Interpolation:

The process of estimating the value of y, for any intermediate value of x, is called interpolation.

Extrapolation:

The method of computing the value of y, for a given value of x, lying outside the table of values of x is known as extrapolation.

Muller's Method:

In Muller's method, f (x) = 0 is approximated by a second degree polynomial; that is by a quadratic equation that fits through three points in the vicinity of a root. The roots of this quadratic equation are then approximated to the roots of the equation f (x) 0.This method is iterative in nature and does not require the evaluation of derivatives as in Newton.

Raphson method. This method can also be used to determine both real and complex roots

of f(x) = 0.

Suppose, x_{i-2}, x_{i-1}, x_i

be any three distinct approximations to a root

Of f (x) = 0.

$$f(x_{i-2}) = f_{i-2}, f(x_{i-1}) = f_{i-1}$$

$$f(x_i) = f_i.$$

Newton -Raphson Method

This method is one of the most powerful method and well known methods, used for finding a root of f(x)=0 the formula many be derived in many ways the simplest way to derive this formula is by using the first two terms in Taylor's series expansion of the form,

$$f(x_{n+1}) = f(x_n) + (x_{n+1} - x_n) f'(x_n)$$

setting $f(x_{n+1}) = 0$ *gives,*

$$f(x_n) + (x_{n+1} - x_n) f'(x_n) = 0$$

thus on simplification, we get,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad for\ n = 0, 1, 2\ldots$$

Example

Find a real root of the equation $x3 - x - 1 = 0$ using Newton - Raphson method, correct to four decimal places.

Solution

$f(x)=x^3 - x - 1$

$f(1)=1^3 - 1 - 1 =-1<0$

$f(2)=2^3 - 2 - 1 = 8-2-1=5>0$

so the root lies between 1 *and* 2

here $f'(x)=3x^2-1$ *and* $f''(x)=6x$

$f'(1)=3*1^2-1=2$

$f'(x)=3*2^2-1=11$

here

$f''(1)=6$

$f''(2)=6(2)=12$

here $f(2)$ *and* $f''(2)$ *have the same signs so* $x_0=2$

Numerical Analysis:

Introduction We begin this chapter with some of the basic concept of representation of numbers on computers and errors introduced during computation. Problem solving using computers and the steps involved are also discussed in brief.

Number (s) System (s)

In our daily life, we use numbers based on the decimal system. In this system, we use ten symbols 0, 1,…,9 and the number 10 is called the base of the system. Thus, when a base N is given, we need N different symbols 0, 1, 2, …,(N – 1) to represent an arbitrary number. The number systems commonly used in computers are

Base, n	number
2	**Binary**
8	**Octal**
10	**Decimal**
16	**Hexadecimal**

An arbitrary real number, a can be written as

$$a = a_m N^m + a_{m-1} N^{m-1} + \cdots + a_1 N^1 + a_0 + a_{-1} N^{-1} + \cdots + a_{-m} N^{-m}$$

In binary system, it has the form,

$$a = a_m 2^m + a_{m-1} 2^{m-1} + \cdots + a_1 2^1 + a_0 + a_{-1} 2^{-1} + \cdots + a_{-m} 2^{-m}$$

The decimal number 1729 is represented and calculated

$$(1729)_{10} = 1\times10^3 + 7\times10^2 + 2\times10^1 + 9\times10^0$$

While the decimal equivalent of binary number 10011001 is

$$1\times2^0 + 0\times2^{-1} + 0\times2^{-2} + 1\times2^{-3} + 1\times2^{-4} + 0\times2^{-5} + 0\times2^{-6} + 1\times2^{-7}$$

$$= 1 + \frac{1}{8} + \frac{1}{16} + \frac{1}{128}$$

$$= (1.1953125)_{10}$$

Electronic computers use binary sy p stem whose base is 2. The two symbols used in this system are 0 and 1, which are called binary digits or simply bits. The internal representation of any data within a computer is in binary form. However, we prefer , data input and output of numerical results in decimal system. Within the computer, the arithmetic is carried out in binary form.

Q) Conversion of decimal number 47 into its binary equivalent.

Sol)

2	47	Remainder
2	23	1
2	11	1
2	5	1
2	2	1
2	1	0
	0	1 Most significant bit

$$(47)_{10} = (101111)_2$$

Q) Binary equivalent of the decimal fraction 0.7625.

Sol.

		Product	Integer
0.7625	x2	1.5250	1
0.5250	x2	1.0500	1
0.05	x2	0.1	0
0.1	x2	0.2	0
0.2	x2	0.4	0
0.4	x2	0.8	0
0.8	x2	1.6	1
0.6	x2	1.2	1
0.2	x2	0.4	0

$(0.7625)_{10} = (0.11....11(0011))_2$

Conversion $(59)_{10}$ into binary and then into octal.

Sol.

2	29	1
2	14	1
2	7	0
2	3	1
2	1	1
	0	1

$$(59)_{10} = (11011)_2$$

$$(111011)_2 = 111011 = (73)_8$$

Errors in Computations:

Numerically, computed solutions are subject to certain errors. It may be fruitful to identify the error sources and their growth while classifying the errors in numerical computation. These are

Inherent errors,

Local round-off errors

Local truncation errors

Inherent errors: It is that quantity of error which is present in the statement of the problem itself, before finding its solution. It arises due to the simplified assumptions made in the mathematical modeling of a problem. It can also arise when the data is obtained from certain physical measurements of the parameters of the problem.

Local round-off errors: Every computer has a finite word length and therefore it is possible to store only a fixed number of digits of a given input number. Since computers store information in binary form, storing an exact decimal number in its binary form into the computer memory gives an error. This error is computer dependent. At the end of computation of a particular problem, the final results in the computer, which is obviously in binary form, should be converted into decimal form-a form understandable to the user-before their print out. Therefore, an additional error is committed at this stage too. This error is called local round-off error.

$(0.7625)_{10} = (0.110000110011)_2$

If a particular computer system has a word length of 12 bits only, then the decimal number 0.7625 is stored in the computer memory in binary form as 0.110000110011. However, it is equivalent to 0.76245. Thus, in storing the number 0.7625, we have committed an error equal to 0.00005, which is the round-off error; inherent with the computer system considered. Thus, we define the error as Error = True value – Computed value Absolute error, denoted by |Error|,

While, the relative error is defined as

Relative error= |Error| \ |True value|

Local truncation error: It is generally easier to expand a function into a power series using Taylor series expansion and evaluate it by retaining the first few terms. For example, we may approximate the function f (x) = cos x by the series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

If we use only the first three terms to compute *cos x* for a given *x*, we get an approximate answer. Here, the error is due to truncating the series. Suppose, we retain the first *n* terms, the truncation error (TE) is given by

$$\text{TE} \le \frac{x^{2n+2}}{(2n+2)!}$$

Polynomial

An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$ where n is a positive integer and $a_0, a_1, a_2 + ...a_n$ are real constants, such type of expression is called an nth degree polynomial in x if $a_0 \neq 0$

Algebraic equation:

An equation f(x)=0 is said to be the algebraic equation in x if it is purely a polynomial in x.

For example

$x^5 + x^4 + 3x^2 + x - 6 = 0$ It is a fifth order polynomial and so this equation is an algebraic equation.

$x^3 - 6 = 0$

$x^6 - 7x = 0$

$y^4 - 4y^3 + 3y^2 - y - 2 = 0$ *polynomial in y*

$t^4 - 6t^2 - 21 = 0$ *polynomail in t*

These all are the examples of the polynomial or algebraic equations.

Transcendental equation

An equation is said to be transcendental equation if it has logarithmic, trigonometric and exponential function or combination of all these three.

For example

$e^x - 5x - 3 = 0$ it is a transcendental equation as it has an exponential function

$e^x - \sin x = 0$

$\ln x - \sin x = 0$

$2\sec^2 x - \tan x - e^x = 0$

These all are the examples of transcendental equation.

Root of an equation

For an equation f(x) =0 to find the solution we find such value which satisfy the equation f(x)=0,these values are known as the roots of the equation .

A value a is known as the root of an equation f(x) =0 if and only if f (a) =0.

Properties of an Algebraic equation

1. Complex roots occur in the pairs. That is ,If (a+ib) is a root of f(x)=0 then (a-ib) is also a root of the equation
2. if x=a is a root of the equation f(x)=0 a polynomial of nth degree ,then (x-a) is a factor of f(x) and by dividing f(x) by (x-a) we get a polynomial of degree n-1.

Descartes rule of signs

This rule shows the relation ship between the signs of coefficients of an equation and its roots.

"The number of positive roots of an algebraic equation f(x) =0 with real coefficients can not exceed the number of changes in the signs of the coefficients in the polynomial f(x) =0.similarly the number of negative roots of the equation can not exceed the number of changes in the sign of coefficients of f (-x) =0"

Numerical methods for solving either algebraic or transcendental equation are classified into two groups

Direct methods

Those methods which do not require any information about the initial approximation of root to start the solution are known as direct methods.
The examples of direct methods are Graefee root squaring method, Gauss elimination method and Gauss Jordan method. All these methods do not require any type of initial approximation.

Iterative methods

These methods require an initial approximation to start.

Bisection method, Newton raphson method, secant method, jacobie method are all examples of iterative methods.

How to get an initial approximation?

The initial approximation may be found by two methods either by graphical method or analytical method
Graphical method

The equation f(x)=0 can be rewritten as $f_1(x) = f_2(x)$ and initial approximation of f(x) may be taken as the abscissa of the point of intersection of graphs of
$y = f_1(x)\ and\ y = f_2(x)$
for example $f(x) = x - \sin x - 1 = 0$
so this may be written as $x - 1 = \sin x$ Now we shall draw the graphs of
$y = x - 1\ and\ y = \sin x$

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Example

Solve the system of equations by Gaussian elimination method with partial pivoting.

$x+y+z=7$

$3x+3y+4z=24$

$2x+y+3z=16$

Solution

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 7 \\ 24 \\ 16 \end{pmatrix}$$

To start with, we observe that the pivot element

$a_{11}=1(\neq 0)$.

However, a glance at the first column reveals that the numerically largest element is 3 which is in second row. Hence *R12*

Thus the given equation takes the form after partial pivoting

$$\begin{bmatrix} 3 & 3 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 24 \\ 7 \\ 16 \end{pmatrix}$$

Stage I (Reduction to upper triangular form):

$$\begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 0 & -1 & \frac{1}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix}$$

Stage II (Back substitution):

$z=3$

$-y+1=0 \quad \text{or} \quad y=1$

$x+1+4=8 \ \text{or} \ x=3$