Let

$$SSM = \begin{cases} \dot{x} = A\overline{x} + B\overline{u} \\ y = C\overline{x} \end{cases}$$

where $A : [nsv \times nsv], B : [nsv \times ni], B : [no \times nsv]$ then

$$y = CA^{-1}\dot{x} + CA^{-1}B\overline{u}$$

where

- nsv - number of state vars;
- ni - number of input signals;
- no - number of output signals.

Let $A = diag(a_i)$ then

$$y = \begin{pmatrix} \sum_{i=0}^{nsv} \frac{c_{0,i}x_i}{a_i} \\ \sum_{i=0}^{nsv} \frac{c_{1,i}x_i}{a_i} \\ \vdots \\ \sum_{i=0}^{nsv} \frac{c_{no,i}x_i}{a_i} \end{pmatrix} + \begin{pmatrix} \sum_{i=0}^{ni} u_i(\sum_{j=0}^{nsv} \frac{b_{j,i}c_{0,j}}{a_j}) \\ \sum_{i=0}^{ni} u_i(\sum_{j=0}^{nsv} \frac{b_{j,i}c_{1,j}}{a_j}) \\ \vdots \\ \sum_{i=0}^{ni} u_i(\sum_{j=0}^{nsv} \frac{b_{j,i}c_{no,j}}{a_j}) \end{pmatrix}.$$

Let

$$V_n(v,i) : \mathbb{R} \times \{0,\ldots,n\} \to \mathbb{R}^n$$

for example

$$V_3(a,1) = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix}, \qquad V_3(a,2) = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}, \qquad V_2(a,0) = \begin{pmatrix} a \\ 0 \end{pmatrix}$$

Let $\mathfrak{L}(y) = \sqrt{(y - y_{true})^2}$ then gradients by each element of matrixies A, B, C:

$$\frac{\partial \mathfrak{L}(y)}{\partial c_{r,v}}(x,u) = \frac{y - y_{true}}{\sqrt{(y - y_{true})^2}}(V_{no}(\frac{x_v}{a_v}, u) + V_{no}(\sum_{i=0}^{ni} \frac{u_i b_{v,i}}{a_v}, u))$$

$$\frac{\partial \mathfrak{L}(y)}{\partial a_v}(\overline{x},\overline{u}) = \frac{y - y_{true}}{\sqrt{(y - y_{true})^2}}(\begin{pmatrix} -\frac{c_{0,v}x_v}{a_v^2} \\ -\frac{c_{1,v}x_v}{a_v^2} \\ \vdots \\ -\frac{c_{no,v}x_v}{a_v^2} \end{pmatrix} + \begin{pmatrix} -\sum_{i=0}^{ni} \frac{u_i b_{v,i} c_{0,v}}{a_v^2} \\ -\sum_{i=0}^{ni} \frac{u_i b_{v,i} c_{1,v}}{a_v^2} \\ \vdots \\ -\sum_{i=0}^{ni} \frac{u_i b_{v,i} c_{no,v}}{a_v^2} \end{pmatrix})$$

$$\frac{\partial \mathfrak{L}(y)}{\partial b_{r,v}}(\overline{x},\overline{u}) = \frac{y - y_{true}}{\sqrt{(y - y_{true})^2}} \begin{pmatrix} \frac{u_v c_{0,r}}{a_r} \\ \vdots \\ \frac{u_v c_{no,r}}{a_r} \end{pmatrix}$$

If apply duality of SSM:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad \Leftrightarrow \quad \begin{cases} \dot{x} = M^{-1}AMx + M^{-1}Bu \\ y = CMx \end{cases} \qquad \forall M : M \in \mathbb{R}[nsv \times nsv]$$

$$y = CM(M^{-1}AM)^{-1}\dot{x} + CM(M^{-1}AM)^{-1}M^{-1}Bu$$

$$y = \begin{pmatrix} \sum_{i=0}^{no}\sum_{j=0}^{no} \dfrac{c_{0,i}m_{i,j}x_j}{a_i} \\ \sum_{i=0}^{no}\sum_{j=0}^{no} \dfrac{c_{1,i}m_{i,j}x_j}{a_i} \\ \vdots \\ \sum_{i=0}^{no}\sum_{j=0}^{no} \dfrac{c_{no,i}m_{i,j}x_j}{a_i} \end{pmatrix} + \begin{pmatrix} \sum_{i=0}^{ni}\sum_{j=0}^{nsv} \dfrac{b_{j,i}c_{0,i}u_i}{a_i} \\ \sum_{i=0}^{ni}\sum_{j=0}^{nsv} \dfrac{b_{j,i}c_{1,i}u_i}{a_i} \\ \vdots \\ \sum_{i=0}^{ni}\sum_{j=0}^{nsv} \dfrac{b_{j,i}c_{no,i}u_i}{a_i} \end{pmatrix}$$

According to the set of admissible SSM's, the gradients over the matrix elements will look like this:

$$\frac{\partial \mathfrak{L}_m(y)}{\partial c_{r,v}}(x,u) = \frac{y - y_{true}}{\sqrt{(y - y_{true})^2}}(V_{no}(\sum_{j=0}^{no} \frac{m_{v,j}x_j}{a_v}, r) + V_{no}(\sum_{j=0}^{nsv} \frac{b_{j,i}u_i}{a_i}, r))$$

$$\frac{\partial \mathfrak{L}_m(y)}{\partial a_v}(\overline{x},\overline{u}) = \frac{y - y_{true}}{\sqrt{(y - y_{true})^2}}(\begin{pmatrix} -\sum_{j=0}^{no} \dfrac{c_{0,v}m_{v,j}x_j}{2a_v} \\ -\sum_{j=0}^{no} \dfrac{c_{1,v}m_{v,j}x_j}{2a_v} \\ \vdots \\ -\sum_{j=0}^{no} \dfrac{c_{no,v}m_{v,j}x_j}{2a_v} \end{pmatrix} + \begin{pmatrix} -\sum_{j=0}^{nsv} \dfrac{b_{j,v}c_{0,v}u_v}{a_v} \\ -\sum_{j=0}^{nsv} \dfrac{b_{j,v}c_{1,v}u_v}{a_v} \\ \vdots \\ -\sum_{j=0}^{nsv} \dfrac{b_{j,v}c_{no,v}u_v}{a_v} \end{pmatrix})$$

$$\frac{\partial \mathfrak{L}_m(y)}{\partial b_{r,v}}(\overline{x},\overline{u}) = \frac{y - y_{true}}{\sqrt{(y - y_{true})^2}} \begin{pmatrix} \dfrac{c_{0,v}u_r}{a_r} \\ \vdots \\ \dfrac{c_{no,v}u_r}{a_r} \end{pmatrix}$$

After all problem state of VI for optimize SSM model:

$$
\begin{cases}
\langle \dfrac{\partial \mathfrak{L}(y)}{\partial c_{r,v}}(x,u), c_{r,v} - c^*_{r,v} \rangle \geq 0, & \forall r, v : r \in \{0, no\}, v \in \{0, nsv\} \\
\langle \dfrac{\partial \mathfrak{L}(y)}{\partial b_{r,v}}(x,u), b_{r,v} - b^*_{r,v} \rangle \geq 0, & \forall r, v : r \in \{0, nsv\}, v \in \{0, ni\} \\
\langle \dfrac{\partial \mathfrak{L}(y)}{\partial a_{v}}(x,u), a_{v} - a^*_{v} \rangle \geq 0, & \forall v \in \{0, nsv\}
\end{cases}
$$

For simpler calculations, we will use the idea underlying the stochastic gradient method.

So SSM is a function as:

$$
SSM(u_i, x_i) : \mathbb{R}^{ni} \times \mathbb{R}^{ni} \to \mathbb{R}^{ni}
$$

If we apply on set with bs pairs $(u_i, x_i) \in \mathbb{R}^{ni} \times \mathbb{R}^{ni}$ and equivalents for them y^i_{pred} (result of output of given SSM) and y^i_{true} (expected results). We can apply for parameter p (an element of one matrix from A, B, C matrixes):

$$
\nabla \overline{\mathfrak{L}_p} = \frac{1}{bs} \sum_{i=0}^{bs} \frac{\partial \mathfrak{L}}{\partial p}(y^i_{pred}, y^i_{true}, u_i, x_i)
$$

for greater rigidity

$$
\nabla \overline{\mathfrak{L}_p} = \frac{1}{bs} \sum_{i=0}^{bs} |\frac{\partial \mathfrak{L}}{\partial p}(y^i_{pred}, y^i_{true}, u_i, x_i)|
$$

After this transformation we got next system:

$$
\begin{cases}
\langle \nabla \overline{\mathfrak{L}_{c_{r,v}}}, c_{r,v} - c^*_{r,v} \rangle \geq 0, & \forall r, v : r \in \{0, no\}, v \in \{0, nsv\} \\
\langle \nabla \overline{\mathfrak{L}_{b_{r,v}}}, b_{r,v} - b^*_{r,v} \rangle \geq 0, & \forall r, v : r \in \{0, nsv\}, v \in \{0, ni\} \\
\langle \nabla \overline{\mathfrak{L}_{a_{v}}}, a_{v} - a^*_{v} \rangle \geq 0, & \forall v \in \{0, nsv\}
\end{cases}
$$

To remove the system of equations, we can complete the matrices A, B, C to shape $[h \times h]$ for the same shape by adding zeros rows and columns. This will not affect the calculations, since the added values will be zeroed out. After that modification we can sum equations in one:

$$\langle \sum_{r=0}^{h}\sum_{v=0}^{h} |\nabla\overline{\mathfrak{L}_{c_{r,v}}}| + |\nabla\overline{\mathfrak{L}_{b_{r,v}}}| + |\nabla\overline{\mathfrak{L}_{a_v}}|, \quad \sum_{v=0}^{h}\sum_{v=0}^{h} |c_{r,v} - c^*_{r,v}| + |b_{r,v} - b^*_{r,v}| + |a_v - a^*_v| \rangle \geq 0,$$

$$\begin{gathered}
\forall c_{r,v} : r \in \{no, h\}, v \in \{nsv, h\}, c_{r,v} = 0, \\
\forall b_{r,v} : r \in \{nsv, h\}, v \in \{ni, h\}, b_{r,v} = 0, \\
\forall a_v : v \in \{0, h\}, a_v = 0,
\end{gathered}$$

$$\begin{gathered}
\forall \nabla\overline{\mathfrak{L}_{c_{r,v}}}, r \in \{no, h\}, v \in \{nsv, h\}, \overline{\mathfrak{L}_{c_{r,v}}} = \overline{0}, \\
\forall \nabla\overline{\mathfrak{L}_{b_{r,v}}} : r \in \{nsv, h\}, v \in \{ni, h\}, \nabla\overline{\mathfrak{L}_{b_{r,v}}} = \overline{0}, \\
\forall \nabla\overline{\mathfrak{L}_{a_v}} : v \in \{0, h\}, \nabla\overline{\mathfrak{L}_{a_v}} = \overline{0}
\end{gathered}$$