Лекция №4 20.10.24г

# Детерминанти

$$A \in M_n(F) \longrightarrow \underset{\Delta}{\underset{\parallel}{\det A}} \in F$$

$$\left|\begin{array}{l} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \dots\dots\dots \\ a_{n1}x_1 + \dots + a_{nn}x_n = b_n \end{array}\right. \Leftrightarrow A = (a_{ij}) \in M_n(F)$$

$$\Updownarrow$$

$$\left|\begin{array}{l} (\Delta)\, x_1 = \Delta_1 \\ \dots\dots \\ (\Delta)\, x_n = \Delta_n \end{array}\right. \qquad \Delta = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

$$\begin{array}{cc} & \text{i-ти стълб} \\ \Delta_i = & \begin{vmatrix} a_{11} & \dots & b_1 & \dots & a_{1n} \\ & & b_2 & & \\ & & \vdots & & \\ a_{n1} & \dots & b_n & \dots & a_{nn} \end{vmatrix} \end{array}, \quad i = \overline{1,n}$$

$$\Delta = \det A \overset{def}{=} \sum (-1)^{[i_1, i_2, \dots, i_n]} a_{1i_1} a_{2i_2} \dots a_{ni_n} \in F,$$

където сумирането е по всички $n!$ на брой пермутации $\sigma \in S_n$ и $[i_1, \dots, i_n]$ – брой на инверсиите в перм.

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}.$$

$$\begin{vmatrix} a_{11} & & * \\ & a_{22} & \\ 0 & & \ddots \\ & & & a_{nn} \end{vmatrix} = a_{11}a_{22}\dots a_{nn}$$

**Тв.** $a_{k_1 j_1} a_{k_2 j_2} \dots a_{k_n j_n}$, $(-1)$

$$[k_1, k_2, \dots, k_n] + [j_1 \dots j_n] = [1, \dots, n] + [\sigma_1 \dots \sigma_n]$$

**Тв:** $\det A^t = \det A$

**Сл** Всички св-ва на $\det A$ по редове са в сила и по стълбове и обратно.

$$\begin{vmatrix} \diagdown & 0 \\ * & \end{vmatrix} = \begin{vmatrix} a_{11} & * \\ 0 & a_{nn} \end{vmatrix} = a_{11} \dots a_{nn}$$

свва на det-ите

1) $\begin{vmatrix} & \\ 0 \cdots 0 \\ & \end{vmatrix} = 0$ $\quad A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

$i \begin{vmatrix} 0 \end{vmatrix} = 0$

2) $\begin{vmatrix} a_i \\ \vdots \\ a_i \end{vmatrix} = \begin{array}{cc} i & j \\ a_i & a_j \end{array} = 0;$

3) $\begin{vmatrix} \lambda a_i \end{vmatrix} = \lambda \begin{vmatrix} a_i \end{vmatrix};$ $\quad \det \lambda A = \lambda^n \det A$

4) $L_j' = L_j + \lambda L_i$

$\begin{array}{c} i \\ j \end{array} \begin{vmatrix} a_i \\ a_j \end{vmatrix} \underset{\uparrow}{=} \begin{vmatrix} a_i \\ a_j + \lambda a_i \end{vmatrix}$

5) $\begin{vmatrix} a_i \\ \lambda a_i \end{vmatrix} = 0$

6) $\begin{vmatrix} a_i' + a_i'' \end{vmatrix} = \begin{vmatrix} a_i' \end{vmatrix} + \begin{vmatrix} a_i'' \end{vmatrix}$

7) $\begin{vmatrix} a_i \\ a_j \end{vmatrix} = - \begin{vmatrix} a_j \\ a_i \end{vmatrix}$

8) $L_k' = L_k + \lambda_1 L_1 + \lambda_2 L_2 + \ldots + \lambda_{k-1} L_{k-1} +$
$+ \lambda_{k+1} L_{k+1} + \ldots + \lambda_n L_n$

Ако $L_s = \lambda L_1 + \mu L_2 + \ldots + \gamma L_k$

$\Rightarrow \det A = 0$

Сл.

$$\begin{vmatrix} 0 & & a_{1n} \\ & \cdots & * \\ a_{n1} & & \end{vmatrix} = \begin{vmatrix} * & & a_{1n} \\ & \cdots & 0 \\ a_{n1} & & \end{vmatrix} =$$

$$= (-1)^{\frac{n(n-1)}{2}} \begin{vmatrix} a_{n1} & \cdots & \\ 0 & a_{n-1\,2} & \cdots \\ 0 & \ddots & \\ & & a_{1n} \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} a_{n1} a_{n-1\,2} \cdots a_{1n}$$

Зад

Інакше

$$\det A = \begin{vmatrix} a & b & \cdots & b \\ b & a & & b \\ \vdots & b & \ddots & \\ & & a & b \\ b & \cdots & b & a \end{vmatrix}_{n\times n} = \begin{vmatrix} (a-b)+b & \\ 0+b & \\ 0+b & \cdots \\ \vdots & \\ 0+b & \end{vmatrix} = \begin{vmatrix} a+(n-1)b & a+(n-1)b & \cdots & a+(n-1)b \\ b & a & \cdots & b \\ & & \ddots & b \\ b & & & a \end{vmatrix}$$

$a \neq b$, $b = 0 \Rightarrow \det A = a^n$, $b \neq 0$

$$L_1' = \sum_{i=1}^{n} L_i$$

$$= (a+(n-1)b) \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ b & a & b & \cdots & b \\ & & \ddots & & b \\ & b & & & a \end{vmatrix} \underset{\substack{L_k' = L_k - bL_1 \\ k=\overline{2,n}}}{=} (a+(n-1)b) \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ & a-b & & & 0 \\ & & a-b & & \\ & 0 & & \ddots & \\ & & & & a-b \end{vmatrix}$$

$$\underset{\text{Дна}}{=} (a+(n-1)b)(a-b)^{n-1}$$

**Пр:**

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & 2 & 5 & 7 & 11 \\ 0 & 3 & 1 & 2 & 5 \\ 0 & -3 & 2 & 1 & -5 \\ 0 & 1 & 2 & 3 & 4 \end{vmatrix} \underset{L_2' = L_1 + L_2}{=} \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 4 & 8 & 11 & 16 \\ 0 & 3 & 1 & 2 & 5 \\ 0 & -3 & 2 & 1 & -5 \\ 0 & 1 & 2 & 3 & 4 \end{vmatrix} =$$

$$= - \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 4 & 8 & 11 & 16 \\ 0 & 3 & 1 & 2 & 5 \\ 0 & -3 & 2 & 1 & -5 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & & & \\ 0 & 0 & & & \\ 0 & 0 & & & \\ 0 & 0 & & & \end{vmatrix} = \sim$$

**Извод:**

ф-ла на Крамер

$$\Delta x_1 = \Delta_1$$
$$\Delta x_2 = \Delta_2$$
$$\Delta x_3 = \Delta_3$$
$$\vdots$$
$$\Delta x_n = \Delta_n$$

$$\Delta = \det A = \sum^{\in F} \sim$$

$$A = (a_{ij})$$

$$\Delta_i = \begin{vmatrix} & b_1 & \\ & b_2 & \\ a_{ij} & \vdots & a_{ij} \\ & b_n & \end{vmatrix}$$

$i$-ти стълб

# Адюнгирани количества. Поддетерминанти. Ф-ли на Крамер

Нека $A \in M_n(F)$, $\det A = \Delta \in F$

p-ти ред и q-ти стълб $\to a_{pq}$

$$\det A = \sum_{n!\ \text{съб}} \cdots = \underbrace{\begin{pmatrix} \text{адюнг} \\ \text{к-во на} \\ \text{ел-та } a_{pq} \end{pmatrix}}_{A_{pq}} a_{pq} + \cdots$$

В сила следните формули:

развиване на $\det A$ по p-ти ред:

$$\det A = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} \quad \text{по } p=1$$

$$\det A = a_{p1}A_{p1} + a_{p2}A_{p2} + \cdots + a_{pn}A_{pn} = \sum_{i=1}^{n} a_{pi}A_{pi}$$

развиване на $\det A$ по q-ти стълб:

$$\det A = a_{1q}A_{1q} + a_{2q}A_{2q} + \cdots + a_{nq}A_{nq} = \sum_{i=1}^{n} a_{iq}A_{iq}$$

Нека $A \in M_n(F)$, $\det A \in F$ и p-ти ред и q-ти стълб, т.е. $a_{pq}$

$$\begin{array}{cc} & q \\ p & \begin{vmatrix} & \vdots & \\ \cdots & a_{pq} & \cdots \\ & \vdots & \end{vmatrix} \end{array}$$

$$\Delta_{pq} = \begin{vmatrix} a_{11} & \cdots & a_{1\,q-1} & a_{1\,q+1} & \cdots & a_{1n} \\ a_{p-1\,1} & \cdots & a_{p-1\,q-1} & a_{p-1\,q+1} & \cdots & a_{p-1\,n} \\ a_{p+1\,1} & \cdots & a_{p+1\,q-1} & a_{p+1\,q+1} & \cdots & a_{p+1\,n} \\ \cdots & & \cdots & \cdots & & \\ a_{n1} & \cdots & a_{n\,q-1} & a_{n\,q+1} & \cdots & a_{nn} \end{vmatrix}_{(n-1)\times(n-1)}$$

поддетерминанта

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 1 & 2 & 3 \\ 1 & 2 & 3 & 7 \\ 0 & 1 & 2 & 3 \end{vmatrix} \qquad \begin{vmatrix} 1 & 3 & 4 \\ 5 & 2 & 3 \\ 0 & 2 & 3 \end{vmatrix} = \Delta_{32}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{21}a_{31} + a_{21}a_{32}a_{13} - a_{13}a_{22}a_{31} - a_{32}a_{23}a_{11} - a_{33}a_{12}a_{21}$$

$$= (\overset{A_{32}}{a_{21}a_{13} - a_{23}a_{11}})a_{32} + \cdots$$

**Th** $A_{pq} = (-1)^{p+q} \Delta_{pq}$

**Сл** В сила следните ф-ли за развитие на $\det A$ по:

1) $p$-ти ред:

$$\det A = \sum_{j=1}^{n} a_{pj} A_{pj} = \sum_{j=1}^{n} (-1)^{p+j} a_{pj} \Delta_{pj}$$

2) $q$-ти стълб:

$$\det A = \sum_{i=1}^{n} a_{iq} A_{iq} = \sum_{i=1}^{n} (-1)^{i+q} a_{iq} \Delta_{iq}$$

**Пример:**

$$\begin{vmatrix} 1 & 2 & 3 & 2 & 1 \\ 5 & 1 & -3 & 1 & 4 \\ 7 & 3 & 2 & 3 & 7 \\ 1 & 0 & 2 & 4 & 5 \\ -2 & 0 & 3 & -3 & -4 \end{vmatrix} \underset{\text{по } L_2}{=} (-1)^{2+1} 5 \begin{vmatrix} 2 & 3 & 2 & 1 \\ 3 & 2 & 3 & 7 \\ 0 & 2 & 4 & 5 \\ 0 & 3 & -3 & -4 \end{vmatrix} + (-1).1. \begin{vmatrix} 1 & 3 & 2 & 1 \\ 7 & 2 & 3 & 7 \\ 1 & 2 & 4 & 5 \\ -2 & 3 & -3 & -4 \end{vmatrix} +$$

$$+(-1)^{2+3}(-3)\begin{vmatrix}1&2&2&1\\7&3&3&7\\1&0&4&5\\-2&0&-3&-4\end{vmatrix}+(-1)^{2+4}\cdot 1\begin{vmatrix}1&2&3&1\\7&3&2&7\\1&0&2&5\\-2&0&3&-4\end{vmatrix}+$$

$$+(-1)^{2+5}\cdot 4\begin{vmatrix}1&2&3&2\\7&3&2&3\\1&0&2&4\\-2&0&3&-3\end{vmatrix}.$$

Символ на Кронекер:

$$\delta_{ij} := \begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}$$

Тв: В сила са следните ф-ли:

$$\sum_{k=1}^{n} a_{pk}A_{rk} = \begin{cases}\det A, & p=r\\ 0, & p\neq r \quad \text{фалшиво развитие}\end{cases}$$

$$= \delta_{pr}\det A$$

$$\sum_{k=1}^{n} a_{kq}A_{kr} = \delta_{qr}\det A$$

# Детерминанта на Вандермонд

$$W(\alpha_1,\alpha_2,\dots,\alpha_n)=\begin{vmatrix}1&1&1&\dots&1\\ \alpha_1&\alpha_2&\alpha_3&\dots&\alpha_n\\ \alpha_1^2&\alpha_2^2&\alpha_3^2&\dots&\alpha_n^2\\ \dots&&&&\dots\\ \alpha_1^{n-1}&\alpha_2^{n-1}&\alpha_3^{n-1}&\dots&\alpha_n^{n-1}\end{vmatrix}_{n\times n}=$$

$$L_k'=L_k-\alpha_1L_{k-1},\quad k=\overline{2,n}$$

$$=\begin{vmatrix}1&1&1&\dots&1\\ 0&\alpha_2-\alpha_1&\alpha_3-\alpha_1&\dots&\alpha_n-\alpha_1\\ 0&\alpha_2(\alpha_2-\alpha_1)&\alpha_3(\alpha_3-\alpha_1)&\dots&\alpha_n(\alpha_n-\alpha_1)\\ \vdots&&&&\\ 0&\alpha_2^{n-2}(\alpha_2-\alpha_1)&\alpha_3^{n-2}(\alpha_3-\alpha_1)&\dots&\alpha_n^{n-2}(\alpha_n-\alpha_1)\end{vmatrix}\underset{S_1}{\overset{\text{по}}{=}}(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)\dots(\alpha_n-\alpha_1)\begin{vmatrix}1&1&\dots&1\\ \alpha_2&\alpha_3&\dots&\alpha_n\\ \dots&&&\\ \alpha_2^{n-2}&\alpha_3^{n-2}&\dots&\alpha_n^{n-2}\end{vmatrix}_{(n-1)\text{ ред}}=$$

$$W(\alpha_1,\alpha_2,\dots,\alpha_n)=(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)\dots(\alpha_n-\alpha_1)W(\alpha_2,\alpha_3,\dots,\alpha_n)=$$

$$=(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)\dots(\alpha_n-\alpha_1)(\alpha_3-\alpha_2)(\alpha_n-\alpha_2)\dots(\alpha_n-\alpha_2).$$

$$W(\alpha_1,\alpha_2,\dots,\alpha_n)=\dots=(\alpha_2-\alpha_1)\dots(\alpha_n-\alpha_2)\dots(\alpha_n-\alpha_{n-1}).$$

$$\underbrace{\begin{vmatrix}1&1&1\\ \alpha_{n-2}&\alpha_{n-1}&\alpha_n\\ \alpha_{n-2}^2&\alpha_{n-1}^2&\alpha_n^2\end{vmatrix}}_{W(\alpha_{n-2},\alpha_{n-1},\alpha_n)}=\dots(\alpha_{n-1}-\alpha_{n-2})(\alpha_n-\alpha_{n-2})\begin{vmatrix}1&1\\ \alpha_{n-1}&\alpha_n\end{vmatrix}=$$

$$=(\alpha_2-\alpha_1)\dots(\alpha_n-\alpha_{n-1})\quad\Rightarrow$$

$$W(\alpha_1, \alpha_2, \ldots, \alpha_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_n^2 \\ \cdots & & \cdots & \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{vmatrix} =$$

$$= \prod_{1 \le j < i \le n} (\alpha_i - \alpha_j)$$