1 Първа и втора основна граница

1.1 Добавки

1. $\lim\limits_{x\to 0} \dfrac{\arcsin x}{x} = 1$

2. $\lim\limits_{x\to 0} \dfrac{\operatorname{arctg} x}{x} = 1$

3. $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{x} = 2$

   Решение: $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{x} = \lim\limits_{x\to 0} \dfrac{e^x - 1}{x} + \lim\limits_{x\to 0} \dfrac{e^{-x} - 1}{-x} = 2$

4. $\lim\limits_{x\to 0} \dfrac{\ln\left(x + \sqrt{1+x^2}\right)}{x} = 1$

   Решение: $\lim\limits_{x\to 0} \dfrac{\ln\left(x + \sqrt{1+x^2}\right)}{x} = \lim\limits_{x\to 0} \dfrac{x + \sqrt{1+x^2} - 1}{x} = 1 + \lim\limits_{x\to 0} \dfrac{x}{1 + \sqrt{1+x^2}} = 1$

1.2 Задачи от контролни

1.2.1 Пресметнете границите:

1. $\displaystyle\lim_{x\to 0}\frac{\ln(1+\sin 6x)}{x}$

2. $\displaystyle\lim_{x\to 0}\frac{\ln^2(1+\arcsin 6\sqrt{x})}{x}$

3. $\displaystyle\lim_{x\to 0, x<0}\frac{\sqrt{\ln(1+\operatorname{arctg} 3x^2)}}{x}$

4. $\displaystyle\lim_{x\to 0}\frac{\ln\left(\cos x+\operatorname{tg} 6x^2\right)}{x^2}$

5. $\displaystyle\lim_{x\to 0}\frac{\ln\left(\sqrt{1-x^2}+\sin 6x^2\right)}{x^2}$

Решение: Използваме втора основна граница $\displaystyle\lim_{u\to 0}\frac{\ln(1+u)}{u}=1$, граница на съставна функция и съответната вариация на първа основна граница:

1. $\displaystyle\lim_{x\to 0}\frac{\ln(1+\sin 6x)}{x}=\lim_{x\to 0}\frac{\ln(1+\sin 6x)}{\sin 6x}.\lim_{x\to 0}\frac{\sin 6x}{6x}.\lim_{x\to 0}\frac{6x}{x}=6$.

2. $$\lim_{x\to 0} \frac{\ln^2(1+\arcsin 6\sqrt{x})}{x} = \lim_{x\to 0}\left(\frac{\ln(1+\arcsin 6\sqrt{x})}{\arcsin 6\sqrt{x}}\right)^2 . \lim_{x\to 0}\left(\frac{\arcsin 6\sqrt{x}}{6\sqrt{x}}\right)^2 . \lim_{x\to 0}\frac{36x}{x} = 36\ .$$

3. $$\lim_{x\to 0, x<0} \frac{\sqrt{\ln(1+\operatorname{arctg} 3x^2)}}{x} = \lim_{x\to 0, x<0} -\sqrt{\frac{\ln(1+\operatorname{arctg} 3x^2)}{x^2}} = -\lim_{x\to 0, x<0}\sqrt{\frac{\operatorname{arctg} 3x^2}{x^2}} = -\sqrt{3}\ .$$

4. $$\lim_{x\to 0} \frac{\ln\left(\cos x + \operatorname{tg} 6x^2\right)}{x^2} = \lim_{x\to 0}\frac{\ln\left(\cos x + \operatorname{tg} 6x^2\right)}{\cos x - 1 + \operatorname{tg} 6x^2} . \lim_{x\to 0}\frac{\cos x - 1 + \operatorname{tg} 6x^2}{x^2} = -\frac{1}{2} + 6 = \frac{11}{2}\ .$$

5. $$\lim_{x\to 0} \frac{\ln\left(\sqrt{1-x^2} + \sin 6x^2\right)}{x^2} = \lim_{x\to 0}\frac{\ln\left(\sqrt{1-x^2} + \sin 6x^2\right)}{\sqrt{1-x^2} - 1 + \sin 6x^2} . \lim_{x\to 0}\frac{\sqrt{1-x^2} - 1 + \sin 6x^2}{x^2} =$$
$$= -\frac{1}{2} + 6 = \frac{11}{2}\ .$$

1.2.2 Пресметнете границите:

1. $$\lim_{x\to 0}\frac{e^{\operatorname{tg} 6x} - 1}{x}$$

2. $$\lim_{x\to 0}\frac{\left(e^{\arcsin 6\sqrt{x}} - 1\right)^2}{x}$$

3. $\displaystyle \lim_{x\to 0, x<0} \frac{\sqrt{e^{\operatorname{arctg} 3x^2}-1}}{x}$

4. $\displaystyle \lim_{x\to 0} \frac{e^{\cos x+\sin 6x^2}-e}{x^2}$

5. $\displaystyle \lim_{x\to 0} \frac{e^{\sqrt{1-x^2}+\operatorname{tg} 6x^2}-e}{x^2}$

Решение: Използваме втора основна граница $\displaystyle \lim_{u\to 0} \frac{e^u-1}{u} = 1$, граница на съставна функция и съответната вариация на първа основна граница:

1. $\displaystyle \lim_{x\to 0} \frac{e^{\operatorname{tg} 6x}-1}{x} = \lim_{x\to 0} \frac{e^{\operatorname{tg} 6x}-1}{x} . \lim_{x\to 0} \frac{\operatorname{tg} 6x}{6x} . \lim_{x\to 0} \frac{6x}{x} = 6$.

2. $\displaystyle \lim_{x\to 0} \frac{\left(e^{\arcsin 6\sqrt{x}}-1\right)^2}{x} = \lim_{x\to 0} \left(\frac{e^{\arcsin 6\sqrt{x}}-1}{\arcsin 6\sqrt{x}}\right)^2 . \lim_{x\to 0} \left(\frac{\arcsin 6\sqrt{x}}{6\sqrt{x}}\right)^2 . \lim_{x\to 0} \frac{36x}{x} = 36$.

3. $\displaystyle \lim_{x\to 0, x<0} \frac{\sqrt{e^{\operatorname{arctg} 3x^2}-1}}{x} \lim_{x\to 0, x<0} -\sqrt{\frac{e^{\operatorname{arctg} 3x^2}-1}{x^2}} = -\lim_{x\to 0, x<0} \sqrt{\frac{\operatorname{arctg} 3x^2}{x^2}} = -\sqrt{3}$.

4. $\lim\limits_{x\to 0} \dfrac{e^{\cos x+\sin 6x^2}-e}{x^2} = e\lim\limits_{x\to 0} \dfrac{e^{\cos x+\sin 6x^2-1}-1}{\cos x-1+\sin 6x^2}.\lim\limits_{x\to 0} \dfrac{\cos x-1+\sin 6x^2}{x^2} = e\left(-\dfrac{1}{2}+6\right) = \dfrac{11e}{2}$.

5. $\lim\limits_{x\to 0} \dfrac{e^{\sqrt{1-x^2}+\operatorname{tg} 6x^2}-e}{x^2} = e\lim\limits_{x\to 0} \dfrac{e^{\sqrt{1-x^2}+\operatorname{tg} 6x^2-1}-1}{\sqrt{1-x^2}-1+\operatorname{tg} 6x^2}.\lim\limits_{x\to 0} \dfrac{\sqrt{1-x^2}-1+\operatorname{tg} 6x^2}{x^2} =$
$= e\left(-\dfrac{1}{2}+6\right) = \dfrac{11e}{2}$.

2 Граници с $(f(x))^{g(x)}$

2.1 Пресметнете границите:

1. $\lim\limits_{x\to 0} (1+x)^{\operatorname{cotg} x}$

2. $\lim\limits_{x\to \frac{\pi}{2}} \left(\dfrac{2x}{\pi}\right)^{\operatorname{tg} x}$

Решение: Имаме неопределеност 1^{∞}. В този случай, $\lim\limits_{x\to b} (f(x))^{g(x)} = \lim\limits_{x\to b} e^{g(x)\ln f(x)} = e^{A}$, където $A = \lim\limits_{x\to b} g(x)\ln f(x) = \lim\limits_{x\to b} g(x)\left(f(x)-1\right)$:

1. $\lim\limits_{x\to 0} (1+x)^{\operatorname{cotg} x} = e$, защото $\lim\limits_{x\to 0} x \operatorname{cotg} x = \lim\limits_{x\to 0} \dfrac{x}{\sin x} . \lim\limits_{x\to 0} \cos x = 1$.

2. $\lim\limits_{x\to \frac{\pi}{2}} \left(\dfrac{2x}{\pi}\right)^{\operatorname{tg} x} = e^{-\frac{2}{\pi}}$, защото $\lim\limits_{x\to \frac{\pi}{2}} \left(\dfrac{2x}{\pi} - 1\right) \operatorname{tg} x = \lim\limits_{x\to \frac{\pi}{2}} \dfrac{2}{\pi} \cdot \dfrac{\left(x-\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}-x\right)} . \lim\limits_{x\to \frac{\pi}{2}} \sin x = -\dfrac{2}{\pi}$.

2.2 Пресметнете границите:

1. $\lim\limits_{x\to 0} \dfrac{(1+x)^{x} - 1}{x^2}$

2. $\lim\limits_{x\to 0} \dfrac{1-(\cos x)^{\sin x}}{x^3}$

Решение: Използваме дефиницията $(f(x))^{g(x)} = e^{g(x)\ln f(x)}$, (вероятно) втора основна граница $\lim\limits_{u\to 0} \dfrac{e^{u}-1}{u} = 1$, граница на съставна функция:

1. $\lim\limits_{x\to 0}\dfrac{(1+x)^{\,x}-1}{x^2}=\lim\limits_{x\to 0}\dfrac{e^{\,x\ln(1+x)}-1}{x^2}=\lim\limits_{x\to 0}\dfrac{e^{\,x\ln(1+x)}-1}{x\ln(1+x)}.\lim\limits_{x\to 0}\dfrac{x\ln(1+x)}{x^2}=1\ .$

2. $\lim\limits_{x\to 0}\dfrac{1-(\cos x)^{\sin x}}{x^3}=\lim\limits_{x\to 0}\dfrac{1-e^{\sin x\ln(\cos x)}}{x^3}=-\lim\limits_{x\to 0}\dfrac{e^{\sin x\ln(\cos x)}-1}{\sin x\ln(\cos x)}.\lim\limits_{x\to 0}\dfrac{\sin x\ln(\cos x)}{x^3}=$

$=-\lim\limits_{x\to 0}\dfrac{\sin x}{x}.\lim\limits_{x\to 0}\dfrac{\ln(1+\cos x-1)}{\cos x-1}.\lim\limits_{x\to 0}\dfrac{\cos x-1}{x^2}=\dfrac{1}{2}\ .$

3 Символът *o* малко

3.1 Дефиниция

Нека $f(x)$ е дефинирана в околност $(a-\delta\,,\ a+\delta)$ (евентуално без a) на точката a и $f(x)$ е безкрайно малка в точката a, т.е. $\lim\limits_{x\to a} f(x)=0$.

Казваме, че $g(x)=o\,(f(x))$ (по-точно $g(x)\in o\,(f(x))$), ако $\lim\limits_{x\to a}\dfrac{g(x)}{f(x)}=0$.

3.2 Основно свойство

Ако $\lim\limits_{x\to a} \dfrac{g(x)}{f(x)} = L \neq 0$, то $o\,(g(x)) = o\,(f(x))$.

3.3 Скали за сравняване

- Основна: $|x-a|^p\ ,\ p>0$
- $p>q \ \Rightarrow\ |x-a|^p = o\,(|x-a|^q)$
- Допълнителна: $|x-a|^p\,|\ln|x-a||^q\ ,\ p>0$
- В $+\infty$: $x^p\ ,\ x^p\,(\ln x)^q\ ,\ p<0$
- $f(x) = o\,(1) \ \Leftrightarrow\ \lim\limits_{x\to a} f(x) = 0$

3.4 Аритметични действия

- Събиране: $o\,((x-a)^p) + o\,((x-a)^q) = o\left((x-a)^{\min(p,\,q)}\right)$
- Умножаване с константа: $b\neq 0 \ \Rightarrow\ o\,(b(x-a)^p) = o\,((x-a)^p)$

- Умножение: $o\left((x-a)^p\right).o\left((x-a)^q\right) = o\left((x-a)^{p+q}\right)$

  $(x-a)^p.o\left((x-a)^q\right) = o\left((x-a)^{p+q}\right)$

- Деление: $p \leq q \Rightarrow \dfrac{o\left((x-a)^q\right)}{(x-a)^p} = o\left((x-a)^{q-p}\right)$

3.5 Асимптотично представяне на основните „елементарни“ функции в $a=0$

3.5.1 степени

1. $\sqrt{1+x} = 1 + \dfrac{x}{2} + o(x)$, $\sqrt{1+x} = 1 + \dfrac{x}{2} - \dfrac{x^2}{8} + o\left(x^2\right)$

2. $\sqrt[k]{1+x} = 1 + \dfrac{x}{k} + o(x)$, $\sqrt[k]{1+x} = 1 + \dfrac{x}{2} - \dfrac{(k-1)\,x^2}{2k^2} + o\left(x^2\right)$

3. $(1+x)^\alpha = 1 + \dbinom{\alpha}{1}x + \dbinom{\alpha}{2}x^2 + \cdots + \dbinom{\alpha}{n}x^n + o\left(x^n\right)$

3.5.2 експонента и логаритъм

1. $$e^x = 1 + x + o\left(x\right), \quad e^x = 1 + x + \frac{x^2}{2} + o\left(x^2\right)$$

$$e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + o\left(x^n\right)$$

2. $$\ln\left(1 + x\right) = x + o\left(x\right), \quad \ln\left(1 + x\right) = x - \frac{x^2}{2} + o\left(x^2\right)$$

$$\ln\left(1 + x\right) = x - \frac{x^2}{2} + \cdots + \frac{\left(-1\right)^{n-1} x^n}{n} + o\left(x^n\right)$$

3. $$\ln\left(x + \sqrt{x^2 + 1}\right) = x + o\left(x\right), \quad \ln\left(x + \sqrt{x^2 + 1}\right) = x - \frac{x^3}{6} + o\left(x^4\right)$$

3.5.3 тригонометрични и обратни тригонометрични функции

1. $$\sin x = x + o\left(x\right), \quad \sin x = x - \frac{x^3}{6} + o\left(x^4\right)$$

$$\sin x = x - \frac{x^3}{6} + \cdots + \frac{\left(-1\right)^n x^{2n+1}}{\left(2n+1\right)!} + o\left(x^{2n+2}\right)$$

2. $\cos x = 1 - \dfrac{x^2}{2} + o\left(x^2\right)\,, \quad \cos x = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} + o\left(x^5\right)$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + o\left(x^{2n+1}\right)$$

3. $\operatorname{tg} x = x + o\,(x)\,, \quad \operatorname{tg} x = x + \dfrac{x^3}{3} + o\left(x^4\right)$

4. $\arcsin x = x + o\,(x)\,, \quad \arcsin x = x + \dfrac{x^3}{6} + o\left(x^4\right)$

5. $\operatorname{arctg} x = x + o\,(x)\,, \quad \operatorname{arctg} x = x - \dfrac{x^3}{3} + o\left(x^4\right)$

$$\operatorname{arctg} x = x - \frac{x^3}{3} + \cdots + \frac{(-1)^n\, x^{2n+1}}{2n+1} + o\left(x^{2n+2}\right)$$

3.6 Задачи от контролни

3.6.1 Пресметнете границите:

1. $$L_1 = \lim_{x\to 0} \frac{\sqrt[3]{3x+1} - \sqrt[4]{4x+1}}{\operatorname{arctg}\,(x\ln(x+1))}\,.$$

Решение: $$L_1 = \lim_{x\to 0} \frac{\sqrt[3]{3x+1} - \sqrt[4]{4x+1}}{x\ln(x+1)} =$$

$$= \lim_{x\to 0} \frac{1+x+\binom{\frac{1}{3}}{2}(3x)^2 + o\left(9x^2\right) - \left(1+x+\binom{\frac{1}{4}}{2}(4x)^2 + o\left(16x^2\right)\right)}{x^2} =$$

$$= \lim_{x\to 0} \frac{-x^2+\frac{3}{2}x^2+o\left(x^2\right)}{x^2} = \frac{1}{2} + \lim_{x\to 0}\frac{o\left(x^2\right)}{x^2} = \frac{1}{2}\,.$$

2. $$L_2 = \lim_{x\to 0} \frac{\ln\left(2\,x^2-2\,x+1\right) + \arcsin 2\,x}{\left(\sqrt{1-4x}-1\right)(1-\cos 2\,x)}\,.$$

Решение:

$$\left(\sqrt{1-4x}-1\right)(1-\cos 2\,x)\;=\left(\frac{-4x}{2}+o\,(-4x)\right)\left(\frac{(2x)^2}{2}+o\left((2x)^2\right)\right)\;=-4x^3+o\left(x^3\right)\;;$$

$$\ln\left(2\,x^2-2\,x+1\right)\;+\;\arcsin 2\,x\;=$$

$$=\;2\,x^2-2x-\frac{\left(2\,x^2-2\,x\right)^2}{2}+\frac{\left(2\,x^2-2\,x\right)^3}{3}+o\left(x^3\right)+2x+\frac{(2x)^3}{6}+o\left(x^3\right)\;=$$

$$=\;2\,x^2-2\left(x^2-x\right)^2+\frac{8}{3}\left(x^2-x\right)^3+\frac{4x^3}{3}+o\left(x^3\right)\;=\frac{8x^3}{3}+o\left(x^3\right)\;;$$

$$L_2\;=\;\lim_{x\to 0}\frac{\dfrac{8x^3}{3}+o\left(x^3\right)}{-4x^3+o\left(x^3\right)}\;=-\frac{2}{3}\;.$$

Алтернатива с правило на Лопитал:

$$L_2\;=\;-\lim_{x\to 0}\frac{\ln\left(2\,x^2-2\,x+1\right)+2x\;+\;\arcsin 2\,x-2x}{4\,x^3}\;=$$

$$=\;-\lim_{x\to 0}\frac{\dfrac{4x-2}{2\,x^2-2\,x+1}\;+\;2}{12\,x^2}\;-\;\lim_{x\to 0}\frac{\dfrac{2}{\sqrt{1-4x^2}}-2}{12x^2}\;=\;-\frac{4}{12}-\;\frac{4}{12}\;=\;-\frac{2}{3}$$

3. $$L_3 = \lim_{x\to 0} \frac{\sqrt[3]{3x+1} + \sqrt[5]{1-5x} - 2}{\operatorname{arctg}\,(x\ln(x+1))}\,.$$

Решение: $$L_3 = \lim_{x\to 0} \frac{\sqrt[3]{3x+1} + \sqrt[5]{1-5x} - 2}{x\ln(x+1)} =$$

$$= \lim_{x\to 0} \frac{1 + x + \binom{\frac{1}{3}}{2}(3x)^2 + o\left(9x^2\right) + 1 - x + \binom{\frac{1}{5}}{2}(-5x)^2 + o\left(25x^2\right) - 2}{x^2} =$$

$$= \lim_{x\to 0} \frac{-x^2 - 2\,x^2 + o\left(x^2\right)}{x^2} = -3 + \lim_{x\to 0}\frac{o\left(x^2\right)}{x^2} = -3\,.$$

Алтернатива с правило на Лопитал: $$L_3 = \lim_{x\to 0} \frac{\sqrt[3]{3x+1} - 1 - x + \sqrt[5]{1-5x} - 1 + x}{x^2} =$$

$$= \lim_{x\to 0} \frac{\dfrac{1}{\sqrt[3]{(3x+1)^2}} - 1}{2\,x} - \lim_{x\to 0} \frac{\dfrac{1}{\sqrt[5]{(1-5x)^4}} - 1}{2\,x} = \lim_{x\to 0} \frac{\dfrac{-2}{\sqrt[3]{(3x+1)^5}}}{2} - \lim_{x\to 0} \frac{\dfrac{4}{\sqrt[5]{(1-5x)^9}}}{2} =$$

$$= -1 - 2 = -3\,.$$

4. $$L_4 = \lim_{x\to 0} \frac{\left(1+6\,x^2\right)^{-x^2} - \sqrt{1+6\,x^4}}{\left(x\ \arcsin x\right)^2}\,.$$

Решение: $$L_4 = \lim_{x\to 0} \frac{\left(1+6\,x^2\right)^{-x^2} - 1 + 1 - \sqrt{1+6\,x^4}}{x^4} =$$

$$= \lim_{x\to 0} \frac{e^{-x^2 \ln\left(1+6\,x^2\right)} - 1}{x^4} + \lim_{x\to 0} \frac{-6\,x^4}{x^4\left(1+\sqrt{1+6\,x^4}\right)} = \lim_{x\to 0} \frac{-x^2 \ln\left(1+6\,x^2\right)}{x^4} - 3 = -9\,.$$

5. $$L_5 = \lim_{x\to 0} \frac{\ln\left(2\,x^2+2\,x+1\right) - \arcsin 2\,x}{\operatorname{tg} x - \sin x}\,.$$

Решение: $$\lim_{x\to 0} \frac{\ln\left(2\,x^2+2\,x+1\right) - \arcsin 2\,x}{x^3} =$$

$$= \lim_{x\to 0} \frac{\ln\left(2\,x^2+2\,x+1\right) - 2x + 2x - \arcsin 2\,x}{x^3} =$$

$$= \lim_{x\to 0} \frac{\dfrac{4x+2}{2\,x^2+2\,x+1} - 2}{3\,x^2} + \lim_{x\to 0} \frac{2 - \dfrac{2}{\sqrt{1-4x^2}}}{3x^2} = -\frac{4}{3} - \frac{4}{3} = -\frac{8}{3}\,;$$

$$\lim_{x\to 0}\frac{\operatorname{tg} x-\sin x}{x^3}=\lim_{x\to 0}\frac{\dfrac{1}{\cos^2 x}-\cos x}{3x^2}=\lim_{x\to 0}\frac{1-\cos^3 x}{3x^2}=$$

$$=\lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x+\cos^2 x}{3}=\frac{1}{2}\,;$$

$$L_5=\lim_{x\to 0}\frac{\ln\left(2\,x^2+2\,x+1\right)-\arcsin 2\,x}{x^3}\cdot\lim_{x\to 0}\frac{x^3}{\operatorname{tg} x-\sin x}=-\frac{16}{3}\,.$$

6. $$L_6=\lim_{x\to 0}\left(\sqrt[4]{4x+1}+\sqrt[5]{1-5x}-1\right)^{\dfrac{1}{\operatorname{arctg} x^2}}\,.$$

Решение: $$\lim_{x\to 0}\frac{\sqrt[4]{4x+1}+\sqrt[5]{1-5x}-2}{\operatorname{arctg} x^2}=\lim_{x\to 0}\frac{\sqrt[4]{4x+1}+\sqrt[5]{1-5x}-2}{x^2}=$$

$$=\lim_{x\to 0}\frac{1+x+\dbinom{\frac{1}{4}}{2}(4x)^2+o\left(16x^2\right)+1-x+\dbinom{\frac{1}{5}}{2}(-5x)^2+o\left(25x^2\right)-2}{x^2}=$$

$$=\lim_{x\to 0}\frac{-\dfrac{3x^2}{2}-2\,x^2+o\left(x^2\right)}{x^2}=-\frac{7}{2}+\lim_{x\to 0}\frac{o\left(x^2\right)}{x^2}=-\frac{7}{2}\quad\Rightarrow\quad L_6=\frac{1}{\sqrt{e^7}}\,.$$