Math 148 Note Week 1

Area under a Curve

- Consider a bounded function $f : [a, b] \to \mathbb{R}$. We know this function is bounded because of the d**Extreme Value Theorem**. Let P denotes a partition of $[a, b]$:

  $P : a = t_0 < t_1 < \cdots < t_n = b$

  This partition divides $[a, b]$ into subintervals $[t_{i-1}, t_i], 1 \leq i \leq n$.

- Let M, m be the supremum and infimum, ie.

  $M_i = sup\{f(x) : x \in [t_{i-1}, t_i]\}$

  $m_i = inf\{f(x) : x \in [t_{i-1}, t_i]\}$

- By the **Extreme Value Theorem**, if f is continuous, then there exists $c_i, d_i \in [t_{i-1}, t_i]$ such that $M_i = f(c_i), m_i = f(d_i)$.

Upper Riemann Sum and Lower Riemann Sum

- ***Formula.***

  $U(f, p) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$

  $L(f, p) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$

  - The value of these two functions depend on the function f and also the partition p.
  - M_i and m_i is the local max/min on the interval $[t_{i-1}, t_i]$; they are the "height" of the function.
  - Multiplying M_i and m_i with the "width" $(t_i - t_{i-1})$, we get a bunch of areas of rectangles.
  - Taking the sum of all the rectangles, we get an approximation of the area under the curve.
- Note that URS is always an over-estimate; LRS is always an under-estimate. That is,

  $\text{LRS} \leq \text{"Area"} \leq \text{URS}$

  $L(f, p) \leq \int_a^b f(x)dx \text{ (if exists)} \leq U(f, p)$

Refinement

- ***Definition.*** A refinement of partition p is a partition Q which contains all the points of p (and more).
- ***Remark.*** The subintervals don't have to be evenly spaced.

- *Now what would happen if we use a refined partition to $L(f,p)$ and $U(f,0)$?*
  - Since refinement makes $L(f,p)$ and $U(f,p)$ better approximations of the actual area, we have the following inequalities: $U(f,q) \leq U(f,p)$ $L(f,q) \geq L(f,p)$ That is, URS would fall and LRS would rise.

- ***Observation.*** Let p_i, p_j be two partitions of $[a,b]$. Let Q be the common refinement of p_i, p_j, ie. $Q = p_i \cup p_j$. Then we have

  $$L(f,p_i) \leq L(f,Q) \leq U(f,Q) \leq U(f,p_j), \quad \forall i,j \in \cup \textbf{partitions}$$

  Hence, any URS is bounded below by any LRS. From this, we can conclude that

  $$I = \inf\{U(f,p) \mid \textbf{all partition of } [a,b] \textbf{ exists}\}$$
  $$S = \sup\{L(f,p) \mid \textbf{all partition of } [a,b] \textbf{ exists}\}$$

  Moreover,

  $$S = \sup\{L(f,p) : \textbf{all partitions } p\} \leq \int_a^b f(x)dx \leq \inf\{U(f,p) : \textbf{all partitions } p\} = I$$

  Thus, if $S = I$, we have $S = \int_a^b f(x)dx = I$.

- ***Definition.*** We say $f : [a,b] \to \mathbb{R}$ is integrable over $[a,b]$ if $S = \sup\{L(f,p) \mid \textbf{all } p\} = \inf\{U(f,p) \mid \textbf{all } p\} = I$. We write $\int_a^b f = I = S$. When f is positive and continuous, $\int_a^b f$ is the area under f over $[a,b]$
- ***Definition.*** We define the area under f over $[a,b]$ for $f \geq 0$ as $\int_a^b f$, provided that f is integrable.

Riemann Sum

- Given any partition p of $[a,b]$, say $p : a = t_0 < \cdots < t_n = b$. Pick $c_i \in [t_{i-1}, t_i]$, then

  $$R(f,p) = \sum_{i=1}^n f(c_i)(t_i - t_{i-1}).$$

  Note that $L \leq R \leq U$ since $m_i \leq f(c_i) \leq M_i$.
- Moreover, if f is continuous, there exists a choice of $c_i \in [t_{i-1}, t_i]$ such that $M = f(c_i)$ and $m = f(c_i')$

Characterization Theorem

- ***Theorem.*** Suppose $f : [a,b] \to \mathbb{R}$ is bounded. Then f is integrable iff for all $\epsilon > 0$ there exists a partition p^* such that $U(f,p^*) - L(f,p^*) < \epsilon$.

- ***Proof.***
  - Assume for each $\epsilon > 0$ there exists a partition p_ϵ such that $U(f,p_\epsilon) - L(f,p_\epsilon) < \epsilon$. We want to show that $I = S$. Since

$I=\inf\{U(f,p)\mid \text{all } p\}\le U(f,p_\epsilon),\ S=\sup\{L(f,p)\mid \text{all } p\}\ge L(f,p_\epsilon),$

we have $I-S\le U(f,p_\epsilon)-L(f,p_\epsilon)<\epsilon.$

This is true for all ϵ, thus $I=S$, and f is integrable.

- Now assume that f is integrable. Let $\epsilon>0$. We know (given) $I=S$.

  $I=\inf\{U(f,p)\mid \text{all } p\},\ S=\sup\{L(f,p)\mid \text{all } p\}.$

  Uses the definition of supremum and infimum, pick p_1 such that $I\le U(f,p_1)<I+\epsilon/2$ and p_2 such that $S-\epsilon/2<L(f,p_2)\le S$. But $I=S$,so

  $I-\frac{\epsilon}{2}<L(f,p_2)\le I\le U(f,p_1)<I+\frac{\epsilon}{2}.$

  Let p be the common refinement for p_1,p_2. Then

  $U(f,p)-L(f,p)\le U(f,p_1)-L(f,p_2)<\epsilon.$

- ***Application***.
  - Show that $f(x)=x^2$ on $[0,1]$.
  - ***Proof.*** Apply the above theorem.
    - Let $\epsilon>0$. Choose $n\in\mathbb{N}$ such that $\frac{1}{n}<\epsilon$. Let $P_n:=0=t_0<t_1<\cdots<t_{n-1}<t_n=1$ where $t_i=\frac{i}{n}$.
    - Then $M_i=f(t_i)=(\frac{i}{n})^2$, $m_i=f(t_{i-1})=(\frac{i-1}{n})^2$.
      - Then

        $$U(f,P_n)=\sum_{i=1}^{n}M_i(t_i-t_{i-1})=\sum_{i=1}^{n}(\frac{i}{n})^2(\frac{i}{n})=\frac{1}{n^3}\sum_{i=1}^{n}i^2=\frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

      $$L(f,P_n)=\sum_{i=1}^{n}m_i(t_i-t_{i-1})=\sum_{i=1}^{n}(\frac{i-1}{n})^2(\frac{i}{n})=\frac{1}{n^3}\sum_{i=1}^{n-1}i^2=\frac{1}{n^3}\cdot\frac{(n-1)n(2n-1)}{6}$$

    - Then

      $$U(f,P_n)-L(f,P_n)=\frac{(n+1)(2n+1)-(n-1)(2n-1)}{6n^2}=\frac{1}{n}<\epsilon$$

    - By the characterization theorem, $f(x)=x^2$ is integrable.

Uniform Continuity

- ***Definition.*** Continuity.

  Let E be a nonempty subset of $\mathbb{R}$ and $f:E\to\mathbb{R}$. Then f is said to be ***continuous*** at a point $a\in E$ if and only if

  $\forall\epsilon>0\ \exists\delta>0\ st.\ |x-a|<\delta\wedge x\in E\Rightarrow|f(x)-f(a)|<\epsilon$

- ***Definition.*** Uniform continuity.

  Let E be a nonempty subset of $\mathbb{R}$ and $f:E\to\mathbb{R}$. Then f is said to be ***uniformly continuous*** on E if and only if

  $\forall\epsilon>0\ \exists\delta>0\ st.\ |u-v|<\delta\wedge u,v\in E\Rightarrow|f(u)-f(v)|<\epsilon$

- ***Remark***. Difference between continuity and uniform continuity.
  - Continuity is about a point a, such that all points around a behave nicely.
  - Uniform continuity is about every two points in the domain, as long as they satisfy the condition $|u-v|<\delta$.

Prove that $f(x)=x^2$ is not uniformly continuous.

- ***Proof.*** Let $\epsilon=1$. Suppose we can find a δ that works. Consider $x=a, y=a+\frac{\delta}{2}$. $|x-y|<\frac{\delta}{2}$, but $|f(x)-f(y)|=|a^2-a^2-a\delta+\frac{\delta^2}{4}|$. Thus $a\to\infty\Rightarrow|f(x)-f(y)|>\epsilon$.

Prove that $f(x)=\frac{1}{x}$ is not uniformly continuous on $(0,\infty)$.

- ***Proof.*** Let $\epsilon=1$. Suppose we can find δ that works. Let $x=\frac{1}{N}, y=\frac{1}{2N}$, then $|x-y|<\frac{1}{2N}<\delta$ given that N is big enough. But $|f(x)-f(y)|=|\frac{1}{x}-\frac{1}{y}|=|N-2N|=N>1=\epsilon$.

Prove if $f(a,b)\to\mathbb{R}$ and f' is bounded, then f is uniformly continuous.

- ***Proof***. Since f' is bounded, pick μ such that $|f'(x)|\le c$ for all $x\in(a,b)$. Let $\epsilon>0$. Take $\delta=\frac{\epsilon}{\mu}$. Then if $x,y\in(a,b)$, by ***Mean Value Theorem*** $\left|\dfrac{f(x)-f(y)}{x-y}\right|=|f'(t)|\le\mu$ for $t\in[x,y]$. Rewriting this, we have $|f(x)-f(y)|\le\mu|x-y|$. Then $|x-y|<\delta=\frac{\epsilon}{\mu}\Rightarrow|f(x)-f(y)|<\epsilon$.

Prove if $f[a,b]\to\mathbb{R}$ is continuous, then f is uniformly continuous.

- ***Proof.***
  - Suppose false. Let $\epsilon>0$, then δ_ϵ does not exist (fails to exist). That is, $\forall\delta>0\ \exists x,y\in dom f\ st\ |x-y|<\delta\to|f(x)-f(y)|\ge\epsilon$. Think about $\delta=\frac{1}{n},\ n\in\mathbb{N}$.
  - For all $n\in\mathbb{N}$, there exists $x_n,y_n\in\textbf{dom } f=[a,b]\ st.\ |x_n-y_n|<\frac{1}{n}$, but $|f(x_n)-f(y_n)|\ge\epsilon$.
  - Take sequences $(x_n),(y_n)_{n=1}^{\infty}\in[a,b]^{\mathbb{N}}$ (they are bounded sequences), by ***BWT***, there exists a subsequence (x_{n_k}) which converges to $x_0\in[a,b]=\textbf{dom } f$.
  - For $(y_n)_{n=1}^{\infty}$, $x_n-\frac{1}{n}<y_n<x_n+\frac{1}{n}$ for all $n\in\mathbb{N}$. Then $x_{n_k}-\frac{1}{n_k}<y_{n_k}<x_{n_k}+\frac{1}{n_k}$. By ***SQZ*** $(y_{n_k})_{k=1}^{\infty}$ converges to x_0.
  - Now f is continuous at x_0. Since $(x_{n_k})\to x_0$, $f(x_{n_k})\to f(x_0)$. Similarly, $(y_{n_k})\to x_0\Rightarrow f(y_{n_k})\to f(x_0)$. Thus $|f(x_{n_k})-f(y_{n_k})|=0$.
  - But $|f(x_{n_k})-f(y_{n_k})|\ge\epsilon$ for all k. This is a contradiction. QED.

Bounded, Continuous Functions are Integrable.

- ***Theorem.*** If $f:[a,b]\to\mathbb{R}$ is continuous (it's automatically bounded by ***EVT***, and note that integrability only makes sense if f is bounded.), then f is integrable.
- ***Strategy:*** Show that for all $\epsilon>0$ there exists partition p such that $U(f,p)-L(f,p)<\epsilon$.
  - Let p be a partition, $a=t_0<\cdots<t_n=b$.
  - Let $M_i=\sup f|_{[t_{i-1},ti]}=\max f|_{[t_{i-1},ti]}=f(c_i)$ and $m_i=\inf f|_{[t_{i-1},ti]}=\min f|_{[t_{i-1},ti]}=f(d_i)$. We know they are attained because of

continuity and ***EVT***.
  - Then $U(f,p) - L(f,p) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) - \sum_{i=1}^{n} m_i(t_i - t_{i-1}) = \sum_{i=1}^{n}(M_i - m_i)(t_i - t_{i-1})$
  - We know $(t_i - t_{i-1})$ is a fixed value, and we can control M_i and m_i using uniform continuity, so we can make this sum less than ϵ.
- ***Proof.*** Let $\epsilon > 0$.
  - By the above theorem, f is continuous on $[a,b]$ implies it's uniformly continuous on $[a,b]$. That is, $\forall \epsilon\ \exists \delta\ st\ x, y \in [a,b] \wedge |x-y| < \delta \Rightarrow |f(x) - f(y)| < \dfrac{\epsilon}{b-a}$ (we deliberately choose the value $\dfrac{\epsilon}{b-a}$).
  - Take any partition p of $[a,b]$, $p : a = t_0 < \cdots < t_n = b$ with the property that the length of each subinterval $|t_i - t_{i-1}| < \delta$.
  - Now $M_i = \sup f|_{[t_{i-1},ti]} = \max f|_{[t_{i-1},ti]} = f(c_i)$ for some $c_i \in [t_{i-1}, t_i]$, $m_i = \inf f|_{[t_{i-1},ti]} = \min f|_{[t_{i-1},ti]} = f(d_i)$ for some $d_i \in [t_{i-1}, t_i]$, hence $|c_i - d_i| \leq t_i - t_{i-1} < \delta$. Therefore $|f(c_i) - f(d_i)| < \dfrac{\epsilon}{b-a}$.
  - So,

$$
\begin{aligned}
U(f,p) - L(f,p) &= \sum_{i=1}^{n}(M_i - m_i)(t_i - t_{i-1}) \\
&< \sum_{i=1}^{n} \frac{\epsilon}{b-a}(t_i - t_{i-1}) \\
&= \frac{\epsilon}{b-a} \cdot (b-a) = \epsilon
\end{aligned}
$$

  - Hence it is integrable.

Bounded and Monotonic Functions are Integrable.

- ***Theorem.*** If $f : [a,b] \to \mathbb{R}$ is monotonic, then it's integrable.
- ***Proof.***
  - Note that f is bounded by EVT. We prove the case that f is increasing.
  - Let $\epsilon > 0$, take a uniform partition $p_n : a = t_0 < \cdots < t_n$, where $t_j = a + j(\frac{b-a}{n})$.
  - Since f is increasing, for each interval $[t_{i-1}, t_i]$, $M_i = f(t_i), m_i = f(t_{i-1})$.
  - Then

$$
\begin{aligned}
U(f,p) - L(f,p) &= \sum_{i=1}^{n}(M_i - m_i)(t_i - t_{i-1}) \\
&= \frac{b-a}{n} \sum_{i=1}^{n} f(t_i) - f(t_{i-1}) \text{ -- Note that this is a telescoping sum!} \\
&= \frac{b-a}{n} \cdot (f(t_n) - f(t_0)) \\
&= \frac{b-a}{n} \cdot (f(b) - f(a))
\end{aligned}
$$

$= \dfrac{c}{n}$ -- where c is some constant.

- Pick n such that $\dfrac{c}{n} < \epsilon$. Then $U(f, p) - L(f, p) = \dfrac{c}{n} < \epsilon$. Hence f is integrable.

V1.0, 2018-01-14