

Integral Calculus

$$1. \int \ln(x) dx$$

$$= \int \ln(x) \times 1 dx = \ln(x) \int 1 dx - \int \frac{1}{x} \cdot \int 1 dx dx \quad \int u dv = u \int v - \int u \int v'$$

$$= x \ln(x) - x + c$$

$$= (x - 1) \ln(x) + c$$

$$2. \int e^x x^5 dx = \int x^5 e^x dx = x^5 \int e^x - \underbrace{\int 5x^4 \cdot \int e^x dx}_{-x_1}$$

$$= x^5 e^x - 5 \int x^4 e^x$$

$$= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120x e^x$$

$$= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120x e^x - 120e^x$$

$$3 \int (x^5 + 8x^2 + 1) dx = \frac{x^6}{6} + \frac{8}{3}x^3 + x + c$$

$$5. \int \frac{2(x+5)}{x^2+10x+25} d\frac{2}{x+5} = 2 \ln(x+5) + c$$

$$5). \int \sin(3x) \cdot e^{2x} dx \sim \left\{ \int e^{ax} \cdot \sin(bx) dx = \left[\frac{e^{ax}}{a^2+b^2} [a \sin(bx) - b \cos(bx)] \right] \right\} \sim \frac{e^{2x}(2 \sin(3x) - 3 \cos(3x))}{13} \left\{ \int e^{ax} \cos(bx) dx \right\}$$

$$6. \int e^x (\sec(x) + \sec(x) \tan(x)) dx = 8 \int x^2 \cdot \tan^{-1}(x)$$

$$= \int \tan^{-1}(x) x^2 - dx$$

$$= \tan^{-1}(x) \cdot \frac{x^3}{3} - \int \frac{1}{1+x^2} \cdot \frac{x^3}{3} dx \quad x^3 \rightarrow t \quad dt = 3x^2 \cdot dx$$

$$\int \frac{x}{1+x^2} \cdot x^2 dx$$

$$\Rightarrow \frac{1}{2} \int \frac{t-1}{z} dt \Rightarrow \frac{1}{2} \int 1 - \frac{1}{2} \int \frac{1}{t}$$

$$= \frac{1}{2} (x^2 + 1) - \frac{1}{2} \ln(x^2 + 1)$$

$$\Rightarrow \tan^{-1}(x) \cdot \frac{x^3}{3} - \frac{1}{6} (x^2 + 1) + \frac{1}{6} \ln(x^2 + 1) + c$$

$$\int \frac{x+1}{x^2+6x+25} =$$

$$\rightarrow \frac{1}{2} \int \frac{2x+6}{x^2+6x+25} dx - 2 \int \frac{dx}{x^2+6x+25} \quad 2x+6 = a$$

$$b = x+1 - (2x+6)$$

$$t_1 = x^2 + 6x + 68$$

$$du = (2x+6)dx$$

$$\rightarrow \frac{1}{2} \int \frac{du}{u} = -2 \int \frac{dx}{(x+3)^2 + 16}$$

$$\Rightarrow \frac{1}{2} \ln(u) = -2 \int \frac{ds}{s^2 + 16} \Rightarrow \dots = -2 \cdot \frac{1}{8} \int \frac{dp}{p^2 + 1} \quad p^2 = s^2/16$$

$$\Rightarrow \dots = -\frac{1}{2} \tan^{-1}(p) \quad p = s/4$$

$$\frac{dp}{ds} = \frac{1}{4} \quad ds = 4dp$$

$$\Rightarrow \frac{\ln(4)}{2} - \frac{1}{2} \tan^{-1}\left(\frac{5}{4}\right)$$

$$\Rightarrow \frac{\ln(x^2+6x+25)}{2} - \frac{1}{2} \tan^{-1}\left(\frac{x+3}{4}\right)$$

Quadrature

Calculation of area of curve bounded by a boundary finding

Area bounded by a curve $y = f(x)$ within $[a, b]$ in x -axis is

$$A = \int_a^b y dx \equiv \int_a^b f(x) dx$$

Also if the curve is $x = f(y)$, then area within $[a, b]$ in y -axis is:

$$A = \int_a^b x dy = \int_a^b f(y) dy$$

- If two curves $y_1 = f(x)$ $y_2 = \phi(x)$ intersect at the points whose coordinates are (a, \dots) (b, \dots) then the area enclosed by curve is given by. $A = \int_a^b (f(x) - \phi(x)) dx$

1 Find area bounded by the curve, $y^2 = 4ax$ and the ordinat at $x = h$,

$$A) \quad A = \int_0^h \sqrt{4ax} dx = 2\sqrt{a} \sqrt{x} dx = 2\sqrt{a} \cdot \frac{2}{3} \cdot \frac{b}{x^{3/2}}$$

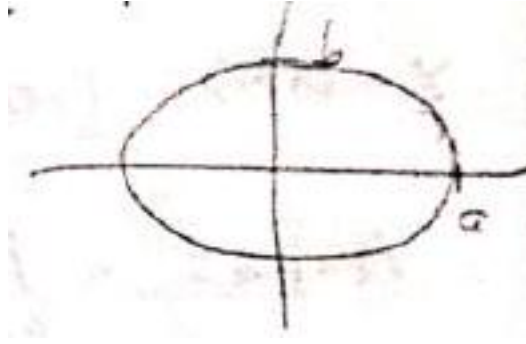
$$a = 6\sqrt{6oh} \Rightarrow \frac{4}{3} \sqrt{a} \cdot h^{3/2}$$

2 Find area enclosed by curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

A.)

$$\text{let } y = 0, \quad \therefore x = a$$

$$x = a, \quad \therefore y = b$$



$\therefore C$

$$\begin{aligned}\therefore y &= b\sqrt{1 - \frac{x^2}{a^2}} = \frac{ab}{a}\sqrt{a^2 - x^2} \\ \therefore A &= 4 \int_0^a y dx = 4 \frac{ab}{a} \int_0^a \sqrt{a^2 - x^2} dx \\ &= 4 \frac{ab}{a} \left[\frac{x}{2} \sqrt{a^2 - x^2} - \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) \right]_0^a \\ &= 4 \frac{ab}{a} \cdot \frac{a^2}{2} \sin^{-1}(1) \\ &= \pi ab\end{aligned}$$

3. Find area enclosed blw lineff and curve $y = x^2 - 6x + 4$

$$A) y = 1 - 2x = \{x^2 - 6x + 4 \Rightarrow x^2 - 4x + 3 = 0$$

$$f = x^2 - 6x + 4, \phi = 1 - 2x$$

$$= 044 \rightarrow 0 \pm 1 \Rightarrow 13$$

$$\therefore \text{Area} = \int_{10}^3 ((f - \phi) dx$$

10/1 4find area bounded by the curve $y = 3x - x^2$ the x -axis and line $x = 3, x = 0$

$$A) \text{Area} = \int_0^3 (3x - x^2) dx$$

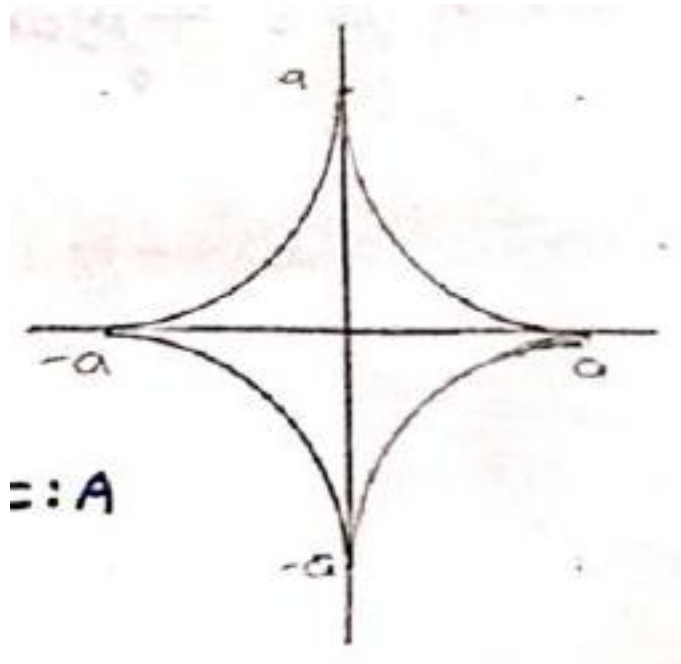
$$= \left[\frac{3}{2}x^2 - \frac{x^3}{3} \right]_0^3 = \frac{27}{2} - 9 = \frac{9}{2}$$

5. Find ane of asteroid: $x^{2/3} + y^{2/3} = a^{2/3}$

$$A) x^{2/2} + y^{2/3} = a^{2/3}$$

$$y = (a^{2/3} - x^{2/3})^{3/2}$$

$$\therefore \text{Area of asteroid} = 4 \int_0^a (a^{2/3} - x^{2/2})^{3/2} dx =: A$$



$$a \sin^3(\theta) = 0 \Rightarrow \theta = 0$$

$$\therefore dx = 3a \sin^2(\theta) \cos(\theta) d\theta$$

$$a \sin^3(\theta) = a \Rightarrow \theta = \pi/2$$

$$\therefore A \equiv 4 \int_0^{\pi/2} (a^{2/3} - a^{2/3} \sin^2(\theta))^{3/2} 3a \sin^2(\theta) \cos(\theta) d\theta$$

$$= 4 \int_0^{\pi/2} a (1 - \sin^2(\theta))^{2/2} \cdot 3a \sin^2(\theta) \cos(\theta) d\theta$$

$$= 12a^2 \int_0^{\pi/2} \cos^3(\theta) \sin^2(\theta) \cos(\theta) d\theta$$

$$= 12a^2 \int_0^{\pi/2} \cos^4(\theta) \sin^2(\theta) d\theta$$

$$= 12a^2 \cdot \frac{3 \times 1 \times 1}{6 \times 4 \times 2} \times \frac{\pi}{2}$$

$$\int_0^{\pi/2} \cos^m \theta \sin^n \theta d\theta$$

$$m, n \in 2\mathbb{N}$$

$$= \frac{(m-1)(m-3) \dots (m-1)(-1)}{(m+2)(m+4) \dots (m+n-1)}$$

$$= \frac{3}{8} a^2 \pi$$

fund $m, n \in 2^{r+1}$

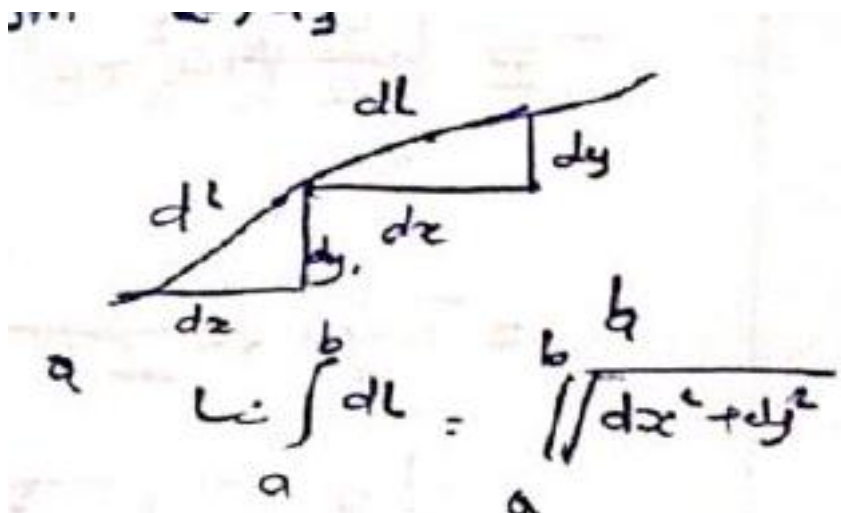
Length (perimeter) of Carve (Rectification)

- let $y = f(x)$. length of carve from $[0, a]$

$$d = \int_0^{\infty} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

If $x = f(y)$

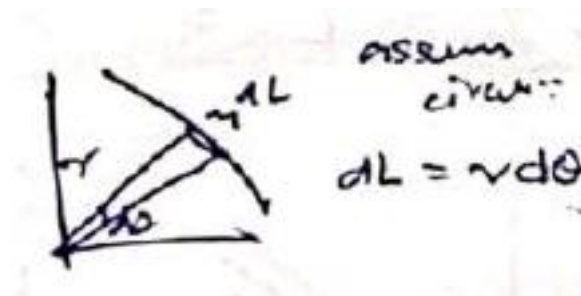
$$L = \int_0^a \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$



$$L = \int_a^b \sqrt{dx^2 + dy^2}$$

- If $x = f(t), y = \phi(t), L = \int_0^a \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$

- If $\gamma' = f(\theta)$, $L = \int_0^a \sqrt{\gamma^2 + \left(\frac{d\gamma}{d\theta}\right)^2} \cdot d\theta$



1. Find the length of the curve $y^2 = 4ax$, measure from the vertex to one extremity of Latus rectum

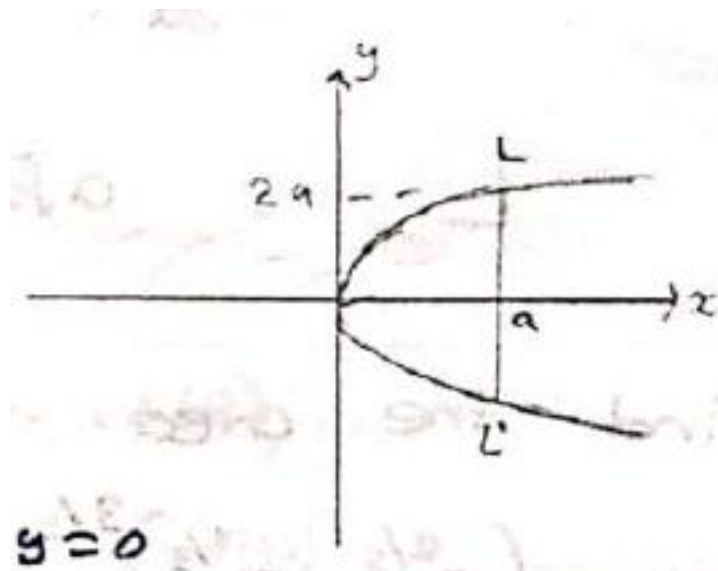
Ans $L = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$

$$y = 2\sqrt{ax}$$

$$\frac{dy}{dx} = \frac{a}{\sqrt{ax}} = \sqrt{\frac{a}{x}}$$

$$\begin{aligned} \therefore L &= \int_0^a \sqrt{1 + \frac{a}{x}} dx \\ &= \int_0^a \frac{\sqrt{x+a}}{\sqrt{x}} dx \end{aligned}$$

$$2. \frac{1}{2} \frac{5}{\sqrt{an}}$$



$$x = 0, y = 0$$

$$x = a, y = 2a$$

$$x = \frac{y^2}{4a} \quad \frac{dx}{dy} = \frac{y}{2a}$$

$$\begin{aligned} L &= \int_0^{2a} \sqrt{1 + \frac{y^2}{4a^2}} dy \\ &= \frac{1}{2a} \cdot \int_0^{2a} \sqrt{(2a)^2 + y^2} dy \end{aligned}$$

$$\sinh^{-1}(x) = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

$$\begin{aligned} &\therefore = \left[\frac{1}{2} \sqrt{4a^2 + y^2} + \frac{(2\partial^2)}{2} \ln \left(y + \sqrt{4a^2 + y^2} \right) \right]_0^{2a} \\ &= \frac{1}{2a} \cdot \left(2a^2 \sqrt{2} + \frac{2a^2}{2a} (2a + 2a\sqrt{2}) \right) \end{aligned}$$

$$L = a\sqrt{2} + \ln(2a(1 + \sqrt{2}))$$

$$L = \frac{1}{2a} \left[\frac{y}{2} \sqrt{y^2 + a^2} + \frac{2a^2}{2} \sinh^{-1} \left(\frac{y}{a} \right) \right]_0^{2a}$$

$$= \frac{1}{2a} \left(\frac{2a}{2} \sqrt{(2a)^2 + (2a)^2} + \frac{2a^2}{2} \sinh^{-1} \left(\frac{2a}{a} \right) \right)$$

$$= \frac{1}{2a} \left(a^2 2\sqrt{2} + 2a^2 \ln(1 + \sqrt{2}) \right)'$$

$$L = a\sqrt{2} + a \ln(1 + \sqrt{2})$$

2. Find the area of astroid $x^{2/3} + y^{2/3} = a^{2/3}$

A) $y = (a^{2/3} - x^{2/3})^{3/2}$

$$\begin{aligned} \text{Area of single piece} &= \int_0^a y \cdot dx \\ &= \int_0^a (a^{2/3} - x^{2/3})^{3/2} dx \end{aligned}$$

let $x = a \sin^3(\theta) \Rightarrow dx = 3a \sin^2(\theta) \cdot \cos(\theta) \cdot d\theta$

\rightarrow bounds are $x = 0 \mapsto \theta = 0$

$$x = a \mapsto \theta = \pi/2$$

$$\therefore \text{Area of single piece} = \int_0^{\pi/2} (a^{2/3} - 0^{2/3} \sin^2 \theta)^{3/2} \cdot 3a \sin^2 \theta \cdot \cos(\theta) \cdot d\theta$$

$$\begin{aligned} &= 3a^2 \int_0^{\pi/2} \cos^4(\theta) \sin^2(\theta) d\theta \\ &= 3a^2 \times \frac{3 \times 1 \times 1}{6 \times 4 \times 2} \times \frac{\pi}{2} = \frac{3\pi}{32} a^2 \end{aligned}$$

$$\therefore \text{Area of astroid} = 4 \times \frac{3\pi a^2}{32} = \frac{3}{8} \pi a^2$$

$$\int_0^{\pi/2} \cos^m(\theta) \sin^n(\theta) d\theta = (m-1)(m-3)(m-5) \cdots (m-\dots) \\ \times (n-1)(n-3)(n-5) \cdots \frac{(n-\dots)}{20} \\ \times \frac{1}{(m+n)(m+n-2)(m+n-4)(m+n-\dots)} \frac{\pi}{2}$$

Area of Polar functions

$$\frac{1}{2} \int_a^b r^2 d\theta$$

1. Find whole area of circle: $r = 2a \cos(\theta)$,

- Cardioid: $r = a(1 - \cos(\theta))$

A) Circle

$$\text{let } r = 0 \Rightarrow \theta = \frac{\pi}{2} \\ r = 2a \Rightarrow \theta = 0 \\ \therefore \text{Area} = 2 \int_0^{\pi/2} \frac{1}{2} \cdot r^2 d\theta = 4a^2 \int_0^{\pi/2} \cos^2(\theta) d\theta \\ = 4a^2 \cdot \frac{1}{2} \pi = \pi a^2 \\ \int_0^{\pi/2} \cos^2(\theta) d\theta = \frac{(n-1)(n-3) \cdots (n-\dots)}{n(n-2)(n-4)(n-\dots)} \times \frac{\pi}{2}$$

Cardioid

$$r = a(1 - \cos(\theta)) \\ r = 0 \Rightarrow \theta = \pi, \therefore \theta \in [0, \pi] \\ \therefore \text{Area} = 2 \int_0^{\frac{\pi}{2}} r^2 d\theta = a^2 \int_0^{\pi} 4(1 - \cos(\theta))^2 d\theta \\ = a^2 \int_0^{\pi} \left(2 \sin^2 \frac{\theta}{2}\right)^2 d\theta = 4a^2 \int_0^{\pi} \sin^4(\theta/2) d\theta \\ \text{let } \phi = \theta/2 \Rightarrow \phi \in [0, \pi/2] \\ \therefore A = 4a^2 \int_0^{\pi/2} \sin^2(\phi) d\phi = 8a^2 \int_0^{\pi/2} \sin^4(\phi) d\phi \\ = 8a^2 \times \frac{3 \times 1}{4 \times 2} \times \frac{\pi}{2} = \frac{3}{2} \pi a^2$$

Surface area of "Curves

- Revolving about x -axis: $\int_a^b 2\pi y ds$ where $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$ in cartesian

$$ds = \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \text{ in parametric}$$

$$ds = \sqrt{r^2 + \left(\frac{dd}{d\theta}\right)^2} d\theta \text{ in polar}$$

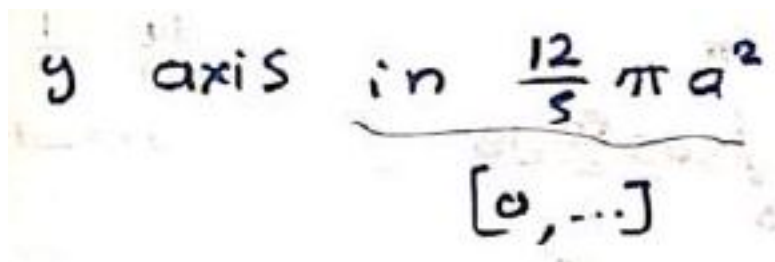
1. Find area of the surface generated by revolving the parabola $y^2 = 4ax$ about x axis from origin to $x = a$. $A = \int_0^a 2\pi y \cdot ds$

$$\frac{dy}{dx} = 2\sqrt{a} \cdot \frac{1}{2} \frac{1}{\sqrt{x}} = \sqrt{\frac{a}{x}} \quad y = 2\sqrt{ax}$$

$$\therefore dS = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{\frac{x+a}{x}} dx$$

$$\begin{aligned} \therefore A &= \int_0^a 2\pi y \cdot ds = 2\pi \int_0^a 2\sqrt{ax} \sqrt{\frac{x+a}{x}} dx \\ &= 4\pi\sqrt{a} \int_0^a \sqrt{x+a} dx \\ &= \frac{8}{3}\pi\sqrt{a} [(x+a)^{3/2}]_0^a \\ &= \frac{8}{3}\pi\sqrt{a} a^{3/2} (2^{3/2} - 1) = \frac{8}{3}\pi a^2 (2^{2/2} - 1) \end{aligned}$$

- 2 Show that the SA. of solid generated by revolving the curve $x = a \cos^3(\theta), y = a \sin^3(\theta)$ about



Handwritten text: y axis in $\frac{12}{5} \pi a^2$ [0, ...]

$$A \quad x = 0 \mapsto \theta = \pi/2, \quad \theta \in [0, \pi/2]$$

$$\begin{aligned} \therefore S &= 2 \cdot \int_0^{\pi/2} 2\pi \cdot x ds \quad ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \\ &= 4\pi \times 3a^2 \int_0^{\pi/2} \cos(\theta) \sin(\theta) d\theta \\ &= 3a \sin(\theta) \cos(\theta) d\theta \\ &= a \cdot 12\pi a^2 \cdot \frac{1}{5} = \frac{12\pi a^2}{5} \end{aligned}$$

- 3 find length of curve $x = t^2, y = t^3$ t $t \in [0, 1]$

A. $\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2$

$$\begin{aligned} \therefore L &= \int_6^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^2 \sqrt{(4t^2 + 9t^4)} dt \\ &= \int_0^1 t \sqrt{4 + 9t^2} dt, \text{ let } 4 + 9t^2 = u \Rightarrow \frac{du}{dt} = 18t, dt = \frac{du}{18} \\ &\equiv \int_4^{13} \sqrt{u} \frac{du}{18} \\ t = [0, 1] &\Rightarrow u = [4, 13] \\ &= \frac{1}{27} [u^{3/2}]_4^{13} = \frac{1}{27} (13^{3/2} - 4^{3/2}) \end{aligned}$$

4. Find the perimeter of cardioid $r = a(1 + \cos(\theta))$

A) Since eqn remains unchanged when $\theta = -\theta$ $\theta \in [0, \pi]$ $L = 2 \int_0^\pi \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta, \quad \frac{dr}{d\theta} = -a \sin(\theta)$

$$\begin{aligned} &= 2 \int_0^\pi \sqrt{a^2 (1 + \cos(\theta))^2 + a^2 \sin^2(\theta)} d\theta \\ &= 2 \int_0^\pi \sqrt{2a^2 (1 + \cos(\theta))} d\theta = 4a \int_0^\pi \cos \frac{\theta}{2} d\theta \\ &= 8a \left[\sin \frac{\theta}{2} \right]_0^\pi = 8a \end{aligned}$$

Volume of Solid of

obtained by revolving $y = f(x)$ about x -axis; $[a, b]$

$$v = \int_a^b \pi y^2 dx$$

1. Prove that the volume of a right circular cone of height h and base radius r is $\frac{1}{3}\pi r^2 h$

A $V = \int_0^h \pi y^2 dx$

$$y = r/hx$$

$$\therefore v = \int_0^h \pi \frac{r^2}{h^2} x^2 dx = \pi \frac{r^2}{h^2} \cdot \frac{1}{3} [x^3]_0^h = \frac{1}{3} \pi r^2 h$$

2. Find the volume formed by revolution of loop of curve $y^2(a+x) = x^2(3a-x)$ about x -axis

le! $y = 0 \Rightarrow x^2(3a-x) = 0 \Rightarrow x = 0, x = 3a$

$$\begin{aligned}
\text{volume} &= \int_0^{3a} \pi y^2 dx = \pi \int_0^{3a} \frac{x^2(3a-x)}{a+x} dx \\
&= \pi \int_0^{3a} \frac{-x^3 + 3ax^2}{a+x} \\
&= \pi \int_0^{3a} \left(-x^2 + 4ax - 4a^2 + \frac{4a^3}{x+a} \right) dx \\
&= \pi \left[-\frac{x^3}{3} + 2ax^2 - \frac{4}{3}a^3x + 4a^3 \ln(x+a) \right]_0^{3a} \\
&= \pi [-9a^3 + 18a^3 - 12a^3 + 4a^3 \ln(4a)] \\
&= a^3 \pi (-3 + 4 \ln(4)) \\
&\quad \frac{-x^2+4ax-4a}{-x^3+3ax^2} \\
&\quad \frac{-x^3-ax^2}{4ax^2} \\
&\quad \frac{4ax^2+4a^2x}{-4a^2x} \\
&\quad \frac{-4a^2x-4a^3}{4a^3}
\end{aligned}$$

Multiple integrals

integrals undergoing

Double integrals

$$1. \underbrace{2}_0 \underbrace{2}_{\text{surdh.}} x_0^{x^2} xy \cdot dy \cdot dx$$

A limits of y are $0 \rightarrow x^2$, limits of $x : 0 \rightarrow 2$

$$\therefore \text{Ans: } \int_0^2 x_5^2 dx = \left[\frac{1}{2} \cdot \frac{x^6}{6} \right]_0^2 = 16/3$$

$$2. \int_0^6 \int_0^{y^3} x^2 y \cdot dy \cdot dx \Rightarrow \int_0^5 \int_0^{y^3} x^2 y \cdot dx \cdot dy$$

$$= \int_0^5 \left[y \cdot \frac{x^3}{3} \right]_0^{y^3} dy = \int_0^5 y^{14}/3 dy = 51/33$$

Triple Integral

$$1. \int_0^1 \int_0^2 \int_0^2 x^2 y z dx dy dz$$

$$\begin{aligned}
fy &= y \\
xyy
\end{aligned}$$

$$A) \int_0^1 \int_0^2 \left(\int_1^2 x^2 \cdot yz \int_1 dx \right) dy \cdot dx.$$

$$\begin{aligned}
&= \int_0^1 \int_0^2 \left[\frac{x^2 y z^2}{2} \right]_{z^1}^2 dy \cdot dx = \int_0^1 \int_0^2 \frac{x^2 y \cdot 4 - x^2 y}{2} dy \cdot dx \\
&= \int_{-0}^{1-2} \int_0^2 \frac{3}{2} x^2 y dy \cdot dx \\
&= \int_0^1 \left[\frac{3x^2 y^2}{4} \right]_0^2 dx \\
&= \int_0^1 3x^2 dx = x^3 \Big|_0^1 = 1
\end{aligned}$$

$$2 \int_{-c}^c \int_{-b}^b \int_{-a}^0 (x^2 + y^2 + z^2) dx dy \cdot dz$$

A)

$$\begin{aligned}
&\int_{-c}^c \int_{-b}^b \left(x^2 z + y^2 z + \frac{z^3}{3} \right)_{-a}^a dy \cdot dx = \int_{-c}^c \int_{-b}^b \left(2ax^2 + 2ay^2 + \frac{2}{3}a^3 \right) dy \cdot dx \\
&= \int_{-1}^c \left[2ax^2 y + \frac{2ay^3}{3} + \frac{2}{3}a^3 y \right]_{-b}^b dx \\
&= \int_{-c}^c \left(4abx^2 + \frac{4ab^3}{3} + \frac{4}{3}a^3 b \right) dx \\
&\Rightarrow \left[\frac{4abx^3}{3} + \frac{4}{3}ab^3 x + \frac{4}{3}a^3 bx \right]_{-c}^c = \frac{8}{3}abc^3 + \frac{8}{3}ab^3 c + \frac{8}{3}a^3 bc \\
&= \frac{8}{3}abc (a^2 + b^2 + c^2)
\end{aligned}$$

$$3 \int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x + y + z) dx dy dz$$

$$\begin{aligned}
&\frac{(x+z)^2}{2} - \frac{(x-z)^2}{2} \\
&= \frac{4xz}{2}
\end{aligned}$$

$$A) \rightarrow \int_{x-x}^{x+1} (x + y + z) dy \Big]$$

$$I = \left[xy + \frac{y^2}{2} + zy \right]_{x-z}^{x+2}$$

$$\begin{aligned}
I &= x(x+z) + \frac{(x+z)^2}{2} + z(x+z) \\
&\quad - \left(x(x-z) + \frac{(x-z)^2}{2} + z(x-z) \right) \\
&= x(2z) + 2xz + z(2z) \\
&= 2xz + 2xz + 2xz^2 \\
&= 4xz + 2z^2
\end{aligned}$$

$$\begin{aligned}
&\int_{-1}^1 \left[\int_0^z (4xz + 2z^2) dx \right] dz \\
&\Rightarrow \int_{-1}^1 (2z^3 + 2z^3) dz = \left[\frac{4z^4}{4} \right]_{-1}^1 = 0
\end{aligned}$$

A) $\int_0^1 \int_0^x \int_0^{x+y} (2x + y - z) dz dy dx$

$$\begin{aligned}
 I_3 &= \int_0^{x+2y} (2x + y - z) dz \\
 &= \left[2xz + yz - \frac{z^2}{2} \right]_0^{x+2y} = 2x^2 + 2xy + xy + y^2 - \frac{(x+y)^2}{2} \\
 &= 2x^2 + 3xy + y^2 - \frac{x^2 + y^2 + 2xy}{2} \\
 &= \frac{3x^2 + y^2 + 4xy}{2}
 \end{aligned}$$

$\therefore I_2 = \int_0^\infty I_3 dy = \frac{1}{2} \left[3x^2y + \frac{y^3}{3} + 2xy^2 \right]_0^\infty$
 $= \frac{1}{2} \left(3x^3 + \frac{x^3}{3} + 2x^3 \right)$

$$\begin{aligned}
 \frac{1}{6} (9x^3 + 6x^3 + x^3) &= \frac{16x^3}{6} = \frac{8}{3}x^3 \\
 I &= \int_0^1 \frac{8}{3}x^3 dx = \left[\frac{2}{3}x^4 \right]_0^1 = \frac{2}{3}
 \end{aligned}$$

5. $\int_0^1 \int_0^x \int_0^{x+y} (2x + y - 1) dz dy dx$

$$\begin{aligned}
 &\int_0^1 \left(\int_0^x (2x^2 + 2xy + xy + y^2 - x - y) dy \right) dx \\
 &= \int_0^1 \int_0^x (2x^2 + 3xy + y^2 - x - y) dy dx \\
 &= \int_0^1 [2x^2y + 3/2xy^2 + y^3/3 - xy - y^2/2]_0^x dx \\
 &= \int_0^1 \left(2x^3 + \frac{3}{2}x^3 + \frac{x^3}{3} - x^2 - \frac{x^2}{2} \right) dx \\
 &= \left[\frac{x^4}{2} + \frac{3x^4}{8} + \frac{x^4}{12} - \frac{x^3}{3} - \frac{x^3}{6} \right]_0^1 = \frac{1}{2} + \frac{3}{8} + \frac{1}{12} - \frac{1}{3} - \frac{1}{6} \\
 &= 11/24
 \end{aligned}$$

A $\int_0^1 \int_0^x [2xz + yz - z]_0^{x+y} dy dx$

$$\begin{aligned}
& 6 \int_1^2 \int_x^{4/x} \int_0^{xy} xy \cdot dx dx dy \\
& \int_1^2 \int_x^{4/x} [xyz]_0^{xy} dx dy = \int_1^2 \int_x^{6/x} x^2 y^2 dy \cdot dx \\
& = \int_1^2 \left[\frac{x^2 y^3}{3} \right]_x^{4/x} dx \\
& = \int_1^2 \frac{x^2}{3} \left(\frac{64}{x^3} - x^3 \right) dx = -4 \\
& = \int_1^2 \left[\frac{64}{3} \cdot \frac{1}{x} - \frac{x^5}{3} \right] dx \\
& = \left[\frac{64}{3} \ln(x) - \frac{x^6}{18} \right]_1^2 = \frac{64}{3} \ln(2) - \frac{64}{18} - \frac{64}{3} \ln(1) + \frac{1}{18} \\
& = \frac{64}{3} \ln(2) - \frac{63}{18} = \frac{64}{3} \ln(2) - \frac{7}{2}
\end{aligned}$$

7. $\int_0^4 \int_0^{2\sqrt{2}} \int_0^{\sqrt{4x-x^2}} dy \cdot dx dy$

$$\begin{aligned}
& = \int_0^4 \int_0^{2\sqrt{2}} \sqrt{4z-x^2} dy \cdot dx dx = \\
& = \int_0^4 \int \sqrt{2\sqrt{2}} \sqrt{(2\sqrt{z})^2 - x^2} dx \cdot dz \\
& = \int_0^4 \left[\frac{x}{2} \sqrt{4z-x^2} + 2z \sin^{-1} \left(\frac{x}{2\sqrt{2}} \right) \right]_0^{2\sqrt{2}} dz \\
& = \int_0^4 (\sqrt{2}\sqrt{4z-8} + \pi z) dx \\
& = \int_0^4 (2\sqrt{2}\sqrt{z-2} + \pi z) dz = \\
& = \left[2\sqrt{2} \cdot \frac{2}{3} (z-2)^{3/2} + \frac{\pi z^2}{2} \right]_0^4 = \frac{4}{3} \sqrt{2} 2^{3/2} + 8\pi - \frac{4\sqrt{2}}{2} (-2)^{3/2}
\end{aligned}$$

8. Calculate.

$\iint r^3 dr d\theta$ over the area included between
circles $\gamma_1 = 2 \sin(\theta)$, $\gamma_2 = 4 \sin(\theta)$

A) $r_1 = 2 \sin(\theta) \rightarrow \theta \in [0, \pi]$

$$\therefore \int_0^\pi \int_{r=2\sin(\theta)}^{r=4\sin(\theta)} r^3 dr d\theta$$

$$\begin{aligned}
& = \int_0^\pi \left[\frac{r^4}{4} \right]_{2\sin(\theta)}^{4\sin(\theta)} d\theta = \int_0^\pi \left(\frac{4^4}{4} \cdot \sin^4(\theta) - \frac{2^4}{4} \sin^4(\theta) \right) d\theta \\
& \quad := \int_0^\pi 60 \sin^4(\theta) d\theta = 2 \cdot 6 \int_0^{\pi/2} \sin^4(\theta) d\theta \\
& \quad = 2 \cdot 60 \cdot \left[\frac{(4-1)(4-3)}{4(4-2)} \right] \frac{\pi}{2} = 120 \cdot \frac{3\pi}{4} = \frac{45}{2} \pi
\end{aligned}$$

$$= \int_0^\pi \left[\frac{\gamma^4}{4} \right]_{2\sin(\theta)}^{4\cos(\theta)} d\theta = \int_0^\pi (64 \cdot \cos^4(\theta) - 4\sin^4(\theta)) d\theta$$

$$= \int_0^\pi [64\cos^4(\theta) - 4 \cdot (1 - \cos^2(\theta))^2] d\theta$$

$$= \int_0^\pi (64\cos^4(\theta) - 4 + 8\cos^2(\theta) - 4\cos^4(\theta)) d\theta$$

$$= \int_0^\pi (68\cos^4(\theta) - 8\cos^2(\theta) - 4) d\theta$$

Change of order

Here we find new limits by comparing actual limits with using sketch.

- Change the order of integration:

$$\int_0^1 \int_0^x f(x, y) dy \cdot dx$$

4. Here limits of $y \geq \{[a \rightarrow \text{is variable limit: } 0 \leq y \leq x]$

limit of x is constant: $0 \leq x \leq 1$

By changing the order of integration. limits of x become variable & that of y becomes constant

\therefore limits of $x = y \leq x \leq 1$

$y : 0 \leq y \leq 1 <$

\therefore cultered int:

$$\int_0^1 \int_y^1 f(x, y) dx \cdot dy$$

2.) Change order and eval:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dy \cdot dx := I$$

A)

$$y \in [0, \sqrt{1-x^2}], x \in [0, 1]$$

$$\text{or } 0 \leq y \leq \sqrt{1-x^2},$$

$$0 \leq x \leq 1$$

$$\sqrt{1-x^2} \Big|_{0 \rightarrow 1} \Rightarrow [\sqrt{1}, 0]$$

$$\therefore 0 \leq y \leq [1, 0] \rightarrow 0 \leq y \leq 1$$

$$y = \sqrt{1-x^2} \Rightarrow x = \sqrt{1-y^2} \Big\} [1, 0]$$

$$0 \leq x \leq \sqrt{1-y^2}$$

$$\begin{aligned} I &\equiv \int_0^1 \int_0^{\sqrt{1-y^2}} y^2 \cdot dx \cdot dy \\ &= \int_0^1 y^2 \cdot \sqrt{1-y^2} dy. \\ \text{clet } y &= \sin(u), \therefore dy = \cos(u) du, u = \sin^{-1}(y), \therefore [0, 1] \mapsto [0, \pi/2] \\ \therefore I &= \int_0^{\pi/2} \sin^2(u) \sqrt{1-\sin^2(u)} \cos(u) du \\ &0\pi/2 \end{aligned}$$

$$= \int_0^{\pi/2} \sin^2(u) \cos^2(u) du = \int_0^{\pi/2} \sin^2(u) (1-\sin^2(u)) du$$

$$\begin{aligned} &= \int_0^{\pi/2} \sin^2(u) du - \int_0^{\pi/2} \sin^4(u) du \\ &= \frac{1}{2} \times \pi/2 - \frac{3 \times 1}{4 \times 2} \times \frac{\pi}{2} = \pi/16 \end{aligned}$$

$$3. \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy dx = I$$

$$4) \rightarrow y \in \left[\frac{x^2}{4a}, 2\sqrt{ax} \right], \Rightarrow \frac{x^2}{4a} \leq y \leq 2\sqrt{ax} \quad \begin{aligned} &\frac{x^2}{4a} \leq \frac{y^2}{4a} \leq x \\ &x \leq 2\sqrt{ay} \leq 2^{3/2} (ax)^{3/4} \end{aligned}$$

$$\begin{aligned} x \in [0, 4a] &\Rightarrow 0 \leq x \leq 4a \\ &\left. \begin{aligned} &\frac{y^2}{4a} \leq x \leq 2\sqrt{ay} \\ &0 \leq y \leq 4a \end{aligned} \right\} \\ I &\equiv \int_0^{4a} \left\{ \int_{y^2/4a}^{2\sqrt{ay}} dx \right\} dy \end{aligned}$$

$$= \int_0^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a} \right) dy = \int_0^{4a} 2\sqrt{a} \sqrt{xy} dy - \int_0^{4a} \frac{y^2}{4a} dy$$

$$= 2\sqrt{a} \left[\frac{2y^{3/2}}{3} \right]_0^{4a} - \left[\frac{y^3}{12a} \right]_0^{4a} = \frac{4}{3} \sqrt{a} 4^{5/2} \cdot a^{5/2} - \frac{64a^3}{12a}$$

$$= \frac{32}{3} a^2 - \frac{16}{3} a^2 = \frac{16}{3} a^2$$

$$4 \int_0^a \int_y^a \frac{x}{x^2+y^2} dx \cdot dy$$

$$\rightarrow 0 \leq 0 \leq a.$$

$$y \leq x \leq a$$

$$I = \int_0^a \int_y^a \frac{x}{x^2+y^2} dx \cdot dy \quad \text{let } u = x^2 + y^2, \therefore \frac{dy}{dx} = 2x, dx = \frac{du}{2x}$$

$$= \int_0^a \int_{y^2+a^2}^{\frac{1}{2a}} dy \cdot dy. \quad \therefore x \rightarrow [y, a] \mapsto \underbrace{[2y^2, y^2+a^2]}_{u-\text{space}}$$

$$= \int_0^a \left[\frac{1}{2} \ln(u) \right]_{2y^2}^{y^2+a^2} dx y = \int_0^a \frac{1}{2} \ln \left(\frac{y^2+a^2}{2y^2} \right) dy \leftarrow \text{by parts}$$

$$= \frac{1}{2} \int_0^a \ln \left(\frac{y^2+a^2}{2y^2} \right) dy$$

$$\frac{d_4}{dy} = \frac{2y^2}{y^2+a^2} \times \frac{-2y^2 \cdot 2y - (y^2+a^2)4y}{4y^4} = \frac{-2a^2}{y^3+a^2y}$$

$$\begin{aligned}\therefore I &= \frac{1}{2} \left\{ \ln \left(\frac{y^2+a^2}{2y^2} \right) y - \int \frac{-2a^2 y}{y^3+a^2 y} dy \right\}_0^a dy \\ \text{for integral: } \int \frac{-2a^2 y}{y^3+a^2 y} &= \int \frac{-2a^2}{y^2+a^2} = -2a^2 \int \frac{1}{y^2+a^2} \\ &= -\frac{2a^2}{a^2} \int \frac{1}{(y/a)^2+1} dy \\ \text{let } u &= y/a. \therefore dy = du \cdot a\end{aligned}$$

$$\begin{aligned}\rightarrow &= -2a \int \frac{dy}{u^2+1} = -2a \tan^{-1}(u) = -2 \tan^{-1} \left(\frac{y}{a} \right) \\ &= \frac{1}{2} \left[\ln \left(\frac{y^2+a^2}{2y^2} \right) \cdot y + 2a \tan^{-1} \left(\frac{y}{a} \right) \right]_0^9 \\ &= \frac{1}{2} \ln(-1) \cdot a + d \cdot \tan^{-1}(1) \\ &= a \cdot \frac{\pi}{4}\end{aligned}$$