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Домашна работа №3

1зад. а) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}} \operatorname{arctg} \frac{1}{2\sqrt{n}} \sim (-1)^n \cdot \frac{1}{2n}$

$\operatorname{arctg} \frac{1}{2\sqrt{n}} \sim \frac{1}{2\sqrt{n}}$

$\left.\begin{array}{l} \frac{1}{2n} > 0 \\ \frac{1}{2n} \text{ - монотонно намаляваща} \\ \frac{1}{2n} \to 0 \end{array}\right\} \Rightarrow$ Редът е усл. сх.

б) $\sum_{n=1}^{\infty} (-1)^n \sin \frac{n^2+(-1)^n}{2n^2+5n+2011} \sim (-1)^n \frac{n^2+(-1)^n}{2n^2+5n+2011} \to \frac{(-1)^n}{2} \Rightarrow$

$\sin \frac{n^2+(-1)^n}{2n^2+5n+2011} \sim \frac{n^2+(-1)^n}{2n^2+5n+2011}$

$\Rightarrow$ Редът е разходящ.

в) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n+1}-\sqrt{n-1}}{\sqrt{n+69}} = \sum_{n=1}^{\infty} (-1)^n \frac{(\cancel{n}+1-\cancel{n}+1)}{\sqrt{n+69}\,(\sqrt{n+1}+\sqrt{n-1})} =$

$= \sum_{n=1}^{\infty} (-1)^n \frac{2}{\sqrt{n+69}\,(\sqrt{n+1}+\sqrt{n-1})}$

Прилича на $\frac{1}{n} \Rightarrow$ Редът е усл. сх.

г) $\sum_{n=0}^{\infty} \binom{e-\pi}{2n}$

Знаем, че $\left.\begin{array}{l} e \sim 2.72 \\ \pi \sim 3.14 \end{array}\right\} \Rightarrow e-\pi < 0$

Ще положим $e-\pi = -\alpha$.

Тогава:

$\sum_{n=0}^{\infty} \binom{e-\pi}{2n} = \sum_{n=0}^{\infty} \binom{-\alpha}{2n}$

$\binom{-\alpha}{2n} = \frac{(-\alpha)\dots(-\alpha-2n+1)}{(2n)!} = \frac{(-1)^{2n}\,\overbrace{\alpha \dots (2n-1+\alpha)}^{\forall \text{ членове са положителни}}}{(2n)!}$

$\frac{a_{n+1}}{a_n} = \frac{(2n+\alpha)(2n+1+\alpha)}{(2n+1)(2n+2)} \to 1$

$\lim_{n\to\infty} n\left(\frac{4n^2+6n+2-4n^2-2n(1+2\alpha)\dots}{4n^2+2n(1+2+\alpha)+\dots}\right) \to 4-4\alpha > 1 \Rightarrow$

$\Rightarrow$ Редът е разходящ

д) $\sum_{n=0}^{\infty} \binom{e-\pi}{3n}$

Знаем, че $\left.\begin{array}{l} e \sim 2.72 \\ \pi \sim 3.14 \end{array}\right\} \Rightarrow e-\pi \sim -0.42 < 0$

Ще положим $e-\pi = -\alpha$

Тогава:

$$\sum_{n=0}^{\infty} \binom{e-\pi}{3n} = \sum_{n=0}^{\infty} \binom{-\alpha}{3n}$$

$$\binom{-\alpha}{3n} = \frac{(-\alpha)\dots(-\alpha-3n+1)}{(3n)!} = \frac{(-1)^{3n}\alpha\dots(3n-1+\alpha)}{(3n)!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\cancel{(-1)^{3n+3}\alpha\dots(3n-1+\alpha)}(3n+2+\alpha)}{\cancel{(3n+3)!}} \cdot \frac{\cancel{(3n)!}}{\cancel{(-1)^{3n}\alpha\dots(3n-1+\alpha)}}\right| =$$

$(3n+1)(3n+2)(3n+3)$

$$= \left|\frac{(-1)^3(3n+2+\alpha)}{(3n+1)(3n+2)(3n+3)}\right| \text{ - сходящ}$$

$\sum_{n=0}^{\infty} \binom{e-\pi}{3n}$ е абс. сх.

2зад. а) $\sum_{n=0}^{\infty} \frac{(3n)!}{n!(2n+1)!}(x-3)^n = \sum_{n=0}^{\infty} \frac{(3n)!}{n!(2n+1)!} t^n$

$x-3=t$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\cancel{(3n+3)!}\,|t|^{n+1}}{(n+1)!\cancel{(2n+3)!}} \cdot \frac{\cancel{n!}\cancel{(2n+1)!}}{\cancel{(3n)!}\,\cancel{|t|^n}} =$$

$(3n+3)(3n+2)(3n+1)$; $(2n+3)(2n+2)$

$$= \frac{(3n+3)(3n+2)(3n+1)|t|}{(n+1)(2n+3)(2n+2)} \sim \frac{27}{4}|t| \Rightarrow |t| = \frac{4}{27}$$

$$x = \frac{4}{27} + 3 = \frac{85}{27}$$

$$\boxed{R = \frac{85}{27}}$$

При $x = \frac{85}{27}$:

$$\sum_{n=0}^{\infty} \frac{(3n)!}{n!(2n+1)!}\left(\frac{85}{27}-3\right)^n = \sum_{n=0}^{\infty} \frac{(3n)!}{n!(2n+1)!}\cdot\left(\frac{4}{27}\right)^n$$

$$\frac{a_{n+1}}{a_n} = \frac{(3n+3)!}{(n+1)!(2n+3)!}\left(\frac{4}{27}\right)^{n+1} \cdot \frac{n!(2n+1)!}{(3n)!}\cdot\left(\frac{27}{4}\right)^n =$$

$$= \frac{4(3n+3)(3n+2)(3n+1)}{27(n+1)(2n+3)(2n+2)} \longrightarrow \frac{108}{81} > 1 \Rightarrow \text{разх.}$$

При $x = -\frac{85}{27}$:

$$\sum_{n=0}^{\infty} \frac{(3n)!}{n!(2n+1)!}\left(-\frac{85}{27}-3\right)^n = \sum_{n=0}^{\infty} \frac{(3n)!}{n!(2n+1)!}\left(-\frac{166}{27}\right)^n$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|-\frac{27(3n+3)(3n+2)(3n+1)}{166(n+1)(2n+3)(2n+2)}\right| \Rightarrow \text{абс. сх.}$$

б) $\sum_{n=0}^{\infty} \frac{x^n}{n5^n}\sqrt{\binom{3n+1}{n+1}} = \sum_{n=0}^{\infty} \frac{x^n}{n.5^n}\sqrt{\frac{(3n+1)!}{(n+1)!(2n)!}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = |x|\frac{n}{5(n+1)}\sqrt{\frac{(3n+4)!(n+1)!(2n)!}{(3n+1)!(n+2)!(2n+2)!}} \longrightarrow$$

$$\to \frac{|x|n}{5(n+1)}\sqrt{\frac{(3n+2).3.\cancel{(n+1)}(3n+4)}{(n+2)(2n+1).2.\cancel{(n+1)}}} \to \frac{|x|3\sqrt{3}}{10}$$

$$\boxed{R = \frac{10}{3\sqrt{3}}}$$

За $x = \frac{10}{3\sqrt{3}}$:

$$\frac{a_{n+1}}{a_n} = \frac{n}{\cancel{5}(n+1)}\sqrt{\frac{(3n+2)3.(3n+4)}{(2n+1).\cancel{2}.(n+2)}}\cdot\frac{\overset{2}{\cancel{10}}}{3\sqrt{3}} = \frac{n}{n+1}\sqrt{\frac{(6n+4)(3n+4)}{(6n+3)(3n+6)}} =$$

$$= \frac{n}{n+1}\sqrt{\frac{18n^2+36n+16}{18n^2+45n+18}} \to b_n^{-1}$$

При Раабе: $\lim\limits_{n\to\infty} n(b_n - 1)$

Цел: $b_n = 1 + \underline{\alpha}\cdot\frac{1}{n}$

$$\frac{n+1}{n}\sqrt{\frac{18n^2+36n+16}{18n^2+45n+18}} \sim 1+\frac{\alpha}{n}$$

$$\frac{n+1}{n} = 1+\frac{1}{n}$$

$$\sqrt{\frac{18n^2+36n+16}{18n^2+45n+18}} \sim 1+\frac{\beta}{n}$$

За да намерим α: $\left(1+\frac{1}{n}\right)\left(1+\frac{\beta}{n}\right) \sim \frac{1+\beta}{n}$

$$\frac{18n^2+45n+18}{18n^2+36n+16} \sim 1+\frac{2\beta}{n}$$

$$18n^3+45n^2+18n \sim (n+2\beta)(18n^2+36n+18)$$

Пред n^2: $45 = 36(\beta+1)$

$$\beta+1 = \frac{5}{4} \Rightarrow \beta = \frac{1}{4}$$

$$b_n = 1+\frac{5}{4n}$$

$$\Rightarrow \lim_{n\to\infty} n\left(\cancel{1}+\frac{5}{4n}-\cancel{1}\right) \to \frac{5}{4} \Rightarrow \text{сх.}$$

3зад. а) $\operatorname{arctg}\frac{1-x}{1+x}$

$$\operatorname{arctg}x = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + C, \quad C=0$$

$$\frac{\pi}{4} - x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^9}{9} + O(x^{10})$$

$\Rightarrow \operatorname{arctg}\frac{1-x}{1+x} =$

б) $\ln\frac{2x+1}{x^2-4x+4} = \ln(2x+1) - \ln(x^2-4x+4) =$

$$= \ln(1+2x) - \ln(1+(x^2-4x+3))$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{(-1)^{n-1} x^n}{n}$$

$$\ln(1+2x) = 2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \frac{32x^5}{5} - \frac{32x^6}{3} + o(x^7)$$

$$\ln(1+(x^2-4x+3)) = \ln 4 - x - \frac{x^2}{4} - \frac{x^3}{12} - \frac{x^4}{32} - \frac{x^5}{80} - \frac{x^6}{192} + o(x^7)$$

$$-\ln 4 + 3x - \frac{7x^2}{4} + \frac{11x^3}{4} - \frac{127x^4}{32} + \frac{513x^5}{80} - \frac{2047x^6}{192} + o(x^7)$$

в) $\frac{1}{x^2+x+1} = \frac{1}{1+(x^2+x)} = 1-(x^2+x)+(x^2+x)^2-(x^2+x)^3+\dots+(-1)(x^2+x)^n$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n$$

4зад. а) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = e^{2x}$, той като $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

б) $f(x) \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!} = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$$

Полагаме x^2

$$h(x) = \sum_{n=0}^{\infty} \frac{(-1) x^{2n}}{(2n)!}$$

$h(x) = \cos x$

$g(x) = \cos x^2$

$f(x) = x \cdot \cos x^2$

в) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{3n+1} = f(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{3n+1}$

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{3n+1}$$

$$f(x) = x\left(x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \dots + \frac{(-1)^n x^{3n+1}}{3n+1}\right)$$

5зад. $\lim_{x\to 0} \frac{e^{\operatorname{arctg} x} - e^{\sin x}}{\ln\sqrt{\frac{1+x}{1-x}} - \operatorname{arctg}(\sin x)}$

$$e^{\operatorname{arctg} x} = 1 + \operatorname{arctg} x + \frac{(\operatorname{arctg} x)^2}{2!} + \frac{(\operatorname{arctg} x)^3}{3!} + o(x^3) =$$

$$= 1 + \left(x - \frac{x^3}{3} + \frac{x^5}{5}\right) + \frac{\left(x - \frac{x^3}{3} + \frac{x^5}{5}\right)^2}{2} + \frac{\left(x - \frac{x^3}{3} + \frac{x^5}{5}\right)^3}{6} + o(x^3) =$$

$$= 1 + x - \frac{x^3}{3} + \frac{1}{2}(x^2 - x^3) + \frac{1}{6}x^3 = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} + o(x^3)$$

$e^{\sin x} = 1 + x + \frac{x^2}{2} + o(x^4)$

$e^{\operatorname{arctg} x} - e^{\sin x} = \cancel{1} + \cancel{x} + \cancel{\frac{x^2}{2}} - \frac{x^3}{6} - \cancel{1} - \cancel{x} - \cancel{\frac{x^2}{2}} = -\frac{x^3}{6} + o(x^3)$

$\ln\sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + o(x^4)$

$\operatorname{arctg}(\sin x) = x - \frac{x^3}{2} + o(x^4)$

$\ln\sqrt{\frac{1+x}{1-x}} - \operatorname{arctg}(\sin x) = \cancel{x} + \frac{x^3}{3} - \cancel{x} + \frac{x^3}{2} = \frac{5}{6}x^3 + o(x^4)$

$$\lim_{x\to 0} \frac{-\frac{x^3}{6} + o(x^3)}{\frac{5}{6}x^3 + o(x^4)} = -\frac{1}{5}$$