Notes on Generating Functions

Math 239: Introduction to Combinatorics

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1 Formal Power Series

1.1 Introduction

A **formal power series** is a generalization of a polynomial, where the number of terms is allowed to be infinite; this implies giving up the possibility of replacing the variable in the polynomial with an arbitrary number. One way to view a FPS is an *infinite ordered sequence of numbers*. In this sense, the power of the variable are used only to indicate the *order of the coefficients*, so that the coefficient of x^5 is the fifth term in the sequence.

In combinatorics, formal power series provide representations of *numerical sequences* and *multisets*, and for instance allow concise expressions for recursively defined sequences regardless of whether the recursion can be explicitly solved. This is known as the method of **generating functions**.

1.2 Definition

1.2.1 Formal Power Series

Let $(a_0, a_1, a_2, \ldots)$ be a sequence of rational numbers [1] . Then

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n \geq 0} a_n x^n$$

is called a **formal power series**. We say that a_n is the **coefficient** of x^n in $A(x)$ is write $a_n = [x^n]A(x)$. Note that each a_n must be a finite number for $A(x)$ to be a FPS.

1.2.2 Ring of Formal Power Series, Addition and Multiplication

The set of all formal power series in X with coefficients in a *commutative ring* R form another ring that is written $R[[X]]$, and called the **ring of formal power series** in the variable X over R. For $A(x) = \sum_{n \geq 0} a_n x^n \in R[[x]]$ and $B(x) = \sum_{n \geq 0} b_n x^n \in R[[x]]$, addition and multiplication are well defined:

$$\begin{aligned} A(x) + B(x) &= \sum_{n \geq 0} (a_n + b_n) x^n \\ A(x) \times B(x) &= \sum_{n \geq 0} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n \end{aligned}$$

The multiplication rule can be derived as follows:

$$\begin{aligned} A(x)B(x) &= \left(\sum_{i \geq 0} a_i x^i \right) \left(\sum_{j \geq 0} b_j x^j \right) \\ &= \sum_{i \geq 0} \sum_{j \geq 0} a_i b_j x^{i+j} && k \leftarrow i, n \leftarrow i + j = k + j \\ &= \sum_{n \geq 0} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n \end{aligned}$$

1.3 Inverse

1.3.1 Definition

Let $A(x)$ and $B(x)$ be formal power series. We say that $B(x)$ is the **inverse** of $A(x)$ if

$$A(x)B(x) = 1;$$

we denote this by $B(x) = A(x)^{-1}$ or $B(x) = \frac{1}{A(x)}$.

1.3.2 Theorem: Existence of Inverse

A formal power series has an inverse if and only if it has a non-zero constant term.

Proof. Let $A(x) = \sum_{n\geq 0} a_n x^n$ be a formal power series. Then a power series $B(x) = \sum_{n\geq 0} b_n x^n$ is the inverse of $A(x)$ if and only if it satisfies the linear equation $A(x)B(x) = 1$. Suppose to the contrary $[x^0]B(x) = 0$ and $A(x)B(x) = 1$. Then

$$\begin{aligned}
1 &= A(x)B(x) \\
&= b_0 A(x) + b_1 x A(x) + b_2 x^2 A(x) \\
&= 0A(x) + b_1 x A(x) + b_2 x^2 A(x) \\
&= b_1 x A(x) + b_2 x^2 A(x)
\end{aligned}$$

Observe RHS does not have a non-zero constant term, i.e., $[x^0](b_1 x A(x) + b_2 x^2 A(x)) = 0$, because each term of $A(x)$ is multiplied by a non-zero power of x from $B(x)$. This leads to a contradiction as LHS has a non-zero term, namely 1. Similarly, $A(x)$ cannot have an inverse if it does not have a non-zero constant term, which can be proved in a mirror argument. □

1.4 Composition

1.4.1 Definition

The **composition** of a formal power series $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x)$ is defined by

$$A(B(x)) = \sum_{n\geq 0} a_n (B(x))^n = a_0 + a_1 B(x) + a_2 (B(x))^2 + \cdots.$$

Unlike polynomials, however, this operations is not always defined. See theorem below.

1.4.2 Theorem: Existence of Composition

Let $A(x)$ and $B(x)$ be formal power series. Then the composition $A(B(x))$ is a formal power series if the constant term of $B(x)$ is equal to zero.

Proof. Let $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ and $B(x) = xC(x)$ (so that $[x^0]B(x) = 0$). Then

$$\begin{aligned} A(B(x)) &= A(xC(x)) \\ &= a_0 + a_1 xC(x) + a_2 (xC(x))^2 + \cdots \\ &= a_0 + a_1 xC(x) + a_2 x^2 (C(x))^2 + \cdots \end{aligned}$$

For each $k \geq 0$, note that $a_k x^k (C(x))^k$ is a formal power series (since it is the product of formal power series). Moreover, for each $n < k$, we have $[x^n](a_k x^k (C(x))^k) = 0$. Therefore,

$$\begin{aligned} [x^n]A(B(x)) &= [x^n](a_0 + a_1 xC(x) + a_2 x^2 (C(x))^2 + \cdots) \\ &= [x^n](a_0 + a_1 xC(x) + a_2 x^2 (C(x))^2 + \cdots a_n x^n (C(x))^n). \end{aligned}$$

Now, $a_0 + a_1 xC(x) + a_2 x^2 (C(x))^2 + \cdots a_n x^n (C(x))^n$ is a formal power series (as it is a sum of formal power series), so $[x^n]A(B(x))$ is well-defined. □

1.5 Remarks

1.5.1 Proposition: Coefficients after Multiplication

Let $A(x) = \sum_{n\geq 0} a_n A(x)$ *be any formal power series, then*

$$[x^n]x^k A(x) = \begin{cases} [x^{n-k}]A(x) & n \geq k \\ 0 & n < k \end{cases}$$

Proof. To see this, consider

$$A(x) = 1 + 2x + 4x^2 + \cdots = \sum_{n\geq 0} 2^n x^n$$

We recognize that $[x^n]A(x) = 2^n$. Next, we multiply $A(x)$ by x^3 (so $k = 3$ in the formula above):

$$x^3 A(x) = x^3 + 2x^4 + 4x^5 + \cdots = \sum_{n\geq 0} 2^n x^{n+3}$$

It looks as if we shifted all coefficients to the right by a factor of $k = 3$:

Power of x	a_k in $A(x)$	a_k in $x^3A(x)$	Remarks
x^0	$2^0 = 1$	0	x^0 no longer in $x^3A(x)$
x^1	$2^1 = 2$	0	x^1 no longer in $x^3A(x)$
x^2	$2^2 = 4$	0	x^2 no longer in $x^3A(x)$
x^3	$2^3 = 8$	$2^0 = 1$	x^3 now has coeff that previously belongs to x^0
x^4	$2^4 = 16$	$2^2 = 4$	x^4 now has coeff that previously belongs to x^1
x^5	$2^5 = 32$	$2^3 = 8$	x^5 now has coeff that previously belongs to x^2
...	...	...	...
x^i	2^i	2^{i-3}	x^i now has coeff that previously belongs to x^{i-3}

Hence, when we are extracting the nth coefficient from $x^kA(x)$, if $n < k$, $[x^n]x^kA(x)$ would be 0 because the term x^n no longer exists after $A(x)$ has multiplied with x^k; otherwise, the coefficients can be seen as shifted towards right for k positions and what we are looking for (i.e., a_n for x^n) was previously belonged to x^{n-k}. □

Example. Let $A(x) = (1 + 2x)^2 = 1 + 4x + 4x^2$ and $B(x) = \sum_{i\geq 0} 2^i x^i$. Then

$$\begin{aligned}
[x^n]A(x)B(x) &= [x^n](1 + 4x + 4x^2)B(x) \\
&= [x^n]B(x) + 4[x^n]xB(x) = 4[x^n]x^2B(x) \\
&= 2^n + 4[x^{n-1}]B(x) + 4[x^{n-2}]B(x) \\
&= 2^n + 2^2 \cdot 2^{n-1} + 2^2 \cdot 2^{n-2} \\
&= 2^n + 2^{n+1} + 2^n \\
&= 2^{n+2}.
\end{aligned}$$

1.5.2 Problem-Solving: Recurrence Relation

Example. What is the result for $\dfrac{1-x}{1-2x-3x^2}$? Is it a formal power series?

Solution. Suppose this is equal to $A(x) = \sum_{n\geq 0} a_n x^n$. Cross-multiply, we get

$$\begin{aligned}
1-x &= (1-2x-3x^2)A(x) \\
&= A(x) - 2xA(x) - 3x^2A(x) \\
&= (a_0 + a_1x + a_2x^2 + \cdots) - 2x(a_0 + a_1x + a_2x^2 + \cdots) - 3x^2(a_0 + a_1x + a_2x^2 + \cdots) \\
&= (a_0 + a_1x + a_2x^2 + \cdots) - (2a_0x + 2a_1x^2 + 2a_2x^3 + \cdots) - (3a_0x^2 + 3a_1x^3 + 3a_2x^4 + \cdots) \\
&= a_0 + (a_1 - 2a_0)x + (a_2 - 2a_1 - 3a_0)x^2 + (a_3 - 2a_2 - 3a_1)x^3 + \cdots \\
&= a_0 + (a_1 - 2a_0)x + \sum_{n\geq 2}(a_n - 2a_{n-1} - 3a_{n-2})x^n
\end{aligned}$$

By equating the coefficients on either side of the equation, we have equality if and only if

- $a_0 = 1$
- $a_1 - 2a_0 = -1 \implies a_1 - 2 = -1 \implies a_1 = 1$
- $(a_n - 2a_{n-1} - 3a_{n-2}) = 0$ or $a_n = 2a_{n-1} + 3a_{n-2}$ for all $n \geq 2$
  - $a_2 = 2a_1 + 3a_0 = 2 + 3 = 5$
  - $a_3 = 2a_2 + 3a_1 = 2 \cdot 5 + 3 \cdot 1 = 13$
  - $\cdots$

Hence, the result is a formal power series: $1 + x + 5x^2 + 13x^3 + \cdots$ where the coefficients are defined recursively. $\square$

2 Generating Series

2.1 Introduction

A **generating function** is a way of encoding an *infinite sequence of numbers* (a_n) by treating them as the *coefficients* of a *power series*. This formal power series is the generating function. Unlike an ordinary series, this formal series is allowed to diverge, meaning that hte generating function is not always a true function and the "variable" is actually an indeterminate.

2.2 Definition

2.2.1 Definition: Generating Series

Let S be a set of configurations with a weight function

$$\begin{aligned} w : S &\to \mathbb{N}_0 \\ \sigma &\to w(\sigma), \end{aligned}$$

The **generating series** for S with respect to w is defined by

$$\Phi_S(s) = \sum_{\sigma \in S} x^{w(\sigma)}.$$

2.2.2 Remarks: Weight Function

One way to look at the weight function is that,

$$\left\{ \quad S \quad \right\} \quad \xrightarrow{w} \quad \left\{ \quad \mathbb{N}_0 \quad \right\},$$

the weight function w assigns each element $\sigma \in S$ to a non-negative integer $n \in \mathbb{N}_0$. The coefficients a_k of $\Phi_S(x)$ counts how many times an $n \in \mathbb{N}_0$ appears in the image set.

2.2.3 Connections: Generating Series vs. Formal Power Series

By collecting like-powers of x in $\Phi_S(x)$, we get

$$\Phi_X(s) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{k \geq 0} \left(\sum_{\sigma \in S, w(\sigma) = k} 1 \right) x^k = \sum_{k \geq 0} a_k x^k,$$

where a_k denotes the number of elements in S with weight k. In other words, the coefficient of x^k in $\Phi_S(x)$ counts the *number of elements of weight k in S.

2.3 Trivial Examples

2.3.1 k-Subsets

Let S be the set of all subsets of $\{1, \ldots, n\}$ and $w(\sigma) = |\sigma|$. Then the number of elements S of weight k is $\binom{n}{k}$, the number of k element subsets of $\{1, \ldots, n\}$:

$$\Phi_S(x) = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots = \sum_{k \geq 0} \binom{n}{k} x^k$$

2.3.2 Binary Strings with k 1's

Let S be the set of length-n binary strings, i.e., $S = \{0, 1\}^n$, and $w(b) = \sum_i b_i$. Then the number of elements S with weight k is $\binom{n}{k}$, the number of binary strings with k 1's.

$$\Phi_S(x) = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots = \sum_{k \geq 0} \binom{n}{k} x^k$$

2.3.3 Two Dices

Let $S = \{1, \ldots, 6\}^2$, $w : (a, b) \mapsto a + b$. Then

$$\begin{aligned}
\Phi_S(x) &= \sum_{a,b=1}^{6} x^{a+b} \\
&= x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^5 + 6x^7 + 5x^8 + \cdots + x^{12} \\
&= \left(\sum_{i=1}^{6} x^i \right)^2
\end{aligned}$$

We can interpret the result as, for example, there are three ways for us to get a weight of 4.

Now, if we restrict $S = \{(a, b) : a \in \{1, 3, 5\}, b \in \{2, 4, 6\}$, with the same weight function, then

$$\begin{aligned}
\Phi_S(x) &= \sum_{a,b=1}^{6} x^{a+b} \\
&= x^3 + 2x^5 + 3x^7 + 2x^9 + x^{11} \\
&= (x^1 + x^3 + x^5)(x^2 + x^4 + x^6)
\end{aligned}$$

This factorization will come back in the future sections.

2.3.4 General Binary Strings

Let $S = \bigcup_{n \geq 0} \{0, 1\}^n$ denote the set of all binary strings (of all lengths) and for each binary string $b \in S$, $w(b)$ returns the length of b. We then have

$$\Phi_S(x) = 1 + 2x + 4x^2 + \cdots = \sum_{n \geq 0} (2x)^n = \frac{1}{1 - 2x}$$

That is, there is 1 way to build a string of length 0 (empty string), 2 ways to build a binary string of length 1 (0 and 1), 4 ways to build a binary string of length 2, etc.

2.4 Sum Lemma, Product Lemmas, Geometric Series

2.4.1 The Sum Lemma

If $A \cup B = S$ *and* $A \cap B = \varnothing$*, then* $\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$.

2.4.2 The Product Lemma

Let A *and* B *be sets of configurations with weight functions* α *and* β *respectively. If* $w(\sigma) = \alpha(a) + \beta(b)$ *for each* $\sigma = (a, b) \in A \times B$*, then* $\Phi_{A\times B}(x) = \Phi_A(x)\Phi_B(x)$.

More generally, if $A_1, \cdots, A_k$ *are sets, then* $A_1 \times \cdots \times A_k$ *denotes the set of all* k*-tuples* $(a_1, a_2, \ldots, a_k)$ *where* $a_i \in A_i$ *for all* i*. Now suppose that* α_i *is a weight function for* A_i *and that* w *is a weight function for* $A_1 \times \cdots \times A_k$*. If* $w(\sigma) = \alpha_1(a_1) + \cdots + \alpha_k(a_k)$ *for each* k*-tuples* $\sigma = (a_1, \cdots, a_k)$*, then* $\Phi_{A_1 \times \cdots A_k}(x) = \Phi_{A_1}(x) \cdots \Phi_{A_k}(x)$.

2.4.3 Geometric Series

Recall the formula for (infinite) geometric series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

Plugging in $x = X$, we get

$$1 + X + X^2 + X^3 + \cdots = \frac{1}{1-X}.$$

For example,

$$1 + x^3 + x^6 + x^9 + \cdots = \frac{1}{1-x^3}.$$

2.5 Product Lemma Applications

2.5.1 Binary String Sums

Exercise. Let k, n be fixed non-negative integers. How many solutions are there to $t_1 + \cdots + t_n = k$, where $t_1, \ldots, t_n = 0$ or 1?

Solution. The solutions are n-tuples $(t_1, \ldots, t_n)$ where $t_i \in \{0, 1\}$ for each $i = 1, \ldots, n$. We know the answer is $\binom{n}{k}$, but we can also use the product lemma. Since we are dealing with the set $\{0, 1\}^n$, and $w(t_i) = t_i$, we get

$$\Phi_{\{0,1\}}(x) = 1x^0 + 1x^1 + 0x^2 + \cdots = 1 + x$$

because there is one way to get 0 and one way to get 1. Extending this to length n, we get

$$\Phi_{\{0,1\}^n}(x) = \left(\Phi_{\{0,1\}}(x)\right)^n = (1+x)^n$$

and the number of solutions is

$$[x^k](1+x)^n = \binom{n}{k}. \quad \square$$

2.5.2 Generalized Strings

Exercise. Let S be the set of all k-tuples $(a_1, \ldots, a_k)$ where each $a_i \in \mathbb{N}_0$. The weight of k-tuple $\sigma = (a_1, \ldots, a_k)$ is defined as $w(\sigma) = a_1 + \cdots + a_k$. Show that $\Phi_S(x) = (1-x)^{-k}$.

Solution. Let $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Now $S = \mathbb{N}_0 \times \cdots \times \mathbb{N}_0 = \mathbb{N}_0^k$. Then, by the product lemma,

$$\Phi_S(x) = (\Phi_{\mathbb{N}_0}(x))^k.$$

Now, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, so there exists 1 way to get each $n \in \mathbb{N}_0$:

$$\Phi_{\mathbb{N}_0}(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

Therefore,

$$\Phi_S(x) = (\Phi_{\mathbb{N}_0}(x))^k = \left(\frac{1}{1-x}\right)^k. \quad \square$$

2.5.3 Negative Binomial Theorem

Exercise. For any positive integer k, show that

$$(1-x)^{-k} = \sum_{n \geq 0} \binom{n+k-1}{k-1} x^n$$

Solution. Let S_n be the set of all k-tuples $(a_1, \ldots, a_k) \in \mathbb{N}_0^k$ with $a_1 + \cdots + a_k = n$. By **2.5.2**, there exists $[x^n](1-x)^{-k}$ number of k-tuples satisfying this property.

Let T_n denote the set of all binary strings of length $n+k-1$ with $k-1$ 1's. Then we know

$$|T_n| = \binom{n+k-1}{k-1}.$$

We will establish a bijection between S_n and T_n, and hence prove that

$$[x^n](1-x)^{-k} = |S_n| = |T_n| = \binom{n+k-1}{k-1}.$$

Define $f : S_n \to T_n$ as follows: For a k-tuple $A = (a_1, \ldots, a_k) \in S_n$,

$$f(A) = 0^{a_1} 1 0^{a_2} 1 \cdots 1 0^{a_k}$$

where 0^{a_i} represents a_i 0's in a row. Notice that $f(A)$ is a string with exactly $k-1$ 1's, and the length of the string $a_1 + \cdots + a_k + (k-1) = n + k - 1$. Therefore $f(A) \in T_n$. It is trivial to show there exists an inverse of f which maps a binary string of length n with exactly $k-1$ 1's to a k-tuple. This shows that f is a bijection between S_n to T_n. $\square$

2.5.4 Coin Value

Exercise. Let $n \in \mathbb{N}_0$. Suppose we have an unlimited supply of Canadian nickels, dimes, quarters, loonies, and toonies. How many ways can we make n cents using these coins?

Solution. Let $C_k = \{0, k, 2k, 3k, \ldots\}$. Each collection of coins can be represented by a 5-tuple $(n, d, q, l, t) \in C_5 \times C_{10} \times C_{25} \times C_{100} \times C_{200}$. Let $S = C_5 \times C_{10} \times C_{25} \times C_{100} \times C_{200}$. We define the weight of any 5-tuple in S to be $w(n, d, q, l, t) = n + d + q + l + t$, which represents the value of this collection of coins in cents.

Using the weight function $\alpha_k(a) = a$ for the set C_k, we see that the generating series for C_k is

$$\Phi_{C_k}(x) = 1 + x^k + x^{2k} + x^{3k} + \cdots = \frac{1}{1 - x^k}.$$

For our set S, the set function $w(n, d, q, l, t) = \alpha_t(n) + \alpha_{10}(d) + \alpha_{25}(q) + \alpha_{100}(l) + \alpha_{200}(t)$. Applying the product lemma, we get the generating series for S:

$$\begin{aligned}\Phi_S(x) &= \Phi_{C_5}(x) \cdot \Phi_{C_{10}}(x) \cdot \Phi_{C_{25}}(x) \cdot \Phi_{C_{100}}(x) \cdot \Phi_{C_{200}}(x) \\ &= \frac{1}{(1 - x^5)(1 - x^{10})(1 - x^{25})(1 - x^{100})(1 - x^{200})}.\end{aligned}$$

The number of collections that are valued at n cents is then $[x^n]\Phi_S(x)$. $\square$

1. *Any particular reason for the use of rational number?* ↩