Integral Calculus

1. $\int \ln(x) dx$

$= \int \ln(x) \times 1dx = \ln(x) \int 1dx - -\int \frac{1}{x} \cdot \int 1dxdx \quad \int u0 = u \int v - \int a \int f$

$$= x\ln(x) - x + c$$
$$= (x-1)\ln(x) + c$$

2. $\int e^x x^5 dx = \int x^5 e^x dx = x^5 \int e^x - \underbrace{\int 5x^4 \cdot \int^x dx}_{-x_1}$

$$= x^5 e^x - 5\int x^4 e^x$$
$$= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^3 e^x + 120xe^x$$
$$= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120xe^x - 120e^x$$

$3 \int \left(x^5 + 8x^2 + 1\right) dx = \frac{x^6}{6} + \frac{8}{3}x^3 + x + c$

5. $\int \frac{2(x+5)}{x^2+10x+25} d\frac{2}{x+5} = 2\ln(x+5) + c$

5). $\int \sin(3x) \cdot e^{2x} dx \sim \left\{\int e^{ax} \cdot \sin(bx) dx = \left[\frac{e^{ax}}{a^2+b^2}[a\sin(bx) - b\cos(bx)]\right\} \sim \frac{e^{2x}(2\sin(3x) - 3\cos(3x))}{13} \left\{\int e^{ax\cos(}\right.\right.$

6. $\int e^x(\sec(x) + \sec(x)\operatorname{ton}(x))dx \ 8\int x^2 \cdot \tan^{-1}(x)$

$$= \int \tan^{-1}(x) x^2 - dx$$
$$= \tan^{-1}(x) \cdot \frac{x^3}{3} - \int \frac{1}{1+x^2} \cdot \frac{x^3}{3} dx \qquad x^3 \to t \qquad dt = 3x^2 \cdot dx$$
$$\int \frac{x}{1+x^2} \cdot x^2 dx$$
$$\Rightarrow \quad \frac{1}{2} \int \frac{t-1}{z} dt \Rightarrow \frac{1}{2} \int 1 - \frac{1}{2} \int \frac{1}{t}$$
$$= \frac{1}{2}\left(x^2+1\right) - \frac{1}{2}\ln\left(x^2+1\right)$$
$$\Rightarrow \tan^{-1}(x) \cdot \frac{x^3}{3} - \frac{1}{6}\left(x^2+1\right) + \frac{1}{6}\ln\left(x^2+1\right) + c$$
$$\int \frac{x+1}{x^2+6x+25} =$$
$$\to \frac{1}{2} \int \frac{2x+6}{x^2+6x+25} dx - 2\int \frac{dx}{x^2+6x+25} \qquad 2x+6 = a \qquad b = x+1-(2x+6)$$
$$t_1 = x^2 + 6x + 68$$

$$du = (2x+6)dx$$

$$\to \frac{1}{2}\int \frac{du}{u} - 2\int \frac{dx}{(x+3)^2+16}$$

$$\Rightarrow \frac{1}{2}\ln(u) - \frac{2}{16}\int \frac{ds}{s^2+16} \Rightarrow \ldots - \frac{2}{16}\int \frac{4dp}{p^2+1} \qquad p^2 = s^2/16$$

$$p = s/4 \qquad \frac{dp}{ds} = \frac{1}{4} \qquad ds = 4dp$$

$$\Rightarrow \ldots -\frac{1}{2}\tan^{-1}(p)$$

$$\Rightarrow \frac{\ln(4)}{2} - \frac{1}{2}\tan^{-1}\left(\frac{5}{4}\right)$$
$$\Rightarrow \frac{\ln\left(x^2+6x+25\right)}{2} - \frac{1}{2}\tan^{-1}\left(\frac{x+3}{4}\right)$$

Quadrature
wapticolise of area of curve bounded by a Gundary finding
Area bounded by a curve $y = f(x)$ within $[a, b]$ in x-axis is

$$A = \int_a^b y dx \equiv \int_a^b f(x) dx$$

Also if the curve is $x = f(y)$, then area within $[a, b]$ in y-ayis is:

$$A = \int_a^b x dx y = \int_a^b f(y) dy$$

- If two curves $y_1 = f(x) \quad y_2 = \phi(x)$ intersects at the points whuse coordinates are $(a, .).(b, \ldots)$ then the area enclosed by curve is givenby. $A = \int_a^b (f(x) - \phi(x)) dx$

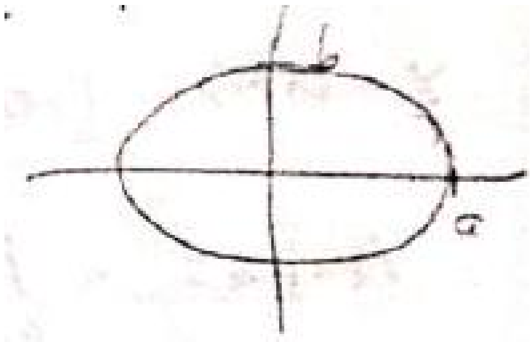
1 Find area bounded by the curve, $y^2 = 4ax$ and the ordinatif at $x = h$,

A) $\quad A = \int_0^h \sqrt{4ax} dx \Big| = 2\sqrt{a}\sqrt{x} dx = 2\sqrt{a} \cdot \frac{2}{3} \cdot \frac{b}{x^{3/2}}$

$$\text{a } = 6\sqrt{6oh} \quad \Rightarrow \frac{4}{3}\sqrt{a} \cdot h^{3/2}$$

2 Find area erelused by curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
A.)

$$\text{let } y = 0, \quad \therefore x = 4$$
$$x = a, \quad \therefore y = b$$



$\therefore c$

$\therefore y = b\sqrt{1-\frac{x^2}{a^2}} = \frac{ab}{a}\sqrt{a^2-x^2}$

$\therefore A = 4\int_0^a ydx = 4\frac{ab}{a}\int_0^a \sqrt{a^2-x^2}dx$

$= 4\frac{ab}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} - \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^c$

$= 4\frac{ab}{a}\cdot\frac{a^2}{2}\sin^{-1}(1)$

$= \pi ab$

3. Find area enclused blw lineff and curve $y = x^2 - 6x + 4$
   A) $y = 1 - 2x = \{x^2 - 6x + 4 \Rightarrow x^2 - 4x + 3 = 0$

$f = x^2 - 6x + 4, \phi = 1 - 2x$
$= 044 \rightarrow 0 \pm 1 \Rightarrow 13$
$\therefore$ Area $= \int_{10}^3((f-\phi)dx$
10/1 4find area bounded by the curve $y = 3x - x^2$ the x-axis and line $x = 3, x = 0$
A) Area $= \int_0^3 (3x - x^2)\, dx$

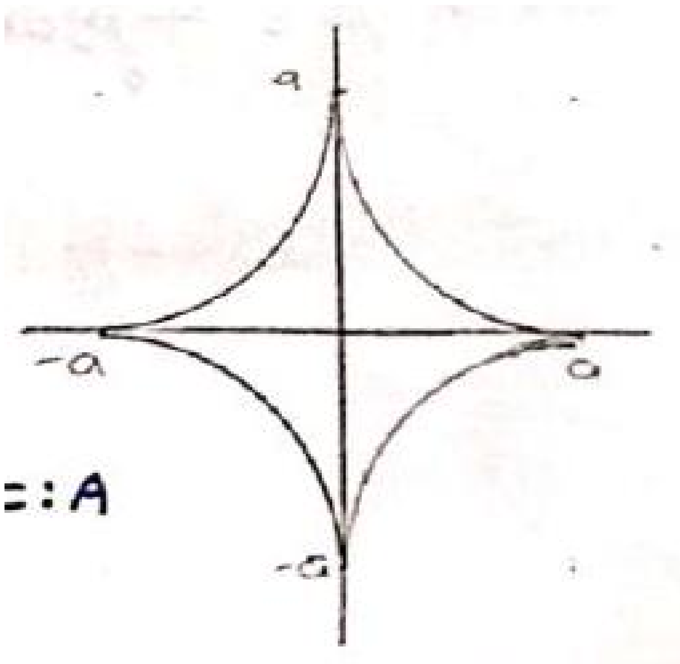
$$= \frac{3}{2}x^2 - \frac{x^3}{3}\Big]_0^3 = \frac{27}{2} - 9 = \frac{9}{2}$$

5. Find ane of asteroid: $x^{2/3} + y^{2/3} = a^{2/3}$

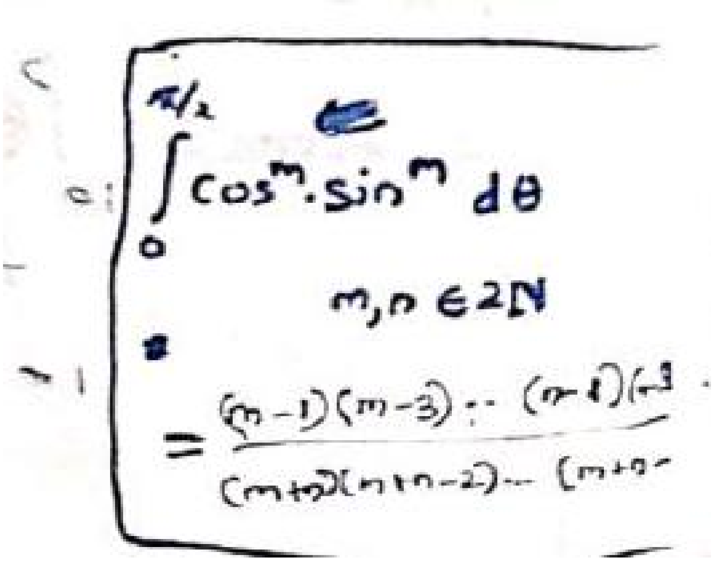
A) $x^{2/2} + y^{2/3} = a^{2/3}$

$$y = \left(a^{2/3} - x^{2/3}\right)^{3/2}$$

$\therefore$ Area of asteroid $= 4\int_0^a \left(a^{2/3} - x^{2/2}\right)^{3/2} dx =: A$



$$a\sin^3(\theta) = 0 \Rightarrow \theta = 0$$
$$\therefore dx = 3a\sin^2(\theta)\cos(\theta)d\theta$$
$$a\sin^3(\theta) = a \Rightarrow \theta = \pi/2$$
$$\therefore A \equiv 4\int_0^{\pi/2} \left(a^{2/s} - a^{2/3}\sin^2(\theta)\right)^{3/2}\Bigg)\, 3a\sin^2(\theta)\cos(\theta)d\theta$$
$$= 4\int_0^{\pi/2} a\left(1 - \sin^2(\theta)\right)^{2/2} \cdot 3a\sin^2(\theta)\cos(\theta)d\theta$$
$$= 012a^2\int_0^{\pi/2} \cos^3(\theta)\sin^2(\theta)\cos(\theta)d\theta$$
$$= 12a^2\int_0^{\pi/2} \cos^4(\theta)\sin^2(\theta)d\theta$$
$$= 12a^2 \cdot \frac{3\times 1\times 1}{6\times 4\times 2} \times \frac{\pi}{2}$$



$$= \frac{3}{8}a^2\pi$$
$$\text{fund } m,n \in 2^{r+1}$$

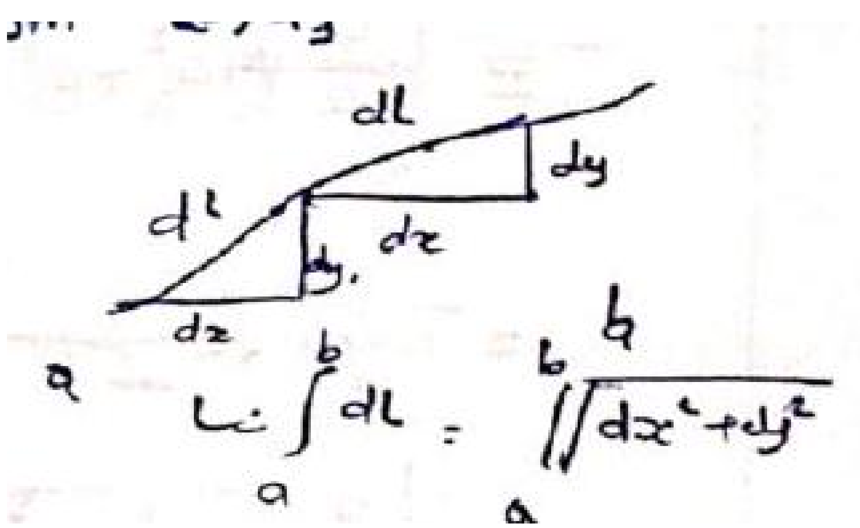
Length (perimeter) of Carve (Rectification)

- let $y = f(x)$. length of carve from $[0, a]$

$$d = \int_0^\infty dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

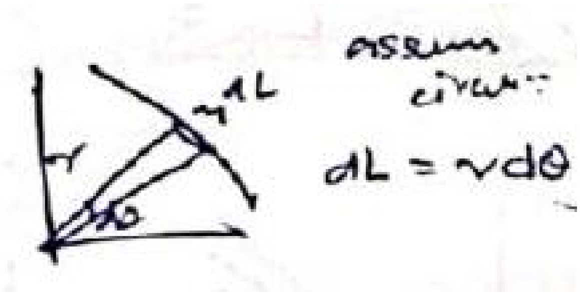
If $x = f(y)$

$$L = \int_0^a \sqrt{1+\left(\frac{dx}{dy}\right)^2}dy$$



- If $x = f(t), y = \phi(t), L = \int_0^a \sqrt{\left(\frac{dx}{db}\right)^2 + \left(\frac{dy}{db}\right)^2} \cdot dt$

- If $\gamma' = f(\theta)$, $\quad L = \int_0^a \sqrt{\gamma^2 + \left(\frac{d\gamma}{d\theta}\right)^2} \cdot d\theta$

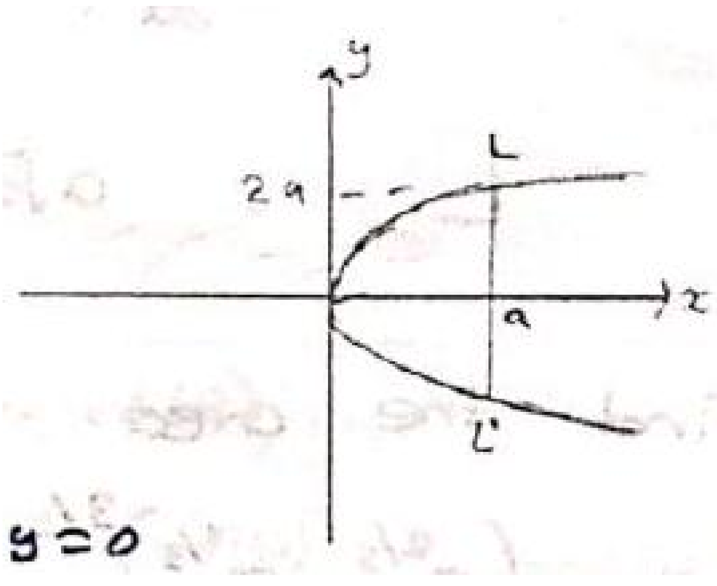


1. Find the length of the carve $y^2 = 4ax$, measure from the vertex to one extremity of Lotus rectum

$\dot{A} \quad L = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$

$$y = 2\sqrt{ax}$$
$$\frac{dy}{dx} = \frac{a}{\sqrt{ax}} = \sqrt{\frac{a}{x}}$$
$$\therefore L = \int_0^a \sqrt{1 + \frac{a}{x}} dx$$
$$= \int_0^a \frac{\sqrt{x+a}}{\sqrt{x}} dx$$

2. $\frac{1}{2}\frac{5}{\sqrt{\text{an}}}$



$$
\begin{aligned}
&x=0, y=0 \\
&x=a, y=2a \\
&x=\frac{y^2}{4a} \quad \frac{dx}{dy}=\frac{y}{2a} \\
&L=\int_0^{2a}\sqrt{1+\frac{y^2}{4a^2}}dy \\
&=\frac{1}{2a}\cdot\int_0^{2a}\sqrt{(2a)^2+y^2}dy
\end{aligned}
$$

$$
\begin{aligned}
&\sinh^{-1}(x)=\ln\left(x+\sqrt{x^2+a^2}\right) \\
&\therefore=\left[\frac{1}{2}\sqrt{4a^2+y^2}+\frac{(2\partial^2}{2}\ln\left(y+\sqrt{4a^2+y^2}\right)\right]_0^{2a} \\
&=\frac{1}{2a}\cdot\left(2a^2\sqrt{2}+\frac{2a^2}{2a}(2a+2a\sqrt{2})\right) \\
&L=a\sqrt{2}+\ln(2a(1+\sqrt{2})) \\
&L=\frac{1}{2a}\left[\frac{y}{2}\sqrt{y^2+a^2}+\frac{2a^2}{2}\sinh h^{-1}\left(\frac{y}{a}\right)\right]_0^{24}
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{1}{2a}\left(\frac{2a}{2}\sqrt{(2a)^2+(2a)^2}+\frac{2a^2}{2}\sinh^{-1}\left(\frac{2a}{2a}\right)\right) \\
&=\frac{1}{2a}\left(a^2 2\sqrt{2}+2a^2\ln(1+\sqrt{2})\right)' \\
&L=a\sqrt{2}+a\ln(1+\sqrt{2})
\end{aligned}
$$

2. Find the area of astroid $x^{2/3}+y^{2/3}=a^{2/3}$
   A) $y=\left(a^{2/3}-x^{2/3}\right)^{3/2}$

$$
\begin{aligned}
\text{Area of single piece } &=\int_0^a y\cdot dx \\
&=\int_0^a\left(a^{2/3}-x^{2/3}\right)^{3/2}dx
\end{aligned}
$$

let $x=a\sin^3(\theta)\Rightarrow dx=3a\sin^2(\theta)\cdot\cos(\theta)\cdot d\theta$
$\rightarrow$ bounds are $x=0\mapsto\theta=0$

$$x=a\mapsto\theta=\pi/2$$

$\therefore$ Area of single piece $=\int_0^{\pi/2}\left(a^{2/3}-0^{2/3}\sin^2\theta\right)^{3/2}\cdot 3a\sin^2\theta\cdot\cos(\theta)\cdot d\theta$

$$
\begin{aligned}
&=3a^2\int_0^{\pi/2}\cos^4(\theta)\sin^2(\theta)d\theta \\
&=3a^2\times\frac{3\times1\times1}{6\times4\times2}\times\frac{\pi}{2}=\frac{3\pi}{32}a^2
\end{aligned}
$$

$\therefore$ Area of astroid $=4\times\frac{3\pi a^2}{32}=\frac{3}{8}\pi a^2$

$$
\begin{aligned}
\int_0^{\pi/2} \cos^m(\theta)\sin^n(\theta)d\theta =& (m-1)(m-3)(m-5)\cdots(m-\cdots) \\
& \times (n-1)(n-3)(n-5)\cdots\frac{(n-\cdots)}{20} \\
& \times \frac{1}{(m+n)(m+n-2)(m+n-4)(m+n-\cdots)}\frac{\pi}{2}
\end{aligned}
$$

Area of Polar functions

$$
\frac{1}{2}\int_a^b \tau^2 d\theta
$$

1. Find whole area of circle: $r = 2\,\mathrm{acos}(\theta)$,

- Cardioid: $r = a(1-\cos(\theta))$

A) Circle

$$
\begin{aligned}
&\text{let } r = 0 \Rightarrow \theta = \frac{\pi}{2} \\
&r = 2a \Rightarrow \theta = 0 \\
&\therefore \text{ Area } = 2\int_0^{\pi/2} \frac{1}{2}\cdot r^2 d\theta = 4a^2\int_0^{\pi/2}\cos^2(\theta)d\theta \\
&= 4a^2\cdot\frac{1}{2}\pi = \pi a^2 \\
&\int_0^{\pi/2}\cos^2(\theta)d\theta = \frac{(n-1)(n-s)\ldots(n-\ldots)}{n(n-2)(n-4)(n-\cdots)}\times\frac{\pi}{2}
\end{aligned}
$$

Cardiad

$$
\begin{aligned}
&r = a(1-\cos(\theta)) \\
&r = 0 \Rightarrow \theta = \pi \quad ,\therefore \theta \in [0,\pi] \\
&\therefore \text{ Area } = 2\int_0^{\frac{1}{2}} r^2 d\theta = a^2\int_0^{\pi} 4(1-\cos(\theta))^2 d\theta \\
&= a^2\int_0^{\pi}\left(2\sin^2\frac{\theta}{2}\right)^2 d\theta = 4a^2\int_0^{\pi}\sin^4(\theta/2)d\theta \\
&\text{let } \phi = \theta/2 \Rightarrow \phi \in [0,\pi/2]
\end{aligned}
$$

$\therefore A = 4a^2\int_0^{\pi/2}\int\sin^2(\phi)d\phi = 8a^2\int_0^{\pi/2}\sin^4(\phi)d\phi$

$$
= 8a^2\times\frac{3\times 1}{4\times 2}\times\frac{\pi}{2} = \frac{3}{2}\pi c^2
$$

Surface area of "Curves

- Revolving about x-axis-: $\int_a^b 2\pi y ds$ where $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx$ in cartesian

$$ds = \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \text{ in parametric}$$

$$ds = \sqrt{r^2 + \left(\frac{dd}{d\theta}\right)^2} d\theta \text{ in polar}$$

1. Find avea of the surface generated by revolving the parabola $y^2 = 4ax$ about x axi from origin to $x = a$ A. $\quad A = \int_0^a 2\pi y \cdot ds$

$$\frac{dy}{dx} = 2\sqrt{a} \cdot \frac{1}{2}\frac{1}{\sqrt{x}} = \sqrt{\frac{a}{x}} \quad y = 2\sqrt{ax}$$

$$\therefore dS = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{\frac{x+a}{x}} dx$$

$$\begin{aligned}\therefore A = \int_0^a 2\pi y \cdot ds &= 2\pi \int_0^a 2\sqrt{ax}\sqrt{\frac{x+a}{x}} dx \\ &= 4\pi\sqrt{a}\int_0^a \sqrt{x+a} dx \\ &= \frac{8}{3}\pi\sqrt{a}\left[(x+a)^{3/2}\right]_0^a \\ &= \frac{8}{3}\pi\sqrt{a}a^{3/2}\left(2^{3/2} - 1\right) = \frac{8}{3}\pi a^2\left(2^{2/2} - 1\right)\end{aligned}$$

2 Show that the SA. of selicd generated by revolving the curve $x = a\cos^3(\theta), y = a\sin^3(\theta)$ about

y axis in $\underline{\frac{12}{5}\pi a^2}$

$[0, \ldots]$

A $x = 0 \mapsto \theta = \pi/2, \quad \theta \in [0, \pi/2]$

$$\therefore s = 2 \cdot \int_0^{\pi/2} 2\pi \cdot x ds \quad ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

$$= 4\pi \times 3a_0^2 \int_0^{\pi/2} \cos(\theta)\sin(\theta) d\theta$$

$$= 3a\sin(\theta)\cos(\theta) d\theta$$

$$= a \cdot 12\pi a^2 \cdot \frac{1}{5} = \frac{12\pi a^2}{5}$$

3 find length of carve $x = t^2, y = t^3$ t $t0, 1]$

A. $\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2$

$$\therefore L = \int_6^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^2 \sqrt{(4t^2 + 9t^4)} dt$$

$= \int_0^1 t\sqrt{4 + at^2} dt$, let $4 + 9t^2 = u \Rightarrow \frac{du}{dt} = 18t, dt = 18d \cdot dy$
$\equiv \int_4^{13} \sqrt{u} \frac{du}{18}$
$t = [0, 1] \Rightarrow u = [4, 13]$

$$= \frac{1}{27} \left[u^{3/2}\right]_4^{1/3} = \frac{1}{27} \left(13^{3/2} - 4^{3/2}\right)$$

4. Find the perimeter of cardioid $r = a(1 + \cos(\theta))$

A) Sine eqn remaims unchanged when $\theta = -\theta$ $\theta \in [0, \pi]$ $L = 2 \iint_0^\pi \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta, \quad \frac{dr}{d\theta} = -a \sin(\theta)$
$= 2 \int_0^\pi \sqrt{a^2 \left(1 + \cos(\theta)^2 + a^2 \sin^2(\theta)\right.} d\theta$
$= 2 \int_0^\pi \sqrt{2a^2(1 + \cos(\theta))} d\theta = 4a \int_0^\pi \cos \frac{\theta}{2} d\theta$
$= 8a \left[\sin \frac{\theta}{2}\right]_0^\pi = 8a$

Volume of Solid od
obtalned by revolving $y = f(x)$ aboul x-ax.s;[a,b]

$$v = \int_a^b \pi y^2 dx$$

1. Prove that the volume of a right circular cone of height h and base radius r is $1/3\pi r^2 h$

A $\quad V = \int_0^h \pi y^2 dx$

$$y = r/hx$$

$\therefore v = \int_0^h \pi \frac{r^2}{h^2} x^2 dx = \pi \frac{r^2}{h^2} \cdot \frac{1}{3} \left[x^3\right]_0^h = \frac{1}{3} \pi r^2 h$

2. Find the volume formed by revolution of loop of carve $y^2(a+x) = x^2(3a-x)$ about x-axis

le! $y = 0 \Rightarrow x^2(3a - x) = 0 \Rightarrow x = 0, x = 3a$

$$
\begin{aligned}
\text{volume } &= \int_0^{3a} \pi y^2 dx = \pi \int_0^{3a} \frac{x^2(3a-x)}{a+x} dx \\
&= \pi \int_0^{3a} \frac{-x^3+3ax^2}{a+x} \\
&= \pi \int_0^{3a} \left(-x^2 + 4ax - 4a^2 + \frac{4a^3}{x+a} \right) dx \\
&= \pi \left[-\frac{x^3}{3} + 2ax^2 - \frac{4}{3}a^3 x + 4a^3 \ln(x+a) \right]_0^{3a} \\
&= \pi \left[-9a^3 + 18a^3 - 12a^3 + 4a^3 \ln(4a) \right] \\
&= a^3 \pi (-3 + 4\ln(4))
\end{aligned}
$$

$\frac{-x^2+4ax-4a}{-x^3+3ax^2}$
$\frac{-x^3-ax^2}{4ax^2}$
$\frac{4ax^2+4a^2x}{-4a^2x}$
$\frac{-4a^2x-4a^3}{4a^3}$

Multiple integrals

integrats undergoing

Double integrals

1. $\underbrace{2}_{0} \underbrace{2}_{\text{surdh.}} x_0^{x^2} xy \cdot dy \cdot dx$

   A limits of y are $0 \to x^2$, limits of $x : 0 \to 2$

$$\therefore \text{ Ans: } \int_0^2 x_5^2 dx = \left[\frac{1}{2} \cdot \frac{x^6}{6} \right]_0^2 = 16/3$$

2. $\int_0^6 \int_0^{y^3} x^2 y \cdot dy \cdot dx \Rightarrow \int_0^5 \int_0^{y^3} x^2 y \cdot dx \cdot dy$

$$= \int_0^5 \left[y \cdot \frac{x^3}{3} \right]_0^{y^3} dy = \int_0^5 y^{14}/3x dy = 51/33$$

Triple Integral

1. $\int_0^1 \int_0^2 \int_0^2 x^2 yzdxdydz$

$$
\begin{aligned}
&fy = y \\
&xyy
\end{aligned}
$$

   A)
   $\int_0^1 \int_0^2 \left(\int_1^2 x^2 \cdot yz \int_1 dx \right) dy \cdot dx.$

$= \int_0^1 \int_0^2 \left[\frac{x^2yz^2}{2}\right]_{z^1}^2 dy \cdot dx = \int_0^1 \int_0^2 \frac{x^2y \cdot 4 - x^2y}{2} dy \cdot dx$
$= \int_{-0}^{1-2} \int_0^2 \frac{3}{2}x^2ydy \cdot dx$

$$= \int_0^1 \left[\frac{3x^2y^2}{4}\right]_0^2 dx$$
$$= \int_0^1 3x^2dx = x^3\Big|_0^1 = 1$$

$2 \int_{-c}^c \int_{-b}^b \int_{-a}^0 \left(x^2+y^2+z^2\right) dxdy \cdot dz$
A)

$$\int_{-c}^c \int_{-b}^b \left(x^2z + y^2z + \frac{z^3}{3}\right)_{-a}^a dy \cdot dx - = \int_{-c}^c \int_{-b}^b \left(2ax^2 + 2ay^2 + \frac{2}{3}a^3\right) dy \cdot dx$$
$$= \int_{-1}^c \left[2ax^2y + \frac{2ay^3}{3} + \frac{2}{3}a^3y\right]_{-b}^b dx$$
$$= \int_{-c}^c \left(4abx^2 + \frac{4ab^3}{3} + \frac{4}{3}a^3b\right) dx$$
$$\Rightarrow \left[\frac{4abx^3}{3} + \frac{4}{3}ab^3x + \frac{4}{3}a^3bx\right]_{-c}^c = \frac{8}{3}abc^3 + \frac{8}{3}ab^3c + \frac{8}{3}a^3bc$$
$$= \frac{8}{3}abc\left(a^2+b^2+c^2\right)$$

$3 \int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z)dxdydz$

$$\frac{(x+z)^2}{2} - \frac{(x-z)^2}{2}$$
$$= \frac{4xz}{2}$$

A) $\rightarrow \int_{x-x}^{x+1} (x+y+z)dy\Big]$

$$I = \left[xy + \frac{y^2}{2} + zy\right]_{x-z}^{x+2}$$

$$\begin{aligned} I =& x(x+z) + \frac{(x+z)^2}{2} + z(x+z) \\ &- \left(x(x-z) + \frac{(x-z)^2}{2} + z(x-z)\right) \\ =& x(2z) + 2xz + z(2z) \\ =& 2xz + 2xz + 2xz^2 \\ =& 4xz + 2z^2 \end{aligned}$$

$$\int_{-1}^1 \left[\int_0^z \left(4xz + 2z^2\right) dx\right] dz$$
$$\Rightarrow \int_{-1}^1 \left(2z^3 + 2z^3\right) dz = \left[\frac{4z^4}{4}\right]_{-1}^1 = 0$$

& $\int_0^1 \int_0^x \int_0^{x+y} (2x+y-z)\,dz\,dy\,dx$

A)

$$
\begin{aligned}
I_3 &= \int_0^{x+2y} (2x+y-z)\,dz \\
&= \left[2xz + yz - \frac{z^2}{2}\right]_0^{x+y} = 2x^2 + 2xy + xy + y^2 - \frac{(x+y)^2}{2} \\
&= 2x^2 + 3xy + y^2 - \frac{x^2+y^2+2xy}{2} \\
&= \frac{3x^2+y^2+4xy}{2}
\end{aligned}
$$

2 1

2

$$\therefore I_2 = \int_0^x I_3\,dy = \frac{1}{2}\left[3x^2y + y^3/3 + 2xy^2\right]_0^x$$

$$= \frac{1}{2}\left(3x^3 + \frac{x^3}{3} + 2x^3\right)$$

$$\int_1^2 \int_x^{4/x} \int_0^{xy} xy.$$

$$\int_0^4 \int_0^{2\sqrt{2}} \int_0^{\sqrt{4-x^2}} dy\,dx\,dz$$

$$
\begin{aligned}
&\frac{1}{6}\left(9x^3 + 6x^3 + x^3\right) = \frac{16x^3}{6} = \frac{8}{3}x^3 \\
&I = \int_0^1 \frac{8}{3}x^3\,dx = \left[\frac{2}{3}x^4\right]_0^1 = \frac{2}{3}
\end{aligned}
$$

5. $\int_0^1 \int_0^x \int_0^{x+y} (2x+y-1)\,dz\,dy\,dx$

$$
\begin{aligned}
&\int_0^1 \left(\int_0^x \left(2x^2 + 2xy + xy + y^2 - x - y\right) dy\right) dx \\
&= \int_0^1 \int_0^x \left(2x^2 + 3xy + y^2 - x - y\right) dy\,dx \\
&= \int_0^1 \left[2x^2y + 3/2xy^2 + y^3/3 - xy - y^2/2\right]^x dx \\
&= \int_0^1 \left(2x^3 + \frac{3}{2}x^3 + \frac{x^3}{3} - x^2 - \frac{x^2}{2}\right) dx \\
&= \left[\frac{x^4}{2} + \frac{3x^4}{8} + \frac{x^4}{12} - \frac{x^3}{3} - \frac{x^3}{6}\right]_0^1 = \frac{1}{2} + \frac{3}{8} + \frac{1}{12} - \frac{1}{3} - \frac{1}{6} \\
&= 11/24
\end{aligned}
$$

A $\int_0^1 \int_0^x [2xz + yz - z]_0^{x+y}\,dy\,dx$

$$6\int_1^2\int_x^{4/x}\int_0^{xy} xy\cdot dxdxdy$$

$$\int_1^2\int_x^{4/x}[xyz]_0^{xy}dxdy=\int_1^2\int_x^{6/x}x^2y^2dy\cdot dx$$

$$=\int_1^2\left[\frac{x^2y^3}{3}\right]_x^{4/x}dx$$

$$=\int_1^2\frac{x^2}{3}\left(\frac{64}{x^3}-x^3\right)dx=-4$$

$$=\int_1^2\left[\frac{64}{3}\cdot\frac{1}{x}-\frac{x^5}{3}\right]dx$$

$$=\left[\frac{64}{3}\ln(x)-\frac{x^6}{18}\right]_1^2=\frac{64}{3}\ln(2)-\frac{64}{18}-\frac{64}{3}\ln(1)+\frac{1}{18}$$

$$=\frac{64}{3}\ln(2)-\frac{63}{18}=\frac{64}{3}\ln(2)-\frac{7}{2}$$

7. $\int_0^4\int_0^{2\sqrt{2}}\int_0^{\sqrt{4x-x^2}}dy\cdot dxdy$

$$=\int_0^4\int_0^{2\sqrt{2}}\sqrt{4z-x^2}dy\cdot dxdx=$$

$$=\int_0^4\int\sqrt{2\sqrt{2}}\sqrt{(2\sqrt{z})^2-x^2}dx\cdot dz$$

$$=\int_0^4\left[\frac{x}{2}\sqrt{4z-x^2}+2z\sin^{-1}\left(\frac{x}{2\sqrt{2}}\right)\right]_0^{2\sqrt{2}}dz$$

$$=\int_0^4(\sqrt{2}\sqrt{4z-8}+\pi z)dx$$

$$=\int_0^4(2\sqrt{2}\sqrt{z-2}+\pi z)dz=$$

$$=\left[2\sqrt{2}\cdot\frac{2}{3}(z-2)^{3/2}+\frac{\pi z^2}{2}\right]_0^4=\frac{4}{3}\sqrt{2}2^{3/2}+8\pi-\frac{4\sqrt{2}}{2}(-2)^{3/2}$$

8. Calculate.

$\iint r^3drd\theta$ ever the area included between circles $\gamma_T=2\sin(\theta),\gamma_2=4\sin(\theta)$

A) $r_1=2\sin(\theta)\to\theta\in[0,\pi]$

$\therefore\int_0^\pi\int_{r=2\sin(t)}^{r=4\sin(\omega)}r^3dr.d\theta$

$$=\int_0^\pi\left[\frac{r^4}{4}\right]_{2\sin(\omega)}^{4\sin(\theta)}d\theta=\int_0^\pi\left(\frac{4^4}{4}\cdot\sin^4(\theta)-\frac{2^4}{4}\sin^4(\theta)\right)d\theta$$

$$:=\int_0^\pi 60\sin^4(\theta)d\theta=2\cdot 6\int_0^{\pi/2}\sin^4(\theta)d\theta$$

$$=2.60\cdot\left[\frac{4-1)(4-3)}{4(4-2)}\right]\frac{\pi}{2}=120\cdot\frac{3\pi}{4}=\frac{45}{2}\pi$$

$$= \int_0^{\pi} \left[\frac{\gamma^4}{4}\right]_{2\sin(\theta)}^{4\cos(\theta)} = d\theta = \int_0^{\pi} \left(64 \cdot \cos^4(\theta) - 4\sin^4(\theta)\right) d\theta$$

$$= \int_0^{\pi} \left[64\cos^4(\theta) - 4 \cdot \left(1 - \cos^2(\theta)\right)^2\right] d\theta$$

$$= \int_0^{\pi} \left(64 \cancel{\sin}^4(\theta) - 4 + 4\cos^4(\theta) - 8\cos^2(\theta)\right) d\theta$$

cos

$= \int_0^4 (68\cos^4(\theta) - 8\cos^2(\theta) - 4)\, d\theta$

Change of order

Here we find new limits by comparing actual limits $w\%$ using sketch.

- Change the ordor of integration:

$$\int_0^1 \int_0^x f(x,y)dy \cdot dx$$

4. Here limits of $y \geq \{[a \rightarrow$ is variable limit: $0 \leq y \leq x$

limit of x is constant: $0 \leq x \leq 1$

By changing the order if integration. limits of x become variable & that of y becomes constant

$\therefore$ limits of $x = y \leq x \leq 1$

$y : 0 \leq y \leq 1 <$

$\therefore$ cultered int:

$$\int_0^1 \int_y^1 f(x,y)dx \cdot dy$$

2.) Change order and eval:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dy \cdot dx := I$$

A)

$$y \in \left[0, \sqrt{1-x^2}\right], x \in [0,1]$$
$$\text{or } 0 \leq y \leq \sqrt{1-x^2},$$
$$0 \leq x \leq 1$$
$$\left.\sqrt{1-x^2}\right|_{0\rightarrow 1} \Rightarrow [\sqrt{1}, 0]$$
$$\therefore 0 \leq y \leq [1,0] \rightarrow 0 \leq y \leq 1$$
$$\left.y = \sqrt{1-x^2} \Rightarrow x = \sqrt{1-y^2}\right\} [1,0]$$

$$0 \leq x \leq \sqrt{1-y^2}$$

$I \equiv \int_0^1 \int_0^{\sqrt{1-y^2}} y^2 \cdot dx \cdot dy$
$= \int_0^1 y^2 \cdot \sqrt{1-y^2} dy.$
clet $y = \sin(u), \therefore dy = \cos(u)du, u = \sin^{-1}(y), \therefore [0,1] \mapsto [0, \pi/2]$
$\therefore I = \int^{\pi/2} \sin^2(u) \sqrt{1-\sin^2(u)} \cos(u) du$
$0\pi/2$

$$= \int_0^{\pi/2} \sin^2(u) \cos^2(u)\, du = \int_0 \sin^2(u)(1-\sin^2(u)) du$$

$= \int_0^{\pi/2} \sin^2(u) du - \int^{\pi/2} \sin^4(u) du$
$= \frac{1}{2} \times \pi/2 \to -\frac{3\times1}{4\times2} \times \frac{\pi}{2} = \pi/16$
3. $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dydx = I$

$0 \quad x^2/4a$

4) $\longrightarrow y \in \left[\frac{x^2}{4a}, 2\sqrt{ax}\right] \quad , \Rightarrow \quad \frac{x^2}{4a} \leq y \leq 2\sqrt{ax}$ $\quad \frac{x^4}{64a^3} \leq \frac{y^2}{4a} \leq x$
$\longrightarrow x \leq 2\sqrt{ay} \leq 2^{3/2}(ax)^{3/4}$

$x \in [0, 4a] \quad \Rightarrow \quad 0 \leq x \leq 4a \quad \longrightarrow \quad \frac{y^2}{4a} \leq x \leq 2\sqrt{ay}$

$$I \equiv \int_0^{4a} \left\{ \int_{y^2/4a}^{2\sqrt{ay}} dx \right\} dy \qquad \longrightarrow \quad 0 \leq y \leq 4a$$

$= \int_0^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy = \int_0^{4a} 2\sqrt{a}\sqrt{xy} dy - \int_0^{4a} \frac{y^2}{4a} dy$
$= 2\sqrt{a} \left[\frac{2y^{3/2}}{3}\right]_0^{4a} - \left[\frac{y^3}{12a}\right]_0^{4a} = \frac{4}{3}\sqrt{a} 4^{5/2} \cdot a^{5/2} - \frac{64a^3}{12a}$
$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$
$4 \int_0^a \int_y^a \frac{x}{x^2+y^2} dx \cdot dy$
$\rightarrow \quad 0 \leq 0 \leq\leq a.$
$y \leq x \leq a$
$I = \int_0^a \int_y^a \frac{x}{x^2+y^2} dx \cdot dy \qquad$ let $u = x^2 + y^2, \therefore \frac{dy}{dx} = 2x, dx = \frac{du}{2x}$
$= \int^a \int^{y^2+a^2} \frac{1}{2a} dy.dy. \qquad \therefore x \to [y, a] \longmapsto \underbrace{\left[2y^2, y^2 + a^2\right]}_{u-\text{ spaue}}$
$= \int_0^a \left[\frac{1}{2}\ln(u)\right]_{2y^2}^{y^2+a^2} dxy = \int_0^a \frac{1}{2} \ln\left(\frac{y^2+a^2}{2y^2}\right) dy \leftarrow$ by parts
$= \frac{1}{2} \int_0^9 \ln\left(\frac{y^2+a^2}{2y^2}\right) dy$
$\frac{d_4}{dy} = \frac{2y^2}{y^2+a^2} \times \frac{-2y^2 \cdot 2y - \left(y^2+a^2\right)4y}{4y^4} = \frac{-2a^2}{y^3+a^2y}$

$\therefore I = \frac{1}{2}\left\{\ln\left(\frac{y^2+a^2}{2y^2}\right) y - \int \frac{-2a^2 y}{y^3+a^2 y} dy\right]_0^a dy$
for integral: $\int \frac{-2a^2 y}{y^3+a^2 y} = \int \frac{-2a^2}{y^2+a^2} = -2a^2 \int \frac{1}{y^2+a^2}$
$= -\frac{2a^2}{a^2} \int \frac{1}{(y/a)^2+1} dy$
let $u = y/a. \therefore dy = du \cdot a$

$$\begin{aligned}
\rightarrow &= -2a \int \frac{dy}{u^2+1} = -92 \tan^{-1}(u) = -2\tan^{-1}\left(\frac{y}{a}\right) \\
&= \frac{1}{2}\left[\ln\left(\frac{y^2+a^2}{2y^2}\right) \cdot y + 2a \tan^{-1}\left(\frac{y}{a}\right)\right]_0^9 \\
&= \frac{1}{2} \ln(-1) \cdot a + d \cdot \tan^{-1}(1) \\
&= a \cdot \frac{\pi}{4}
\end{aligned}$$