

Limits & Derivatives

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1 Limits

1.1 Some algebra

- $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \left(\lim_{x \rightarrow a} f(x) \right) \left(\lim_{x \rightarrow a} g(x) \right)$
- $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$

1.2 Basics

- $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$
- $\lim_{x \rightarrow a} \frac{x^m - a^m}{x^n - a^n} = \left(\frac{m}{n} \right) a^{m-n}$
- $\lim_{x \rightarrow 0} \frac{(x+a)^n - a^n}{x} = na^{n-1}$

1.3 Trigonometric limits

- $\lim_{x \rightarrow 0} \tan(x) = \lim_{x \rightarrow 0} \frac{x}{\cos(x)} = 0$
- $\lim_{x \rightarrow 0} \frac{\tan(x)}{x} = \lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{x}{\sin(x)} = \lim_{x \rightarrow 0} \frac{x}{\tan(x)} = 1$
- $\lim_{x \rightarrow 0} \frac{\sin(ax)}{x} = \lim_{x \rightarrow 0} \frac{\tan(ax)}{x} = a$
- $\lim_{x \rightarrow 0} \frac{x}{\sin(ax)} = \lim_{x \rightarrow 0} \frac{x}{\tan(ax)} = \frac{1}{a}$
- $\lim_{x \rightarrow 0} \frac{\sin(ax)}{\sin(bx)} = \frac{a}{b}$
- $\lim_{x \rightarrow 0} \frac{1 - \cos(mx)}{x} = 0$
- $\lim_{x \rightarrow 0} \frac{1 - \cos(mx)}{x^2} = \frac{m^2}{2}$

1.4 Exponential & logarithmic limits

- $\lim_{x \rightarrow 0} \frac{e^{ax} - 1}{x} = a$
- $\lim_{x \rightarrow 0} \frac{\ln(1+ax)}{x} = a$ $\left[\ln(x) = \log_e(x) \right]$

1.5 Left & Right limit

$$\lim_{x \rightarrow a} f(x) = l \quad \text{if and only if} \quad \lim_{x \rightarrow a^-} = \lim_{x \rightarrow a^+} = l$$

In other words, the limit of a function at a point a has a definite value if and only if both limits from left and right side are equal.

2 Derivatives

2.1 Differentiation from first principle

A Method of finding derivative from the definition.

Consider a function $f(x)$. We say $f(x)$ is differentiable at $x = a$, if $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ exists.

And we write it as $f'(a)$ or $\frac{df}{dx}$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

2.2 Algebra of derivatives

- $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
- $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$
- $\frac{d}{dx}(f \cdot g) = f \cdot \frac{d}{dx}g + g \cdot \frac{d}{dx}f$ [Product rule]
- $\frac{d}{dx}(f \cdot g \cdot h) = gf \cdot \frac{d}{dx}h + gh \cdot \frac{d}{dx}f + fh \cdot \frac{d}{dx}g$ [Product rule]
- $\frac{d}{dx} \left[\frac{f}{g} \right] = \frac{g \cdot \frac{d}{dx}f - f \cdot \frac{d}{dx}g}{g^2}$ [Quotient rule]

2.3 Derivative of some simple functions

- $\frac{d}{dx}x = 1$
- $\frac{d}{dx}kx = k$
- $\frac{d}{dx}x^n = nx^{n-1}$

2.4 Derivative of Trigonometric functions

- $\frac{d}{dx} \sin(x) = \cos(x)$
- $\frac{d}{dx} \cos(x) = -\sin(x)$
- $\frac{d}{dx} \tan(x) = \sec^2(x)$
- $\frac{d}{dx} \cot(x) = -\operatorname{cosec}(x)$
- $\frac{d}{dx} \sec(x) = \sec(x) \tan(x)$
- $\frac{d}{dx} \operatorname{cosec}(x) = -\operatorname{cosec}(x) \cot(x)$

2.5 Some other results

These are some results obtained by various methods. Results given below are only for reference

- $\lim_{x \rightarrow 0} \left[\frac{ax + b}{cx + d} \right] = \frac{b}{d}$
- $\frac{d}{dx} \sin(ax) = a \cos(ax)$
- $\frac{d}{dx} \cos(ax) = -a \sin(ax)$
- $\frac{d}{dx} \sin^2(x) = \sin(2x)$
- $\frac{d}{dx} \left[\frac{x}{x+1} \right] = \frac{1}{(x+1)^2}$
- $\frac{d}{dx} \left[\frac{x+1}{x-1} \right] = -\frac{2}{(x-1)^2}$
- $\frac{d}{dx} \left[\frac{\sin(x)}{x} \right] = \frac{x \cos(x) - \sin(x)}{x^2}$

2.6 Some needless proofs..

$$\begin{aligned}
 \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} &= \lim_{h \rightarrow 0} \frac{(a+h)^n - a^n}{a+h-a} \\
 &= \lim_{h \rightarrow 0} \frac{(a^n + {}^nC_1 a^{n-1} h + {}^nC_2 a^{n-2} h^2 + \dots + h^n) - a^n}{h} \\
 &= \lim_{h \rightarrow 0} \frac{na^{n-1} h + {}^nC_2 a^{n-2} h^2 + \dots + h^n}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(na^{n-1} + {}^nC_2 a^{n-2} h + \dots + h^{n-1})}{h} \\
 &= na^{n-1}
 \end{aligned}$$