STAT-333 Highlights

Applied Probability / Stochastic Processes I

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Contents

1 Discrete-Time Markov Chain

1.1 Discrete-Time Markov Chain

- A discrete-time stochastic process $\{X_n\}_{n=0,1,\ldots}$ is called a **discrete time Markov chain** if for any $n \in \mathbb{N}$ and any states $j, i, i_{n-1}, \ldots, i_0 \in S$, the **Markov property** holds:

$$\Pr(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = \Pr(X_{n+1} = j \mid X_n = i).$$

- The **transition probabilities** P_{ij} from i to j are recorded in the **transition matrix**

$$P = [P_{ij}]_{i,j \in S} = [\Pr(X_{n+1} = j \mid X_n = i)]_{i,j \in S}.$$

- The transition matrix satisfies $P_{ij} \geq 0$ for all $i, j \in S$ and $\sum_{j \in S} P_{ij} = 1$ for all $j \in S$.
- The n-step transition matrix is given by $P^{(n)} = P^n$.

1.2 Chapman-Kolmogorov Equation

- $P^{(m+n)} = P^{(m)} P^{(n)}$, or equivalently, $\Pr(X_{m+n} = j \mid X_0 = i) = (P^{(m)} P^{(n)})_{ij}$.
- $P_{ij}^{m+n} = \sum_{k \in S} P_{ik}^{(m)} P_{jk}^{(n)} \geq P_{i\ell}^{(m)} P_{\ell j}^{(n)}$ for a fixed $\ell \in S$.

1.3 Distribution of X_n

- The vector $\mu_n = (\mu_n(0), \mu_n(1), \ldots)^T$ where $\mu_n(i) = \Pr(X_n = i)$ gives the **distribution** of X_n.
- The case where $n = 0$ is known as the **initial distribution** of a DTMC, denoted $\mu = \mu_0$.
- Let $n \in \mathbb{N}$. We have $\mu_n = \mu P^n$.

1.4 Expectation of Functions of X_n

- Let $f : S \to \mathbb{R}, i \mapsto f(i)$ be a function of X_n, or equivalently, $f = [f(0), f(1), \ldots]^T$. Then

$$\mathbb{E}[f(X_n)] = \mu P^n f^T.$$

1.5 Stationary Distribution and Stationary Measure

- A probability distribution $\pi = (\pi_0, \pi_1, \ldots)^T$ is called **stationary** if $\pi = \pi P$ and $\sum_{i \in S} \pi_i = 1$.
- A row vector $\mu^* = (\mu^*(0), \mu^*(1), \ldots)$ is called a **stationary measure** if $\mu^* \geq \mathbf{0}$ and $\mu^* P = \mu^*$.
- For a irreducible and recurrent DTMC, the row vector

$$\mu_x(y) = \sum_{n=0}^{\infty} \Pr_x(X_n = y, T_x > n), \quad y \in S$$

defines a stationary measure with $0 < \mu_x(y) < \infty$ for all $y \in S$.

1.6 Communicating Class, Irreducibility, and Closedness

- Let $x, y \in S$. We say x **communicates** to y, denoted $x \to y$, iff $\rho_{xy} > 0$.
- A set $C \subseteq S$ is a **communicating class** if $i \leftrightarrow j$ for all $i, j \in C$ and $i \not\leftrightarrow j$ for all $i \in C$, $j \notin C$.
- A set $A \subseteq S$ is **irreducible** if $i \leftrightarrow j$ for all $i, j \in A$, i.e., A is a communicating class.
- A DTMC is **irreducible** if S is irreducible, i.e., all of its states are in the same class.
- A set $A \subseteq S$ is **closed** if $i \not\to j$, or equivalently, $P_{ij} = 0$, whenever $i \in A$, $j \notin A$.

1.7 Recurrence and Transience

- Let $T_y := \min\{n \geq 1 \mid X_n = y\}$ be the time of the first (re)visit to state $y \in S$.
- Let $\Pr_x(A) := \Pr(A \mid X_0 = x)$ be the probability function conditioned on $X_0 = x$.
- Let $\rho_{xy} := \Pr_x(T_y < \infty)$ be the probability that the DTMC ever revisits y if it starts at x.
- A state $y \in S$ is **recurrent** if $\rho_{yy} = 1$ and is **transient** if $\rho_{yy} < 1$.
- If $\rho_{xy} > 0$ but $\rho_{yx} < 1$, then x is transient.
- If x is recurrent and $\rho_{xy} > 0$, then $\rho_{yx} = 1$.

1.8 Recurrence and Transience as Class Properties

- A finite closed set has at least one recurrent state.
- A finite closed class must be recurrent.
- An irreducible DTMC with a finite state space is recurrent.
- The state space S can be written as a disjoint union $S = T \,\dot\cup\, R_1 \,\dot\cup\, R_2 \,\dot\cup\, \cdots$ where T is the set of all transient states (not necessarily a class) and R_i's are closed recurrent classes.

1.9 Strong Markov Property

- The process $\{X_{T_y+k}\}_{k=0,1,\ldots}$ behaves like the DTMC with initial state y, i.e., we can forget about the history and restart from state y, provided that the DTMC is time-homogeneous.

1.10 Total Number of Visits

- Let $T_y^k = \min\{n \geq T_y^{k-1} \mid X_n = y\}$ be the time of the kth (re)visit to state $y \in S$.
- $\Pr_y(T_y^k < \infty) = \Pr(T_y^k < \infty \mid X_0 = y) = (\rho_{yy})^k$.
- Let $N(y)$ be the total number of visits to y.
- For any $x, y \in S$,

$$\mathbb{E}_x[N(y)] = \frac{\rho_{xy}}{1 - \rho_{yy}} = \sum_{n=1}^{\infty} P_{xy}^n.$$

1.11 Classifying States

- y is transient $\iff \rho_{yy} < 1 \iff \rho_{yy}^k \to 0 \iff \mathbb{E}_y[N(y)] < \infty \iff \sum_{n=1}^{\infty} P_{yy}^n < \infty$.
- y is recurrent $\iff \rho_{yy} = 1 \iff \rho_{yy}^k \to 1 \iff \mathbb{E}_y[N(y)] = \infty \iff \sum_{n=1}^{\infty} P_{yy}^n = \infty$.

1.12 Periodicity

- The **period** of a state $x \in S$ is given by $d(x) = \gcd\{n \geq 1 : P_{xx}^n > 0\}$.
- If $d(x) = 1$, the state x is said to be **aperiodic**. A MC is **apariodic** if all states are aperiodic.
- States in the same class have the same period, i.e., $x \leftrightarrow y \implies d(x) = d(y)$.

1.13 IRAS: Irreducibility, Recurrence, Aperiodicity, and Stationarity

- If y is aperiodic, then there exists $n_0 \in \mathbb{N}$ such that $P_{yy}^n > 0$ for all $n \geq n_0$.
- If π is a stationary distribution with $\pi(y) > 0$, then y is recurrent.
- If y is transient, then $\pi(y) = 0$ for any stationary distribution π.
- An irreducible MC with a stationary distribution π is recurrent, i.e., $I \wedge S \implies R$.

1.14 Limiting Behavior

- For an irreducible and aperiodic MC with a stationary distribution π, the limiting transition probability to state y converges to $\pi(y)$, i.e., $I \wedge A \wedge S \implies \forall x, y \in S : P_{xy}^n \xrightarrow{n\to\infty} \pi(y)$.
- The limiting transition probability and distribution does not depend on the starting state.
- Let $N_n(Y)$ be the number of visits to y up to time n.
- The long run fraction of time spent in state y is $1/\mathbb{E}_y[T_y]$, i.e.,

$$\frac{N_n(y)}{n} \xrightarrow{n\to\infty} \frac{1}{\mathbb{E}_y[T_y]}.$$

- We call $\mathbb{E}_y[T_y]$ the expected cycle length as it records the revisit time to y after starting at y.