Module - 1

Ordinary Differential Equations (ODE)

Differential equation (DE)

An equation involving derivative(s) of dependent variable with respect to independent variable(s) is called differential equation.

e.g: $x\dfrac{dy}{dx} + y = 0$

- **Order of DE**: Order of highest order derivative of dependent variable, with respect to independent variable involved in the equation.
- **Degree of DE**: If differential equation is a polynomial equation of derivative, the highest power (positive integral index) of highest order derivative is its degree.

ODE

A differential equation involving derivortives of dependent variable with respect to only one independent variable is called ODE

e.g:

$$3\left(\frac{d^2y}{dx^2}\right)^4 + \left(\frac{dy}{dx}\right)^5 = 0$$

This has degree 4 and order 2

1^{st} order differential equation

General form : $f\,(x, y, y') = 0$

or $y' = g(x, y)$

e.g: $y^2 = \sqrt{x^2 + y^2}$

1^{st} Order Linear Differential equation

General form:

$y' + P(x)y = Q(x)$ where functions $P\&Q$ are continuous in some interval `I`

General solution:

$$yF = \int FQdx + C$$

$$\text{where F} = e^{\int Pdx}$$

F is also known as **Integration factor**

1 solve $y' + 2xy = x$

Answer
Given equation is an 1st order LDE comparing the general form: $y' + P(x)y = Q(x)$ we get: $P = 2x, Q = x$

$\therefore$ Integrating factor $F = e^{\int P \cdot dx} = e^{\int 2x \cdot dx} = e^{x^2}$

$$\therefore \text{Solution: } y \cdot F = \int FQdx + C$$

$$ye^{x^2} = \int e^{x^2} x \; dx + C$$

let $t = x^2$

$$\therefore 2x = \frac{dt}{dx} \quad \text{or} \quad xdx = \frac{1}{2}dt$$

$$\therefore \int e^{x^2} x \; dx = \int \frac{1}{2} e^t dt$$

$$= \frac{1}{2} e^t = \frac{1}{2} e^{x^2}$$

$$\therefore ye^{x^2} = \frac{1}{2} e^{x^2} + C$$

$$\therefore y = e^{-x^2} \left(\frac{1}{2} e^{x^2} + C \right)$$

$$\boxed{y = \frac{1}{2} + e^{-x^2} C}$$

2 Solve $\dfrac{dy}{dx} + 2y\tan(x) = \sin(x)$

Answer Here $P(x) = 2\tan(x), Q(x) = \sin(x)$

$$
\begin{aligned}
\text{Integrating factor (IF) } F &= e^{\int P\cdot dx}\\
&= e^{\int 2\tan(x)dx}\\
&= e^{2\cdot \ln|\sec(x)|} \qquad\qquad \boxed{\int \tan(x)dx = \ln|\sec(x)|}\\
&= \left(e^{\cdot \ln|\sec(x)|}\right)^2\\
&= |\sec(x)|^2 \equiv \sec^2(x)
\end{aligned}
$$

$\therefore$ Solution:

$$
\begin{aligned}
y\cdot\sec^2(x) &= \int \sin(x)\cdot\sec^2(x)dx + C\\
&= \int \cdot\sec(x)\tan(x)dx + C\\
&= \sec(x) + C\\
\therefore y &= \frac{\sec(x)}{\sec^2(x)} + \frac{C}{\sec^2(x)}\\
y &= \cos(x) + \cos^2(x)\cdot C
\end{aligned}
$$

3 Find solution of initial value problem:

$$
x^2y' - xy = x^4\cos(2x), \qquad\qquad y(\pi) = 2\pi
$$

Answer

$$
\begin{aligned}
x^2y' - xy &= x^4\cos(2x)\\
\therefore y'_{-} - x^{-1}y &= x^2\cos(2x)
\end{aligned}
$$

$$P(x) = -x^{-1}, Q(x) = x^2\cos(2x)$$

$$\therefore I.F = e^{\int p(x)\cdot dx} = e^{-\int x^{-1}dx} = e^{-\ln(x)} = x^{-1} = \frac{1}{x}$$

General Solution:

$$
\begin{aligned}
y\cdot IF &= \int Q\cdot IF\cdot dx + C\\
y\cdot\frac{1}{x} &= \int x^2\frac{1}{x}\cos(2x)\cdot dx + C\\
&= \int x\cos(2x)dx + C\\
&= \frac{x\sin(2x)}{2} - \frac{1}{2}\int \sin(2x)dx + C\\
&= \frac{x\sin(2x)}{2} + \frac{\cos(2x)}{4} + C\\
\therefore y &= \frac{x^2\cdot\sin(2x)}{2} + \frac{x\cos(2x)}{4} + xC \qquad (1)
\end{aligned}
$$

It is given that $y(\pi) = 2\pi$

$$\begin{aligned}
\therefore 2\pi &= \frac{\pi^2 \cdot \overbrace{\sin(2\pi)}^{0}}{2} + \frac{\pi \cdot \cos(2\pi)}{4} + \pi \cdot C \\
\therefore 2 &= \frac{1}{4} + c \\
&\Rightarrow c = 2 - \frac{1}{4} = \frac{7}{4}
\end{aligned}$$

Substitute in Eq. (1) to get specific solution:

$$\boxed{y = \frac{x^2 \sin(2x)}{2} + \frac{x \cos(2x)}{4} + \frac{7}{4} x}$$

4 Solve $y' - 2xy = 2$

Answer

Let $P(x) = -2x, Q(x) = 2x$

$I.F = e^{\int -2x \cdot dx} = e^{-x^2}$

$\therefore$ Solution: $y \cdot e^{-x^2} = \int 2x \cdot e^{-x^2} dx + c$
Let $-x^2 = t \Rightarrow dx = \dfrac{dt}{-2x}$

$$\begin{aligned}
\therefore \int 2xe^{-x^2} dx + C = - \int e^t dt &= -e^t + C \\
&= -e^{-x^2} + C
\end{aligned}$$

$$\begin{aligned}
\therefore y \cdot e^{-x^2} &= -e^{-x^2} \cdot C \\
\therefore y &= -1 + e^{x^2} \cdot C
\end{aligned}$$

5 Solve $xy' - 2y = -x$

Answer

$$\begin{aligned}
xy' - 2y &= -x \\
\therefore y' - \frac{2}{x} \cdot y &= -1
\end{aligned}$$

$\therefore P(x) = -\frac{2}{x}, Q(x) = -1$
IF $= e^{\int Px} = e^{\int -2/x \cdot dx} = e^{-2\ln(x)} = x^{-2}$
$\therefore$ Solution:

$$
\begin{aligned}
y \cdot x^{-2} &= \int x^{-2} \times (-1) \cdot dx + C \\
&= \frac{x^{-1}}{-1} \times -1 + C \\
&= \frac{1}{x} + C
\end{aligned}
$$

$$\therefore \boxed{y = x + Cx^2}$$

6 Solve $xy' + 2y = \frac{\cos(x)}{x}$

<u>Answer</u>

$$
\begin{aligned}
xy' + 2y &= \frac{\cos(x)}{x} \\
y' + \frac{2}{x} \cdot y &= \frac{\cos(x)}{x^2}
\end{aligned}
$$

$$P(x) = \frac{2}{x}, Q(x) = \frac{\cos(x)}{x^2},\ \text{IF } = e^{\int P \cdot dx} = e^{\int 2/x \ dx} = e^{\ln\left(x^2\right)} = x^2$$

Solution: $y \cdot IF = \int IF \cdot Q \cdot dx + C$

$$
\begin{aligned}
y \cdot x^2 &= \int x^2 \cdot \frac{\cos(x)}{x^2} \cdot dx + C \\
&= \int \cos(x) dx + C \\
y \cdot x^2 &= \sin(x) + C \\
&\Rightarrow \boxed{y = \frac{\sin(x)}{x^2} + \frac{C}{x^2}}
\end{aligned}
$$

7 Solve $y' + \frac{2y}{x} = \frac{4}{x}$. where $y(1) = 6$

<u>Answer</u>

$$P(x) = \frac{2}{x}, Q(x) = \frac{4}{x}, \quad \text{IF } = e^{\int Pdx} = e^{\int 2/x \cdot dx} = x^2$$

$\therefore$ Solution:

$$y \cdot x^2 = \int x^2 \cdot \frac{4}{x} dx + C = 2x^2 + C$$

$$\therefore y = 2 + \frac{C}{x^2}$$

Given $y(1) = 6$

$$\therefore \quad 6 = 2 + \frac{C}{1} \Rightarrow C = 4$$

$$y = 2 + \frac{4}{x^2}$$

Variable Separable Equation

A differential equation of the form $m(x, y)dx + n(x, y)dy = 0$ is a variable separable equation if it can be expressed in the form: $f(x)dx + g(y)dy = 0$

1 Solve : $\frac{dy}{dx} = \frac{y}{x}$

Answer

$$\frac{dy}{dx} = \frac{y}{x}$$
$$\Rightarrow dy \cdot x = dx \cdot y$$
$$\Rightarrow \frac{dx}{x} = \frac{dy}{y}$$

Integrate both sides:

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$
$$\begin{aligned} \ln(y) &= \ln(x) + \ln(C) \\ &= \ln(xC) \\ &\Rightarrow \boxed{\text{y = xC}} \end{aligned}$$

2 Solve: $(y + 2)dx + y(x + 4)dy = 0$

Answer Divide by $(y + 2)(x + 4)$

$$\frac{1}{x+4}dx + \frac{y}{y+2}dy = 0$$

Integrating both sides:

$$\begin{aligned} \int \frac{dx}{x+4} + \int \frac{y}{y+2}dy &= \ln(C) \qquad \text{[ln(C) is used to make further steps easier]} \\ \int \frac{dx}{x+4} + \int \frac{y+2-2}{y+2}dy &= \ln(C) \\ \int \frac{dx}{x+4} + \int \left[\frac{y+2}{y+2} + \frac{-2}{y+2}\right] dy &= \ln(C) \\ \int \frac{dx}{x+4} + \int \left[1 + \frac{-2}{y+2}\right] dy &= \ln(C) \end{aligned}$$

$$\begin{aligned}\ln(x+4)+y-2\ln(y+2) &= \ln(C)\\ y &= \ln(C)+2\ln(y+2)-\ln(x+4)\end{aligned}$$

$$\therefore y = \ln\left[C\cdot\frac{(y+2)^2}{x+4}\right]$$

3 solve $3x\sin(y)\cdot dx+(x^2+1)\cdot\cos(y)\cdot dy=0$

Answer

Divide by $\sin(y)\cdot(x^2+1)$

$$\therefore \frac{x}{x^2+1}dx+\frac{\cos(y)}{\sin(y)}dy=0$$

Integrate both sides:

$$\int\frac{x}{x^2+1}dx+\int\frac{\cos(y)}{\sin(y)}dy=\ln(C)$$

$$\begin{aligned}\text{Let } t = x^2+1 &\Rightarrow dx=\frac{dt}{2x}\\ u=\sin(y) &\Rightarrow du = dy\cos(y)\end{aligned}$$

$$\begin{aligned}\therefore\int\frac{dt}{2t}+\int\frac{du}{u} &= \ln(C)\\ \frac{1}{2}\ln(t)+\ln(u) &= \ln(C)\\ \frac{1}{2}\ln\left(x^2+1\right)+\ln(\sin(y)) &= \ln(C)\end{aligned}$$

$$\begin{aligned}\ln(\sin(y)) &= \ln\left[\frac{C}{\left(x^2+1\right)^2}\right]\\ \sin(y) &= \frac{C}{\left(x^2+1\right)^2}\\ &\text{or}\\ y &= \sin^{-1}\left[\frac{C}{\left(x^2+1\right)^2}\right]\end{aligned}$$

4 Solve $\tan(\theta)dr+2r\cdot d\theta=0$

Answer

$$\begin{aligned}
\frac{dr}{2r} + \frac{d\theta}{\tan(\theta)} &= 0\\
\int \frac{dr}{2r} + \int \underbrace{\frac{d\theta}{\tan(\theta)}}_{\cot(\theta)d\theta} &= \ln(C)\\
\frac{1}{2}\ln(r) + \ln(\sin(\theta)) &= \ln(C)\\
\ln(\sqrt{x}) + \ln(\sin(\theta)) &= \ln(C)\\
\ln(\sqrt{r}) &= \ln\left[\frac{C}{\sin(\theta)}\right]\\
\sqrt{r} &= \frac{C}{\sin(\theta)}\\
\Rightarrow r &= \frac{C}{\sin^2(\theta)}\\
r &= C \cdot \csc^2(\theta)
\end{aligned}$$

5 Solve $4xy\ dx + (x^2+1)\,dy = 0$

Answer

$$\begin{aligned}
4xy\ dx + \left(x^2+1\right) dy &= 0\\
\Rightarrow \frac{4x}{x^2+1}dx + \frac{dy}{y} &= 0\\
2\int \frac{2x}{x^2+1}dx + \int \frac{dy}{y} &= \ln(C)\\
2\ln(x^2+1) + \ln(y) &= \ln(C)\\
\Rightarrow y &= \frac{C}{(x^2+1)^2}
\end{aligned}$$

Homogeneous Differential Equation

An differential equation that can be reduced into the form: $\dfrac{dy}{dx} = F\left(\dfrac{y}{x}\right)$ is called homogerious differential equation. This can be solved by putting $y = vx$ and hence reducing to variable separable form.

1. Solve $2xy \cdot \frac{dy}{dx} - y^2 + x^2 = 0$

$$2xy\frac{dy}{dx} = y^2 - x^2 \Rightarrow \frac{dy}{dx} = \frac{y^2-x^2}{2xy} - (1)$$

pat $y = vx$ $v = \frac{y}{x}$

$$
\begin{aligned}
&\therefore (1) \equiv v + x \cdot \frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x^2v} = \frac{(v^2-1)}{2v} = \frac{(y^2/x^2-1)}{2 \cdot x/xx} \\
&x \cdot \frac{dv}{dx} = \frac{v^2-1}{2v} - v = \frac{v^2-1-2v^2}{2v} \\
&x \cdot \frac{dv}{dx} = \frac{-(1+v^2)}{yv} \\
&\frac{2v}{(c1+v^2)} \cdot dv = \frac{dx}{x} \\
&\therefore -\int \frac{2v}{(1+v^2)} dv = \int \frac{dx}{x} + \ln(c) \\
&= -\ln\left(v^2+1\right) = \ln(x) + \ln(c) \\
&x \equiv \ln\left(v^2+1\right) = -\ln(x) + \ln(c) \\
&\therefore v^2 + 1 = \frac{c}{x} \\
&\frac{y^2}{x^2} + 1 = \frac{c}{x} \Rightarrow y^2 + x^2 = cx
\end{aligned}
$$

let $y = vx, \quad \therefore v = \frac{y}{x}$

$$\therefore \frac{dy}{dx} = 1 + \frac{vx}{x} = 1 + v$$

$= v + x \cdot \frac{dv}{dx} = 1 + v \Rightarrow x \cdot \frac{dv}{dx} = 1 \Rightarrow \frac{dx}{x} = dv$

$$\therefore \int \frac{dx}{x} = \int dv + C$$

$$= \ln(x) = v + c$$

$$y = \ln\left(\frac{x}{D}\right) \cdot x$$

Bernoüllis's Differential Equation

A differential equation of form $y' + p(x)y = Q(x)y^n . n \in \mathbb{R}/\{0,1\}$ called Bernocilli's Differential equation

Method to solve:

i.) Divide by y^n

$$y^{-n} \cdot y' + p(x)y^{1-n} = Q(x) - (1)$$

2. put $z \leq y^{1-n}, \therefore \frac{dz}{dx} = (1-n)y^{-n} \cdot \frac{dy}{dx} \Rightarrow \underbrace{y^{-n} \cdot \frac{dy}{dx} = \frac{1}{1-n} \cdot \frac{dz}{dx}}_{(2)}$

3. Substitute (2) in (1):

$(1) \rightarrow \frac{1}{1-n} \cdot \frac{dz}{dx} + P(x) \cdot z = Q(x)$

$$\Rightarrow z' + (1-n)P(x) \cdot z = (1-n)Q(x)$$

This is FLDE. in dependent variable z
$\therefore$ Solution:

$$Z \cdot (I \cdot F) = \int (1-n)Q(x) \cdot IF \cdot dx + C \quad , IF = e^{\int (1-n) \cdot P(x) \cdot dx}$$

Solve following:
a) $y' + 2y = y^2$
A. Divide by y^2 :

$$y^{-2} \cdot y' + 2y^{-1} = 1$$

put $z = y^{-1} \quad \therefore z\frac{dz}{dx} = -y^{-2} \cdot \frac{dy}{dx}$
$\therefore$ (1) becomes:
$-\frac{dz}{dx} + 2z = 1 \Rightarrow \frac{dz}{dx} - 2x = -1$
This is FLDE.
$\therefore P(x) = -2, \quad Q(x) = -1$
$IF = e^{\int Pdx} = e^{\int -2 \cdot dx} = e^{-2x}$
$\therefore$ General Solution:

$$\therefore z \cdot e^{-2x} = \int -e^{-2x} dz + c/2$$

$$zxe^{-2x} = \frac{1}{2}e^{-2x} + \frac{c}{2}$$

$$\text{now } z = y^{-1}$$

$$\therefore \quad y^{-1} \cdot e^{-2x} = \frac{1}{2}e^{-2x} + \frac{c}{2}$$

$$\therefore y = \frac{2}{1 + e^{2x} \cdot c}$$

Divide by y^4 :

$$y^{-4} \cdot \frac{dy}{dx} - y^{-3} \cdot \tan(x) = \sec(x) - (1)$$

let $z = y^{-3}, \quad \frac{dz}{dx} = -3.y^{-4} \cdot \frac{dy}{dx}$
multiply θ by -3 & substitute $\frac{dz}{dx}$

$$+3, \quad z\tan(x) = -3\sec(x)$$

$$\frac{dz}{dx}$$

$$P(x) = 3\tan(x) \quad Q(x) = -3\sec(x)$$

$$I : F = e^{\int P \cdot dx} = e^{3\int \tan x \cdot dx} = e^{\ln\left(\sec^3(x)\right)} = \sec^3(x)$$

multiply (2) by I.F.
$\sec^3(x) \cdot \frac{dz}{dx} + 3 \cdot \tan(x) \cdot \sec^3(x)z = -3\sec^4(x)$
Apply reverse product rule: $uv' + vu' = (4v)'$

$$\Rightarrow \frac{d}{dx}\left(\sec^3(x)\cdot z\right) = -3\sec^4(x)$$

$\int sc.^4 \to \int \mathrm{se}^2 \cdot \mathrm{sc}^2$
Integrate both sides: $\to (1+t^2)\, sc^2$

$$\sec^3(x)\cdot z = -3\int \sec^4(x)\cdot dx + c \tag{x}$$

$= \int(1+\infty)dt$

$$= -3\left[\tan(x) + \frac{\tan^3(x)}{3}\right] + c$$

$\Rightarrow \frac{\sec^3(x)}{y^3} = -3\tan(x) = \tan^3(x) + c$
$\therefore y = \frac{1}{\mathrm{sos}^5(x)\cdot\sqrt[3]{-3\tan(x) - \tan^3(x) + c}}$
d) $\frac{dy}{dx} + \tan(x)\tan(y) = \cos(x)\cdot\sec(y)$
A) This is B.D.E.
Divide by $\sec(y)$
$\cos(y)\frac{dy}{dx} + \tan(x)\sin(y) = \cos(x) \quad -(1)$
let $z = \sin(y)$. $\quad \frac{dz}{dx} = \cos(y)\cdot\frac{dy}{dx}$.
Substitute this in (1)
$\frac{dz}{dx} + \tan(x)\cdot z = \cos(x)$
let IF $= e^{\int \tan(x)\cdot dx} = e^{\ln(\sec(z))} = \sec(x)$
multiply both sides by T.F.
$\sec(x)\cdot\frac{dz}{dx} + \tan(x)\cdot\sec(x)\cdot z\cdot = 1$
$\equiv \frac{d}{dx}(\sec(x)\cdot z) = 1 \quad \Rightarrow \quad \sec(x)\cdot z = x + c \Rightarrow \sec(x)\cdot\sin(y) = x + c.$
$\therefore y = \sin^{-1}(\cos(x)\cdot(x+c))$
d) $\frac{dy}{dx} + x\cdot\sin(2y) = x^3\cdot\cos^2(y)$
Divide by $\cos^2(y)$:
$\sec^2(y)\cdot\frac{dy}{dx} + \frac{x\cdot 2\sin(y)\cos(y)}{\cos^2(y)} = x^3$
$\sec^2(y)\frac{dy}{dx} + 2x\cdot\tan(y) = x^3 \quad -10$
let $z = \tan(y) \quad \frac{dy}{dx} = \sec^2(y)\cdot\frac{dy}{dx}$
substitute this in (1):
$\frac{dz}{dx} + 2x\cdot z = x^3$
let I.F $e^{\sqrt{2x}\cdot dx} = e^{x^2}$, demultiply both sides by I.F $e^{x^2}\cdot\frac{dz}{dx} + z\cdot 2x\cdot e^{x^2} = x^3 e^{x^2}$
$\Rightarrow \frac{d}{dx}\left(e^{x^2}\cdot z\right) = x^3\varepsilon e^{x^2}$

$e^{x^2} z = \int x^3 \cdot e^{x^2} dx + c \longrightarrow$ let $t = x^2$, $dt = 2x\,dx$

$\downarrow$

$\frac{1}{2}\int t \cdot e^{t}$

$= \frac{1}{2} e^{x^2}(x^2 - 1) + c$

$= \tan(y) \cdot e^{x^2} = \frac{1}{2} e^{x^2}(x^2-1) + c$

$= \frac{1}{2}\left(t \cdot \int e^t dt - \int \int e^t dt \cdot dt\right)$

$= \frac{1}{2}\left(t \cdot e^t - e^t\right) = \frac{1}{2} e^t (t-1)$

$y = \tan^{-1}\left(\frac{1}{2}(x^2-1) + e^{-x^2} \cdot c\right)$

1.) $\frac{dy}{dx} + y = xy^3$ **2)** $\frac{dy}{dx} - y\tan(x) = y^2 \cdot \sec(x)$

3) $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x}\ln(x)$

4) $\frac{dy}{dx} + xy = x^3y^3$

Partial Differentiation

If $z = f(x,y)$ it can be differentiated partially wort x ory $\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$, Here we treat y as constant $z_x = \frac{\partial z}{\partial x}$

e.y: $z(x,y) = x^2 + y^2 + 2xy$,

$z_y = \frac{\partial z}{\partial x}\ \frac{\partial z}{\partial x} = 2x + 0 + 2y \quad 0$

or wirt. y by: $\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$ [we treat x constaint] $\frac{\partial z}{\partial y} = 0 + 2y + 2x$

find $\frac{\partial f}{\partial x} \& \frac{\partial f}{\partial y}$ in following:

a) $f(x,y) = x^3 + 3x^2y + xy^3$

A) $\frac{\partial f}{\partial x} = 3x^2 + 6xy + y^3 \quad , \frac{\partial f}{\partial y} = 0 + 3x^2 + 3xy^2$

b) $f(x,y) = 2x\cos(y) + 3x^2y$

A) $\frac{\partial f}{\partial x} = 2\cos(y) + 6xy \quad \frac{\partial f}{\partial y} = -2x\sin(y) + 3x^2$

c)

$$f(x,y) = x\tan^{-1}\left(\frac{y}{x}\right)$$

$$f_x = \frac{1}{1 + y^2/x^2} \cdot \frac{\partial}{\partial x} \cdot \left(\frac{y}{x}\right) = \frac{y}{x^2} \cdot \frac{x^2}{x^2+y^2} \cdot -\frac{1}{x^2} = \frac{-y}{x^2+y^2}$$

$\Delta f_y = \frac{1}{1+y^2/x^2} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$

d) $f(x,y) = x^3 - x^2\sin(y) - y$

$f_x = 3x^2 - 2x\sin(y)$. $\quad f_y = -x^2\cos(y) - 1$

Higher Order Partial Derivative

- $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$

- $\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=f_{yy}$

$\cdot \frac{\partial^2 f}{\partial x \partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=f_{xy}$

- $\frac{\partial^2 f}{\partial y \partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=f_{yx}$

- If $f(x, y)$ is continuous function, $f_{xy}=f_{yx}$

1. find I & II order partial derivatives of
   a) $t=x^2 y$
   A) $f_x=2xy.\ f_y=x^2, f_{xy}=2x, f_{yx}=2x, f_{xx}=2y, f_{yy}=0$
   b) $x^3 f(x, y)=x^3 \sin(y)$

$f_x=3x^2 \sin(y), f_{xx}=6x \sin(y), f_{yx}=3x^2 \cos(y)$
$f_y=x^3 \cos(y), \quad f_{yy}=-x^3 \sin(y), \quad f_{xy}=-3x^2 \sin(y)$

Differentials

If $z=f(x, y), dz, dx, dy$ are known as differentials. in z, z, y respectively.
cex $dz=\frac{\partial z}{\partial x} dx+\frac{\partial z}{\partial y} dy$
1.) find the differentiols in f of if $f=\frac{x^3}{3}-xy^2$

$$
\begin{aligned}
df &= \frac{df}{\partial x} \frac{\partial f}{\partial x} dx+\frac{\partial f}{\partial y} dy \\
df &= \left(x^2-y^2\right) dx-2xy dy
\end{aligned}
$$

Exact Differential Equation

- A Differential equation of the form $M(x, y) dx+N(x, y) dy=0$ is sard to be exact differential equation.

such that $\frac{\partial \mu}{\partial x}=m(x, y), \& \frac{\partial \mu}{\partial y}=N(x, y)$
ie, $M dx+N dy=\frac{\partial \mu}{\partial x} dx+\frac{\partial \mu}{\partial y} dy=d \mu$

$$
\therefore \text { Solution is } \int d \mu \Rightarrow \mu(x, y)=c
$$

Methnd to find exact or not

- If Diff. eqn. is exact then $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

eg: $(1-x) \cdot dx-(1+y) dy=0$
$M=1-x, \quad N=-(1+y)$
$\frac{\partial M}{\partial y}=0, \quad \frac{\partial N}{\partial x}=0 \quad \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \quad \therefore$ Eqn. is exact.
Method to solve E.D.E
Solution: $\int M dx+\int[$ Terms in N not containing $x] dy=c$
Solve $(1-x) dg-(1+y) dy=0$
A) Solution: $\int(1-x) dx+\int-(1+y) dy=c / 2 \quad$ (soy)

$$
\begin{aligned}
x-\frac{x^2}{2}-y-\frac{y^2}{2}&=c/2\\
\Rightarrow\quad x-y&=\frac{x^2+y^2}{2}+c/2\\
2(x-y)&=x^2+y^2+c
\end{aligned}
$$

2. $(3x^2+4xy)\,dx+(2x^2+2y)\,dy=0$

$$
\begin{aligned}
&M=3x^2+4xy, \quad \frac{\partial M}{\partial y}=4x\\
&N=2x^2+2y \quad , \quad \frac{\partial N}{\partial x}=4x \quad \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\Rightarrow \text{ eqn. is Exact}
\end{aligned}
$$

Solution: $\int M\cdot dx+\int[\text{ terms in } N \text{ not containing } x]dy=c$

$$
\begin{gathered}
\int\left(3x^2+4xy\right)dx+\int 2y\cdot dy=c\\
x^3+2x^2y+y^2=c
\end{gathered}
$$

3. Why condition for exactness is $\frac{\partial M}{\partial x}=\frac{\partial N}{\partial y}$?

A) for E.DE. $\exists u(x,y):\frac{\partial u}{\partial y}=N(x,y)\cdot\frac{\partial u}{\partial x}=m(x,y)$
consider $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial N}{\partial x},\quad \frac{\partial^2 u}{\partial y\partial x}=\frac{\partial M}{\partial y}$
since. $u(x,y)$ represents a family of curve and
it is continuous, $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}\Rightarrow\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

2. Anodian woy]:

$m=3x^2+4xy,(1)\quad N=2x^2+2y-(2)$
$\frac{\partial M}{\partial H}=4x$
$\nrightarrow\cdot m\ 0,4xy+\Psi(x)$
$=4xy+\psi(x)-15$
comparing (1) & (3) $\psi(x)=3x^2$

4. $(2x\cos(y)+3x^2y)\,dx+(x^3-x^2\sin y-y)\,dy=0$

$$
\begin{aligned}
&M=2x\cos(y)+3x^2y && N=x^3-x^2\sin(y)-y\\
&\tfrac{\partial M}{\partial y}=-2x\sin(y)+3x^2, && \tfrac{\partial N}{\partial x}=3x^2-2x\sin(y)\\
&\tfrac{\partial M}{\partial y}=\tfrac{\partial N}{\partial x}\Rightarrow \text{ eqn. is EDE}
\end{aligned}
$$

Solution: $\int(2x\cos(y)+3x^2y)\,dx+\int -ydy\pm c$

$$
x^2\cos(y)+x^3y-\frac{y^2}{2}=c
$$

Hw:

1. Solve:

a) $\left(5x^4+3x^2y^2-2xy^3\right)dx+\left(2x^3y-3x^2y^2-5y^4\right)dy=0$
b) $(2xy+y-\tan(y))dx+\left(x^2-x\tan^2(y)+\sec^2(y)+2y\right)dy=0$

2. (another uy).

$$M=3x^2+4xy, N=2x^2+2y$$

Define $f(x,y)$ f $f_x=m.f_y=N$
Then solution is given by $f(x,y)=c_1$

1. Integrate f_x with respect toge to find $f(x,y)$:

$$f(x,y)=\int\left(3x^2+4yx\right)dx=x^3+2x^2y+\psi(y)$$

Differentiate this curty to find $\psi(y)$:

$$\partial f_y=2x^2+\frac{d\psi}{dy}$$

Substitute $f_y=N, \quad (\text{by } def.)$

$$2x^2+\frac{d\psi}{dy}=2x^2+2y\Rightarrow\frac{d\psi}{dy}=2y$$

Integrate $d\psi$ sixy urty: $\psi(y)=y^2$
substitute $\psi(y)$ in $f(x,y)$:

$$f(x,y)=x^3+y^2+2x^2y$$

The Solution is $f(x,y)=c$:

$$:\frac{x^3+y^2+2x^2y=c}{2}$$
$$x^4+y^2+2x^2y=x^4-x^3+c$$
$$\Rightarrow\left(y+x^2\right)^2=x^4-x^3+c$$
$$y+x^2=\pm\sqrt{x^4+x^3+c}$$
$$y=-x^2\pm\sqrt{x^4-x^3+c}$$

$?\frac{dy^4}{dx}+\frac{xy}{1-x^2}=x\sqrt{y}$
divide by $\sqrt{y}$.

$$y^{-1/2}\frac{dy}{dx}+\frac{xy^{1/2}}{1-x^2}=x \tag{1}$$

let $z=y^{1/2}, \quad \frac{dz}{dx}=\frac{1}{2}y^{-1/2}\frac{dy}{dx}$
$\therefore$ (1) become:

$$\frac{dz}{dx}+\underbrace{\frac{dx}{2\left(1-x^2\right)}}_{p}\cdot z=\frac{1}{2}x.$$

let $f = e^{\frac{1}{e}\int \frac{x}{1-x^2}dx} = \left(1-x^2\right)^{-1/4}$ $\quad -\frac{1}{2}\int \frac{x}{1-x^2}dx\ t = 1-x^2\ dt = dx\cdot(-2x)$

$$s:\mathbb{Z}\cdot F = \int Q\cdot F\cdot dx + c = -2x\cdot dx$$

$$\Psi\cdot\left(1-x^2\right)^{-1/4} = \int \frac{1}{2}x\cdot\left(1-x^2\right)^{-1/4}dx + c \quad -\frac{1}{4}\int\frac{dt}{t} \Rightarrow\Rightarrow n(t)$$

$$= \quad \frac{1}{2}\int x\cdot\left(1-x^2\right)^{-1/4}dx + c \quad \text{- (2)}$$

$\therefore$ Solution is: $y\cdot F = \int Q\cdot F\cdot dx + c$
let $t = 1-x^2, dt = -2xdx$
$\therefore$ (2) becomes:

$$y\cdot\left(1-x^2\right)^{-1/4} = -\frac{1}{4}\int t^{-1/4}dt + c$$

$$= -\frac{1}{4}x\frac{t^{3/4}}{-1/4+1} = -\frac{1}{3}\cdot t^{3/4} + c$$

$$z\cdot\left(1-x^2\right)^{-1/4} = -\frac{1}{3}\left(1-x^2\right)^{3/4} + c$$

$$z = -\frac{1}{3}\left(1-x^2\right) + c\cdot\left(1-x^2\right)^{1/4}$$

$$z = \sqrt{y}$$

$$\therefore y = \left[\sqrt[4]{c\cdot(1-x^2)} - \frac{1}{3}\left(1-x^2\right)\right]^2$$

$$\frac{dy}{dx} + x\cdot\sin(2y) = x^3\cdot\cos^2(y)$$

$$\equiv \frac{dy}{dx} + x\cdot 2\cdot\sin(y)\cos(y) = x^3\cdot\cos^2(y)$$

Divide by $\cos^2(y)$:

$$\sec^2(y)\cdot\frac{dy}{dx} + 2x\cdot\tan(y) = x^3$$

let $z = \tan(y) = \frac{dz}{dx} = \sec^2(y)\cdot\frac{dy}{dx}$.

$$\frac{dz}{dx} + 2x\cdot z = x^3$$

I.F $= e^{\sqrt{2x}} = e^{x^2}$, &multiply by it:

$$e^{x^2}\frac{dz}{dx} + 2x\cdot e^{x^2}\cdot z = x^3\cdot e^{x^2}$$

$$\Rightarrow \quad e^{x^2}\cdot z = \int x^3\cdot e^{x^2}dx^+ + c$$

$$= \frac{1}{2}e^{x^2}\left(x^2-1\right) + c$$

$$\therefore \tan(y) = \frac{1}{2}\left(x^2-1\right) + e^{-x^2}\cdot c$$

3. $\frac{dy}{dx} + y\tan(x) = y^3\cdot\sec(x)$

EDE -Hw-1

$$\underbrace{\left(5x^4+3x^2y^2-2xy^3\right)}_{M}dx+\underbrace{\left(2x^3y-3x^2y^2-5y^4\right)}_{N}dy=0$$

$$M_{xy}=6x^2y-6xy^2, N_y=6x^2y-6xy^2$$

$$M_y=N_y \quad \therefore \text{ EDE}$$

$\therefore$ So ln: $\int_{\text{cor}} mdx+\int(N\ \cancel{x})dxy=C$

$$=x^5-y^5+x^3y^2-x^2y^3=c$$

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$$\left(3x^2+6xy^2\right)dx+\left(6x^2y+4y^3\right)dy=0$$

Solve:
a) $[\cos(x)\tan(y)+\cos(x+y)]dx+\left[\sin(x)\cdot\sec^2(y)+\cos(x+y)\right]dy=0$
A) $M=\cos(x)\tan(y)+\cos(x+y)\cdot\frac{\partial M}{\partial y}=\cos(x)\cdot\sec^2(y)-\sin(x+y)$

$$N=\sin(x)\sec^2(y)+\cos(x+y),\frac{\partial N}{\partial x}=\sec^2(y)\cdot\cos(x)-\sin(x+y)$$

$\frac{\partial M}{\partial y}=\frac{\partial r}{\partial x}\Rightarrow$ Equation is exact.
$\therefore$ Solution is:

$$\int_{y\cdot\text{ cons}}M\cdot dx+\underbrace{\int(\text{tem}\,s\,\text{in}\,N\text{ not containing }x)dy\,d}_{0}=c$$

$$\tan(y)\int\cos(x)dx+\int\cos(x+y)dx\quad=c$$

$$\tan(y)\cdot\sin(x)+\sin(x+y)=c$$

1. $(y\cos(x)+1)dx+\sin(x)dy=0$
2. $\left(\sec(x)\tan(x)\tan(y)-e^x\right)dx+(\sec(x)\sec^2(y)dy=0$

Linear D.E with Constant Coeffis.
It is eqn of form.

$$a_0\cdot\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}\frac{dy}{dx}+a_ny=\phi(x)$$

where $a_i\in\mathbb{R}$,
If $\phi(x)=0:$ to solve this we have to change the equation to symbolic form. is $(a_0D^n+Da,D^{n-1}+\cdots)\,s=$ 0\$ ItS Auxillary equation is: $(a_\theta m^n+a_1m^{n-1}+\cdots)$ ky $=0$
From the auxillory equation we get the roots, $m_1,m_2,\cdots$ Now we procede by following rales.
(which depends on nature of roots.

Roots	complimentary f_x
1 Roots are Rd equal	$(c_1 + c_2 x)\, e^{m_1 x}$
$m_1 = m_2$	$c_1 e^{m_1 x} + c_2 e^{m_2 x}$
$m_1 \neq m_2$	
2) $m_1 = m_2 = m_3$	$(c_1 + c_2 x + c_3 x^2)\, e^{m_1 x}$
3) $m_1 \neq m_2 \neq m_3.$	$(c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e_3 x$
4)$m_1 = m_2 \neq m_3$	$(c_1 + c_2 2 e^{m_1 x} + c_3 e^{m_3 x}$
4) $\mathbb{I} : \alpha \pm i\beta$	$e^{\alpha x}\,(c_1 \cos(\beta x) + c_2 \sin(\beta x))$

- From the nature of roots, we get complimentary function, Hence the Solution is:

$$y = C \cdot F$$

1. Solve $\frac{d^2y}{dx} + \frac{\int}{dx} + 6y = 0$

Symbolic form: $(D^2 + 5D + 6)\, y = 0 \Rightarrow (D+3)(D+2) = 0$
$\therefore$ roots are: $m = -3, -2$
Real & distinct.
$\therefore$ complimentary function is : $c_1 \cdot e^{m \cdot x} + c_2 \cdot e^{m_2 x}$

$$= c_1 e^{-2x} + c_2 e^{-3x}$$

$\therefore$ Solution is: $\quad y = c_1 e^{-2x} + c_2 e^{-3x}$
2 Solve $(D^3 + 1)\, y = 0$
$\rightarrow D^3 = -1 \quad \Rightarrow \quad$ roots ar: $, -1, \frac{1}{2} \pm \frac{\sqrt{2}}{2} i$
usang: $(a+b)\,(a^2 - ab + b^2)$

$$\rightarrow (D+1)\left(D^2 - D + 1\right) = 0$$
$$\Rightarrow \quad D + 1 = 0 \Rightarrow \text{ root } = -1$$
$$D^2 - D + 1 = 0 \Rightarrow \text{ root } = \frac{1 \pm \sqrt{3} i}{2}$$

$CF: \quad e^{1/2x}\left(c_1 \cos\left(\frac{\sqrt{3}}{2} x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2} x\right)\right) + c_3 \cdot e^{-x}$
To find particular seder integral $\frac{\phi(e)}{5}$
Case

$$I : \phi(x) = e^{ax}, \text{ put } D = a$$

e.g: $\frac{d^2y}{dx} - 13\frac{dy}{dx} + 12y = e^{-2x}$

$$\rightarrow \underbrace{\left(D^2 - 12D + 12\right) y}_{=0 \rightarrow \text{ roots}} = e^{-2x}$$
$$\therefore C \cdot F = c_1 e^{x} + c_2 e^{12x}$$

$\therefore$ Porticulor integral : $PI = \frac{e^{-2x}}{D^2 - 13D + 12} \quad , D = -2,$

$$\Rightarrow \frac{e^{-2x}}{4 + 26 + 12} = \frac{e^{-2x}}{42}$$

$\therefore$ Solution: $y = CF + PI$

$$= c_1 e^x + c_2 e^{12x} + \frac{e^{-2x}}{42}$$

$$6D^2 y - D_y - 2y = e^{4x} \quad , \quad 6$$

$$\therefore \text{ Auy-fx } = 6D^2 - D - 2, \text{ rooks } = \frac{+1 \pm \sqrt{1 + 4 \times 6 \times 2}}{12} = \frac{1 \pm 7}{12} \Rightarrow \frac{2}{3} - \frac{1}{2}$$

$$\therefore CF = c_1 e^{2/3x} + c_2 e^{1/2x}$$

$$PI = \frac{e^{4x}}{6D^2 - D - 2} = \frac{e^{4x}}{6 \times 16 - 4 - 2} = \frac{e^{4x}}{90}$$

$\therefore$ Solution:

$$y = CF + PF$$
$$= c_1 e^{\frac{2}{3}x} + c_2 e^{\frac{1}{2}x} + \frac{e^{4x}}{90}$$
$$y = c_1 \sqrt[3]{e^x}^2 + c_2 \sqrt{e^x} + e^{4x}/90$$

Particular Integral
case 2 : $\phi(x) = \cos(ax)$ or $\sin(ax)$, put $D^2 = a - a^2$?. Solve $(0^2 + 4)\, y = \cos(3x)$
Aux. $f_x = D^2 + 4$, roots $= \pm 2i$
$\therefore CF = e^{ox}\left(c_1 \cdot \sin(2x) + c_2 \cdot \cos(2x)\right) = c_1 \cdot \sin(2x) + c_2 \cdot \cos(22)$
$PI = \frac{\cos(3x)}{D^2+4} = \frac{\cos(3x)}{-9+4} = \frac{\cos(3x)}{-5}$
$\therefore y = CF + PI$
$= c_1 \sin(2x) + c_2 \cdot \cos(2x) - \frac{\cos(3x)}{5}$
II? $(D^2 - 3D + 2)\, y = \sin(3x)$
Aux. $f_x : D^2 - 3D + 2 \rightarrow$ routs : $1, 2$
$\therefore CF = c_1 e^x + c_2 e^{2x}$
$D^2 = -9$
$PI = \frac{\sin(3x)}{D^2 - 3D + 2} =$

$$= -\frac{\sin(3x)}{67 + 3D} = \frac{-\sin(52)}{3D + 7}$$
$$= \frac{-\sin(3x)(30 - 7)}{9D^2 - 49}$$
$$= \sin(3x) \cdot (3D - 7)$$
$$= +81 + 49$$
$$= \frac{\sin(3x)(3D - 7)}{130}$$
$$= \frac{1}{130}\left(\frac{3D \cdot \sin(3x)}{\frac{d \sin(x)}{dx}} - 7 \cdot \sin(3x)\right)$$
$$= \frac{1}{130}(9\cos(3x) - 7\sin(30))$$

$\therefore$ Solution: $c_1 e^x + c_2 e^{2x} + \frac{1}{130}(9\cos(3x) - 7\sin(3x))$
$(D^2 - 2D - 8)\, y = 4\cos(2x) + e^{4x}$
Aus. $f_n = D^2 - 2D - 8 \Rightarrow (D - 4)(D + 2) \Rightarrow$ roots $24, -2,$
$\therefore C \cdot F = c_1 e^{4x} + c_2 e^{-2x}$

$$
\begin{aligned}
PI_1 &= \frac{4\cos(2x)}{D^2-2D-8} \quad D^2=-4 \\
&= \frac{4\cos(2x)}{-4-2D-8} = -\frac{4\cos(2x)}{-2D+12} \\
\Rightarrow & \frac{-2\cos(2x)(D-6)}{(D+6)(D-6)} = \frac{-2\cos(2x)(D-6)}{D^2-36} \\
&= \frac{2\cos(2x)(D-6)}{40} \\
&= \frac{\cos(2x)(D-6)}{20} \\
&= \frac{D\cdot\cos(2x)-6\cos(2x)}{20} \\
&= -\frac{\sin(2x)+3\cos(2x)}{10}
\end{aligned}
$$

$$
\begin{aligned}
PI_2 &= \tfrac{e^{4x}}{D^2-2D-8} & D=4 \quad *: 1ff(x)=e^{ax} \\
&= \tfrac{e^{4x}}{16-8-8} \quad \tfrac{1}{f(a)}=0, \tfrac{1}{f(D)}e^{ax} = \tfrac{x}{\phi} & \\
&= \tfrac{e^{4x}}{0} & \tfrac{1}{0} \quad f(x)=\cos(ax)\&\tfrac{1}{f((^2)}=0
\end{aligned}
$$

$$
\begin{aligned}
PI_2 &= \tfrac{e^{4x}}{(D-4)(D+2)} & * \text{ If } f(x)=\sin(ax)d\tfrac{1}{f(D^2)}=0 \\
&= \tfrac{1}{D-4}\times\tfrac{e^{4x}}{D+2} & \tfrac{1}{f(D)}\sin(ax) = \tfrac{-x\cdot\cos(ax)}{29} \\
\Rightarrow &= \tfrac{xe^{4x}}{4+2} = \tfrac{xe^{4x}}{6} &
\end{aligned}
$$

$\therefore$ Solution is: $CF+PI_1+PI_2$

$$
= c_1e^{4x}+c_2e^{-2x}-\frac{\sin(2x)+3\cos(2x)}{10}+\frac{xe^{4x}}{6}
$$

2. $(D^2-9)\,y = 1+5e^{4x}+2e^{3x}$

A)

$$
\begin{aligned}
\text{Aus } F_n &= D^2-9 = 3,-3 \\
CF &= c_1e^{3x}+c_2e^{-3x} \\
PI_1 &= \frac{e^{0x}}{D^2-9} = D^2=0 \\
&= \frac{1}{-9} \quad D=4 \\
PI_2 &= \frac{5e^{4x}}{D^2-9} \quad \frac{5e^{4x}}{7} = \frac{1}{(D-3)}\cdot\frac{2e^{3x}}{(D+3)} \\
&= \frac{2e^{3x}}{D^2-9} = \frac{2xe^{3x}}{6} = \frac{xe^{3x}}{3}
\end{aligned}
$$

$\therefore$ Solution: $c_1e^{3x}+c_2e^{-3x}-\frac{1}{9}+\frac{5e^{4x}}{7}+\frac{xe^{3x}}{3}$
3. $(0^2+16)\,y=\cos(4x)$
Case 3: $\phi(x)=x^m$
To find PI. $\frac{1}{f(D)}\phi(x)$, take $[f(D)]^{-1}\phi(x)$
$\rightarrow$ expand binomially, neglecting higher powers of D, (upto m^{th} power)

$$
\begin{aligned}
(1+x)^n &= 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots \\
(1+x)^{-1} &= 1-x+x^2-x^3+x^4+\cdots \\
(1+x)^{-2} &= 1-2x+3x^2 4-4x^3+\cdots
\end{aligned}
$$

1. $(D^2+D+1)\,y=x^2$

Aux. f n $=D^2+D+1$, roots: $\quad \frac{-1\pm\sqrt{-3}}{2}=\frac{-1\pm i\sqrt{3}}{2}$

$$EF = e^{-\frac{1}{2}x}\left(C_1\cos(\sqrt{3}x)+C_2\sin(x\sqrt{3})\right)$$

$$
\begin{aligned}
PI = \frac{x^2}{D^2+D+1} &= \left(1+\left(D+D^2\right)\right)^{-1}x^2 \\
&= \left(1-\left(D+D^2\right)+\left(D+D^2\right)^2-\left(D+D^2\right)^3+\cdots\right)x^2 \\
&= \left[1-D-D^2+D^2+2D^3+D^4\right]x^2 \\
&= x^2-D\left(x^2\right)-D^2\left(x^2\right)+D^2\left(x^2\right)+2D^3\left(x^2\right)+D^4\left(x^2\right) \\
&= x^2-D\left(x^2\right)+2D^3\left(x^2\right)+D^4\left(x^2\right) \\
&= x^2-2x+0+0=x^2-2x
\end{aligned}
$$

$\therefore$ Solution: $y=cF+PI=e^{-\frac{1}{2}x}\left(c_1\cos(x\sqrt{3})+c_2\sin(x\sqrt{3})\right)+x^2\to -p$

$$
\begin{aligned}
&\left(D^2+2D+1\right)y=2x+x^2 \\
&-1 \\
&\therefore F=c_1e^{-x}+c_2e^{-x}x
\end{aligned}
$$

$$
\begin{aligned}
&PI_1=\frac{2x}{D^2+2D+1}=\left(D^2+2D+1\right)^{-1}(2x) \\
&=(D+1)^{-2}(2x) \\
&=\left(1-2D+3D^2\right)2x \\
&=1-2D(2x)+3D^2(2x) \\
&=2x-4+6=2x-4 \\
&PI_2=\frac{x^2}{(D+1)^2}=(D+1)^{-2}\left(x^2\right) \\
&=\left(1-2D+3D^2\right)x^2 \\
&=x^2-2D\left(x^2\right)+3D^2\left(x^2\right) \\
&=x^2-4x+6=x^2-4x+6
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \text{Solution } &= y=c_1e^{-x}+x^2 \nrightarrow -2x+2+c_1e^{-x} \\
&= e^{-x}\left(c_1+c_2x\right)+x^2-2x+2
\end{aligned}
$$

$$
\begin{aligned}
(2D^2-5D+3)\,y &= \cos(3x)\cos(2x) \\
&= \tfrac{1}{2}(\cos(5x)-\cos(x)) && C_HC_B \\
2D^2-5D+3 &= \tfrac{5\pm\sqrt{25-24}}{4}\Rightarrow\tfrac{3}{2},1 && S_1S_2= \\
\therefore CF &= C_1e^{3/2x}+C_2e^x && S_1C_2= \\
PI_1 &= \tfrac{1}{2}\tfrac{\cos(5x)}{2D^2-5D+3} && C_1= \\
&= \tfrac{1}{2}\tfrac{\cos(5x)}{10-5D+3}=-\tfrac{1}{2}\tfrac{\cos(5x)}{5D-13}=\tfrac{1}{2}\tfrac{\cos(5x)(5D+13)}{25D^2-16q}
\end{aligned}
$$

$$C_A C_B = \frac{1}{2}[C(A+B) + C(A-B)]$$

$$\begin{aligned}
&= -\frac{1}{2}\frac{\cos(5x)(5D+13)}{125-169} = \frac{\cos(58)(50+13)}{88} \\
&= \frac{5D(\cos(5x)) + 13\cos(5x)}{88} \\
&= \frac{13\cos(5x) - 25\sin(5x)}{88} \\
PI_2 &= \frac{1}{2}\frac{\cos(x)}{2D^2 - 50 + 3} \ \vec{z}
\end{aligned}$$

$\therefore$ solution: $C_1 e^{3/2x} + c_2 e^x + \frac{1}{5668}(47\cos(5x) + 25\sin(5x))$
$(D^2 - 4D + 3)\, y = \sin(3x)\cos(2x)$
A 2^{nd} $O.DE.$ has complimentary fo \$ particular integral compl.fn is of form: $Ae^m x + Be^{m_2 x} + BCe^m x \ldots$ where m_1, m_2, m_3 are roots of auzillory f_n. for imaginary roots:
let a_{5bi} be the root,
$\therefore$ Solution is: $Ae^{(a+bi)x} + Be^{(a-bi)x}$

$$\begin{aligned}
&\Rightarrow Ae^{ax}e^{bix} + Be^{ax}e^{-bix} \\
&= e^{ax}\left(Ae^{bix} + Be^{-bix}\right) \\
&= e^{ax}(A(\cos(bx) + i\sin(bx)) + B(\cos(bx) - i\sin(bx)) \\
&= e^{ax}((A+B)(\cos(bx) + (A-B)i\sin(bx))
\end{aligned}$$

$$\begin{aligned}
&\to e^{ax}(c \cdot \cos(bx) \\
&\sin(3x)\cos(2x) \\
&= \frac{1}{2}\sin(5x) + \frac{1}{2}\sin^{\sin}(x) \\
&\text{Aux. } f_n = D^2 - 4D + 3, \text{ roots } = 1, 3 \\
&= (D-1)(D-3) \\
&\therefore CF = c_1 e^x + c_2 e^{3x}
\end{aligned}$$

$$\begin{aligned}
PI_1 &= \frac{\sin(5x)}{2\,(D^2 - 4D + 3)} \quad , D^2 = 5 \\
&\Rightarrow \frac{\sin(5x)}{13 - 8D} \Rightarrow \frac{\sin(5x)(13+8D)}{169 - 64D^2} = -\frac{\sin(5x)(13+8D)}{151} \\
&= -\frac{13\sin(5x) - 40\cos(5x)}{151} \\
PI_2 &= \frac{\sin(x)}{2\,(D^2 - 4D + 3)} \Rightarrow CD^2 = 1 \\
&\Rightarrow \quad \frac{\sin(x)}{5 - 4D} \Rightarrow \frac{\sin^2(x)(5+4D)}{25 - 160^2} = \frac{\sin(x) + 4\cos(x)}{9}
\end{aligned}$$

$\therefore$ Solution: $y = CF + PI_1 + PI_2$

$$= c_1 e^x + c_2 e^{3x} - \frac{13\sin(5x) + 40\cos(5x)}{91} + \frac{5\cos(x) - 4\sin(x)}{9}$$

Case $\forall$ T $-e^{ax} f(x)$
$\lambda\,(D^2 + 30 + 2)\, y = e^{2x}\sin(x)$

C.F in $y = c_1e^{-1} + c_2e^{-2x}$

$$P.I, = \frac{e^{2x}\sin(x)}{D^2+3x+2} = e^{2x}\cdot\frac{\sin(x)}{D^2+3D+2} \overset{D=D+a}{\sin(x)} \cdot e^{2x}$$

$$\begin{aligned}
&\frac{\sin(x)}{D^2+4D+4+3D+6+2}e^{23} \\
&= \frac{\overline{\sin(x)}}{D^2+7x+12}e^{2x} \\
&D^2 = -(1) \\
&\rightarrow \frac{\sin(x)}{70+11}e^{221} \\
&\Rightarrow \frac{\sin(x)(7D\bar{a}11)}{\cos D^2 - 121}e^{22} \\
&\Rightarrow \frac{\sin(x)(70-11)}{-170}\left(e^{2x}\right) \Rightarrow \frac{7\cos(x) - 11\cos\sin(x)}{9}\cdot e^{2x}
\end{aligned}$$

-Solution.

$$y = c_1e^{-x} + c_2e^{-2x} + \frac{11\sin(x) - 7\cos(x)}{170}e^{2x}$$

Ho

$$\left\{\begin{array}{l} \left(D^2+4D+5\right)y = 12e^{-12}\cdot\cos(x) \\ \left(D^2-2D+1\right)y = xe^{2x} \end{array}\right.$$

e.

$$\therefore CF = c_1e^{-x} + c_2xe^x$$

$$\left.\begin{aligned}
PI &= e^{2x}\cdot\frac{x}{D^2-2D+1} = D^2
\end{aligned}\right\}$$

$$\begin{aligned}
&ye^{2x} - \left(D^2-2D+1\right)x \\
&\quad\Rightarrow e^{2x}\cdot\left(D^2-2D+1\right)^{21}x \\
&\quad\Rightarrow e^{2x}\cdot(D-1)^{-2}\cdot x \\
&\quad\Rightarrow e^{2x}\cdot\left(1+2D-3D^2\right)x \\
&\quad\Rightarrow e^{2x}\cdot(x+2) \\
&\quad\Rightarrow \frac{x}{(D-1+2)^2}\cdot e^{2x} \\
&\quad\Rightarrow (D+1)^{-2}x\cdot e^{2x} \\
&\quad\quad\left(1+2D+3D^2\right)x\cdot e^{2x} \\
&\quad\Rightarrow (x-2)e^{2x}
\end{aligned}$$

$\therefore$ Solution: $y = c_1e^x + c_2e^x + (x-2)e^{2x}$

$$
\begin{aligned}
&\left(D^3-3D_x^2+3D-3\right)y=x^2e^x\\
&\Rightarrow (D-1)^3 \Rightarrow D=1\\
&\therefore \quad c=c_1e^x+c_2e^x\cdot x+c_2x^2e^x\\
&PI=e^x\cdot\frac{x^2}{(Q-1)^3},\quad D\to D+1\\
&\Rightarrow e^x\frac{x^2}{D^3}\Rightarrow e^x\cdot\left(D^{-3}\right)x^2\\
&e^x\cdot\left(D^{-2}\right)\frac{x^3}{3}=e^x\cdot\left(D^{-1}\cdot\frac{24}{12}\right)=e^x\cdot\frac{25}{650}\\
&D^{-3}+(D+1-1)^{-3}\to(1+(D-1))^3x^2\\
&\to\left(1+3(D-1)+3(D-1)^2\right)\gtrless^2\\
&=x^2+3(D-1)x^2+3\left(D^2-2D+1\right)x^2\\
&\Rightarrow x^2+(3D-3)x^2+\left(3D^2-6D+3\right)x^2\\
&\Rightarrow x^2+-3x^2+6\\
&x^2+6x^3+6-12x \qquad (1)\\
&\left(D^2-2D+1\right)y=x\cdot e^x\sin(x)\\
&C\cdot f=\mathrm{se}\left(c_1+c_2x\right)e^x\\
&PI=e^x\cdot\frac{x\sin(x)}{(D-1)^2}\overset{D\to D+1}{2}e^x\cdot\frac{x\sin(x)}{D^2}\\
&=e^x\cdot D^1(x\cdot\sin(x))
\end{aligned}
$$

$$
\begin{aligned}
&e^x\cdot D^{-4}(-x\cdot\cos(x)+\sin(x))\\
&e^x\cdot(-x\cdot\sin(x)+\cos(x)+\cos(x)\\
&\Rightarrow e^x\cdot(x\sin(x)+2\cos(x))\\
&\therefore\ \text{soln}\quad y=\left(c_1+c_2x-x\sin(x)-2\cos(x)\right)e^x
\end{aligned}
$$

Cauchy's LDE
General form:

$$a_0x^n\cdot\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_xx^{dy}dx+y=\phi(x)$$

is reduced to LDE with Const. coett by substituting

$$
\begin{aligned}
&x=e^t,\text{ or }t=\ln(x)\quad\frac{d}{dt}=D,\because x^2\cdot\frac{d^2y}{dx^2}\Rightarrow D(D-1)y\\
&x\frac{dy}{dx}=Dy\quad:\quad x^3\frac{d^3y}{dx^3}\to D(D-1)(D-2)y
\end{aligned}
$$

Solve:
1). $x^2\cdot\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=\ln(x)$
This if Cauchy's LDG $\therefore$ put $x=e^t,\Rightarrow\sin(x)=t$
Redue to LDE with const-eoeth.

$$\begin{aligned}(1)\quad \rightarrow & D(D-1)y - Dy + y = t \\ & \left(D^2-2D+1\right)y = t \\ & \therefore CF = \left(c_1+c_2t\right)e^t\end{aligned}$$

$$\begin{aligned}P_I &= \frac{t}{(D-1)^2} \rightarrow \\ &= \frac{t}{D^2-2D+1} = \left(1+\left(D^2-2D\right)\right)^{-2} = \\ &= \left(1-D^2+2D\right)t \\ &= t+2 \\ &= \ln(x)+2\end{aligned}$$

$\therefore$ solution: $y = \left(c_1 + c_2\ln(x)\right)e^{\ln(x)} + \ln(x) + 2$

$$\begin{aligned}&= \left.\oint_1 + r_2\ln(x)\right]x + \ln(x) + 2 \\ &= c_1x + \ln(x)\left(1+xc_2\right) + 2\end{aligned}$$

2. $x^2 \cdot y'' - 4xy' + 6y = x^2$

put $x = e^t, t = \ln(x)$

$$\begin{aligned}& D(D-1)y - 4Dy + 6y = e^{2t} \\ & \left(D^2-5D+6\right)y = e^{2t} \\ & D = \frac{e^{2t}}{D^2-5D+6}, \quad CF = c_1e^{+3x} + c_2e^{-12x} \\ & \therefore PI_1 = 2 \\ & \Rightarrow \frac{e^{2t}}{4-10+6} = \\ & \frac{+5\pm\sqrt{25-25}}{2} = \frac{+5\pm 1}{2} \\ & = +2, -3 \\ & \frac{e^{2t}}{(D-3)(D-2)} \Rightarrow \frac{1}{D-2}\cdot\frac{te^{2t}}{D-3} = -te^{2t}\end{aligned}$$

$\therefore$ Solution: $y = c_1e^{3t} + c_2e^{2t} + te^{2t}$

$$= c_1e^{\ln(x)^2} + c_2e^{2\cdot\ln(x)} - \ln(x)e^{2\ln(x)}$$

$$\begin{aligned}&= c_1x^3 + c_2x^2 - \ln(x)x^2 \\ y &= c_1x^3 + x^2\left(c_2 - \ln(x)\right)\end{aligned}$$

HT
3. $x^2y'' - 2xy' - 4y = x^4$.
4. $x^2y'' + 4xy' + 2y = x^2 + x^{-2}$
In 10
H.

Simultaneous LDE
It contains two or more dependent variables (soy x, y.. and one inclependent variable (say t)

1. $\frac{dx}{dt} = 7x - y, \quad \frac{dx}{dt} - \frac{dy}{dt} = 5(x - y)$
   A)

$$
\begin{aligned}
& \frac{dx}{dt} - 2x = y = 0 \rightarrow (D-7)x + y = 0 \qquad (1) \\
& \frac{dx}{dt} - \frac{dy}{5x} - \frac{dy}{dx} + 5y = 0 \rightsquigarrow (D-5)x + -(D-5)y = 0 \qquad (2) \\
& (D-5) \times 0 : \\
& (D-7)(D-5)x + (D-5)y = 0 \\
& + [(D-5)\dot{x} - (D-5)y = 0] \\
& \underbrace{\left(D^2 - 11D + 30\right)}_{\text{aux ff}} x = 0 \\
& = \\
& \therefore \text{ costs of aux ff } z \\
& 5, 6 \\
& \therefore C \cdot F_{12} x = C, e^{5t} + C_2 e^{6t}
\end{aligned}
$$

$$D = \frac{d}{db}$$

$\therefore$ (1) becomv.

$$
\begin{aligned}
& (D-7)\left(c_1 e^{st} + c_2 e^{st}\right) + y = 0 \\
& \Rightarrow = 5c_1 e^{st} + 6c_2 e^{6t} - 7c_1 e^{st} - 7c_2 e^{-st} = -y \\
& \Rightarrow y = 2c_1 e^{st} + c_2 e^{6t}
\end{aligned}
$$

$$\& \quad x = c_1 e^{5t} + c_2 e^{6t}$$

$2\frac{dx}{dt} + 2x - 3y = t \quad , \quad \frac{dy}{dt} - 3x + 2y = e^{2t}$
A)

$$
\begin{aligned}
& (D+2)x - 3y = t \\
& (D+2) \text{ y } - 3x = e^{2t} \qquad (2) \\
& (1) \ \times 3 : 8 \\
& 3x(D+2) - 9y = 3t \qquad (3) \\
& + \left(-3x(D+2) + (D+2)^2 y = (D+2)e^{2t}\right) \qquad (9)
\end{aligned}
$$

(3) + (4)

$$
\begin{aligned}
& y\left((D+2)^2 - 4\right) = 3t + 4e^{2t} \\
& y\left(D^2 + 4D - 5\right) = 3t + 4e^{2t} \\
& = y(D+5)(D-1) \cdot \\
& \therefore \quad \text{ roots } = -5, +1 \\
& \therefore CF_{\text{an } y} c_1 e^{-5t} + C_2 e^{+t}
\end{aligned}
$$

$$
\begin{aligned}
PI_{y_1} =& \frac{3t}{(D^2+4D-5)} \\
&= -\frac{3}{5} \times \left(1 + \left(\frac{D^2}{-5} - \frac{4}{5}D\right)\right)^{-1} t \\
&= -\frac{3}{5} \times \left(1 - \frac{D^2}{-5} + \frac{4}{5}Dx^2 + \cdots\right) t \\
&= -\frac{3}{5}\left(t + 0 + \frac{4}{5}\right) \\
PI_{y_1} =& \frac{4e^{2t}}{(D+5)(D-1)} \\
\Rightarrow & \frac{4}{7}t - \frac{12}{25} \\
\Rightarrow y &= CF_y + PI_{y_1} + PI_{y_2} \\
\therefore y &= C_1e^{-5t} + C_2e^t - \frac{3}{5}t + \frac{4}{7}e^{2t} - \frac{12}{25}
\end{aligned}
$$

Substitute in (2):

$$
\begin{aligned}
&(D+2)y - e^{2t} = 3x \Rightarrow x = \frac{1}{3}\left[(D+2)y - \frac{t}{2}e^{2t}\right] \\
&x = \frac{1}{3}\Bigg(-5c_1e^{-5t} + c_2e^t - \frac{3}{5} + \frac{8}{7}e^{2t} \\
&+2c_1e^{-5t} + 2c_2e^t - \frac{6}{5}t + \frac{8}{7}e^{2t} - \frac{24}{25} - e^{2t}\Bigg) \\
&= \frac{1}{3}\left(-3c_1e^{-5t} + 3c_2e^t + \frac{9}{7}e^{2t} - \frac{6}{5}t - \frac{39}{25}\right)
\end{aligned}
$$

$$
x = -c_1e^{-5t} + c_2e^t + \frac{3}{7}e^{2t} - \frac{2}{5}t - \frac{13}{25}
$$

3.

$$
\begin{aligned}
\frac{dx}{dt} + 2y &= -\sin(t), \\
\frac{dy}{dt} &= 2x + \cos(t)
\end{aligned}
$$

A) $\frac{dx}{dt} + 2y = -\sin(t)$-1

$$
\begin{aligned}
&\frac{dx}{dt} + 2y = -\sin(t) - (D)(D-2)x + (D+2)y = \cos(t) - \sin(t) \\
&\frac{dy}{dt} - 2x = \cos(t) \quad \text{(D)}
\end{aligned}
$$

(1) $\times 2$,

$$
2x \cdot 0 + 4y = -2\sin(t)
$$

(2) $\times 0$.

$$
D^2 \cdot y - 2x \cdot D = D \cdot \cos(t) - (4)
$$

(3) + (4):

$$
\begin{aligned}
&D^2 \cdot y + 4y = -3\sin(t) \\
&\frac{d^2y}{dt} + 4y = -3\sin(t) \\
&\therefore \quad ye^{4t} = \int e^{4t}x - 3\sin(t)dt = -3\int \sin(t)e^{4t} \cdot dt
\end{aligned}
$$

$$
\begin{aligned}
I &= -3\int \sin(t)e^{4t}dt \quad I_2 = \cos(t)4e^{44} + \underbrace{4\int \sin(t)\ldots}_{} \\
&= -3(\sin(t) \times 4e^{4t} - 4\underbrace{\cos(t)\cdot e^{4t}}_{I}) \quad = 40\cos(t)e^{4t} + 4I \\
&= -8\sin(t)e^{4t} + 32\cos(t)e^{4t} + 32I \\
\therefore &= \frac{e^{4t}}{4^2+1}(4\sin(t) - \cos(4t)
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad &= \frac{e^{4t}}{4^2+1}(4\sin(t) - \cos(\theta t) \\
&\text{or } \left(p^2+4\right)y = -3\sin(t) \\
&\therefore y = (c_1\cos(2t) + c_2\sin(2t)) \\
&PI = \frac{-3\sin(t)}{D^2+4}, \quad D^2 = -1 \\
&\Rightarrow -\frac{3}{3}\sin(t) = -\sin(t) \\
&\therefore y = c_1\cos(2t) + c_2\sin(2t) - \sin(t)
\end{aligned}
$$

Substitute this in (2):

$$
\begin{aligned}
&\Rightarrow x = -e^{2t}\left(\cos^5\left(-c_1+c_2\right)\right. \\
&\Rightarrow x = -2e^{2t}\left(c_1+c_2\right)\cos(2t) + \left(c_2-c_1\right)\sin(2E)) - \tfrac{1}{16} - \cos(t) \\
2 &= -\frac{1}{2}\left(\cos(t) - 2c_1\sin(2t) + 2c_2\cos(2t) - \sin(t)\right)
\end{aligned}
$$

Hew 1. $\frac{dy}{dt} + 2y + x = \sin(t) \quad \frac{dx}{dt} - 4y - 2x = \cos(t)$ H)

$$
\begin{aligned}
Dy + 2y + x &= \sin(t) \\
Dx - 4y - 2x &= \cos(t)
\end{aligned}
$$

$\mathcal{C} \times \bar{\partial} D:$

$$
\begin{aligned}
&- D^2y + 2Dy - Dx = -\cos(t) \\
&\oplus -4y + Dx - 42x = \sin(t) \\
&- \left(+D^2 + 2D + 4\right)y - 2x = \sin(t) - \cos(t)
\end{aligned}
$$

(20) 2

$$\therefore + \frac{2Dy + 4y + 2x = 2\sin(t)}{4D^2 + 3)}$$

$$- D^2 y = 3\sin(t) - \cos(t)$$

$$\therefore y = \int (\cos(t) - 3\sin(t)) dt dt$$

$$= \int (\sin(t) + 3\cos(t)) dt$$

$$y = 3\sin(t)\cos(t) \quad y = 2\sin(t) + ct + 2$$

$\therefore$ from (0: $8\sin(t) + B3\cos(t) + 266\sin(t) - 2\cos(t) + x = \sin(t)$ $\therefore x = -6\sin(t) - \cos(t) \quad x = -3\sin(t) - 2\cos(\theta$

$$-2ct + 2c_2 + c$$