

2. Partial Differentiation

Partial Differentiation Let $z = f(x, y)$, be a function of x, y . Then partial derivative w.r. x is denoted by: $\frac{\partial z}{\partial x}$. It means differentiating z w.r. x while keeping y as constant. Similarly $\frac{\partial z}{\partial y}$ represents partial derivative of z w.r. y keeping x constant. In all the cases:

$$\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \equiv \frac{\partial z^2}{\partial x \partial y} = \frac{\partial}{\partial y} \cdot \frac{\partial z}{\partial x} \equiv \frac{\partial z^2}{\partial y \partial x}$$

1. Find $1^{17} dz^{\text{nd}}$ partial derivative of $z = x^3 + y^3 - 3axy$ s.)

$$\begin{aligned} \frac{\partial z}{\partial x} &= 3x^2 - 3ay, \quad \frac{\partial z^2}{\partial x^2} = 6x, \quad \frac{\partial \cdot \partial z^2}{\partial x \partial y} = 3a \\ \frac{\partial z}{\partial y} &= 3y^2 - 3ax, \quad \frac{\partial z^2}{\partial y^2} = 6y, \quad \frac{\partial z^2}{\partial y \partial x} = 3a \\ * \text{ here: } \frac{\partial z^2}{\partial x \partial y} &= \frac{\partial z^2}{\partial y \partial x} \end{aligned}$$

2. If $v = (x^2 + y^2 + z^2)^{-1/2}$, find $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}$ A.

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= \frac{\partial v}{\partial x} = -\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} \cdot 2x, \quad \frac{\partial x^2}{\partial / 2} = -zx \cdot (x^2 \\ \frac{\partial v}{\partial y} &= -\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} \cdot 2y \\ \frac{\partial v}{\partial z} &= -\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} \cdot 2z \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 t}{\partial x^2} &= -(x^2 + y^2 + z^2)^{-3/2} + x \cdot \frac{3}{2} (x^2 + y^2 + z^2)^{5/2} \times 2x \\ &= \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \\ \frac{\partial^2 v}{\partial y^2} &= \frac{3y^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \\ \frac{\partial^2 v}{\partial z^2} &= \frac{3z^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \\ \therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} &= \frac{3}{(x^2 + y^2 + z^2)^{7/2}} \left(\frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)} - 1 \right) \end{aligned}$$

- If $u = x^2 \tan^{-1}(7x) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$ find $\frac{\partial^2 u}{\partial x \partial y}$ and

show that $u_{xy} = u_{yx}$

$$\begin{aligned} \text{A) } \frac{\partial u}{\partial y} &= x^2 \cdot \frac{1}{1+(y/x)^2} \cdot \frac{1}{x} - 2y \tan^{-1}\left(\frac{x}{y}\right) - y^2 \cdot \frac{1}{1+(\frac{x}{y})^2} \cdot \frac{-1}{y^2} \\ &= x^2 \cdot \frac{x^2}{x^2 + y^2} \cdot \frac{1}{x} + y^2 \cdot \frac{y^2}{x^2 + y^2} \cdot \frac{x}{y^2} - 2y \tan^{-1}\left(\frac{x}{y}\right) \\ &\quad \frac{x^3}{x^2 + y^2} + \frac{xy^2}{x^2 + y^2} - 2y \tan^{-1}\left(\frac{x}{y}\right) = x - 2y \tan^{-1} \end{aligned} \quad (\text{x})$$

$$\begin{aligned} \frac{\partial u}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(x - 2y \tan^{-1}\left(\frac{x}{y}\right) \right) \\ &= 1 - 2y \cdot \frac{1}{1+(\frac{x}{y})^2} \cdot \frac{1}{y} = 1 - \frac{2y^2}{x^2 + y^2} = \frac{x^2 - y^2}{x^2 + y^2} \\ \frac{\partial u}{\partial x} &= 2x \cdot \tan^{-1}\left(\frac{y}{x}\right) + x^2 \cdot \frac{1}{1+(\frac{y}{x})^2} \times \frac{y}{x^2} - y^2 \cdot \frac{1}{1+(\frac{x^2}{y^2})} \cdot \frac{1}{y} \\ &= 2x \tan^{-1}\left(\frac{y}{x}\right) - \frac{yx^2}{x^2 + y^2} - \frac{y^3}{x^2 + y^2} \\ &= 2x \tan^{-1}\left(\frac{y}{x}\right) - y \end{aligned}$$

$$\frac{\partial u}{\partial y \partial x} = 2x \frac{1}{1+y^2 x^2} \frac{1}{x} \phi - 1$$

$$= \frac{2xx^2}{x^2 + y^2} - 1 \Rightarrow \frac{x^2 - y^2}{x^2 + y^2}$$

$$\therefore \frac{\partial u}{\partial y \partial x} = \frac{\partial u}{\partial x \partial y}$$

Total derivative / Differential Coeff.

let $u = f(x, y)$ is a function of x, y

let $x = \phi(t), y = \varphi(t)$

which gives: $u = f(\phi(t), \varphi(t))$

Hence u becomes function of ' t ' alone

Then ordinary derivative: $\frac{du}{dt}$ is known as total differential coeff/Total derivative

This total derivative can also be obtained without Substitution

a $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$ chain rule

case 1: If $t = x$,

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dt} \equiv \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

case 2: u is constant: $\frac{du}{dt} = 0$

$$\begin{aligned} \longrightarrow &= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \\ \Rightarrow \frac{dy}{dx} &= -\frac{\partial u}{\partial x} \cdot \frac{\partial}{\partial y} \end{aligned}$$

- $u = x^3 y^4 z^2, x = t^2, y = t^3, z = t^4$. Find $d, y/d$ chasio rule & substitution:

$$\begin{aligned} \frac{du}{dt} &= \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt} \\ &= 3x^2 y^4 z^2 \times 2t + 4x^3 y^3 z^2 \times 3t^2 + 2x^3 y^4 z \times ut^3 \\ &= 6t \cdot x^2 y^4 z^2 + 12t^2 x^3 y^3 z^2 + 8t^3 x^3 y^4 z \end{aligned}$$

$$m \equiv 6t \cdot t^4 \cdot t^{12} \cdot t^8 + 12t^2 \cdot t^6 \cdot t^9 \cdot t^{88} + 8t^6 \cdot t^3 \cdot 7^2 \cdot t^4$$

$$= 6t^{25} + 12t^{25} + 8t^{25} \Rightarrow$$

$$26t^{25}$$

By substitution:

$$u \equiv t^6 \cdot t^{12} \cdot t^8 = t^{26}.$$

$$\therefore \frac{du}{dt} = 26t^{25}$$

- find $\frac{dy}{dx}$. if $ax^2 + 2hxy + by^2 = 1$

Find $\frac{dy}{dx}$ if $u = \sin(x^2 + y^2)$, where $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
A)

$$\begin{aligned}\frac{dy}{dx} &= -\frac{\partial u}{\partial x} / \frac{\partial u}{\partial y} \\ &= -(2ax + 2hy) / (2by + 2hx) \\ &= -\frac{ax + by}{hx + by} \\ &= \cos(x^2 + y^2) \left(2x + 2y \cdot \frac{dy}{dx} \right).\end{aligned}$$

$$\frac{dy}{dx} = \frac{1}{2b\sqrt{1-\frac{x^2}{a^2}}} \cdot b\left(0 - \frac{2x}{a^2}\right) = -\frac{bx^2}{a\sqrt{a^2-x^2}}$$

$$\therefore \frac{du}{dx} = \cos(x^2 + y^2) \cdot \left(2x - \frac{2xy}{a\sqrt{a^2 - x^2}} \right)$$

$$2) \frac{du}{dx} =$$

Euler's Theorem

let $f(x, y)$ be a homogeneous function of degree ' n ' then "euler's theorem:

$$x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} = n \cdot f \quad \text{or} \quad \sum_{i \in y} i \frac{\partial f}{\partial i} = n \cdot f$$

Proof

$$\text{Let } u = x^n f(y/x)$$

$$\therefore \frac{\partial u}{\partial x} = x^n f'(y/x) \cdot \frac{-y}{x^2} + nx^{n-1} f\left(\frac{y}{x}\right)$$

$$x \cdot \frac{\partial u}{\partial x} = -x^{n-1} y f'\left(\frac{y}{x}\right) + nx^n f\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right) \cdot \frac{1}{x} + 0$$

$$y \cdot \frac{\partial u}{\partial y} = x^{n-1} y f'\left(\frac{y}{x}\right)$$

$$\begin{aligned}\therefore x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} &= nx^n f\left(\frac{y}{x}\right) - x^{n-1} y f'\left(\frac{y}{x}\right) + x^{n-1} y f'\left(\frac{y}{x}\right) \\ &= n \cdot u\end{aligned}$$

Verify Euler's theorem if $t(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz$
 $+ 2azx + 2bxy$

$$x \cdot \frac{\partial t}{\partial x} = 2(2ax^2 + 2gz + 2hy) = 2ax^2 + 2gxz + 2bxy$$

$$y \cdot \frac{\partial t}{\partial y} = y(2by + 2fz) = 2by^2 + 2fyz$$

$$z \cdot \frac{\partial t}{\partial z} = 2(2cz + 2fy + 2gx) = 2cz^2 + 2fyz + 2gxz$$

$$\therefore x \cdot \frac{\partial t}{\partial x} + y \cdot \frac{\partial t}{\partial y} + z \cdot \frac{\partial t}{\partial z} = 2ax^2 + 2by^2 + 2cz^2 + 3fyz + 4yxz + 4hxy = 2 \cdot 4$$

$$= 2(\dots)$$

\therefore Euler's theorem is verified

2. If $u = \sin^{-1}((x + 2y + 3z)/(x^8 + y^8 + z^8))$, find $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z}$. $u = \sin^{-1}\left(\frac{x+2y+3z}{x^8+y^8+z^8}\right)$

This is not homogeneous function. To make it homogeneous, take $\sin \theta$.

$$\cos \sin(4) = \frac{x+2y+3z}{x^8+y^8+z^8} \equiv \frac{x \cdot \left(1 + \frac{2y}{x} + \frac{3z}{x}\right)}{x^8 \left(1 + \left(\frac{y}{x}\right)^8 + \left(\frac{z}{x^2}\right)^8\right)}$$

$$\equiv \underbrace{x^{-7}}_{\text{degree} = -7} \cdot f\left(\frac{y}{x}\right)$$

By Euler's theorem: $x \cdot \frac{\partial \omega}{\partial x} + y \cdot \frac{\partial \omega}{\partial y} + z \cdot \frac{\partial \omega}{\partial z} = -7 \cdot \omega$

$$\Rightarrow x \cdot \cos(\omega) \cdot \frac{\partial u}{\partial x} + y \cdot \cos(u) \frac{\partial u}{\partial y} + z \cdot \cos(u) \cdot \frac{\partial u}{\partial z} = -7 \cdot \sin(\omega)$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z} = -7 \frac{\sin(u)}{\cos(u)} = -7 \tan(u)$$

$$= -7 \cdot \tan\left(\sin^{-1}\left(\frac{x + 2y + 3z}{x^8 + y^8 + z^8}\right)\right)$$

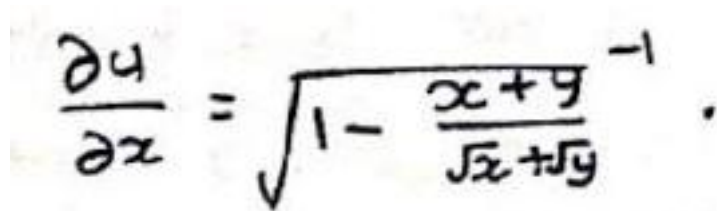
verify Euler's theorem, if $u = \sin^{-1}((x + y)/(\sqrt{x} + \sqrt{y}))$ Hence find $x \partial u / \partial x + 2y \partial u / \partial y$

$$u = \sin^{-1}\left(\frac{x + y}{\sqrt{x} + \sqrt{y}}\right) \Rightarrow \sin(u) = \frac{x + y}{\sqrt{x} + \sqrt{y}} \equiv x^{1/2} \times \frac{1 + \frac{y}{x}}{1 + \sqrt{y}}$$

let $\varepsilon = \sin(u)$, \therefore By Euler's theorem

$$x \cdot \frac{\partial \varepsilon}{\partial x} + y \cdot \frac{\partial \varepsilon}{\partial y} + q \cdot \frac{\partial \varepsilon}{\partial z} = \cos(u) \left[x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + \frac{\partial y}{\partial x} \right] = \frac{1}{2} \sin(u) \therefore x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{1}{2} \tan(u) =$$

$$\frac{1}{2} \tan\left(\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)\right) u = \sin^{-1}(\dots) \Rightarrow u = \sin^{-1}\left(\sqrt{x} - \sqrt{y} \quad \frac{1}{\sqrt{1-y^2}}\right)$$



Kf Zet =

If $u = \log [(x^4 + y^4) / (x + y)]$, find $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y}$

$$\text{let } x = e^u = \frac{x^4 + y^4}{x + y} \equiv x^3 \cdot \frac{1 + \left(\frac{y}{x}\right)^4}{1 + y/x} \quad \therefore \text{deg:3}$$

\therefore By Euler's theorem: $x \cdot \frac{\partial x}{\partial x} + y \cdot \frac{\partial x}{\partial y} = 3 \cdot x$

$$= x \cdot e^4 \cdot \frac{\partial u}{\partial x} + y \cdot e^4 \cdot \frac{\partial u}{\partial y} = 3 \cdot e^u$$

$$\Rightarrow x \cdot e \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 3$$

Errors & approximations

If δx and δy represents small increments in x and y then our new function is:

$$f(x + \delta x, y + \delta y)$$

Hence expanding $f(x + \delta x, y + \delta y)$ by Taylor's series by supposing δx and δy be small, so that their products, squares and higher powers can be neglected then, the error in the function denoted as δf is $\delta f = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y$.

*: If δy is error made in calculating y , the relative error in y represented by:

$$\frac{\delta y}{y}$$

percentage error in y : $\frac{\delta y}{y} \times 100$

Radius of a sphere is found to be 10 cm, with a possible error of 0.2 cm, what is the relative error in the computed volume

A) $r = 10$ cm, $\delta r = 0.2$

Volume (v) = $\frac{4}{3}\pi r^3$

error in computing volume: $\delta V = \frac{\partial V}{\partial r} \cdot \delta r$

$$= 4\pi r^2 \cdot \delta r$$

$$= 4\pi \times 10^2 \times 0.2$$

$$= 251.3 \text{ cm}^3$$

volume ($r = 10$), $v = \frac{4}{3}\pi r^3 = 4188.8 \text{ cm}^3$

\therefore Be

\therefore Relative error: $\frac{\delta V}{V} = \frac{251.3}{4188.8} = 0.060$

- The diameter and altitude of a can in the shape of a right circular cylinder measured as 4 cm & 6 cm respectively. The possible error in each measurement is 0.1 cm find approximately the maximum possible error in the value computed for volume and lateral surface area

$$\begin{aligned}
\text{volume} &= \pi \left(\frac{D}{2} \right)^2 \cdot h = V = \frac{\pi}{4} D^2 \cdot h \\
\therefore \partial \delta v &= \frac{\partial v}{\partial D} \delta D + \frac{\partial V}{\partial h} \cdot \delta h \quad (\text{approx.}) \\
&= \frac{\pi}{2} D \cdot L \cdot \delta D + \frac{\pi}{4} D^2 \cdot \delta h \\
&= \frac{\pi}{20} (4 \times 6 + 4^2) = 2\pi \\
LSA &= \pi DK, 2 : S \\
\therefore \delta S &= \frac{\partial S}{\partial D} \cdot \delta D + \frac{\partial S}{\partial h} \cdot \delta h \quad (\text{approx}) \\
&= \pi \times 0.1(h + 0) \\
&= \pi \text{cm}^2 \text{ (approx.)}
\end{aligned}$$

3. The focal length of mirror is given by the formula:

$$\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$$

If equal errors k are made in the determination of u and v , so that the relative error in the f is given by: $k \cdot \left(\frac{1}{v} + \frac{1}{u} \right)$

$$\begin{aligned}
\frac{2}{f} &= \frac{1}{v} - \frac{1}{u} \\
\therefore f &= 2 \cdot \frac{uv}{u-v} \equiv 2 \left(\frac{1}{v} - \frac{1}{u} \right)^{-1} \\
\delta f &= \frac{\partial f}{\partial u} \delta u + \frac{\partial f}{\partial v} \cdot \delta v \\
&= k \left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} \right) \\
&= 2k \cdot \left(- \left(\frac{1}{v} - \frac{1}{u} \right)^{-2} \left(+ \frac{1}{u^2} \right) - \left(\frac{1}{v} - \frac{1}{u} \right)^{-2} \left(- \frac{1}{v^2} \right) \right) \\
\therefore \frac{\delta f}{f} &= \frac{1}{v} \left(- \left(\frac{1}{v} - \frac{1}{u} \right)^{-1} \cdot \frac{1}{u^2} + \left(\frac{1}{v} - \frac{1}{u} \right)^{-1} \frac{1}{v^2} \right) \\
&= \sin \left(- \frac{u^v}{u^2} \cdot \frac{1}{u^2} + \frac{uv}{u-v} \cdot \frac{1}{v^2} \right) \\
&= \left\{ \frac{u^2 - v^2}{(uv)^2} \cdot \frac{uv}{u-v} \right\} \\
&= k \cdot \left\{ \frac{u+v}{uv} \right\} \\
\frac{\delta f}{\delta} &= b_r k \cdot \left(\frac{1}{u} + \frac{1}{v} \right) \\
\frac{2}{5} &= \frac{1}{2} - \frac{1}{4} \\
- \frac{2}{f^2} \delta f &= - \frac{1}{v^2} \delta v + \frac{1}{u^2} \delta u
\end{aligned}$$

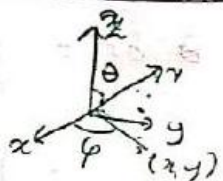
4. If the I HP required to propel a steamer varies as the cube of the velocity and square of its length, prove that a 3% increase in velocity and 5% increase in length will require, an increase of about 17% in HP. A)

$$H = k \cdot v^3 \cdot l^2.$$

$$\delta v A = 0.03 \delta 2 / 2 = 0.04$$

$$\frac{\delta H}{H} = \frac{k \cdot 3v^2 \cdot 2^2 \delta v + k \cdot 2v^3 2 \delta 2}{k \cdot v^3 \eta^2} \quad 3 \frac{\delta v}{V} + 2 \frac{\delta 2}{82} = 3 \times 0.03 + 2 \times 0.4 = 0 \Leftrightarrow$$

Coordinate Systems $G(x, y, z) \mapsto S(x, \theta, y), \gamma = \sqrt{x^2 + y^2 + z^2}$

$$\begin{aligned}
 x &= r \sin(\theta) \cos(\varphi) \\
 y &= r \sin(\theta) \sin(\varphi) \\
 z &= r \cos(\theta)
 \end{aligned}
 \quad \theta = \cos^{-1}\left(\frac{z}{r}\right), \quad \varphi = \text{sgn}(y) \cos^{-1}\frac{x}{\sqrt{x^2+y^2}}$$


1. Relation b/w cartesian and polar co-ordinates:

$$(x, y) \mapsto (r \cos(\theta), r \sin(\theta)), \text{ where } r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}$$

Here (x, y) changed to (r, θ)

$$\text{or } J\left(\frac{x, y}{r, \theta}\right) = r$$

2. b/w cartesian and cylindrical co-ordinates:

$$(x, y, z) \mapsto (r \cos(\theta), r \sin(\theta), z)$$

Here (x, y, z) changed to (r, θ, z)

$$\dots J\left(\frac{x, y, z}{r, \theta, z}\right) = r$$

3. Cartesian & spherical coordinates: $[(x, y, z) \mapsto (r, \theta, \phi)]$

$$(x, y, z) \mapsto (r \sin(\theta) \cos(\phi), r \sin(\theta) \sin(\phi), r \cos(\theta)) \quad J\left(\frac{x, y, z}{r, \theta, \phi}\right) = r^2 \cdot \sin(\theta)$$

Convert $x^2 + y^2 = 4x$ into polar curve

$$x \mapsto r \cos(\theta), y \mapsto r \sin(\theta)$$

$$r^2 (\cos^2(\theta) + \sin^2(\theta)) = 4r \cos(\theta)$$

$$\text{or } r = 4 \cos(\theta)$$

Express the point: $(x, y, z) = (1, -\sqrt{3}, 2)$ in cylindrical coordinates

$$(x, y, z) \mapsto (r \cos(\theta), r \sin(\theta), z)$$

$$r = \sqrt{x^2 + y^2} = \sqrt{1 + 3} = 2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(-\sqrt{3}) = -60^\circ$$

$$(x, y, z) \equiv (r, \theta, z) \equiv (2, -60^\circ, 2) \equiv \left(2, -\frac{\pi}{3}, 2\right)$$

Transform equation in spherical coordinates,

$$x^2 + y^2 + z^2 = 2z^2$$

$$\gamma^2 \sin^2(\theta) \cos^2(\phi)$$

$$= (\Delta) \sin^2(\phi) + \gamma^2 \cos^2(\phi) = 2\gamma^2 \cos^2(\phi)$$

$$\sin^2(\theta) (\cos^2(\theta) + \sin^2(\theta)) + \cos^2(\theta) = 2\cos^2(\theta)$$

$$2\cos^2(\theta) = 1$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\theta = 45^\circ$$

Jacobian

If u, v are functions of x, y then $\partial \left(\frac{u, v}{x, y} \right)$ or $J \left(\frac{u, v}{x, y} \right)$ is known as Jacobian

$$\partial^x \left(\frac{u, v}{x, y} \right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \partial \frac{(u, v)}{(x, y)} = \frac{\partial(u, v)}{\partial(x, y)}$$

If Jacobian J is $\partial \left(\frac{u, v}{x, y} \right)$, then

$$J' = 2 \left(\frac{x, y}{u, v} \right)$$

$$J' = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

and vice-versa

In all cases. $J \cdot J' = 1$

Functionally Dependent

functions u, v are said to be functionally dependent, then $J = 0$. And it is functionally independent if $J \neq 0$ if $x = uv$ and $y = 4/v$, then prove that $JJ' = 1$ $x = uv, y = \frac{u}{v}$

$$\therefore u = \sqrt{xy}, v = \sqrt{\frac{x}{y}}$$

$$\therefore J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{y}}{2\sqrt{x}} & \frac{\sqrt{x}}{2\sqrt{y}} \\ \frac{1}{2\sqrt{xy}} & \frac{-\sqrt{x}}{2y\sqrt{y}} \end{vmatrix}$$

$$\therefore J = -\frac{1}{4y} - \frac{1}{4y} = -\frac{1}{2y}$$

$$\begin{aligned}
\therefore J' &= \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right| = \left| \begin{array}{cc} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{array} \right| = -\frac{u}{v} - \frac{u}{v} = -2\frac{u}{v} \\
&= -2\frac{\sqrt{xy}}{\sqrt{x/y}} \\
&= -2 \cdot \sqrt{x} \cdot \sqrt{y} \cdot \frac{\sqrt{y}}{\sqrt{x}}
\end{aligned}$$

$$\therefore J \cdot J' = -\frac{1}{2y}x - 2y = 1$$

- If $rt - s^2 < 0$. neither minimum nor maximum exists - If $rt - s^2 = 0$, Further verification is needed.

Langrangis methud

For finding max/min of 3 or more variables find max/min of f , wrtconitruint Working rule

- let $F = f(x, y, z) + \lambda \phi(x, y, z)$
- Simplifying equations: $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$ we get \times rel.blw. x, y, z and substituting in any of equation, we get their values

Examine following functions for extreme value

a) $f(x, y) = x^4 + y^4 + 4xy - 2x^2 - 2y^2$

$$f(x, y) \rightarrow \lambda \beta(x, y)$$

A)

$$\frac{\partial f}{\partial x} = 4x^3 + 4y - 4x, \quad \frac{\partial f}{\partial y} = 4y^3 + 4x - 4y^2$$

$$\text{let } \frac{\partial f}{\partial x} = 0 \Rightarrow 4x^3 + 4y - 4x = 0 \Rightarrow y = x - x^3 - 1 \quad \frac{\partial f}{\partial y} = 0 \Rightarrow 4y^3 + 4x - 4y = 0 \Rightarrow x = y - y^3 - (2)$$

$$\text{let } r = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 4$$

$$t = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$$

$$\therefore 7 + -5^2 = (12x^2 - 4)(12y^2 - 4) - 16$$

$$(1) + (2): -y^3 - x^3 + x + y = x + y$$

$$\Rightarrow x^3 + y^3 = 0 \Rightarrow x = -y$$

Substitute these in ω :

$$-x = x - x^3 \Rightarrow x^2 = 2x \Rightarrow x = +\sqrt{2}$$

substitute in (2):

$$-y = y - y^3 \Rightarrow y = \pm\sqrt{2}$$

$$\text{for } x = \pm\sqrt{2}, y = \pm\sqrt{2}, \quad \gamma t - s^2 = 384$$

to

$$\therefore r + -5^2 > 0$$

$$r = 20 > 0$$

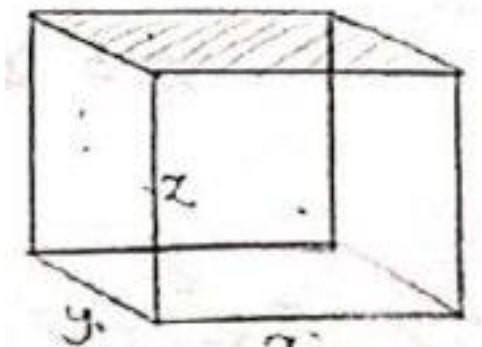
$$\therefore \text{minima exists at } x = \pm\sqrt{2}, y = \pm\sqrt{2}$$

$$\therefore \text{minimum value is } +(\pm\sqrt{2}, \pm\sqrt{2}) = 8 - 8$$

b) $f(x) = x^5 - 5x^4 + 5x^3 - 1$

2. A rectangular box open at the top is to have the volume 32ft^3 . find dimensions of box requiring least amount of material for its construd $v = xyz = 32\text{ft}^3$

$$LSA := A = xy + 2xz + 2yz$$



let $E = v$

let $f = xyz - 32$

$F = f + \lambda A$

$= xyz - 32 + \lambda (xy + 2xz + 2yz)$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow yz + \lambda(y + 2z) = 0 - 10 \frac{\partial F}{\partial z} = 0 \Rightarrow xy + \lambda(2y + 2x) = 0$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow xz + \lambda(x + 2z) = 0 - 2$$

#

x. (1) $x - (2) y ::$

$$\Rightarrow 2 \times 2(x - y) = 0 \Rightarrow \xrightarrow{\rightarrow y=0} y$$

(2). $y - (9) z$

$$\Rightarrow \lambda(xy + zzy - 2yz - zz) = 0$$

$$x\lambda(y + 2z) = 0 \Rightarrow \begin{cases} x = 0 \\ y = +2z \end{cases}$$

$$\therefore x = y, y = 2z, z = y/2$$

Substitute in $x_y z = 32$

$$\therefore 32 = y \cdot y \cdot y/2 \Rightarrow y = 64 \Rightarrow y = 24$$

$$32 = x \cdot x \cdot x/2 \Rightarrow x = 4$$

$$32 = 2z \cdot 2z \cdot z \Rightarrow z = 2$$

3 find max & min of points, $A(3, 4, 12)$ from the sphere $x^2 + y^2 + z^2 = 1$

A) Let $P = (a, b, c)$ on point closest to A. $\therefore P = (x_3, \lambda 4, > 12)$



$$\therefore |P| = 1 \Rightarrow 9\lambda^2 + 16\lambda^2 + 144\lambda^2 = 1 \Rightarrow 169\lambda^2 = 1 \Rightarrow \lambda = 1/13$$

$$\therefore = \left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$$

$$\therefore \text{min. distance} = \sqrt{\left(3 - \frac{3}{13}\right)^2 + \left(4 - \frac{4}{13}\right)^2 + \left(12 - \frac{12}{13}\right)^2}$$

$$= \underline{\underline{12}}$$

\therefore max distance on is to point on opposite side = $12 + 2 = 14$ sunnts

let p be a point on sphere $P = (x, y, z)$

$$\therefore \text{dist}(A, P)^2 = (3 - x)^2 + (4 - y)^2 + (12 - z)^2 = D \quad \frac{\partial F}{\partial y} = 0 \Rightarrow$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow -2(4 - y) + 2xy = 0 \Rightarrow y = \frac{4}{\lambda - 1}$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow -2(12 - z) + 2z = 0 \Rightarrow z = \frac{12}{\lambda - 1}$$

$$\therefore x^2 + y^2 + z^2 = 1$$

$$\Rightarrow \frac{1}{(\lambda^2 - 1)^2} (9 + 16 + 144) = 1$$

$$\lambda - 1 = \pm 13 \Rightarrow \lambda = -12 \text{ or } 14$$

for $\lambda = 14$

$$x = \frac{-3}{-13}, y = \frac{4}{-13}, z = \frac{12}{-13}$$

$$\lambda = 14 \quad x = \frac{3}{13}, y = \frac{4}{13}, z = \frac{12}{13}$$

$$\text{let } P = \left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right), Q = \left(\frac{-3}{13}, \frac{-4}{13}, \frac{-12}{13}\right)$$

$$\therefore AP = \sqrt{\left(3 - \frac{3}{13}\right)^2 + \left(4 - \frac{4}{13}\right)^2 + \left(\frac{12}{13} - 12\right)^2} = 12$$

$$AQ = \sqrt{\left(3 + \frac{3}{13}\right)^2 + \left(4 + \frac{4}{13}\right)^2 + \left(12 + \frac{12}{13}\right)^2} = 14$$

\therefore minimum distance = 12, max = 14

4. Find shortest distance from the origin to hyperbola

$$x^2 + 8xy + 7y^2 = 225$$

A) let $P(x, y)$ be a point on hyperbola.

distance from $O \rightarrow P = D = \sqrt{x^2 + y^2}$

$$D^2 = x^2 + y^2$$

let $F = D^2 + \lambda(x^2 + 8xy + 7y^2 - 225)$.

$$\frac{\partial F}{\partial x} = 2x + \lambda(2x + 8y) = 0 \quad (1)$$

$$\frac{\partial F}{\partial y} = 2y + \lambda(14y + 8x) = 0 \quad (2)$$

let $F = D + x(x^2 + y^2 + z^2 - 1) \therefore \frac{\partial F}{\partial x} = 0 \Rightarrow -2(3 - x) + xx = 0 \Rightarrow x = \frac{6}{2\lambda - 2} = \frac{3}{\lambda - 1}$

$$\begin{aligned} x + \lambda(x + 4y) &= 0 - 11 &\Rightarrow (1 + \lambda)x + 4\lambda y &= 0 \\ y + \lambda(7y + 4x) &= 0 - (2) &\Rightarrow (1 + 7\lambda)y + 4\lambda x &= 0 \end{aligned}$$

(3) xy

$$(1 + \lambda)xy + 4\lambda y^2 = 0 \quad - (0) \quad (6) \quad (7) :$$

$$(4) \times x \cdot (1 + 7\lambda)xy + 4\lambda x^2 = 0 - (6)$$

$$(1 + \lambda - 1 - 7\lambda)xy + 4\lambda(y^2 - x^2) = 0$$

$$(-6\lambda)xy + 4\lambda(y^2 - x^2) = 0$$