Lecture 11: Convex Sets and Uniform Continuity

Math 247

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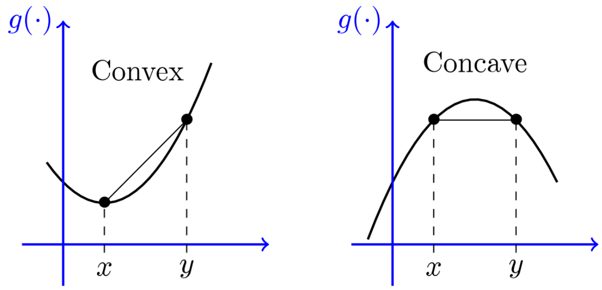
1 Convex Sets

1.1 Convex Curve

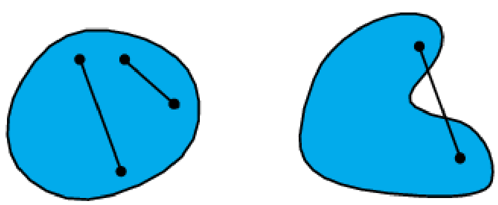


In a Euclidean space, a **convex curve** is a curve which lies completely on one side of each and every one of its tangent lines.

Recall for a twice-differentiable function f, the graph of f is *convex* (or *concave upward*) if $f''(x) > 0$ and *concave* (or *concave downward*) if $f''(x) < 0$.

The graph for $f(x) = x^2$ is convex on its domain $\mathbb{R}$ as $f''(x) = 2 > 0$; the graph for $f(x) = \log(x)$ is concave on its domain $(0, \infty)$ as $f''(x) = -x^{-2} < 0$.

1.2 Convex Set



convex *concave*

In a Euclidean space, a **convex region** is a region where, for every pair of points within the region, every point on the straight line segment that joins the pair of points is also within the region. The boundary of a convex set is always a convex curve.

More precisely, let X be a nonempty subset of $\mathbb{R}^n$. We say that X is **convex** if for all points $\vec{x}, \vec{y} \in X$ and for all $t \in [0, 1]$, the point $\vec{x} + t(\vec{y} - \vec{x})$ is in X.

Theorem. If X is a convex set and $x_1, \ldots, x_k$ are any points in it, then $x = \sum_{i=1}^{k} \lambda_i x_i$ where all $\lambda_i > 0$ and $\sum_{i=1}^{k} \lambda_i = 1$ is also in X.

Proof. Induction on k. See here.

2 Uniform Continuity

2.1 Uniform Continuity

We say that a function f from $A \subset \mathbb{R}^n$ to $\mathbb{R}^m$ is **uniformly continuous** if given $\varepsilon > 0$, there exists $\delta > 0$ such that $\|f(\vec{x}) - f(\vec{y})\| < \varepsilon$ for every $\vec{x}$ and $\vec{y}$ in A satisfying $\|\vec{x} - \vec{y}\| < \delta$.

For a function to be continuous at a point $\vec{a}$ (pointwise continuity), we choose δ after fixing the point $\vec{a}$ and $\varepsilon > 0$. For uniform continuity, the same δ must work for all points $\vec{y}$. Hence, the "uniform". In particular, uniform continuity implies pointwise continuity.

Section 2.2 reviews the $\varepsilon - \delta$ proof we used in Math 147. Since nobody likes δ, we introduce two other techniques for recognizing uniform continuity without hunting for δ's: via *compactness* (Section 2.3) and *Lipschitz functions* (Section 3.1).

2.2 Example: $f(x) = x^2$

2.2.1 Proving $f(x)$ is Uniformly Continuous on a $[a, b] \subsetneq \mathbb{R}$

Consider $f(x) = x^2$ on the domain $[a, b] \subset \mathbb{R}$ for some real numbers $a < b$. We show that f is uniformly continuous on $[a, b]$. Let $\varepsilon > 0$. For any $x, y \in [a, b]$, we have that

$$\begin{aligned} |f(x) - f(y)| = |x^2 - y^2| = |(x + y)(x - y)| &= |x + y||x - y| \\ &\leq (|x| + |y|)|x - y| \\ &\leq 2\max\{|x|, |y|\}|x - y| \\ &\leq 2\max\{|a|, |b|\}|x - y|. \end{aligned}$$

Since $a < b$, either $a \neq 0$ or $b \neq 0$ (or both). Let $M = 2\max\{|a|, |b|\} > 0$ and define $\delta = \varepsilon/M$. Note that this δ does not depend on x or y. We now have that

$$|x - y| < \delta \implies |f(x) - f(y)| \leq M|x - y| < M\varepsilon/M = \varepsilon. \qquad \square$$

2.2.2 Proving $f(x)$ is NOT Uniformly Continuous on $\mathbb{R}$

Consider $f(x) = x^2$ on the domain of $\mathbb{R}$. We show that f is not uniformly continuous here. Suppose to the contrary that it is. Then for every ε, there exists a δ for which

$$|x - y| < \delta \implies |x^2 - y^2| < \varepsilon.$$

In particular, there exists a δ for $\varepsilon = 1$. Let $y = x + \delta/2$ (notice $|x - y| < \delta$). Then

$$|x^2 - y^2| < \varepsilon \implies \left| x^2 - \left(x + \frac{\delta}{2} \right)^2 \right| < 1 \implies \left| x\delta + \frac{\delta^2}{4} \right| < 1$$

for every $x \in \mathbb{R}$. This is a clear contradiction, since we can choose x arbitrarily large. Hence, $f(x)$ is not uniformly continuous on $\mathbb{R}$. $\square$

2.3 Compactness & Continuity $\implies$ Uniform Continuity

Theoerm 11.4.1 Let f be a continuous function from $K \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$. If K is compact, then f is uniformly continuous on K.

Proof. Suppose for contradiction that K is not uniformly continuous. This means that there exists some $\varepsilon > 0$ such that *for all* $\delta > 0$*, there are points* $\vec{x}$ *and* $\vec{y}$ *in* K *satisfying* $\|\vec{x} - \vec{y}\| < \delta$ *and* $\|f(\vec{x}) - f(\vec{y})\| \geq \varepsilon$. (You should be very familiar of the negation of the definition.)

Define a sequence of δ values, $\delta_k = 1/k$, and choose points $\vec{x}_k$ and $\vec{y}_k$ such that

$$\|\vec{x}_k - \vec{y}_k\| < \delta_k \quad \text{and} \quad \|f(\vec{x}_k) - f(\vec{y}_k)\| \geq \varepsilon.$$

By compactness of K, there must be a subsequence $(\vec{y}_{k_j})_{j=1}^{\infty}$ that converges to a point $\vec{c} \in K$.

We now show that the sequence $(\vec{x}_{k_j})_{j=1}^{\infty}$ (using the same k_j as for the subsequence $(\vec{y}_{k_j})_{j=1}^{\infty}$) also converges to $\vec{c}$. Fix ε_0 and choose $N = \lfloor \frac{1}{\varepsilon_0} \rfloor + 1$. Then for $k_j > N$,

$$\|\vec{x}_{k_j} - \vec{y}_{k_j}\| < \delta_{k_j} = \frac{1}{k_j} < \frac{1}{N} = \frac{1}{\lfloor \frac{1}{\varepsilon_0} \rfloor + 1} < \frac{1}{\frac{1}{\varepsilon_0}} = \varepsilon_0.$$

In other words, $\lim_{j\to\infty} \|\vec{x}_{k_j} - \vec{y}_{k_j}\| = 0$ so both subsequences converge to the same point $\vec{c}$.

Since f is continuous on K, in particular at each $\vec{x}_{k_j}$ and $\vec{y}_{k_j}$, by sequential characterization of continuity,

$$\lim_{j\to\infty} f(\vec{y}_{k_j}) = f(\vec{c}) = \lim_{j\to\infty} f(\vec{x}_{k_j})$$

because both subsequences in K converge to $\vec{c}$. By linearity,

$$\lim_{j\to\infty} f(\vec{y}_{k_j}) = \lim_{j\to\infty} f(\vec{x}_{k_j}) \implies \lim_{j\to\infty} \|f(\vec{x}_{k_j}) - f(\vec{y}_{k_j})\| = 0 < \varepsilon.$$

This contradicts our hypothesis that we can always find $\vec{x}$ and $\vec{y}$ in K satisfying $\|\vec{x} - \vec{y}\| < \delta$ and $\|f(\vec{x}) - f(\vec{y})\| \geq \varepsilon$, no matter what δ is given. Hence, if K is compact and f is continuous on K, then f is uniformly continuous on K. $\square$

3 Lipschitz Functions

3.1 Lipschitz Functions

A function f from $A \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$ is called a **Lipschitz function** if

$$\exists C \in \mathbb{R}: \quad \forall \vec{x}, \vec{y} \in A: \quad \|f(\vec{x}) - f(\vec{y})\| < C\|\vec{x} - \vec{y}\|.$$

Any constant C for which this condition is satisfied is called a **Lipschitz constant** for f. The smallest C for which this condition holds is called **the (best) Lipschitz constant**.

Intuitively, a Lipschitz continuous function is limited in how fast it can change: there exists a real number such that, for every pair of points on the graph of this function, the absolute value of the slope of the line connecting them is not greater than this real number.

As a remark, an everywhere differentiable function $g: \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous (with $C = \sup |g'(x)|$) if and only if it has bounded first derivative.

3.2 Lipschitz Functions and Uniform Continuity

Proposition. Every Lipschitz function is uniformly continuous.

Proof. Suppose that f is a Lipschitz function with Lipschitz constant C. Given $\varepsilon > 0$, let $\delta = \varepsilon/C$. Then

$$\|\vec{x} - \vec{y}\| < \delta \implies \|f(\vec{x}) - f(\vec{y})\| \leq C\|\vec{x} - \vec{y}\| < C\delta = \varepsilon. \qquad \square$$

Proposition. Every linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ is uniformly continuous.

Proof. Later.