Lecture 2: Sequences

Math 247 Winter Term 2019

Wednesday, January 9th, David Duan

Contents

2.1 Definition of Sequences and Convergence in $\mathbb{R}^n$

Core Definitions

- An **(infinite) sequence** of vectors, or points, in $\mathbb{R}^n$, is an infinite, enumerated list

$$(\vec{x}_k)_{k=1}^{\infty} = (\vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots)$$

  where $\vec{x}_k \in \mathbb{R}^n$ for all $\vec{k} \geq 1$.

- A sequence of points $(\vec{x}_k)$ **converges** to a point $\vec{a}$ if the following statement is true: Given $\varepsilon > 0$, there exists an integer N such that $\|\vec{x}_k - \vec{a}\| < \varepsilon$ for all $k \geq N$.
- If such a point $\vec{a}$ exists, then we say that $(\vec{x}_k)$ is **convergent** and that $\vec{a}$ is the **limit** of the sequence; we write $\lim_{k\to\infty} \vec{x}_a = \vec{a}$.

Lemma 2.1.1 Let $(\vec{x}_k)_{k=1}^{\infty}$ be a sequence of points in $\mathbb{R}^n$. Then,

$$\lim_{k\to\infty} \vec{x}_k = \vec{a} \iff \lim_{k\to\infty} \|\vec{x}_k - \vec{a}\| = 0.$$

Lemma 2.1.2 Let $(\vec{x}_k)_{k=1}^{\infty}$ be a sequence of points in $\mathbb{R}^n$ where each point is of the form $\vec{x}_k = (x_{k,1}, x_{k,2}, \ldots, x_{k,n})$. Then, the sequence $(\vec{x}_k)$ converges to a point $\vec{a} = (a_1, a_2, \ldots, a_n)$ if and only if

$$\forall 1 \leq i \leq n : \lim_{k\to\infty} x_{k,i} = a_i.$$

Proof.

$\Longrightarrow$: Suppose $(\vec{x}_k)$ converges to $\vec{a}$. We want to show that for each $i \in \{1, 2, \ldots, n\}$ and for all $\varepsilon > 0$, there exists N_i such that $|x_{k,i} - a_i| < \varepsilon$ for all $k \geq N_i$. Let $i \in \{1, 2, \ldots, n\}$ and $\varepsilon > 0$. By convergence of $(\vec{x}_k)$ to $\vec{a}$, we konw that there exists N such that $\|\vec{x}_k - \vec{a}\| < \varepsilon$ for all $k \geq N$. By the definition of Euclidean norm,

$$\|\vec{x}_k - \vec{a}\| = \left(\sum_{j=1}^{n} (x_{k,j} - a_j)^2\right)^{1/2} \geq |x_{k,i} - a_i|$$

Hence, for all $k \geq N_i := N$, we have $|x_{k,i} - a_i| \leq \|\vec{x}_k - \vec{a}\| < \varepsilon$ as required.

$\Longleftarrow$: Let $\varepsilon > 0$ and define $\bar{\varepsilon} = \varepsilon/\sqrt{n}$. For each $i \in \{1, 2, \ldots, n\}$, there exists N_i such that $|x_{k,i} - a_i| < \bar{\varepsilon}$ for all $k \geq N_i$ (convergence of component sequence). Define $N = \max\{N_i\}$ so that $|x_{k,i} - a_i| < \bar{\varepsilon}$ for all $k \geq N$ and for all i. By the definition of the Euclidean norm,

$$\|\vec{x}_k - \vec{a}\| = \left(\sum_{i=1}^{n} (x_{k,i} - a_i)^2\right)^{1/2} < \left(\sum_{i=1}^{n} \bar{\varepsilon}^2\right)^{1/2} = (n \cdot (\varepsilon^2/n))^{1/2} = \varepsilon$$

for all $k \geq N$ as required. $\square$

2.2 Cauchy Sequences

2.3 Completeness