1

Analytic functions

12th Aug 2024 7:20pm

12 Aug

Representation of complex numbers

Cartesian

$z = x + yi$
$r = \sqrt{x^2 + y^2} = |z|$

Polar

if $x = rcos(\theta)$ and $y = rsin(\theta)$

$$\begin{aligned} z &= x + yi \\ &= rcos(\theta) + irsin(\theta) \\ &= re^{i\theta} \qquad [\text{where } \theta = \tan^{-}1\left(\frac{y}{x}\right)] \end{aligned}$$

Core ideas

1. Complex functions and its values

let $\omega = f(x)$ where $z = x + iy$ is a complex function. **1 Find value of $f(z) = z^2 + iz + 2$ at $z = 1 - i$**

<u>**Answer**</u>

$$\begin{aligned} f(1-i) &= (1-i)^2 + i(1-i) + 2 \\ &= 1 - 2i - 1 + i + 1 + 2 \\ &= 3 - i \end{aligned}$$

2 Find real and imaginary part of $f(z) = \ln(z)$

Answer

$$
\begin{aligned}
&f(z) = \ln(z) \\
&\qquad f\left(re^{i\theta}\right) = \ln\left(re^{i\theta}\right) = \ln r + \ln e^{i\theta} \\
&\qquad\qquad\quad = \ln r + i\theta \ln e \\
&\qquad\qquad\quad = \ln r + i\theta \\
&\qquad\qquad\quad = \ln\sqrt{x^2+y^2} + i\tan^{-1}(y/x)
\end{aligned}
$$

Real part $= \ln\sqrt{x^2+y^2}$, imaginary part $= \tan^{-1}(y/x)$

2. Analytic Function (Complex Differentiable Function)

A complex function $f(z)$ i said to be analytic at the point z_0 if $f(3)$ is differentiable at z_0 in some neighbourhood of z_0. $f'(z)$ exists at z if

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

exists
A function $f(z)$ is analytic in a domain $\mathcal{D}$ if it is analytic in all points in $\mathcal{D}$
A function $f(z)$ is entire function if it is analytic in every point z in the complex plane. Eg: $f(z) = e^z$

The function $f(z) = \dfrac{z^2+2}{(z-3)(z+5)}$ fails to be analytical at z=3 and -5.
$\therefore f(z)$ is not an entire function.

3. Singular Points

A point at which complex function $f(z)$ fails to be analytic is called singular point.
Eg:

$$
\begin{aligned}
f(3) &= \frac{3^2+2}{(3-3)(3-5)} && z = 3, 5 \text{ are singular point} \\
f'(z) &= \frac{1}{3} && z = 0 \text{ is a singular point} \\
f(z) &= \frac{1}{z^2+1} && z = +i, -i \text{ ane singular points}
\end{aligned}
$$

4. Canchy - Riemann Equation (CR Equation)

Used to check whether a complex function $f(x)$ is analytic or not.
If $f(z) = u + iv$ is analytic, then u & v must satisfy $C - R$ equation:

$U_x = V_y$ $U_y = -v_x$	$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{dx}$

1 Prove that $f(z) = \bar{z}$ is not analytic

Answer

$$f(z) = \bar{z} = \overline{x + iy} = x - iy = u + iv$$

$$u = x \quad v = -y$$

$$U_x = \frac{du}{\partial x} = 1 \qquad v_y = \frac{\partial v}{\partial y} = -1$$

because $u_x \neq v_y$ $\quad C - R$ equation is not satisfied $\therefore f(z) = \bar{z}$ not analytic.

2 P.T $f(z) = z^2$ is analytic. Also find $f'(z)$ at $z = 1 + i$

Answer

$$\begin{aligned} f(z) &= z^2 \\ &= (x + iy)^2 \\ &= x^2 + i2xy + (iy)^2 \\ &= x^2 + i2xy - y^2 = x^2 - y^2 + i2xy \\ &= u + iv \end{aligned}$$