Limits & Derivatives

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1 Limits

1.1 Some algebra

- $\lim_{x \to a} \big[f(x) + g(x)\big] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
- $\lim_{x \to a} \big[f(x) - g(x)\big] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
- $\lim_{x \to a} \big[f(x) \cdot g(x)\big] = \Big(\lim_{x \to a} f(x)\Big)\Big(\lim_{x \to a} g(x)\Big)$
- $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

1.2 Basics

- $\lim_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$
- $\lim_{x \to a} \dfrac{x^m - a^m}{x^n - a^n} = \left(\dfrac{m}{n}\right) a^{m-n}$
- $\lim_{x \to 0} \dfrac{(x + a)^n - a^n}{x} = na^{n-1}$

1.3 Trigonometric limits

- $\lim_{x \to 0} \tan(x) = \lim_{x \to 0} \dfrac{x}{\cos(x)} = 0$
- $\lim_{x \to 0} \dfrac{\tan(x)}{x} = \lim_{x \to 0} \dfrac{\sin(x)}{x} = 1$
- $\lim_{x \to 0} \dfrac{x}{\sin(x)} = \lim_{x \to 0} \dfrac{x}{\tan(x)} = 1$
- $\lim_{x \to 0} \dfrac{\sin(ax)}{x} = \lim_{x \to 0} \dfrac{\tan(ax)}{x} = a$
- $\lim_{x \to 0} \dfrac{x}{\sin(ax)} = \lim_{x \to 0} \dfrac{x}{\tan(ax)} = \dfrac{1}{a}$
- $\lim_{x \to 0} \dfrac{\sin(ax)}{\sin(bx)} = \dfrac{a}{b}$
- $\lim_{x \to 0} \dfrac{1 - \cos(mx)}{x} = 0$
- $\lim_{x \to 0} \dfrac{1 - \cos(mx)}{x^2} = \dfrac{m^2}{2}$

1.4 Exponential & logarithmic limits

- $\lim_{x \to 0} \dfrac{e^{ax} - 1}{x} = a$
- $\lim_{x \to 0} \dfrac{\ln(1 + ax)}{x} = a \qquad \Big[\ln(x) = \log_e(x)\Big]$

1.5 Left & Right limit

$\lim_{x\to a} f(x) = l$ if and only if $\lim_{x\to a^-} = \lim_{x\to a^+} = l$

In other words, the limit of a function at a point a has a definite value if and only if both limits from left and right side are equal.

2 Derivatives

2.1 Differentiation from first principle

A Method of finding derivative from the definition.

Consider a function $f(x)$. We say $f(x)$ is differentiable at $x = a$, if $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists.

And we write it as $f'(a)$ or $\dfrac{df}{dx}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

2.2 Algebra of derivatives

- $\dfrac{d}{dx}(f(x) + g(x)) = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x)$
- $\dfrac{d}{dx}(f(x) - g(x)) = \dfrac{d}{dx}f(x) - \dfrac{d}{dx}g(x)$
- $\dfrac{d}{dx}(f \cdot g) = f \cdot \dfrac{d}{dx}g \ + \ g \cdot \dfrac{d}{dx}f$ $\quad \left[\textit{Product rule}\right]$
- $\dfrac{d}{dx}(f \cdot g \cdot h) = gf \cdot \dfrac{d}{dx}h \ + \ gh \cdot \dfrac{d}{dx}f \ + fh \cdot \dfrac{d}{dx}g$ $\quad \left[\textit{Product rule}\right]$
- $\dfrac{d}{dx}\left[\dfrac{f}{g}\right] = \dfrac{g \cdot \dfrac{d}{dx}f \ - \ f \cdot \dfrac{d}{dx}g}{g^2}$ $\quad \left[\textit{Quotient rule}\right]$

2.3 Derivative of some simple functions

- $\dfrac{d}{dx}x = 1$
- $\dfrac{d}{dx}kx = k$
- $\dfrac{d}{dx}x^n = nx^{n-1}$

2.4 Derivative of Trigonometric functions

- $\dfrac{d}{dx}\sin(x) = \cos(x)$
- $\dfrac{d}{dx}\cos(x) = -\sin(x)$
- $\dfrac{d}{dx}\tan(x) = \sec^2(x)$
- $\dfrac{d}{dx}\cot(x) = -\mathrm{cosec}(x)$
- $\dfrac{d}{dx}\sec(x) = \sec(x)\tan(x)$
- $\dfrac{d}{dx}\mathrm{cosec}(x) = -\mathrm{cosec}(x)\cot(x)$

2.5 Some other results

These are some results obtained by various methods. Results given below are only for reference

- $\lim_{x \to 0} \left[\frac{ax+b}{cx+d} \right] = \frac{b}{d}$
- $\frac{d}{dx} \sin(ax) = a\cos(ax)$
- $\frac{d}{dx} \cos(ax) = -a\sin(ax)$
- $\frac{d}{dx} \sin^2(x) = \sin(2x)$
- $\frac{d}{dx} \left[\frac{x}{x+1} \right] = \frac{1}{(x+1)^2}$
- $\frac{d}{dx} \left[\frac{x+1}{x-1} \right] = -\frac{2}{(x-1)^2}$
- $\frac{d}{dx} \left[\frac{\sin(x)}{x} \right] = \frac{x\cos(x) - \sin(x)}{x^2}$

2.6 Some needless proofs..

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\begin{aligned}
\lim_{x \to a} \frac{x^n - a^n}{x - a} &= \lim_{h \to 0} \frac{(a+h)^n - a^n}{a+h-a} \\
&= \lim_{h \to 0} \frac{\left(a^n \ + \ {}^nC_1 a^{n-1}h \ + \ {}^nC_2 a^{n-2}h^2 \ + \ \ldots + h^0\right) - a^n}{h} \\
&= \lim_{h \to 0} \frac{na^{n-1}h \ + \ {}^nC_2 a^{n-2}h^2 \ + \ \ldots \ + \ h^n}{h} \\
&= \lim_{h \to 0} \frac{h\left(na^{n-1} \ + \ {}^nC_2 a^{n-1}h \ + \ \ldots \ + \ h^{n-1}\right)}{h} \\
&= na^{n-1}
\end{aligned}
$$