Lecture 1: Euclidean Space

Math 247 Winter Term 2019

Monday, January 7th, David Duan

Contents

1.1 $\mathbb{R}^n$, Inner Products, and Norms

Core Definitions

- We work with the n**-dimensional real vector space**

$$\mathbb{R}^n := \{\vec{x} = (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i \leq i \leq n\}$$

  equipped with usual vector addition and scalar multiplication operations.

- **Euclidean inner product** (also called the *dot product* or *scalar product*):

$$\langle \vec{x}, \vec{y} \rangle := \sum_{i=1}^{n} x_i y_i$$

- **Euclidean norm** of $\vec{x} \in \mathbb{R}^n$:

$$\|\vec{x}\| := \langle \vec{x}, \vec{x} \rangle^{1/2} = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$$

  The Euclidean norm corresponds to the usual notion of *length* for vectors; $\|\vec{x} - \vec{y}\|$ is interpreted as the *distance* between points $\vec{x}$ and $\vec{y}$.

- By **Euclidean space**, we mean $\mathbb{R}^n$ with the structure of space imposed by the Euclidean inner product and norm.

Proposition 1.1.1: Properties of the Euclidean inner product Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. The Euclidean inner product satisfies the following properties:

1. *Positive definite*: $\langle \vec{x}, \vec{x} \rangle \geq 0$ with equality if and only if $\vec{x} = \vec{0}$
2. *Symmetry*: $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$
3. *Bilinearity*: $\langle \alpha\vec{x} + \beta\vec{y}, \vec{z} \rangle = \alpha \langle \vec{x}, \vec{z} \rangle + \beta \langle \vec{y}, \vec{z} \rangle$

Proof. Trivial.

Proposition 1.1.2: Properties of the Euclidean norm Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. The Euclidean norm satisfies the following properties:

1. *Positive definite*: $\|\vec{x}\| \geq 0$ with equality if and only if $\vec{x} = \vec{0}$
2. *Homogeneous*: $\|\alpha\vec{x}\| = |\alpha|\|\vec{x}\|$
3. *Triangle Inequality*: $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

Proof. Trivial, or see **Theorem 1.2.2**.

1.2 Inequalities

Theorem 1.2.1: Cauchy-Schwarz Inequality For any $\vec{x}, \vec{y} \in \mathbb{R}^n$, $|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\|\|\vec{y}\|$.

Proof. The statement is trivial if $\vec{x} = \vec{0}$ or $\vec{y} = \vec{0}$. Suppose $\vec{x} \neq \vec{0}$ and $\vec{y} \neq \vec{0}$ and define the unit vectors

$$\vec{u} = (u_1, u_2, \ldots, u_n) := \frac{\vec{x}}{\|\vec{x}\|}, \quad \vec{v} = (v_1, v_2, \ldots, v_n) := \frac{\vec{y}}{\|\vec{y}\|},$$

For each $i = 1, 2, \ldots, n$,

$$0 \leq (u_i - v_i)^2 = u_i^2 - 2u_i v_i + v_i^2 \implies u_i v_i \leq \frac{1}{2}(u_i^2 + v_i^2).$$

Adding together the inequalities for all i's gives us

$$\sum_{i=1}^{n} u_i v_i \leq \frac{1}{2}\left(\sum_{i=1}^{n} u_i^2 + \sum_{i=1}^{n} v_i^2\right) \implies \langle \vec{u}, \vec{v} \rangle \leq \frac{1}{2}(\|\vec{u}\|^2 + \|\vec{v}\|^2) = 1,$$

since $\|\vec{u}\| = \|\vec{v}\| = 1$ by construction.

Repeat this with the component-wise inequality, we get

$$0 \leq (u_i + v_i)^2 = u_i^2 + 2u_i v_i + v_i^2 \implies u_i v_i \geq -\frac{1}{2}(u_i^2 + v_i^2)$$

and

$$\sum_{i=1}^{n} u_i v_i \geq -\frac{1}{2}\left(\sum_{i=1}^{n} u_i^2 + \sum_{i=1}^{n} v_i^2\right) \implies \langle \vec{u}, \vec{v} \rangle \geq -\frac{1}{2}(\|\vec{u}\|^2 + \|\vec{v}\|^2) = -1.$$

Hence, we have shown that $|\langle \vec{u}, \vec{v} \rangle| \leq 1$, which is the Cauchy-Schwarz Inequality applied to the unit vectors $\vec{u}$ and $\vec{v}$.

Next, observe

$$\|\vec{x}\|\|\vec{y}\| |\langle \vec{u}, \vec{v} \rangle| = \|\vec{y}\| |\langle \|\vec{x}\|\vec{u}, \vec{v} \rangle| = \|\vec{y}\| |\langle \vec{v}, \|\vec{x}\|\vec{u} \rangle| = |\langle \|\vec{y}\|\vec{v}, \|\vec{x}\|\vec{u} \rangle| = |\langle \|\vec{x}\|\vec{u}, \|\vec{y}\|\vec{v} \rangle| = |\langle \vec{x}, \vec{y} \rangle|.$$

Thus, we have

$$|\langle \vec{u}, \vec{v} \rangle| \leq 1 \implies \|\vec{x}\|\|\vec{y}\| |\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{x}\|\|\vec{y}\| \implies |\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\|\|\vec{y}\|$$

as desired. □

Theorem 1.2.2: Triangle Inequality For any two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

Proof.

$$\begin{aligned}
\|\vec{x}+\vec{y}\|^2 &= \langle \vec{x}+\vec{y}, \vec{x}+\vec{y} \rangle \\
&= \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle && \text{By Bilinearity} \\
&\leq \langle \vec{x}, \vec{x} \rangle + |\langle \vec{x}, \vec{y} \rangle| + |\langle \vec{y}, \vec{x} \rangle| + \langle \vec{y}, \vec{y} \rangle && \text{Absolute Values} \\
&\leq \|\vec{x}\|^2 + \|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|\|\vec{x}\| + \|\vec{y}\|^2 && \text{Cauchy-Schwarz} \\
&= (\|\vec{x}\| + \|\vec{y}\|)^2 \\
\implies \|\vec{x}+\vec{y}\| &\leq \|\vec{x}\| + \|\vec{y}\|
\end{aligned}$$

$\square$

1.3 Angles Between Vectors

The Cauchy-Schwarz Inequality can be written as

$$-\|\vec{x}\|\|\vec{y}\| \leq \langle \vec{x}, \vec{y} \rangle \leq \|x\|\|\vec{y}\| \quad \text{or} \quad -1 \leq \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\|\|\vec{y}\|} \leq 1.$$

We can define $\theta \in \mathbb{R}$ by the relation

$$\cos\theta = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\|\|\vec{y}\|}.$$

Observe that

$$\begin{aligned} \|\vec{y} - \vec{x}\|^2 &= \langle \vec{y} - \vec{x}, \vec{y} - \vec{x} \rangle \\ &= \|\vec{y}\|^2 - 2\langle \vec{y}, \vec{x} \rangle + \|\vec{x}\|^2 \\ &= \|\vec{y}\|^2 + \|\vec{x}\|^2 - 2\|\vec{y}\|\|\vec{x}\|\cos\theta, \end{aligned}$$

which is the familiar *Law of Cosines*. Hence, θ is the angle between the two vectors.

This gives us an alternative formula to create the inner product:

$$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\|\|\vec{y}\|\cos\theta$$