That Weird Trigonometry

A list of some weird but seemingly useful trigonometric identities & charts

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1 Angle conversions

- Degree to radian: $x^\circ = \frac{\pi x}{180}$ rad
- Radian to degree: x rad $= \left(\frac{180x}{\pi}\right)^\circ$

2 Basic Identities

- $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{1}{\cot(\theta)}$
- $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{1}{\tan(\theta)}$
- $\sec(\theta) = \frac{1}{\cos(\theta)}$
- $\text{cosec}(\theta) = \csc(\theta) = \frac{1}{\sin(\theta)}$

3 Trigonometric table

Degree	0°	30°	45°	60°	90°	180°	270°	360°
Radian	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined	0	Not defined	0
cosec	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	Not defined	-1	Not defined
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined	-1	Not defined	1
cot	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	Not defined	0	Not defined

4 Pythagorian Relations

- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $\sec^2(\theta) - \tan^2(\theta) = 1$
- $\csc^2(\theta) - \cot^2(\theta) = 1$

5 Sign of trigonometric functions for negative angles

- $\sin(-x) = -\sin(x)$
- $\cos(-x) = \cos(x)$
- $\tan(-x) = -\tan(x)$
- $\csc(-x) = -\csc(x)$
- $\sec(-x) = \sec(x)$
- $\cot(-x) = -\cot(x)$

6 Expansion for trigonometric functions with two angles

- $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
- $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
- $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
- $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
- $\tan(x+y) = \dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$
- $\tan(x-y) = \dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$
- $\cot(x+y) = \dfrac{\cot(x)\cot(y)-1}{\cot(x)+\cot(y)}$
- $\cot(x-y) = \dfrac{\cot(x)\cot(y)+1}{\cot(y)-\cot(x)}$

$$\boxed{\begin{array}{l} c(x+y) = cc - ss \\ c(x-y) = cc + ss \\ s(x+y) = sc + cs \\ s(x-y) = sc - cs \end{array}}^{*}$$

- **With π**
  1. $\sin\left(\frac{\pi}{2}+\theta\right) = \cos(\theta)$
  2. $\sin\left(\frac{\pi}{2}-\theta\right) = \cos(\theta)$
  3. $\sin(\pi-\theta) = \sin(\theta)$
  4. $\sin(\pi+\theta) = -\sin(\theta)$
  5. $\sin(2\pi-\theta) = -\sin(\theta)$
  6. $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin(\theta)$
  7. $\cos\left(\frac{\pi}{2}-\theta\right) = \sin(\theta)$
  8. $\cos(\pi-\theta) = -\cos(\theta)$
  9. $\cos(\pi+\theta) = -\cos(\theta)$
  10. $\cos(2\pi-\theta) = \cos(\theta)$

7 Product formula

- $\sin(x+y) + \sin(x-y) = 2\sin(x)\cos(y)$
- $\sin(x+y) - \sin(x-y) = 2\cos(x)\sin(y)$
- $\cos(x+y) + \cos(x-y) = 2\cos(x)\cos(y)$
- $\cos(x+y) - \cos(x-y) = -2\sin(x)\sin(y)$

$$\boxed{\begin{array}{l} s+s = 2sc \\ s-s = 2cs \\ c+c = 2cc \\ c-c = -2ss \end{array}}^{*}$$

8 Sum formula

- $\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$
- $\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$
- $\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$
- $\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

*

$s + s = 2sc$

$s - s = 2cs$

$c + c = 2cc$

$c - c = -2ss$

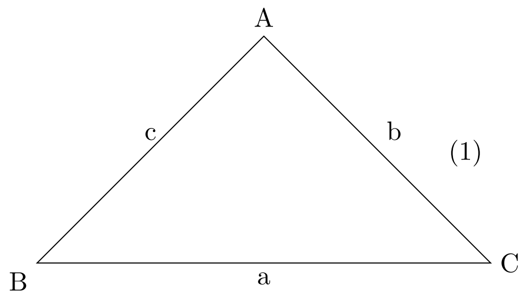
9 Expansion for multiple angles

- $\sin(2x) \;=\; 2\sin(x)\cos(x) \;=\; \frac{2\tan(x)}{1+tan^2(x)}$
- $\sin(3x) = 3\sin(x) - 4\sin^3(x)$
- $\cos(2x) \;=\; \cos^2(x) - \sin^2(x) \;=\; 2\cos^2(x) - 1 \;=\; 1 - 2\sin^2(x) \;=\; \frac{1-\tan^2(x)}{1+\tan^2(x)}$
- $\cos(3x) = 4\cos^3(x) - 3\cos(x)$
- $\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$
- $\tan(3x) = \frac{3\tan(x) - \tan^3(x)}{1-3\tan^2(x)}$
- $\sin^2(x) = \frac{1-\cos(2x)}{2}$
- $\cos^2(x) = \frac{1+\sin(2x)}{2}$

10 Law of sines

In any triangle, sides are proportional to to the sins of the angles

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \qquad or \qquad \frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} \tag{1}$$

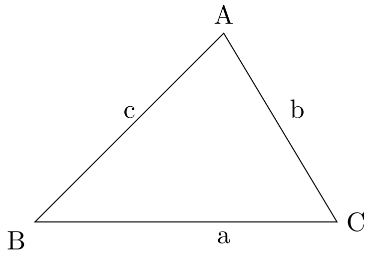


From above equations we also get:

- $\frac{a-b}{c} = \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{C}{2}\right)}$
- $\sin\left(\frac{B-C}{2}\right) = \frac{b-c}{c}\cos\left(\frac{A}{2}\right)$
- $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\left(\frac{C}{2}\right)$

11 Law of cosines

For any ΔABC



$$a^2 = b^2 + c^2 - bc\cos(A)$$
$$b^2 = a^2 + c^2 - ac\cos(B)$$
$$c^2 = a^2 + b^2 - ab\cos(C)$$

also

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos(B) = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$$

From above equations, we also get:

$$\frac{\cos(A)}{a} = \frac{\cos(B)}{b} = \frac{\cos(C)}{c} = \frac{a^2 + b^2 + c^2}{2abc}$$

12 Solutions of some trigonometric equations

- $\sin(x) = 0 \implies x = n\pi,\ n \in \mathbb{Z}$
- $\cos(x) = 0 \implies x = (2n+1)\dfrac{\pi}{2},\ n \in \mathbb{Z}$
- $\sin(x) = \sin(y) \implies x = n\pi + (-1)^n y,\ n \in \mathbb{Z}$
- $\cos(x) = \cos(y) \implies x = n\pi \pm y,\ n \in \mathbb{Z}$
- $\tan(x) = \tan(y) \implies x = n\pi + y,\ n \in \mathbb{Z}$
- $\tan(x) = 0 \implies x = n\pi,\ n \in \mathbb{Z}$

13 Other useful stuffs...

These are results obtained from equations above.

- $\cos(2n\pi + x) = \cos(x),\ \ n \in \mathbb{Z}$
- $\sin(2n\pi + x) = \sin(x),\ \ n \in \mathbb{Z}$
- $\tan\left(\dfrac{\pi}{4} - x\right) = \dfrac{1 - \tan(x)}{1 + \tan(x)}$
- $\tan\left(\dfrac{\pi}{4} + x\right) = \dfrac{1 + \tan(x)}{1 - \tan(x)}$
- $\dfrac{\tan\left(\dfrac{\pi}{4} + x\right)}{\tan\left(\dfrac{\pi}{4} - x\right)} = \left[\dfrac{1 + \tan(x)}{1 - \tan(x)}\right]^2$
- $\sin(x+y)\sin(x-y) = \sin^2(x) - \sin^2(y)$
- $\cos(x+y)\cos(x-y) = \cos^2(x) - \sin^2(y)$
- $\tan(3x)\tan(2x)\tan(x) = \tan(3x) - \tan(2x) - \tan(x)$
- $\tan(4x) = \dfrac{4\tan(x)(1 - \tan^2(x))}{1 - 6\tan^2(x) + \tan^4(x)}$

14 Some *Condensed* trigonometric identities

- $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$
- $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$
- $\tan(x \pm y) = \dfrac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}$
- $\cot(x \pm y) = \dfrac{\cot(x)\cot(y) \mp 1}{\cot(x) \pm \cot(y)}$
- $\tan\left(\dfrac{\pi}{4} \pm x\right) = \dfrac{1 \pm \tan(x)}{1 \mp \tan(x)}$

- *: Table in teal shows some mnemonics to remember the formulae easily