1A R^n and C^n

求证 $\alpha\beta = 1$, $\alpha + \beta = 0$. . .

(3)(8)

exist and unique.

① 写 α 表示式

② 写 β 表示式

③ Suppose $\eta \in C$.

④ 变换后得 $\eta = \beta$.

1B Definition of Vector Space

- $-v = (-1)v$.

  $0 = 0 + 0$.

- 得出 $a\lambda = 0$. 由于 $a \neq 0 \Rightarrow \lambda = 0$.

- existence & uniqueness

1C Subspaces

substance.

① $0 \in U$. (additive identity)

② closed under addition 加法封闭

③ Closed under scalar multiplication 数乘封闭

direct sum $\oplus$

check: $0 = u_1 + u_2 + \cdots + u_m = 0 + 0 + \cdots + 0$ / $U \cap W = \{0\}$

unique way to write.

want: $V = U \oplus W$

① $V = U + W$. 互相包含 ✓.

② $U \cap W = 0$

用 degree.
basis.
linear independent.
span.

- 证 subspace (1)

① $0 \in U$.

② closed under addition $f(u_1 + u_2) = f(u_1) + f(u_2)$

③ closed under scalar multiplication $\lambda f(u) = f(\lambda u)$

- iff (if and only if) (2)

only if $\Rightarrow$ 让条件成立. 得出 only if 结果

if $\Leftarrow$ 假设结果成立、反推条件成立.

- function. (2)

① constant ✓. continuous. ✓

+ . *   + *

② 要求出特征 additive identity

- controdiction. (12)

① 先把 if & only if 证明了 if $u \in w$, if $w \in u$.

② argue by controdiction. (why not?)

$U \cup W$ is a subspace while $U \not\subseteq W$. $W \not\subseteq U$. ← 新条件

$\exists\, u \in U$. $u \notin W$, $w \in W$, $w \notin U$; $u.w \in U \cup W$.

$u + w \in U \cup W$ or $\in W$ or $\in U$. ←

if $u + w \in U$. since $-u \in U$.

$(u+w)+(-u) = w \in U$ contradict.

if $u+w \in W$, since $-w \in W$

$(u+w)+(-w) = u \in W$ contradict

- Even $U_1 \cap U_2 = U_2 \cap U_3 = U_3 \cap U_1 = \{0\}$

¡NOT! A Subspace.

$U_1 = \{(x,y,0)\}$

$U_2 = \{(0,0,z)\}$

$U_3 = \{(0,y,y)\}$

$(0,0,0) = (0,1,0)+(0,0,1)+(0,-1,-1)$

$(0,0,0) = (0,0,0)+(0,0,0)+(0,0,0)$

2A. Span and Linear Independence

- linear combinations.

$v_1, v_2, \dots, v_m$

$\rightarrow a_1v_1 + a_2v_2 + \cdots + a_mv_m$.

- Span is the smallest containing subspace.

证相互包含.

① span is a subspace of V. (3 rules)

② every subspace of V containing each v_j contains span$(v_1 \dots v_m)$ $v_j \in$ span $\supset$ span$(v_1 \dots v_m) \subseteq V$.

③ v_j is a linear combinition of $v_1 \dots v_m$. $v_j \in$ span$(v_1 \dots v_m)$

- span $(v_1 \dots v_m)$ equals V. $\Rightarrow$ spans.

could be linearly dependent.

- $P_m(F)$ degree at most m. dim: $m+1$.

- infinite-dimensional

no list spans V.

- Linearly independent.

$a_1v_1 + \cdots + a_mv_m = 0 \Rightarrow a_1 = \cdots = a_m = 0$.

- Linear Dependence Lemma. from right to left

linearly dependent. exist j (largest element)

$\Rightarrow v_j \in \text{span}(v_1 \ldots v_{j-1})$

$\Rightarrow$ removed j^{th} term.

the remaining list equals $\text{span}(v_1 \ldots v_m)$

(reaching linearly independent list)

$$v_j = -\frac{a_1}{a_j}v_1 - \cdots - \frac{a_{j-1}}{a_j}v_{j-1}$$

↑ use above.

- linearly independent list $\le$ spanning list (length)

$v_1 \ldots v_m$ $w_1 \ldots w_n$.

$v_1, w_1 \ldots w_{n-1}$ add v_1. remove w_{n} ..

$v_2, v_1, w_1 \ldots w_{n-2}$

$v_1 \ldots v_{n-1}, w_1$ spanning list.

$v_1 \ldots v_{n-1}, v_n, w$ linearly dependent

$v_1 \ldots v_{n-1}, v_n$ spanning list.

$v_1 \ldots v_{n-1}, v_n, v_{n+1}$ linearly dependent

contradiction

- contradiction.

if xxx, then wrong result. (2)

- the coefficients of a polynomial are uniquely determined by the polynomial.

- 举实例 $i(1+i)+1(1-i)=0$. (5)

- $v_1 \dots v_m$ linearly independent. $w \in V$.

and $v_1+w, v_2+w \dots v_m+w$ dependent.

prove $w \in \operatorname{span}(v_1 \dots v_m)$

$\Rightarrow a_1v_1+\dots+a_mv_m+(a_1+\dots+a_m)w=0.$

假设来反驳.

$\Rightarrow$ if $a_1+\dots+a_m=0$. $a_1v_1+\dots+a_mv_m=0$. and a_j not all zero.

then $v_1 \dots v_m$ dependent X.

$\therefore a_1+\dots+a_m \neq 0.$

$$w = -\frac{a_1}{a_1+\dots+a_m}v_1 - \dots - \frac{a_m}{a_1+\dots+a_m}v_m \in \operatorname{span}()$$

- proof infinite-dimensional.

① choose $v_2 \notin \operatorname{span} v_1$. $v_3 \notin \operatorname{span}(v_1, v_2) \dots$

找不到有限个.

②. if V is finite

$\exists$ $v_1 \dots v_m$ is a basis. $\exists$ $v_{m+1} \in \operatorname{span}(v_1 \dots v_m)$

but exist $v_{m+1} \notin \operatorname{span}(v_1 \dots v_m)$. contradiction

2B Bases

- basis of V.

$\Rightarrow$ linearly independent (unique)

$\Rightarrow$ spans V. (every) (dimension)

- every subspace is part of a direct sum equal to V.

U is a subspace of V.

exist subspace W. $V = U \oplus W$

$\Rightarrow$ let $u_1 \dots u_m$. $w_1 \dots w_n$. be a basis of V. (linearly independent)

$V = \underbrace{a_1u_1 + \dots + a_mu_m}_{U} + \underbrace{b_1w_1 + \dots + b_nw_n}_{W.}$ spans.

$\Rightarrow$ let $v = a_1u_1 + \dots + a_mu_m = b_1w_1 + \dots + b_nw_n$

$0 = a_1u_1 + \dots + a_mu_m - b_1w_1 - \dots - b_nw_n$.

so. $a_1 = \dots = a_m = b_1 = \dots b_n = 0$. $\Rightarrow v = 0$. $U \cap W = \{0\}$

- prove $V = \text{span}\{u_1 \dots u_m . w_1 \dots w_n\}$. while $V = U \oplus W$ (8)

first proof basis of U and W are linearly independent.

$\Rightarrow$ $v = u + w = a_1u_1 + \dots + a_mu_m + b_1w_1 + \dots + b_nw_n$ every $v \in V$

so $V \in \text{span}\{\dots\}$.

$\Leftarrow$ $\text{span}\{\dots\} \in V$. as $U \in V$. $W \in V$.

so $V = \text{span}\{\dots\}$

2C. Dimension

- dimensions. $\rightarrow$ length of any basis.
- linear independent list / spanning list of right length is a basis.
- $\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$.

proof: $\underbrace{v_1 \dots v_j . \overbrace{u_1 \dots u_m}^{\text{basis of } U_1 \cap U_2}}_{\text{basis of } U_1}$; $w_1 \dots w_k$.

basis of U_2

want to proof all above is a basis of $U_1 + U_2$.

$\Rightarrow$ let. $a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_jv_j + c_1w_1 + \cdots + c_kw_k = 0$

$\downarrow$ $\underbrace{c_1w_1 + \cdots + c_kw_k}_{\in U_2} = -\underbrace{(a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_jv_j)}_{\in U_1}$

$\downarrow$ $c_1w_1 + \cdots + c_kw_k = d_1u_1 + \cdots + d_mu_m \in U_1 \cap U_2$

$\downarrow$ linearly independent ... $\nearrow\nearrow\nearrow$.

a's. b's. c's. all equal 0.

- proof U is a subspace of V. $\dim U = \dim V$ (1)

  then $U = V$.

  证互相包含.

  U. V has same basis. (linear independent list right length = basis).

  $v = a_1u_1 + \cdots + a_mu_m$. (linear combinitions)

- $U = \{p \in \mathcal{P}_4(F) : p(6) = 0\}$ 重点题型. (4-8)

① find dimensions / basis.

则 $p(x) = (x-6)(k_3x^3 + k_2x^2 + k_1x + k_0)$

$\Rightarrow$ basis $x-6$. x^2-6x. x^3-6x^2, x^4-6x^3.

又如 $P(2) = P(5) = P(6)$.

$p(x) = (x-2)(x-5)(x-6)(k_1x + k_0)$

$\Rightarrow$ basis $(x^4 - 13x^3 + 52x^2 - 60x)$, $(x^3 - 13x^2 + 52x - 60)$

also $p(2) = p(5) = p(6) = Z$. so 1 also a part of the basis.

又如 $\int_{-1}^{1} p = 0$. let $Q(x) = \int p(x)$.

$Q(x)\Big]_{-1}^{1} = a_0x + \frac{1}{2}a_1x^2 + \frac{1}{3}a_2x^3 + \frac{1}{4}a_3x^4 + \frac{1}{5}a_4x^5\Big]_{-1}^{1} = 0$.

解出 a 关系式再代回 p.

$p(x) = a_1(x) + a_2(x^2 - \frac{1}{3}) + a_3(x^3) + a_4(x^4 - \frac{1}{5})$

最后还要证 all linearly independent

② extend to the whole basis.

找次数 dimension.

③ 还要证 $P_m(\mathbb{F}) = U \oplus W$. $\begin{cases} P = U + W \\ U \cap W = \{0\} \end{cases}$

- proof $\dim \operatorname{span}(v_1+w \dots v_m+w) \geq m-1$. (9)

$v_i - v_1 \in \operatorname{span}(\dots)$ $2 \leq i \leq m$.

so $v_2 - v_1 \dots v_m - v_1$ linearly independent. V.

- exist 1-dimensional $U_1 \dots U_m \Rightarrow V = U_1 \oplus \dots \oplus U_m$

构造 $U_j = \{av_j : a \in \mathbb{F}\}$. $v = a_1v_1 + \dots + a_mv_m$.

- proof $\dim U_1 \oplus \dots \oplus U_m = \dim U_1 + \dots + \dim U_m$ (16)

$U_i \cap U_j = 0$. $\dim(U_i \cap U_j) = 0$.

用数归做.

① $m=2$ check.

② let $m=k-1$ true. check $m=k$.

let $W = U_1 \oplus \dots \oplus U_{m-1}$, $W + U_m$ $0 = \underset{W}{x} + \underset{U_m}{y}$

as direct sum $0 = 0 + 0$. $W + U_m$ direct sum

3A. The Vector Space of Linear Maps

- linear map $T: V \to W$

① additivity $T(u+v) = Tu + Tv$

② homogeneity $T(\lambda v) = \lambda(Tv)$

proof not. controdict to these two. properties.

- properties.

associativity, identity. distributive properties.

- $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$

$T(x_1 \dots x_n) = (A_{1,1}x_1 + \dots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \dots + A_{m,n}x_n)$

- there exists a unique linear map $T: V \to W$ such that $Tv_j = w_j$ for each $j = 1, \dots, n$

proof. ① existence.

define $T(c_1v_1 + \dots + c_nv_n) = c_1w_1 + \dots + c_nw_n$.

for each j. take $c_j = 1$. others 0. $\Rightarrow Tv_j = w_j$ ✓

② linear map

$T(u+v) = Tu + Tv$. $\quad T(\lambda v) = \lambda Tv$.

③ uniqueness.

let $S(v_j) = w_j$

$S(c_1v_1 + \dots + c_nv_n) = c_1Sv_1 + \dots + c_nSv_n = c_1w_1 + \dots + c_nw_n$

so $S(v) = T(v)$.

($v_1 \dots v_n$ is a basis of V, $\Rightarrow T$ is uniquely determined on V)

- $T \in \mathcal{L}(U, V)$ and $S \in (V, W)$.

$(ST)u = S(Tu)$. $\quad ST \in \mathcal{L}(U, W)$

- T is a linear map. $\Rightarrow T(0) = 0$

- $Tv_1 \dots Tv_m$ linearly independent $\Rightarrow v_1 \dots v_m$ linearly independent (W)

$\Rightarrow$ suppose $c_1 \dots c_m \in \mathbb{F}$.

$c_1Tv_1 + \dots + c_mTv_m = 0$.

$\Downarrow$ $c_1 = \dots = c_m = 0$

$\Downarrow$ $T(\underbrace{c_1v_1 + \dots c_mv_m}_{0.\ \text{by } 3.11.}) = 0$

so $v_1 \dots v_m$ linearly independent

- 1-dimensional.

then $Tu = \lambda u$.

- 举例！（多积累）

$\Rightarrow \varphi(av) = a\varphi(v)$ but φ is not linear.

$\varphi(x, y) = (x^3 + y^3)^{\frac{1}{3}}$

$\Rightarrow \varphi(w+z) = \varphi(w) + \varphi(z)$ but φ is not linear.

$\varphi(a+bi) = a \Rightarrow i\varphi(1) \neq \varphi(i)$.

(complex vector space)

$\Rightarrow ST \neq TS$.

$Sv_k = \begin{cases} v_2 & \text{if } k=1 \\ 0 & \text{if } k \neq 1 \end{cases}$ $\quad Tv_k = \begin{cases} v_1 & \text{if } k=2 \\ 0 & \text{if } k \neq 2 \end{cases}$

$(ST)v_1 = 0 \neq (TS)v_2 = v_1$.

3B. Null Spaces and Ranges.

i all very important!

- $\text{null}\, T = \{v \in V : Tv = 0\}$.

  null T is a subspace of V.

- $\text{range}\, T = \{Tv : v \in V\}$ $\quad$ $\underline{\text{range}\, T = \text{span}(Tv_1, \ldots, Tv_n)}$

  range T is a subspace of W.

- injective (one-to-one)

  $Tu = Tv$ implies $u = v$. $\quad \Rightarrow$ to a small~~er~~ dimensional

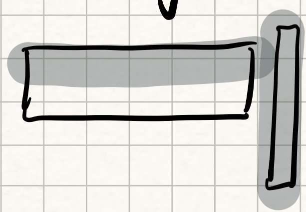
  iff $\text{null}\, T = \{0\}$ $\quad$ 用dim证明

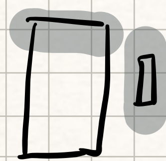
- surjective (onto)

range equals W. ⇒ to a la~~r~~ger dimensional

- $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$.

- homogeneous. Inhomogeneous





free valuables may be insolvable least squares.

- proof injective iff $\operatorname{null} T = \{0\}$.

$\Leftarrow$ $v \in \operatorname{null} T$. $Tv = 0 = T(0)$

as T injective ⇒ $v = 0$.

⇒ if $T(u) = T(v) \Rightarrow T(u-v) = T(0) = 0$.

so $u - v \in \operatorname{null} T = \{0\}$.

$u = v$. ⇒ injective.

- proof $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$

⇒ take $\underbrace{\underbrace{u_1 \dots u_m}_{\text{basis of null } T}, v_1 \dots v_n}_{\text{basis of } V}$.

want to show $\dim \operatorname{range} T = n$.

by proving $Tv_1 \dots Tv_n$ is a basis of range T

⇒ let $v = a_1u_1 + \dots + a_mu_m + b_1v_1 + \dots + b_nv_n$

$Tv = b_1Tv_1 + \dots + b_nTv_n$.

then show $Tv_1 \dots Tv_n$ linearly independent

let $c_1Tv_1 + \dots + c_nTv_n = 0$. ⇒ $T(c_1v_1 + \dots + c_nv_n) = 0$.

$\Rightarrow c_1v_1 + \cdots + c_nv_n \in \text{null } T$.

so $c_1v_1 + \cdots + c_nv_n = d_1u_1 + \cdots + d_mu_m \Rightarrow$ all c's is zero

$\therefore Tv_1 \cdots Tv_n$ linearly independent

$\text{span}(Tv_1 \ldots Tv_n) = \text{range } T \Rightarrow \dim \text{range } T = n$. ✓

- 构造!!!

如: $T \in \{\mathcal{L}(R^5, R^4): \dim \text{null } T > 2\}$, not a subspace

define $T_1(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, 0, 0)$

$T_2(x_1, x_2, x_3, x_4, x_5) = (0, 0, x_3, x_4)$

null T_1: $\{(0, 0, x_3, x_4, x_5)\}$.

null T_2: $\{(x_1, x_2, 0, 0, x_5)\}$.

but $(T_1 + T_2)(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3, x_4)$.

so null $(T_1 + T_2) = \{(0, 0, 0, 0, x_5)\}$

$\Downarrow \dim = 1 < 2$. contradict.

So is not a subspace.

又如: $T: R^4 \to R^4$, range T = null T.

define $T(x_1, x_2, x_3, x_4) = (x_3, x_4, 0, 0)$.

- 求 linearly independent / injective.

let $a_1 \cdots a_n$. prove $a_1v_1 + \cdots + a_nv_n = 0 \Rightarrow a_1 = a_2 = \cdots = a_n = 0$.

- $(T_1T_2 \cdots T_n)v = 0 \Rightarrow$ proof $v = 0$.

as $T_1, \ldots T_n$ injective.

$T_1(T_2 \cdots T_n)v = 0 \quad (T_2 \cdots T_n)v = 0 \quad \ldots \quad T_nv = 0 \Rightarrow v = 0$ (11)

- $F^5 \to F^2$. dim null $T = 3$. the proof dim inequality (15

- injective linear map

$=$ injective $+$ linear map 缺一不可

- 经常要设一个线性映射

$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$

define $R: \operatorname{null} ST \to V.$

$Ru = Tu.$ If $u \in \operatorname{null} ST.$ $(ST)u = S(\underbrace{Tu}_{\text{range } T}) = 0$ $\quad Tu \in \operatorname{null} S$

- if $T_2 = ST_1$

① $\forall v \in V.$ $\quad T_2 v = ST_1 v.$

② basis $\quad T_2 v_i = ST_1 v_i$

- $T \in \mathcal{L}(V, W)$

range $T \subset W.$ $\quad \operatorname{null} T \subset V$

- ① $(ST)u = S(Tu)$ so range $ST \subset$ range S

② $Tv \in$ range T $\quad$ range $(ST) = \underbrace{\operatorname{span}(Su_1 \ldots Su_m) \subset \text{range } T}_{\text{dimension.}}$

$\begin{cases} Tv = a_1u_1 + \cdots + a_m u_m \\ STv = a_1 Su_1 + \cdots + a_m Su_m. \end{cases}$

3C Matrices.

- $Tv_k = A_{1,k} W_1 + \cdots + A_{m,k} W_m.$

- 很多题下，证 $v_1 \ldots v_n$ 是一部分基. $u_1 \ldots u_m$ 是 nullspace 的基.

令 $T(c_1v_1 \ldots c_nv_n) = 0.$

使得 $c_1v_1 + \cdots + c_nv_n = 0 \in \operatorname{null} T$

然后令 $c_1v_1+\cdots+c_nv_n = d_1u_1+\cdots+d_mu_m$.

所以得 $v_1\dots v_n, u_1\dots u_m$ linearly independent.

接着就得 $Tv_1\dots Tv_n$ linearly independent.

然后就可以 extend $Tv_1\dots Tv_n, w_1\dots w_p$ 为基了.

(第一题)之后用3.5 $Tv_j = w_j$，就可证得 $Tu = Su$ for $u\in U$.

可证 surjective (3D.3).

- 设 basis.

extend to the whole space.

3D Invertibility and Isomorphic Vector Spaces

- Invertible. = injective + surjective.
- isomorphism. = invertible linear map (dim equal)

$\mathcal{L}(V,W) \Leftrightarrow F^{m,n}$

- $M(Tv) = M(T)M(v)$
- operator $\mathcal{L}(V) = \mathcal{L}(V,V)$

  ↳ invertible = injective = surjective.

- 要证 $T_1 = T_2S$.

设 basis T_1. T_2.

T_1: $\varepsilon_1 \dots \varepsilon_n$. $v_1 \dots v_m$

T_2: $u_1 \dots u_n$. $w_1 \dots w_m$.

然后 define $\begin{cases} Su_i = \varepsilon_i \\ Sw_j = v_j \end{cases}$ (学会搞基啊兄弟!)

$\begin{cases} T_1 u = T_2 S u \in \text{range } T_2 \\ T_2 u = T_1 S^{-1} u \in \text{range } T_1 \end{cases}$ range equal.

$e_1 \dots e_n$ basis of $F^{n,1}$

$u_1 \dots u_m$ basis of $F^{m,1}$

if $x \in F^{n,1}$

$M(x) = x$

$Tx = M(Tx) = M(T)M(x) = Ax$.

交换时灵活运用等价条件 (16).

$T = aI$. $TS = a(IS) = as = sa = saI = ST$

$\Leftarrow (ST = TS \Rightarrow T = aI.)$

v. Tv linearly dependent $\Rightarrow Tv = av$.

From. $Sv = 0$. $STv = v$. $Su_i = 0$. $i = 1 \dots n$ contridiction

接着证 $a_v = a_w$.

设 v. $w \in V$.

① if v. w dependent.

let $w = bv$. $a_w w = Tw = T(bv) = bTv = ba_v v = a_v(bv) = a_v w$

② v. w independent.

$a_{v+w}(v+w) = T(v+w) = Tv + Tw = a_v v + a_w w$

$a_{v+w} = a_v + a_w$ check v.

- isomorphic.

  dimension equal. linear. surjective

(★可任取S使T成立即可)

3E Products and Quotients of Vector Space.

- $T(u_1 \cdots u_m) = u_1 + \cdots + u_m$

  $T: U_1 \times \cdots \times U_m \to U_1 + \cdots + U_m$.

- $v + U = \{v + u : u \in U\}$. (affine subset)

quotient space $V/U = \{v + U : v \in V\}$ all.

- $\begin{cases} v - w \in U \\ v + U = w + U. \\ (v+U) \cap (w+U) \neq \phi. \end{cases}$

  $(v+U) + (w+U) = (v+w) + U$

  $\lambda(v+U) = \lambda v + U$.

- quotient map $\pi: V \to V/U$.

  $\pi(v) = v + U$

- $\dim V/U = \dim V - \dim U$.

- $\tilde{T} : V/(\text{null}\, T) \to W$

$$\tilde{T}(v + \text{null}\, T) = Tv.$$

- $\tilde{T}$ is a linear map.
- $\tilde{T}$ is injective   let $\tilde{T}(v+\text{null}\,T) = 0 = Tv$ $v \in \text{null}\, T$. $\text{null}\, \tilde{T} = 0$.
- $\text{range}\, \tilde{T} = \text{range}\, T$ definition
- $V/(\text{null}\, T)$ isomorphic $\text{range}\, T$ $(V/\text{null}\, T) \to \underbrace{W}_{\text{range}\, T.}$

- define a linear map

① proof linear map

② injective

③ surjective

- $\lambda v + (1-\lambda) w \in A$. A is a affine subset $\underbrace{x + U}$

$\lambda_1 v_1 + \cdots + \lambda_n v_n$.

$\lambda_1 + \cdots + \lambda_n = 1$.

affine subset.

sol. $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$ $\alpha_1 + \cdots + \alpha_n = 1$

$w = \beta_1 v_1 + \cdots + \beta_m v_m$. $\beta_1 + \cdots + \beta_m = 1$.

$$\underbrace{\lambda v + (1-\lambda) w}_{\text{affine subset}} = (\underbrace{\lambda \alpha_1 + (1-\beta_1)}_{\text{add to one.}}) v_1 + \cdots \in A$$

So A affine subset.

3F Duality

- dual space $V' = \mathcal{L}(V, F)$.

  $\dim V' = \dim V$.

- dual basis

  $$\varphi_j(v_k) = \begin{cases} 1 & \text{if } k=j \\ 0 & \text{if } k \neq j \end{cases}$$

- dual map T'

  $T' \in \mathcal{L}(W', V')$. $\qquad T'(\varphi) = \varphi \circ T$.

  $W' : W \to F$, $V' : V \to F$.

- annihilator U^0.

  $U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}$.

  $\varphi : V \to F$.

- if $e_1\ e_2\ e_3\ e_4\ e_5$ standard basis of R^5

  $\varphi_1\ \varphi_2\ \varphi_3\ \varphi_4\ \varphi_5$ dual basis of $(R^5)'$

then if $U = \operatorname{span}(e_1, e_2)$

$U^0 = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$

- $U \subset V$. then U^0 is a subspace of V'.
- $\dim U + \dim U^0 = \dim V$.
- $\operatorname{null} T' = (\operatorname{range} T)^0$ $\quad T'(\varphi)(v) = 0 = \varphi \circ (Tv)$
- $\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$.
- T injective $\rightarrow$ T' surjective

  $T' \leftarrow . T$
- $M(T') = (M(T))^t$
- dimension of range T equals column rank of

4. Polynomials

- $z = \operatorname{Re} z + (\operatorname{Im} z) i$
- $p(z) = a_0 + a_1 z + \cdots + a_m z^m$.
- $p = sq + r$. all is $\in \mathcal{P}(F)$

  fixed.
- Every nonconstant polynomial with complex

coefficients has a zero.

- $p(x) = c_1(x-\lambda_1)\cdots(x-\lambda_m) \Rightarrow$ over C
  $(x^2+b_1x+c_1)\cdots(x^2+b_Mx+C_M) \Rightarrow$ over R.

5A Invariant Subspaces

- $u \in U$. $Tu \in U$.
- restriction operator. quotient operator

  $T|_U \in \mathcal{L}(U)$ $\quad$ $T/U \in \mathcal{L}(V/U)$

  $T|_U(u) = Tu$ $\quad$ $(T/U)(v+U) = Tv+U$