2. Partial Differentiation

Partial Differentiation Let $z=f(x,y)$, be a function of x,y. Then partial derivative wet. x is denoted by: $\frac{\partial x}{\partial x}\cdot\frac{\partial z}{\partial x}$ It means differentiating $\approx z$ wit x while keeping y as constant. Similarly $\frac{\partial z}{\partial y}$ represents partial derivative of z wot y keeping x constant In all the cases:

$$\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\equiv\frac{\partial z^2}{\partial x\partial y}=\frac{\partial}{\partial y}\cdot\frac{\partial z}{\partial x}\equiv\frac{\partial z^2}{\partial yy\partial x}$$

1. Find $1^{17}dz^{\text{nd}}$ partial derivative of $z=x^3+y^3-3axy$ s.)

$$\frac{\partial z}{\partial x}=3x^2-3ay,\frac{\partial z^2}{\partial x^2}=6x,\quad\frac{\partial\cdot\partial z^2}{\partial x\partial y}=3a$$
$$\frac{\partial z}{\partial y}=3y^2-3ax,\frac{\partial z^2}{\partial y^2}=6y,\quad\frac{\partial z^2}{\partial y\partial x}=3a$$
$$*\text{ here: }\frac{\partial z^2}{\partial x\partial y}=\frac{\partial z^2}{\partial y\partial x}$$

2. If $v=\left(x^2+y^2+z^2\right)^{-1/2}$, find $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}$ A.

$$\frac{\partial^2 y}{\partial x^2}=\frac{\partial v}{\partial x}=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}\cdot 2x,\quad\frac{\partial x^2}{\partial/2}=-zx\cdot\left(x^2\right.$$
$$\frac{\partial v}{\partial y}=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}\cdot 2y$$
$$\frac{\partial v}{\partial z}=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}\cdot 2z$$

$\frac{\partial^2 t}{\partial x^2}=-\left(x^2+y^2+z^2\right)^{-3/2}+x\cdot\frac{3}{2}\left(x^2+y^2+z^2\right)^{5/2}\times 2x$

$=\frac{3x^2}{(x^2+y^2+z^2)^{5/2}}-\frac{1}{(x^2+y^2+z^2)^{3/2}}$

$\frac{\partial^2 v}{\partial y^2}=\frac{3y^2}{(x^2+y^2+z^2)^{5/2}}-\frac{1}{(x^2+y^2+z^2)^{3/2}}$

$\frac{\partial^2 v}{\partial z^2}=\frac{\partial z^2}{(x^2+y^2+z^2)^{5/2}}-\frac{1}{(x^2+y^2+z^2)^{3/2}}$

$\therefore\frac{\partial^2 x}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=\frac{3}{(x^2+y^2+z^2)^{7/}}\left(\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)}-1\right)$

- If $u=x^2\tan^{-1}(7x)-y^2\tan^{-1}\left(\frac{x}{y}\right)$ find $\frac{\partial^2 u}{\partial x\partial y}$ and

show that $u_{xy}=u_{yx}$

A) $\frac{\partial u}{\partial y}=x^2\cdot\frac{1}{1+(y/x)^2}\frac{1}{x}-2y\tan^{-1}\left(\frac{x}{y}\right)-y^2\cdot\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{-1}{y^2}$

$$=x^2\cdot\frac{x^2}{x^2+y^2}\frac{1}{x}+y^2\cdot\frac{y^2}{x^2+y^2}\frac{x}{y^2}-2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{x^3}{x^2+y^2}+\frac{xy^2}{x^2+y^2}-2y\tan^{-1}\left(\frac{x}{y}\right)=x-2y\tan^{-1}\tag{x}$$

$\frac{\partial u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(x-2y\tan^{-1}\left(\frac{x}{y}\right)\right)$

$=1-2y\cdot\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y}=1-\frac{2y^2}{x^2+y^2}=\frac{x^2-y^2}{x^2+y^2}$

$\frac{\partial u}{\partial x}=2x\cdot\tan^{-1}\left(\frac{y}{x}\right)+x^2\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\times\frac{y}{x^2}-y^2\cdot\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{y}$

$$=2x\tan^{-1}\left(\frac{y}{x}\right)-\frac{yx^2}{x^2+y^2}-\frac{y^3}{x^2+y^2}$$
$$=2x\tan^{-1}\left(\frac{y}{x}\right)-y$$

$\frac{\partial u}{\partial y \partial x} = 2x \frac{1}{1+y^2x^2} \frac{1}{x} \phi - 1$

$$= \frac{2xx^2}{x^2+y^2} - 1 \Rightarrow \frac{x^2-y^2}{x^2+y^2}$$

$\therefore \frac{\partial u}{\partial y \partial x} = \frac{\partial u}{\partial x \partial y}$

Total derivative / Differential Coeff.

let $u = f(x,y)$ is a function of x, y
let $x = \phi(t), y = \varphi(t)$

$$\text{which gives: } u = f(\phi(t), \varphi(t))$$

Hence u becomes function of ' t ' alone
Then ordinary derivative: $\frac{du}{dt}$ is known as total differential coeff/Total derivative
This total derivative can also be optained without Substitution
a $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$ chain rute
case 1: If $t = x$,

$$\therefore \quad \frac{du}{dt} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dt} \equiv \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

case 2: u is constant: $\frac{du}{dt} = 0$

$$\longrightarrow = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{dx} = -\frac{\partial u}{\partial x} \cdot / \frac{\partial}{\partial y}$$

- $u = x^3y^4z^2, x = t^2, y = t^3, z = t^4$. Find d,y/d) chasio rule & substitution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot -\frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$
$$= 3x^2y^4z^2 \times 2t + 4x^3y^3z^2 \times 3t^2 + 2x^3y^4z \times ut^3$$
$$= 6t \cdot x^2y^4z^2 + 12t^2x^3u^3z^2 + 8t^3x^3y^4z$$

$m \equiv 6t \cdot t^4 \cdot t^{12} \cdot t^8 + 12t^2 \cdot t^6 \cdot t^9 \cdot t^{88} + 8t^6 \cdot t^3 \cdot 7^2 \cdot t^4$

$$= 6t^{25} + 12t^{25} + 8t^{25} \Rightarrow$$

$$26t^{25}$$

By substitution:
$u \equiv t^6 \cdot t^{12} \cdot +t^8 = t^{26}$.
$\therefore \frac{du}{dt} = 26t^{25}$

- find $\frac{dy}{dx}$.if $ax^2 + 2hxy + by^2 = 1$

Find $\frac{dy}{dx}$ if $u=\sin\left(x^2+y^2\right)$, where $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

A)

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{\partial u}{\partial x} / \frac{\partial u}{\partial y} \\
&= -(2ax+2hy)/(2by+2hx) \\
&= -\frac{ax+by}{hx+by} \\
&= \cos\left(x^2+y^2\right)\left(2x+2y\cdot\frac{dy}{dx}\right).
\end{aligned}$$

$$\frac{dy}{dx} = \frac{1}{2b\sqrt{1-\frac{x^2}{a^2}}}\cdot b\left(0-\frac{2x}{a^2}\right) = -\frac{2x}{a\sqrt{a^2-x^2}}$$

$$\therefore \frac{du}{dx}=\cos\left(x^2+y^2\right)\cdot\left(2x-\frac{2xy}{a\sqrt{a^2-x^2}}\right)$$

$$2)\ \frac{du}{dx}=$$

Euler's Theorem

let $f(x,y)$ be a homogeneous function of degree ' n ' then "euler's theorem:

$$x\cdot\frac{\partial f}{\partial x}+y\cdot\frac{\partial f}{\partial y}=n\cdot f \quad \text{ or } \sum_{i\in y} i\frac{\partial f}{\partial i}=n\cdot f$$

Proof

$$\begin{aligned}
\text{Let } u &= x^n f(y/x) \\
\therefore \frac{\partial u}{\partial x} &= x^n f'(y/x)\cdot\frac{-y}{x^2}+nx^{n-1}f\left(\frac{y}{x}\right) \\
x\cdot\frac{\partial u}{\partial x} &= -x^{n-1}yf'\left(\frac{y}{x}\right)+nx^n f\left(\frac{y}{x}\right) \\
\frac{\partial u}{\partial y} &= x^n f'\left(\frac{y}{x}\right)\cdot\frac{1}{x}+0 \\
y\cdot\frac{\partial u}{\partial y} &= x^{n-1}f'\left(\frac{y}{x}\right) \\
\therefore x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y} &= nx^n f\left(\frac{y}{x}\right)-x^{n-1}yf'\left(\frac{y}{x}\right)+x^{n-1}yf'\left(\frac{y}{x}\right) \\
&= n\cdot u
\end{aligned}$$

Verity Euler's theorem if $t(x, y, x) = ax^2 + by^2 + cz^2 + 2fyz$
$$+ 2azx + 2bxy$$
$x \cdot \frac{\partial \phi x}{\partial x} = 2\left(2ax^2 + 2gz + 2hy\right) = 2ax^2 + 2gxz + 2bxy$
$y \cdot \frac{\partial f}{\partial y} = y(2by + 2fz) = 2by^2 + 2fyz$
$z \cdot \frac{\partial f}{\partial z} = 2(2cz + 2fy + 2gx) = 2cz^2 + 2fyz + 2gxz$
$$\therefore x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} + z \cdot \frac{\partial f}{\partial z} = 2ax^2 + 2by^2 + 2cz^2 + 34f_y z + 4yxz + 4hxy \quad = 2 \cdot 4$$
$$= 2(\ldots)$$
$\therefore$ Euler's theorem is verifred

2. If $u = \sin^{-1}\left((x + 2y + 3z) / \left(x^8 + y^8 + z^8\right)\right)$, find $x \cdot \frac{\partial y}{\partial x} + y \cdot \frac{\partial y}{\partial y} + z \cdot \frac{\partial y}{\partial z}$. $u = \sin^{-1}\left(\frac{x+2y+3z}{x^8+y^8+z^8}\right)$

This is not homogeneous functoon. To malce it homogeneous, take $\sin\theta$.
$\cos\sin(4) = \frac{x+2y+3x}{x^8+y^8+z^8} \equiv \frac{x \cdot \left(1 + \frac{2y}{x} + \frac{3z}{x}\right)}{x^8\left(1 + \left(\frac{y}{x}\right)^8 + \left(\frac{z}{3c^2}\right)\right.}$
$$\equiv \underbrace{x^{-7}}_{\text{degrec } = -7} \cdot f\left(\frac{yz}{x}\right)$$
By Euler's theorem: $x \cdot \frac{\partial \omega}{\partial x} + y \cdot \frac{\partial \omega}{\partial y} + z \cdot \frac{\partial \omega}{\partial x} = -7 \cdot \omega$
$\Rightarrow x \cdot \cos(\omega) \cdot \frac{\partial u}{\partial x} + y \cdot \cos(u) \frac{\partial u}{\partial y} + z \cdot \cos(u) \cdot \frac{\partial u}{\partial z} = -7 \cdot \sin(\omega)$
$\Rightarrow x \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z} = -7 \frac{\sin(u)}{\cos(u)} = -7\tan(u)$
$$= -7 \cdot \tan\left(\sin^{-1}\left(\frac{x + 2y + 3z}{x^3 + y^8 + z^8}\right)\right)$$
verify Euler's theorem, if $u = \sin^{-1}((x + y)/(\sqrt{x} + \sqrt{y}))$ Hence find $x\partial u/\partial x + 2g.\partial u/\partial y$
$$u = \sin^{-1}\left(\frac{x + y}{\sqrt{x} + \sqrt{y}}\right) \Rightarrow \sin(u) = \frac{x + y}{\sqrt{x} + \sqrt{y}} \equiv x^{v_2} \times \frac{1 + \frac{y}{x}}{1 + \sqrt{y}}$$
let $\varepsilon = \sin(u), \therefore$ By Euler's theorem
$x \cdot \frac{\partial \xi}{\partial x} + y \cdot \frac{\partial \varepsilon}{\partial y} + q \cdot \frac{\partial \xi}{\partial z} = \cos(u)\left[x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + \frac{\partial y}{\partial x}\right] = \frac{1}{2}\sin(\omega) \therefore x \cdot \frac{\partial y}{\partial x} + y \cdot \frac{\partial y}{\partial y} = \frac{1}{2}\tan(u) =$
$\frac{1}{2}\tan\left(\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)\right)$ $u = \sin^{-1}(\ldots) \Longrightarrow u = \sin^{-1}\Bigg(\sqrt{x} - \sqrt{y} \quad \frac{1}{\sqrt{1-y^2}}$

$$\frac{\partial u}{\partial x} = \sqrt{1 - \frac{x+y}{\sqrt{x}+\sqrt{y}}}^{\,-1} \cdot$$

Kf Zet $=$

$$\begin{aligned}
&\text{If } u=\log\left[\left(x^4+y^4\right)/(x+y)\right], \text{ find } x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}\\
&\text{let } x=e^u=\frac{x^4+y^4}{x+y}\equiv x^3\cdot\frac{1+\left(\frac{y}{x}\right)^4}{1+y/x}\quad\therefore\ \text{deg:3}\\
&\therefore\ \text{By Euleriss theorem: } x\cdot\frac{\partial x}{\partial x}+y\cdot\frac{\partial x}{\partial y}=3\cdot x\\
&=x\cdot e^4\cdot\frac{\partial u}{\partial x}+y\cdot e^4\cdot\frac{\partial u}{\partial y}=3\cdot e^u\\
&\Rightarrow x\cdot e\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=3
\end{aligned}$$

Errors & approximations

If δx and δy represents small increments is x and y then our new function is:

$$f(x+\delta x, y+\delta y)$$

Hence expanding $f(x+\delta x\neq y+\delta y)$ by Taylor's series by supp supposing δx and δy be small, so othat their products, squares and higher powers can be neglected then, the error in the function clenoted os δf is $\delta f=\frac{\partial f}{\partial x}\delta x+\frac{\partial f}{\partial y}\delta y$.

*: If δy is error made in calculating y, the relative error in y represented by:
$\frac{\delta y}{y}$
percentage error in y : $\frac{\delta y}{y}\times 100$

Radius of a sphere is found to be 10 cm , with a possible error of 0.2 cm , what is the relative error in the computed volume

A) $r=10$ cm$, \delta r=0.2$
Volume (v) $=\frac{4}{3}\pi r^3$
error in computing volume: $\delta V=\frac{\partial V}{\partial F}\cdot\delta\gamma$

$$\begin{aligned}
&=4\pi r^2.8r\\
&=4\pi\times 10^2\times 0.2\\
&=251.3\ \text{cm}^3
\end{aligned}$$

volume $(r=40), v=\frac{4}{3}\pi r^3=4188.8$ cm^3
$\therefore$ Be
$\therefore$ Relotive error: $\frac{\delta V}{V}=\frac{251.3}{4188.8}=0.060$

2. The diameter and altitude of a can in the spope of a right circular cylinders measured as 4 cm&6 cm respectively. The possible error in eoch measurement is 0.1 cm find approximately the maximum possible errov in the volue computed for volume and loteral surface area

$$
\begin{aligned}
&\text{volume } = \pi\left(\frac{D}{2}\right)^2 \cdot h = V = \frac{\pi}{4}D^2 \cdot h \\
&\therefore \partial\delta v = \frac{\partial v}{\partial D}\delta D + \frac{\partial V}{\partial h} \cdot \delta h \quad \text{(approx.)} \\
&= \frac{\pi}{2}D \cdot L \cdot \delta D + \frac{\pi}{4}D^2 \cdot \delta h \\
&= \frac{\pi}{20}\left(4 \times 6 + 4^2\right) = 2\pi \\
&LSA = \pi DK, 2 : S \\
&\therefore \delta S = \frac{\partial S}{\partial D} \cdot \delta D + \frac{\partial S}{\partial h} \cdot \delta h \quad \text{(approx)} \\
&= \pi \times 0.1(h+0) \\
&= \pi \text{cm}^2 \text{ (approx).}
\end{aligned}
$$

3. The focal length of mirror is given by the formula:

$$\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$$

If equal errors k are made in the determination of u and v, sow that the relative error in the f is given by: $k \cdot \left(\frac{1}{v} + \frac{1}{u}\right)$

$$\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$$

$$\therefore f = 2 \cdot \frac{uv}{u-v} \equiv 2\left(\frac{1}{v} - \frac{1}{u}\right)^{-1}$$

$$\delta f = \frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial v} \cdot \delta v$$

$$= k\left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right)$$

$$= 2k \cdot \left(-\left(\frac{1}{v} - \frac{1}{u}\right)^{-2}\left(+\frac{1}{u^2}\right) \quad -\left(\frac{1}{v} - \frac{1}{4}\right)^{2^2}\left(-\frac{1}{v^2}\right)\right)$$

$$\therefore \frac{\delta f}{f} = \frac{1}{v}\left(-\left(\frac{1}{v} - \frac{1}{u}\right)^{-1} \cdot \frac{1}{u^2} + \left(\frac{1}{v} - \frac{1}{u}\right)^{-1}\frac{1}{v^2}\right)$$

$$= \sin\left(-\frac{u^v}{u^2} \cdot \frac{1}{u^2} + \frac{uv}{u-v} \cdot \frac{1}{v^2}\right)$$

$$= \left\{\frac{u^2 - v^2}{(uv)^2} \cdot \frac{uv}{u-v}\right\}$$

$$= \text{ k. } \{ \frac{u+v}{uv}\}$$

$$\frac{\delta f}{\delta} = b_r k \cdot \left(\frac{1}{u} + \frac{1}{v}\right)$$

$$\frac{2}{5} = \frac{1}{2} - \frac{1}{4}$$

$$-\frac{2}{f^2}\delta f = -\frac{1}{v^2}\delta v + \frac{1}{u^2}\delta u$$

4. If the I HP required to propel a steamer varies os the cube of the velocity and square of its length, prove that a 3% increase in velocity and 5% increase in length will require, an increase of about 17% in HP. A)

$$H = k \cdot v^3 \cdot v^2.$$

$$\delta v A = 0.03 \delta 2/2 = 0.04$$

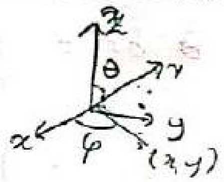
$$\frac{\delta H}{H} = \frac{k \cdot 3v^2 \cdot 2^2 \delta v + k \cdot 2v^3 2\delta 2}{k \cdot v^3 \eta^2} \quad 3\frac{\delta v}{V} + 2\frac{\delta 2}{82} = 3 \times 0.03 + 2 \times 0.4 = 0 \Leftrightarrow$$

Coordinate Systems $G(x, y, z) \mapsto S(x, \theta, y), \gamma = \sqrt{x^2 + y^2 + z^2}$

$x = r\sin(\theta)\cos(\varphi)$ $\qquad \theta = \cos^{-1}\left(\frac{z}{r}\right), \quad \varphi = \mathrm{sgn}(y)\cos^{-1}\frac{x}{\sqrt{x^2+y^2}}$

$y = r\sin(\theta)\sin(\varphi) \qquad \varphi \sim \phi$

$z = r\cos(\theta)$



1. Relation b/w cortesion and polar co-ordinates:

$(x, y) \mapsto (r\cos(\theta), r\sin(\theta))$, where $\gamma = \sqrt{x^2+y^2}$
$\theta = \tan^{-1}$
Here (x, y) changed to (r, θ)

$$\text{or } J\left(\frac{x, y}{r, \theta}\right) = \gamma$$

2. b/w cortesian and cylindrical co-ordinates:

$$(x, y, z) \longmapsto (r\cos(\theta), r\sin(\theta), z)$$

Here (x, y, z) changed to (r, θ, z)

$$\ldots J\left(\frac{x, y, z}{r, \theta, z}\right) = \gamma$$

3. Cartesian & spherical coordinates: $[(x, y, z) :\rightarrow (x, \phi, \phi)]$

$(x, y, z) \longmapsto (r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta))$ $J\left(\frac{x,y,z}{r,\theta,\phi}\right) = r^2 \cdot \sin(\theta)$
Convert $x^2 + y^2 = 4x$ into polor curve
$x \mapsto r\cos(\theta), y \mapsto r\sin(\theta)$

$$\gamma^2\left(\cos^2(\theta) + \sin^2(\theta)\right) = 4r\cos(\theta)$$

$$\text{or } r = 4\cos(\theta)$$

Express the point: $(x, y, z) = (1, -\sqrt{3}, 2)$ in cylindrical corrdinates

$$(x, y, z) \longmapsto (r\cos(\theta) \sim \sin(\theta), z)$$
$$\gamma = \sqrt{x^2+y^2} = \sqrt{1+3} = 2$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(-\sqrt{3}) = -60^\circ$$
$$(x, y, z) \equiv (r, \theta, z) \equiv (2, -60^\circ, 2) \equiv \left(2, -\frac{\pi}{3}, 2\right)$$

Transform equation in spberical coordinates,

$$x^2 + y^2 + z^2 = 2z^2$$

$\gamma^2 \sin^2(\theta) \cos^2(\varphi)$

$$= (\Delta) \sin^2(\phi) + \gamma^2 \cos^2(\phi) = 2\gamma^2 \cos^2(\phi)$$

$$\underbrace{\sin^2(\theta)\left(\underbrace{\cos^2(\phi) + \sin^2(\phi)}_{1}\right) + \cos^2(\theta)}_{1} = 2\cos^2(\theta)$$

$$2\cos^2(\theta) = 1$$

$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$
$\theta = 45°$

Jacubian

If u, v are functions of $x \& y$ then $\partial \frac{(u,v)}{(x,y)}$ or $J\left(\frac{u,u}{x,y}\right)$ is known as Jacobian

$$\partial^x \left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \partial \frac{(u, v)}{(x, y)} = \frac{\partial(u, v)}{\partial(x, y)}$$

If Jacobiar J is $\partial\left(\frac{a,v}{x,y}\right)$, then

$$J' = 2\left(\frac{x,y}{u,v}\right)$$
$$J' = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

and vice-versa
In all coses. $J \cdot J' = 1$

Functionally Dependent

functions u, u are said to be functionally dependent, then $J = 0$. And it is functionally independent if $J \neq$ if $x =$ u.v and $y = 4/v$, then prove that $JJ' = 1$ $x = uv, y = \frac{u}{v}$

$\therefore u = \sqrt{xy}, v = \sqrt{\frac{x}{y}}$

$\therefore J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{y}}{2\sqrt{x}} & \frac{\sqrt{x}}{2\sqrt{y}} \\ \frac{1}{2\sqrt{xy}} & \frac{-\sqrt{x}}{2y\sqrt{y}} \end{vmatrix}$

$\therefore J = -\frac{1}{4y} - \frac{1}{4y} = -\frac{1}{2y}$

$$\therefore J' = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = -\frac{u}{v} - \frac{u}{v} = -2\frac{u}{v}$$

$$= -2\frac{\sqrt{xy}}{\sqrt{x/y}}$$

$$= -2 \cdot \sqrt{x} \cdot \sqrt{y} \cdot \frac{\sqrt{y}}{\sqrt{x}}$$

$$\therefore J \cdot J' = -\frac{1}{2y}x - 2y = 1$$

- If $rt - s^2 < 0.$ neither minimum nor maximum exists - If $rt - s^2 = 0$, Further verification is needed.

Langrangis methud

For finding max/min of 3 or more variables find max /min of f, wrtconitruint Working rule

- let $F = f(x, y, z) + \lambda\dot{\phi}(x, y, z)$
- Simplifying equations: $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$ we get $\times$ rel.blw. x, y, z and substituting in any of equation, we get their values

Examine following functions for extreme value
a) $f(x, y) = x^4 + y^4 + 4xy - 2x^2 - 2y^2$
$f(x, y) \to \lambda\beta(x, y)$
A)
$\frac{\partial f}{\partial x} = 4x^3 + 4y - 4x, \quad \frac{\partial f}{\partial y} = 4y^3 + 4x - 4y^2$
let $\frac{\partial f}{\partial x} = 0 \Rightarrow 4x^3 + 4y - 4x = 0 \Rightarrow y = x - x^3$-1 $\frac{\partial f}{\partial y} = 0 \Rightarrow 4y^3 + 4x - 4y = 0 \Rightarrow x = y - y^3$-(2)
let $r = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 4$$

$t = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$
$\therefore 7 + -5^2 = (12x^2 - 4)(12y^2 - 4) - 16$
$(1) + (2): \quad -y^3 - x^3 + x + y = x + y$

$$\Rightarrow x^3 + y^3 = 0 \quad \Rightarrow \quad x = -y$$

Substiture these in ω :

$$-x = x - x^3 \Rightarrow x^2 = 2x \Rightarrow x = +\sqrt{2}$$

substitute in (2):

$$-y = y - y^3 \Rightarrow y = \pm\sqrt{2}$$

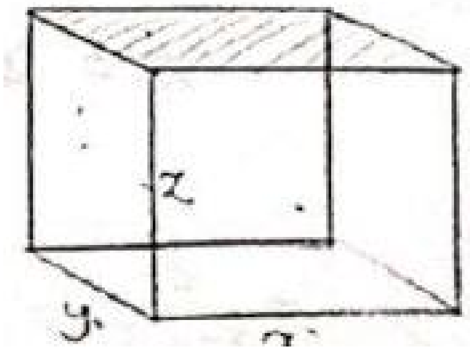
for $x = \pm\sqrt{2}, y = \pm\sqrt{2}, \quad \gamma t - s^2 = 384$
to
$\therefore r + -5^2 > 0$

$r = 20 > 0$
$\therefore$ minima exists at $x = \pm\sqrt{2}, y = \pm\sqrt{2}$
$\therefore$ minimum value is $\quad +(\pm\sqrt{2}, \pm\sqrt{2}) = 8 - 8$

b) $f\left(x \in x^5 - 5x^4 + 5x^3 - 1\right.$

2. A rectangular box open at the top is to have the volume 32ft^3. find dimensions of box requiring least amount of material for its construd $v = xyz = 32\text{ft}^3$

$LSA := A = xy + 2xz + 2yz$



tet $E = v$

let $f = xyz - 32$

$F = f + \lambda A$

$= xyz - 32 + \lambda(xy + 2yz + 2xz)$

$\frac{\partial F}{\partial x} = 0 \Rightarrow yz + \lambda(y + 2z) = 0 - 10 \frac{\partial F}{\partial z} = 0 \Rightarrow xy + \lambda(2y + 2x) = 0$
$\frac{\partial F}{\partial y} = 0 \Rightarrow xz + \lambda(x + 2z) = 0 - 2$
#
x. (1) $x-$ (2) y ::
$\Rightarrow 2 \times 2(x - y) = 0 \quad \Rightarrow \longrightarrow \underset{\rightarrow y=0}{\rightarrow} y$
(2). y-(9) z
$\Rightarrow \lambda(xy + zzzy - 2yz - zz) = 0$

$x\lambda(y - 2z) = 0 \;\begin{cases} \rightarrow x = 0 \\ \rightarrow y = +2z \end{cases}$

$\therefore x = y, y = 2z, z = y/2$

Substitute in $x_y z = 32$

$$\therefore 32 = y \cdot y \cdot y/2 \Rightarrow y = 64 \Rightarrow y = 24$$
$$32 = x \cdot x : x/2 \Rightarrow x = 4$$
$$32 = 2z \cdot 2z : z \Rightarrow 2 = 2$$

3 find max & min of points, $A(3,4,12)$ from the sphere $x^2+y^2+z^2=1$
A) Llet $P=(a,b,c)$ on point closest tuA. $\therefore P=(x3,\lambda 4,>12)$

$\therefore |P| = 1 \Rightarrow 9\lambda^2 + 16\lambda^2 + 144\lambda^2 = 1 \Rightarrow 169\lambda^2 = 1 \Rightarrow \lambda = 1/13$
$\therefore = \left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$
$\therefore$ min. distance $= \sqrt{\left(3-\frac{3}{13}\right)^2 + \left(4-\frac{4}{13}\right)^2 + \left(12-\frac{12}{13}\right)^2}$

$$= \underline{\underline{12}}$$

$\therefore$ max distance on is to point on opposite side $= 12+2 = 1$ sunnts
let p be a point on sphore $P=(x,y,z)$
$\therefore \operatorname{dist}(A,P)^2 = (3-x)^2 + (4-y)^2 + (12-2)^2 = D \ \frac{\partial F}{\partial y} = 0 \Rightarrow$
$\partial y = 0 \Rightarrow -2(4-y) + 2xy = 0 \Rightarrow y = \frac{4}{\lambda - 1}$
$\frac{\partial F}{\partial z} = 0 \Rightarrow -2(12-2) + 2z = 0 \Rightarrow z = \frac{12}{\lambda-1}$

$$\therefore \quad x^2 + y^2 + z^2 = 1$$

$$\Rightarrow \frac{1}{(\lambda^2-1)^2}(9+16+144) = 1$$

$\lambda - 1 = \pm 13 \Rightarrow \lambda = -12$ or 19

for $\lambda = 12$
$x = \frac{-3}{-13}, y = \frac{4}{-13}, z = \frac{12}{-13}$
$\lambda = 14 \quad x = \frac{3}{13}, y = \frac{4}{13}, z = \frac{12}{13}$
let $P = \left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right), Q = \left(\frac{-3}{13}, -\frac{4}{13}, -\frac{12}{13}\right)$

$$\therefore AP = \sqrt{\left(3-\frac{3}{13}\right)^2 + \left(4-\frac{4}{13}\right)^2 + \left(\frac{12}{13}-12\right)^2} = 12$$

$$AQ = \sqrt{\left(3+\frac{3}{13}\right)^2 + \left(4+\frac{4}{13}\right)^2 + \left(12+\frac{12}{13}\right)^2} = 14$$

$\therefore$ minimum distance $= 12$, max $= 14$

4. Find shortest distance from the origin to byporbolu

$x^2+8xy+7y^2=225$
A) let $P(x,y)$ be a point on byperbola.
distance from $0 \rightarrow P=D=\sqrt{x^2+y^2}$

$$D^2=x^2+y^2$$

let $F=D^2+\lambda\left(x^2+8xy+7y^2-225\right)$.

$$\begin{aligned}
& \frac{\partial F}{\partial x}=2x+\lambda(2x+8y)=0-(1) \\
& \frac{\partial F}{\partial y}=2y+\lambda(14y+8x)=0\text{-}(2)
\end{aligned}$$

let $F=D+x\left(x^2+y^2+z^2-1\right) \therefore \frac{\partial F}{\partial x}=0 \Rightarrow -2(3-x)+xx=0 \Rightarrow x=\frac{6}{2\lambda-2}=\frac{3}{\lambda-1}$

$$\begin{array}{ll}
x+\lambda(x+4y)=0-11 & \Rightarrow(1+\lambda)x+4\lambda y=0 \\
y+\lambda(7y+4x)=0-(2) & \Rightarrow(1+7\lambda)y+4\lambda x=0
\end{array}$$

(3) xy

$$\begin{aligned}
& (1+\lambda)xy+4\lambda y^2=0 \quad-(0) \quad (6)\ (7): \\
& \ (4)\ \times x \cdot(1+7\lambda)xy+4\lambda x^2=0-(6) \\
& (1+\lambda-1-7\lambda)xy+4\lambda\left(y^2-x^2\right)=0 \\
& (-6\lambda)xy+4\lambda\left(y^2-x^2\right)=0
\end{aligned}$$