Limits & Derivatives

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1 Limits

1.1 Some algebra

- $\lim_{x\to a}\big[f(x)+g(x)\big] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- $\lim_{x\to a}\big[f(x)-g(x)\big] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
- $\lim_{x\to a}\big[f(x)\cdot g(x)\big] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$
- $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

1.2 Basics

- $\lim_{x\to a}\dfrac{x^n-a^n}{x-a} = na^{n-1}$
- $\lim_{x\to a}\dfrac{x^m-a^m}{x^n-a^n} = \left(\dfrac{m}{n}\right)a^{m-n}$
- $\lim_{x\to 0}\dfrac{(x+1)^n-1}{x} = n$
- $\lim_{x\to 0}\dfrac{(x+a)^n-a^n}{x} = na^{n-1}$

1.3 Trigonometric limits

- $\lim_{x\to 0}\tan(x) = \lim_{x\to 0}\frac{x}{\cos(x)} = 0$
- $\lim_{x\to 0}\frac{\tan(x)}{x} = \lim_{x\to 0}\frac{\sin(x)}{x} = 1$
- $\lim_{x\to 0}\frac{x}{\sin(x)} = \lim_{x\to 0}\frac{x}{\tan(x)} = 1$
- $\lim_{x\to 0}\frac{\sin(ax)}{x} = \lim_{x\to 0}\frac{\tan(ax)}{x} = a$
- $\lim_{x\to 0}\frac{x}{\sin(ax)} = \lim_{x\to 0}\frac{x}{\tan(ax)} = \frac{1}{a}$
- $\lim_{x\to 0}\frac{\sin(ax)}{\sin(bx)} = \frac{a}{b}$
- $\lim_{x\to 0}\frac{1-\cos(mx)}{x} = 0$
- $\lim_{x\to 0}\frac{1-\cos(mx)}{x^2} = \frac{m^2}{2}$

1.4 Exponential & logarithmic limits

- $\lim_{x\to 0}\frac{e^x-1}{x} = 1$
- $\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$ $\qquad\Big[\ln(x) = \log_e(x)\Big]$
- $\lim_{x\to 0}\frac{\ln(1+ax)}{x} = a$

2 Derivatives

2.1 Differentiation from first principle

Method of finding derivative from the definition.

Consider a function $f(x)$. We say $f(x)$ is differentiable at $x = a$, if $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ exists.

And we write it as $f'(a)$ or $\dfrac{df}{dx}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2.2 Algebra of derivatives

- $\dfrac{d}{dx}(f(x) + g(x)) = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x)$
- $\dfrac{d}{dx}(f(x) - g(x)) = \dfrac{d}{dx}f(x) - \dfrac{d}{dx}g(x)$
- $\dfrac{d}{dx}(f \cdot g) = f \cdot \dfrac{d}{dx}g \ + \ g \cdot \dfrac{d}{dx}f$ $\quad \left[\textit{Product rule}\right]$
- $\dfrac{d}{dx}(f \cdot g \cdot h) = gf \cdot \dfrac{d}{dx}h \ + \ gh \cdot \dfrac{d}{dx}f \ + fh \cdot \dfrac{d}{dx}g$ $\quad \left[\textit{Product rule}\right]$
- $\dfrac{d}{dx}\left[\dfrac{f}{g}\right] = \dfrac{g \cdot \dfrac{d}{dx}f \ - \ f \cdot \dfrac{d}{dx}g}{g^2}$ $\quad \left[\textit{Quotient rule}\right]$

2.3 Derivative of some simple functions

- $\dfrac{d}{dx}x = 1$
- $\dfrac{d}{dx}kx = k$
- $\dfrac{d}{dx}x^n = nx^{n-1}$

2.4 Derivative of Trigonometric functions

- $\dfrac{d}{dx}\sin(x) = \cos(x)$
- $\dfrac{d}{dx}\cos(x) = -\sin(x)$
- $\dfrac{d}{dx}\tan(x) = \sec^2(x)$
- $\dfrac{d}{dx}\cot(x) = -\mathrm{cosec}(x)$
- $\dfrac{d}{dx}\sec(x) = \sec(x)\tan(x)$
- $\dfrac{d}{dx}\mathrm{cosec}(x) = -\mathrm{cosec}(x)\cot(x)$

2.5 Some other results

These are some results obtained by various methods. Results given below are only for reference

- $\lim_{x \to 0} \left[\dfrac{ax+b}{cx+d}\right] = \dfrac{b}{a}$
- $\dfrac{d}{dx}\sin(ax) = a\cos(ax)$
- $\dfrac{d}{dx}\cos(ax) = -a\sin(ax)$

- $\dfrac{d}{dx}\sin^2(x) = \sin(2x)$
- $\dfrac{d}{dx}\left[\dfrac{x}{x+1}\right] = \dfrac{1}{(x+1)^2}$
- $\dfrac{d}{dx}\left[\dfrac{x+1}{x-1}\right] = -\dfrac{2}{(x-1)^2}$
- $\dfrac{d}{dx}\left[\dfrac{\sin(x)}{x}\right] = \dfrac{x\cos(x)-\sin(x)}{x^2}$