Exercises on Finite Horizon Dynamics Programming Problems

The Big Picture: In previous lessons, we talked about the situation where a single decision maker (e.g., a consumer) solves a *finite* horizon optimization problem. The most basic macro example was the two-period OLG model. It turns out that this problem is no different—mathematically—to a problem of, say, a microeconomic consumer choosing a bundle of finitely many different commodities to maximize utility subject to her budget constraint. The clever way to think about this in the finite-horizon dynamic decision problem, is to treat each *date-contingent* choice variable as a unique type of good. That is, we're just indexing each type of good by date, or natural time.

The first problem gets you to repeat this method again in a different application: a problem of optimally harvesting a (variable) natural resource over time. For now, let's again solve it the brute-force Lagrange problem way.

We get you to practise again on the finite horizon optimal capital accumulation problem. In this example, think of the decision maker as a benevolent social planner whose incentives aligns with the representation agent. (Equivalently, this of this as Robinson Crusoe as the lone agent in an island, and his allocation problem is that of corn: Eat it today and/or plant it for tomorrow.)

Finally, the last problem that illustrates the consumption-smoothing problem in a life-cycle setting where a consumer is finitely lived. We can think of this example someone who has just entered the labor force at age 15 (date $t = 0$) and retires at age 61 (date $t = T = 46$).

Problem 0 (Level: Easy. Past Exam question.) Ordralfabétix the fisherman discovers and proceeds to own a lake full of fish. This initial stock of fish is $y_0 > 0$. In each period $t \in \{0, 1, 2\}$, he must decide how much to harvest, $x \in \mathbb{R}_+$, which would earn him a profit of $\pi(x)$, where $\pi : \mathbb{R}_+ \to \mathbb{R}$. The unharvested fish each period will grow to a new stock at the start of next period, according to the growth model $f(y - x)$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$. Let $\pi(x) = \ln x$ and $f(z) = z^\alpha$, $\alpha \in (0, 1]$. Solve for Ordralfabétix' optimal fishing strategy (sequence of optimal decision rules) and corresponding sequence of value functions both ways:

- Use our brute-force Lagrange method (i.e., the sequence problem).
- Use the recursive Bellman equation and backward induction approach.

Problem 1 (Level: Easy) *This is a small extension from our simple OLG example.*
Read up and do the exercise in Section 6.2.1 (finite-horizon planner's problem in the RCK economy) of our Class Notes. Describe solution by dynamic programming. (Do parts 1-4.)
Hint: It suffices to show it for the case that $T = 2$, but if you're a sucker for pain, try $T = 9$.

Problem 2 (Level: Medium.) *From experience, the best way to master the theory is to understand it on paper (last exercise) and also to code it up and implement in more generally on computer (this task).*
Try writing your own `Python` method (function) to solve these systems of first-order conditions for the last exercise. Then invoke and execute it for a parametric instance of the same model.
See if you can solve it for $T = 10, 20, 50$. Set $\alpha = 0.33$, $\beta = 1/(1.04)$, $\delta = 0.1$.

Problem 3 (Level: Easy/Medium.) Consider a consumer who ranks sequences of consumption outcomes $\{c_t\}_{t=0}^{T}$ according to this preference function:

$$\sum_{t=0}^{T} \beta^t U(C_t) + \beta^{T+1} V_{T+1}(a_{T+1}),$$

where the per-period utility of consumption is

$$U(C_t) = -(C_t - c^\star)^2.$$

Assume that $V_{T+1}(a_{T+1}) = k(a_{T+1})^2$, where $k > 0$ is some arbitrarily large number. The parameter $c^\star \in (0, \infty)$ is satiation-point consumption. The consumer discounts per period payoffs by discount factor $\beta^t \in (0, 1]$. The consumer faces a sequence of budget constraints:

$$a_{t+1} + C_t = Ra_t + y_t, \qquad t = 0, 1, ..., T,$$

where a_t is the consumer's asset position (positive/negative denotes saving/borrowing), $R > 1$ is some given gross return on the asset, and, y_t is income-endowment. Let us assume that

$$y_t = \bar{y} + \kappa_1 t + \kappa_2 t^{\kappa_3},$$

where the parameters $(\bar{y}, \kappa_1, \kappa_2, \kappa_3)$ are all strictly positive.

1. Combine the sequential budget constraint with the trend process of income.

2. The relevant state vector is $x_t = (1, t, t^{\kappa_3}, a_t)$. Why?

   Hint: Assume $\kappa_3 = 2$ from here onwards. Use the last result you derived and rewrite in as a linear state space form (i.e., a controllable linear difference equation system):

   $$x_{t+1} = Ax_t + Bu_t,$$

   where $u_t = (C_t - c^\star)$ is the (shifted) control/decision variable, and A and B are appropriately conformable matrices.

3. Now derive (by hand) the optimal date- and state-contingent decision rule for the consumer using dynamic programming (with backward induction). Given each date t and state x_t, show that the rule is linear in the state vector:

   $$C_t^* = c^\star + F_t x_t.$$

   Derive the specific formula for F_t! (Show that F_t can be pinned down by using a recursive system of matrix quadratic equations.)

4. Implement an example solution in `Python`. Submit this part of your work as a replicable Jupyter Notebook. Set $R = 1.05$, $\beta = 1/R$, $T = 46$, $c^\star = 2.0$, $k = 10^6$, and $\bar{y} = 0$, $\kappa_1 = T/(T/2)^2$, $\kappa_2 = -1/(T/2)^2$, and $\kappa_3 = 2.0$.

5. Once you have computed $\{F_t : t = 0, 1, ..., T\}$, simulate the time path of consumption, assets, and income and plot them in appropriate diagrams. Discuss your result.

6. What is the consumer's indirect utility at date $t = 34$ if $a_t = 0.3$?