ x^{ij}_{t|t-1} &= E\left[X_t \mid \psi_{t-1}, S_t = j, S_{t-1} = i\right] \end{aligned}$$

In contrast to the single regime, however, in the multiple regimes, $x^{ij}_{t|t}$ and $P^{ij}_{t|t}$ cannot be used the next state prediction due simply to the mismatch both 1) between $x^{ij}_{t|t}$ and $x^{i}_{t-1|t-1}$ and 2) between $P^{ij}_{t|t}$ and $P^{i}_{t-1|t-1}$. To resolve this mismatch problem, Kim (1994) developed a dimension collapsing algorithm.

1.3 Kim(1994)' s Collapsing procedure

Kim (1994) introduces a collapsing procedure (approximation) to reduce the $(M \times M)$ posteriors $\left(x^{ij}_{t|t}\right.$ and $\left.P^{ij}_{t|t}\right)$ into M to complete the above Kalman filter recursion.

$$\begin{aligned} x^{j}_{t|t} &= \frac{\sum_{i=1}^{M} P\left[S_{t-1} = i, S_t = j \mid \psi_t\right] x^{ij}_{t|t}}{P\left[S_t = j \mid \psi_t\right]} \\ P^{j}_{t|t} &= \frac{\sum_{i=1}^{M} P\left[S_{t-1} = i, S_t = j \mid \psi_t\right] \left[P^{ij}_{t|t} + \left(x^{j}_{t|t} - x^{ij}_{t|t}\right)\left(x^{j}_{t|t} - x^{ij}_{t|t}\right)^T\right]}{P\left[S_t = j \mid \psi_t\right]} \end{aligned}$$

To calculate the above approximation, when we calculate $P\left[S_{t-1} = i, S_t = j \mid \psi_t\right]$, $P\left[S_t = j \mid \psi_t\right]$ is easily obtained by summing its M branches from each i states.

$$P\left[S_t = j \mid \psi_t\right] = \sum_{i=1}^{M} P\left[S_{t-1} = i, S_t = j \mid \psi_t\right]$$

Knowing $P\left[S_{t-1} = i, S_t = j \mid \psi_t\right]$ means that we observe time t data since the last time of information set is t and the state is migrated from i to j. For this data information into account, we can think of the marginal probability of state transition by integrating out data.

$$\begin{aligned} P\left[S_{t-1} = i, S_t = j \mid \psi_t\right] &= \frac{f\left(y_t, S_{t-1} = i, S_t = j \mid \psi_{t-1}\right)}{f\left(y_t \mid \psi_{t-1}\right)} \\ f\left(y_t \mid \psi_{t-1}\right) &= \sum_{j=1}^{M} \sum_{i=1}^{M} f\left(y_t, S_{t-1} = i, S_t = j \mid \psi_{t-1}\right) \end{aligned}$$

As can be seen from the above equations, when we know $f(y_t, S_{t-1} = i, S_t = j \mid \psi_{t-1})$ which is the joint density of data and two states, $P[S_{t-1} = i, S_t = j \mid \psi_t]$ and $f(y_t \mid \psi_{t-1})$ are easily obtained. We get the following the joint density.

$$
\begin{aligned}
& f(y_t, S_{t-1} = i, S_t = j \mid \psi_{t-1}) \\
& = f(y_t \mid S_{t-1} = i, S_t = j, \psi_{t-1}) \times P(S_{t-1} = i, S_t = j \mid \psi_{t-1})
\end{aligned}
$$

Now we need to know two parts. The first part is the forecast error given data. (MVN : probability density function of multivariate normal distribution with zero mean and forecast error variance)

$$
\begin{aligned}
& f(y_t \mid S_{t-1} = i, S_t = j, \psi_{t-1}) \\
& = MVN \text{ (forecast error, its variance)}
\end{aligned}
$$

The second part is calculated by the multiplication of the transition probability and the summation of its branches.

$$
\begin{aligned}
& P(S_{t-1} = i, S_t = j \mid \psi_{t-1}) \\
& = P[S_t = j \mid S_{t-1} = i] \times \sum_{k=1}^{M} f(S_{t-2} = k, S_{t-1} = i \mid \psi_{t-1}) \\
& = P_{ij} \times f(S_{t-1} = i \mid \psi_{t-1})
\end{aligned}
$$

In the above equation, as we know, P_{ij} is already known as transition probability matrix and $f(S_{t-2} = k, S_{t-1} = i \mid \psi_{t-1})$ is exactly what we want to find but evaluated at the previous t- 1 time. Therefore for the iteration, $f(S_{-1} = k, S_0 = i \mid \psi_0)$ calls for initialization with the steady state probabilities. Therefore, we can calculate $P[S_{t-1} = i, S_t = j \mid \psi_t]$ and $P[S_t = j \mid \psi_t]$ through the above equations.

1.4 Kim (1994) Filter for Regime Switching State Space model

Kim filtering procedure is summarized in a sequence of equations. Kalman Filtering

$$
\begin{aligned}
x_{t|t-1}^{ij} &= A x_{t-1|t-1}^{i} \\
P_{t|t-1}^{ij} &= A P_{t-1|t-1}^{i} A^T + Q \\
\eta_{t|t-1}^{ij} &= y_t - \mu_j - F x_{t|t-1}^{ij} \\
H_{t|t-1}^{ij} &= F P_{t|t-1}^{ij} F^T + R \\
K^{ij} &= P_{t|t-1}^{ij} F^T \left[H_{t|t-1}^{ij} \right]^{-1} \\
x_{t|t}^{ij} &= x_{t|t-1}^{ij} + K^{ij} \eta_{t|t-1}^{ij} \\
P_{t|t}^{ij} &= \left(I - K^{ij} F \right) P_{t|t-1}^{ij}
\end{aligned}
$$

$$\Downarrow$$

Hamilton Filtering

$$
\begin{aligned}
& f\left(y_t, S_{t-1}=i, S_t=j \mid \psi_{t-1}\right) = N\left(\eta_{t|t-1}^{ij}, H_{t|t-1}^{ij}\right) \times P_{ij} \times P\left(S_{t-1}=i \mid \psi_{t-1}\right) \\
& f\left(y_t \mid \psi_{t-1}\right) = \sum_{j=1}^{M} \sum_{i=1}^{M} f\left(y_t, S_{t-1}=i, S_t=j \mid \psi_{t-1}\right) \\
& P\left[S_{t-1}=i, S_t=j \mid \psi_t\right] = \frac{f\left(y_t, S_{t-1}=i, S_t=j \mid \psi_{t-1}\right)}{f\left(y_t \mid \psi_{t-1}\right)} \\
& P\left[S_t=j \mid \psi_t\right] = \sum_{i=1}^{M} P\left[S_{t-1}=i, S_t=j \mid \psi_t\right]
\end{aligned}
$$

$$\Downarrow$$

Kim's Collapsing

$$
\begin{aligned}
x_{t|t}^{j} &= \frac{\sum_{i=1}^{M} P\left[S_{t-1}=i, S_t=j \mid \psi_t\right] x_{t|t}^{ij}}{P\left[S_t=j \mid \psi_t\right]} \\
P_{t|t}^{j} &= \frac{\sum_{i=1}^{M} P\left[S_{t-1}=i, S_t=j \mid \psi_t\right] \left[P_{t|t}^{ij} + \left(x_{t|t}^{j} - x_{t|t}^{ij}\right) \left(x_{t|t}^{j} - x_{t|t}^{ij}\right)^T \right]}{P\left[S_t=j \mid \psi_t\right]}
\end{aligned}
$$

In particular, three red colored terms $\left(x_{t-1|t-1}^{i}, P_{t-1|t-1}^{i}\right.$, and $\left.P\left(S_{t-1}=i \mid \psi_{t-1}\right)\right)$ are initialized for iteration to be started. Once the iterations get started, these red colored terms are replaced with blue colored terms $\left(x_{t|t}^{j}, P_{t|t}^{j}\right.$, and $\left.P\left(S_t=j \mid \psi_t\right)\right)$ for each iterations.

A R code