Лекция №5

27.10.21г

$$W(a_1, a_2, \dots, a_n) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ a_1^3 & a_2^3 & \dots & a_n^3 \\ \hline a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{vmatrix}_{n\times n} = \prod_{1\le j<i\le n} (a_i - a_j)$$

det на Вандермонд

$$\begin{vmatrix} a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \\ & \vdots & & \\ a_1 & a_2 & \dots & a_n \\ 1 & 1 & \dots & 1 \end{vmatrix} = (-1)^{n} W(a_1, \dots, a_n) = \prod_{1\le i<j\le n} (a_i - a_j)$$

<u>Пр:</u>

$$\begin{vmatrix} a_0^n & a_1^n & \dots & a_n^n \\ a_0^{n-1}b_0 & a_1^{n-1}b_1 & \dots & a_n^{n-1}b_n \\ a_0^{n-2}b_0^2 & a_1^{n-2}b_1^2 & \dots & a_n^{n-2}b_n^2 \\ \hline a_0 b_0^{n-1} & a_1 b_1^{n-1} & \dots & a_n b_n^{n-1} \\ b_0^n & b_1^n & \dots & b_n^n \end{vmatrix}_{(n+1)\times(n+1)} = a_0^n a_1^n \dots a_n^n \cdot$$

$$\cdot \begin{vmatrix} 1 & 1 & \dots & 1 \\ \left(\frac{b_0}{a_0}\right) & \left(\frac{b_1}{a_1}\right) & \dots & \left(\frac{b_n}{a_n}\right) \\ \hline \left(\frac{b_0}{a_0}\right)^n & \left(\frac{b_1}{a_1}\right)^n & \dots & \left(\frac{b_n}{a_n}\right)^n \end{vmatrix} = \prod_{j=0}^{n} a_j^n \cdot W\left(\frac{b_0}{a_0}, \dots, \frac{b_n}{a_n}\right)$$

$=$

$$=\left(\prod_{j=0}^{n} a_j^{n}\right)\cdot \prod_{0\le k<i\le n}\left(\frac{b_i}{a_i}-\frac{b_k}{a_k}\right)$$

формули на Крамер

$$(*)\ \begin{cases} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n=b_2 \\ \text{— — — —} \\ a_{n1}x_1+a_{n2}x_2+\dots+a_{nn}x_n=b_n \end{cases}$$

м. на $(*)$ $A=(a_{ij})_{n\times n}$ $\downarrow$ $\det A=\Delta$

$$\Updownarrow\quad \begin{cases} \Delta x_1=\Delta_1 \\ \Delta x_2=\Delta_2 \\ \vdots \\ \Delta x_n=\Delta_n \end{cases}$$

$$(1)_*\cdot A_{11}+(2)_*\cdot A_{21}+(3)_*\cdot A_{31}+\dots+(n)_*\cdot A_{n1}=\sum_{k=1}^{n} b_k A_{k1}$$

$\Longleftrightarrow$

$$(a_{11}A_{11}+a_{21}A_{21}+\dots+a_{n1}A_{n1})x_1+(\underbrace{a_{12}A_{11}+\dots+a_{n2}A_{n1}}_{ф.р.})x_2+$$

$$+\underbrace{\dots}_{ф.р.}+(\underbrace{a_{1n}A_{11}+\dots+a_{nn}A_{n1}}_{ф.р.})x_n=\sum_{k=1}^{n} b_k A_{k1}$$

$$\Delta x_1=\sum_{k=1}^{n} b_k A_{k1} \underset{\substack{р.\\ по\\ \beta_1}}{=} \begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \vdots & \text{—} & \text{—} & \text{—} \\ b_n & a_{n2} & \dots & a_{nn} \end{vmatrix}=\Delta_1$$

$\Longleftrightarrow \Delta x_1=\Delta_1$

$$(1)\cdot A_{12} + (2)\cdot A_{22} + \ldots + (n)\cdot A_{n2} = \sum_{k=1}^{n} b_k A_{k2}$$

$$\Leftrightarrow \quad \Delta x_2 = \Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} & \ldots & a_{1n} \\ a_{21} & b_2 & a_{23} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & b_n & a_{n3} & & a_{nn} \end{vmatrix}$$

--- Аналогично $\Rightarrow \Delta x_i = \Delta_i, \ i = \overline{3, n}$

$$(*) \Leftrightarrow \begin{cases} \Delta x_1 = \Delta_1 \\ \Delta x_2 = \Delta_2 \\ \vdots \\ \Delta x_n = \Delta_n \end{cases}$$

Ако $\Delta \neq 0$, то

$$\begin{cases} x_1 = \Delta_1 / \Delta \\ x_2 = \Delta_2 / \Delta \\ \vdots \\ x_n = \Delta_n / \Delta \end{cases}$$

формули на Крамер.

<u>Проверка:</u> (в първото урав., аналогично в останалите $(n-1)$ бр. урав.)

$$a_{11}\frac{\Delta_1}{\Delta} + a_{12}\frac{\Delta_2}{\Delta} + \ldots + a_{1n}\frac{\Delta_n}{\Delta} = \frac{1}{\Delta}(\ldots) =$$

$$= \frac{1}{\Delta}\left(a_{11}\sum_{k=1}^{n} b_k A_{k1} + a_{12}\sum_{k=1}^{n} b_k A_{k2} + \ldots + a_{1n}\sum_{k=1}^{n} b_k A_{kn}\right) =$$

$$= \frac{1}{\Delta}\Big[\overbrace{(a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n})}^{\Delta \ (\text{по } L_1)}\, b_1 + \underbrace{(a_{11}A_{21} + a_{12}A_{22} + \ldots + a_{1n}A_{2n})}_{\text{ч. Р}}\, b_2 + \ldots +$$

$+(a_{11}A_{n1}+a_{12}A_{n2}+\dots+a_{1n}A_{nn})b_n] =$

$= \frac{1}{\Delta}[\Delta b_1 + 0b_2 + \dots + 0b_n] = \frac{\Delta b_1}{\Delta} = b_1$

С това доказахме следната Тh.

<u>Th</u> (формули на Крамер): Ако $\Delta \neq 0$ за системата (*), то (*) има единствено решение, получено по формулите на Крамер, т.е $x_i = \frac{\Delta_i}{\Delta}, i=\overline{1,n}$.

<u>Заб. по мат от преди</u>: E_{ij} м. единици

$A = \sum a_{ik}E_{ik}$, $B = \sum b_{kj}E_{kj}$, $AB \neq BA$

$C = AB = (\sum a_{ik}E_{ik})(\sum b_{\ell j}E_{\ell j}) =$

$= \sum a_{ik}(E_{ik}E_{\ell j})b_{\ell j} = c_{ij}$

$E_{ik}E_{\ell j} = \delta_{k\ell}E_{ij}$; $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$

Умножение на детерминанти

Нека $A, B \in M_n(F)$ и $C = AB \in M_n(F)$.

$$\det C = \det A \cdot \det B$$

Първо ще докажем следното помощно твърдение:

<u>Лема:</u> Нека A, B са две квадратни матрици от n-ти и k-ти ред съответно и да образуваме следната м. $D \in M_{n+k}(F)$

$$D = \begin{pmatrix} A & 0 \\ * & B \end{pmatrix}_{(n+k)}$$

Тогава $\det D = \det A . \det B$

<u>Док:</u> Индукция по реда на A, т.е. по n

$n=1$ $A = (a_{11}) \rightarrow \det A = a_{11}$

$$D = \left(\begin{array}{c|ccc} a_{11} & 0 & 0 \dots & 0 \\ \hline & b_{11} & b_{12} & b_{1k} \\ * & - & & - \\ & b_{k1} & b_{k2} - & b_{kk} \end{array}\right)$$

$\det D = |\sim| \underset{\text{по } L1}{\overset{P.}{=}} a_{11} (-1)^{1+1} . \det B$

$= \det A . \det B$

ИП: $\det D = \det A . \det B$ за ред на м. D до $(n+k-1)$

$$D=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & & O & & \\ \dots & & & & & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} & & & & \\ & & & & b_{11} & b_{12} & \dots & b_{1k} \\ & * & & & & & & \\ & & & & \dots & & & \\ & & & & b_{k1} & b_{k2} & \dots & b_{kk} \end{pmatrix}$$

. Тогава $\Delta_{ij} = \det D_{ij}$

$$\det D \underset{\text{по } L_1}{=} (-1)^{1+1} a_{11} \begin{vmatrix} D_{11} & O \\ * & B \end{vmatrix}_{(n+k-1)} + (-1)^{1+2} a_{12} \begin{vmatrix} D_{12} & O \\ * & B \end{vmatrix}_{(n+k-1)} +$$

$$+ \dots + (-1)^{1+n} a_{1n} \begin{vmatrix} D_{1n} & O \\ * & B \end{vmatrix}_{\substack{(n+k-1)\\ \text{ред}}} \overset{\text{ИП}}{=} (-1)^{1+1} a_{11} \Delta_{11} \det B +$$

$$D_{11} = \begin{pmatrix} a_{22} & \dots & a_{2n} \\ \dots & & \dots \\ a_{n2} & \dots & a_{nn} \end{pmatrix}_{(n-1)} \qquad D_{12} = \begin{pmatrix} a_{21} & a_{23} & \dots & a_{2n} \\ \dots & & \dots & \\ a_{n1} & a_{n3} & \dots & a_{nn} \end{pmatrix}_{(n-1)}$$

$$\Delta_{11} = \det D_{11} \qquad \Delta_{12} = \det D_{12}$$

$$+ (-1)^{1+2} a_{12} \Delta_{12} \det B + \dots + (-1)^{1+n} a_{1n} \Delta_{1n} \det B \underset{\text{Т}}{=}$$

$$= \underbrace{(a_{11} A_{11} + a_{12} A_{12} + \dots + a_{1n} A_{1n})}_{\text{р. по } L_1} \det B = \det A \cdot \det B$$

$\Rightarrow$ инд $\Rightarrow$ лем. е док.

Th (за умножение на det) Нека $A, B \in M_n(F)$. Тогава $\det(AB) = \det A \cdot \det B$.

Дво: Образуваме матр D от Лемата, че

$$D = \begin{pmatrix} A & 0 \\ * & B \end{pmatrix}_{2n} \quad \text{като } * = -E_n, \text{ т.е } D = \begin{pmatrix} A & 0 \\ \begin{matrix} -1 & & \\ & -1 & \\ & & \ddots -1 \end{matrix} & B \end{pmatrix}_{2n}$$

$$\det D \overset{\text{Лемата}}{=} \det A \cdot \det B$$

и ще покажем че $\det D = \det C = \det(AB)$

$\det D =$

$$\begin{vmatrix} a_{11} & a_{12} & \sim & a_{1n} & 0 & 0 & \sim & 0 \\ a_{21} & a_{22} & \sim & a_{2n} & 0 & 0 & \cdots & 0 \\ - & - & - & - & - & - & & \\ a_{n1} & a_{n2} & \sim & a_{nn} & 0 & 0 & - & 0 \\ -1 & 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & -1 & \sim & 0 & b_{21} & b_{22} & \sim & b_{2n} \\ - & - & - & & - & & & \\ 0 & 0 & \sim & -1 & b_{n1} & b_{n2} & \sim & b_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} \cdots a_{1n} & & \\ \vdots & & c_{ij} \\ -1 \cdots 0 & & 0 - 0 \\ \vdots & & O \\ 0 \cdots -1 & & 0 - 0 \end{vmatrix} =$$

$$S'_{n+1} = +b_{11}S_1 + b_{21}S_2 + \dots + b_{n1}S_n + S_{n+1}$$

$$k = \overline{1,n}, \quad S'_{n+k} = S_{n+k} + \sum_{j=1}^{n} S_j b_{jk}$$

$$(\ )_{1,n+1} = b_{11}a_{11} + b_{21}a_{12} + \dots + b_{n1}a_{1n} + 0 =$$
$$= \sum_{k=1}^{n} a_{1k} b_{k1} = c_{11} \qquad \boxed{C = AB}$$

$$(\ )_{1,n+2} = b_{12}a_{11} + b_{22}a_{12} + \dots + b_{n2}a_{1n} + 0 =$$
$$= \sum_{k=1}^{n} a_{1k} b_{k2} = c_{12}$$

$\Rightarrow$

$$\det \mathcal{D} = \begin{vmatrix} a_{ij} & c_{ij} \\ -E_n & 0 \end{vmatrix}_{2n} = (-1)^n \begin{vmatrix} -E_n & 0 \\ a_{ij} & c_{ij} \end{vmatrix} =$$

$$L'_k = L_{n+k},\ k=\overline{1,n}$$
$$L'_{n+k} = L_k,\ k=\overline{1,n}$$

Лапласа

$$= (-1)^n \det(-E_n) \cdot \det C = (-1)^n (-1)^n \det C$$
$$\Rightarrow \det \mathcal{D} = \det C = \det(AB) \Rightarrow$$
$$\det \mathcal{D} = \det A \cdot \det B$$

$\det(AB) = \det A \cdot \det B \Rightarrow$ Т-е док.

$$\underset{\text{ред} \times \text{стлб}}{\det(AB)} = \det A \cdot \det B \quad , \quad \det A^t = \det A$$

$$\underset{\text{ред} \times \text{ред}}{\det(A \cdot B^t)} = \det A \cdot \det B^t = \det A \cdot \det B = \det AB$$

$$\underset{\text{стлб} \times \text{стлб}}{\det(A^t \cdot B)} = \det A^t \cdot \det B = \det A \cdot \det B$$

$$\underset{\text{стлб} \times \text{ред}}{\det(A^t \cdot B^t)} = \det A^t \cdot \det B^t = \det A \cdot \det B$$

<u>Пример:</u>

$$\begin{vmatrix} \cos(\alpha_1-\beta_1) & \cos(\alpha_1-\beta_2) & \cos(\alpha_1-\beta_3) & \cos(\alpha_1-\beta_4) \\ \cos(\alpha_2-\beta_1) & \cos(\alpha_2-\beta_2) & \cos(\alpha_2-\beta_3) & \cos(\alpha_2-\beta_4) \\ \cos(\alpha_3-\beta_1) & \cos(\alpha_3-\beta_2) & \cos(\alpha_3-\beta_3) & \cos(\alpha_3-\beta_4) \\ \cos(\alpha_4-\beta_1) & \cos(\alpha_4-\beta_2) & \cos(\alpha_4-\beta_3) & \cos(\alpha_4-\beta_4) \end{vmatrix} =$$

$=$

(виж на следващата страница)

$$\begin{vmatrix} \cos\alpha_1 & \sin\alpha_1 & 0 & 0 \\ \cos\alpha_2 & \sin\alpha_2 & 0 & 0 \\ \cos\alpha_3 & \sin\alpha_3 & 0 & 0 \\ \cos\alpha_4 & \sin\alpha_4 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} \cos\beta_1 & \cos\beta_2 & \cos\beta_3 & \cos\beta_4 \\ \sin\beta_1 & \sin\beta_2 & \sin\beta_3 & \sin\beta_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

$$\underset{0 \quad n \geq 3}{\parallel} \qquad \qquad \underset{0 \quad n \geq 3}{\parallel}$$

$$\cos(\alpha_1-\beta_1) \quad \cos(\alpha_1-\beta_2)$$

$$\begin{vmatrix} \cos(\alpha_1-\beta_1) & \cos(\alpha_1-\beta_2) \\ \cos(\alpha_2-\beta_1) & \cos(\alpha_2-\beta_2) \end{vmatrix} = \ldots \quad \alpha, \beta$$

$n=2$

$$|\cos(\alpha_1-\beta_1)| = \cos(\alpha_1-\beta_1)$$

Обратими матрици

$M_n(F)$, $A+B$, O, $(-A)$, AB, има асоц. з-н, но $AB \neq BA$

$E = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$, $\cdot$? $\exists A^{-1}$

Деф. Една $A \in M_n(F)$ се нарича обратима, ако $\exists \underset{\text{озн}}{A^{-1}} \in M_n(F)$: $AA^{-1} = A^{-1}A = E$.
Матр A^{-1} се нарича обратна м. на м. A.

Св-ва: 1) Ако A - обратима, то A^{-1} е еднозначно определена от A, т.е. A^{-1} е единствена за A обр. м.

Д-во: Да допуснем противното, т.е. че $\exists\, A'$ и A'' обратни на A, т.е.

$$AA' = A'A = E \quad \text{и} \quad AA'' = A''A = E, \text{ тогава}$$

$$\left(\begin{array}{l} A'AA'' = (A'A)A'' = E.A'' = A'' \\ A'AA'' = A'(AA'') = A'.E = A' \end{array}\right. \Rightarrow A' = A''.$$

2) $(AB)^{-1} = B^{-1}A^{-1}$, т.е. ако A и B са обратими матрици, то AB е обратима матрица и е в сила горното р-во.

Д-во: $A \to \exists A^{-1}$, $B \to \exists B^{-1}$. Образуваме $C = B^{-1}A^{-1}$

$$(AB).C = (AB)(B^{-1}A^{-1}) = A\overbrace{(BB^{-1})}^{E}A^{-1} = AA^{-1} = E$$

$$C.(AB) = (B^{-1}A^{-1})(AB) = B^{-1}\underbrace{(A^{-1}A)}_{E}B = B^{-1}B = E$$

$$\Rightarrow \text{def} \Rightarrow C = (AB)^{-1}$$

3) A – обрат., $\exists A^{-1}$: $AA^{-1} = A^{-1}A = E$

$\Rightarrow A^{-1}$ обрат. и $(A^{-1})^{-1} = A$

4) $AA^{-1} = E$, за обр. м. A

$$\det(AA^{-1}) \underset{\substack{\text{Тъ умн} \\ \det}}{=} \det A \cdot \det A^{-1} = \det E = 1$$

$\det A \neq 0$

$$\Rightarrow \det A^{-1} = \frac{1}{\det A}, \quad \det A^{-1} \neq 0$$

F – поле

<u>Def</u>. Една м. $A \in M_n(F)$ наричаме неособена м., ако $\det A \neq 0$. В противен случай, ако $\det A = 0$, то A – особена матр.

<u>Тъ</u> (критерий за обратимост на м.) Една кв. матр. $A \in M_n(F)$ е обратима м. $\Leftrightarrow$ A е неособена матр., т.е

A: $\exists A^{-1} \Leftrightarrow \det A \neq 0$

Док: $\Rightarrow$) A-обратима $\Rightarrow \exists A^{-1} : AA^{-1} = E$

$\Rightarrow$ $\underline{\det A \neq 0}$, $\det A^{-1} \neq 0$, $\Rightarrow A$-неособ.

$\Leftarrow$) Нека $A \in M_n(F)$, неособена м. $\Rightarrow \det A \neq 0$

Образуваме следната матрица

$$X = \begin{pmatrix} A_{11} & A_{21} & & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix}, \quad Y = \frac{1}{\underset{\overset{\neq}{0}}{\det A}} X \quad (A \text{ неособ})$$

$$AX = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & \dots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix} = \det A \cdot E$$

$$= \begin{pmatrix} \det A & & & 0 \\ & \det A & & \\ 0 & & \ddots & \\ & & & \det A \end{pmatrix}$$

$AX = \det A E$

$AY = E$

$\Rightarrow$

Директно се проверява че $YA = E$ $\underline{\quad}$ $Y = A^{-1}$, A обр.

Пр:

$$A=\begin{pmatrix}1 & 2 & 3 & -1\\ 2 & 1 & -3 & 2\\ 3 & 4 & 7 & 2\\ 5 & 1 & 3 & 6\end{pmatrix},\ \underset{\text{проверка (сиркл)}}{\overset{\Delta}{\parallel}}\!\!\det A \neq 0$$

$$A^{-1}=\frac{1}{\Delta}\begin{pmatrix}A_{11} & \\ A_{12} & \sim \\ A_{13} & \\ A_{14} & \end{pmatrix}$$

$$A_{11}=\cdot\begin{vmatrix}1 & -3 & 2\\ 4 & 7 & 2\\ 1 & 3 & 6\end{vmatrix},\quad A_{12}=(-1)^{1+2}\begin{vmatrix}2 & -3 & 2\\ 3 & 7 & 2\\ 5 & 3 & 6\end{vmatrix},\ \sim$$

Сл: $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $\det A = ad-bc\neq 0$

$$A^{-1}=\frac{1}{ad-bc}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \dots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases}$$

$A = (a_{ij})_{m\times n}$

$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}_{m\times 1}$

$AX = b$, $A = (a_{ij})_{n\times n} \in M_n(F)$

$\underline{m = n}$

$A_{n\times n} X_{n\times p} = b = \begin{pmatrix} b_1 \\ \vdots \\ b_p \end{pmatrix}$

Ако A-обратима, т.е. A^{-1}

$A^{-1} / \ AX = b \Rightarrow A^{-1}(AX) = A^{-1}(b)$

$\underbrace{(A^{-1}A)}_{E}X = A^{-1}b \Rightarrow X = A^{-1}b$

т.е решаване матр. урав.

Матрични уравнения

I) $AX = B$, $\underset{\det A \neq 0}{A \in M_n(F)}$, $\overset{?}{X}, B \in F_{n\times p}$

$X = A^{-1}B$

М. уравнение от II тип

II) $XA = B$, $A \in M_n(F)$, $\det A \neq 0 \Rightarrow \exists A^{-1}$

$X = BA^{-1}$

III) $A, B \in M_n(F)$, $\det A \neq 0$, $\det B \neq 0$, те $\exists A^{-1}, B^{-1}$

$AXB = C$ $\Rightarrow$ $X = A^{-1}CB^{-1}$

$A^{-1}/$ $\quad /B^{-1}$ $\quad$ решение

IV) $A \in M_n(F)$, $\det A \neq 0$

$AX = E$ или $XA = E$

$\Rightarrow X = A^{-1}$