Формула за двойно векторно → вектор произведение

$$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c}).\vec{b} - (\vec{b}.\vec{c}).\vec{a}$$
$$(1 \quad 2) \quad 3 = (1 \quad 3) \quad 2 - (2 \quad 3) \quad 1$$

Зад 3 Да се докаже:

$$\underbrace{\vec{a} \times (\vec{b} \times \vec{c})} + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}$$

Реш: $ЛС = -(\vec{b} \times \vec{c}) \times \vec{a} - \underline{(\vec{c} \times \vec{a}) \times \vec{b}} - (\vec{a} \times \vec{b}) \times \vec{c}$
$$\quad\ \ 1 \quad\ 2 \quad\ 3 \qquad\quad 1 \quad\ 2 \quad\ 3 \qquad\quad 1 \quad\ 2 \quad\ 3$$

$= -[(\vec{b}.\vec{a}).\vec{c} - (\vec{c}.\vec{a}).\vec{b}] - [(\vec{c}.\vec{b})\vec{a} - (\vec{a}.\vec{b}).\vec{c}] -$

$- [(\vec{a}.\vec{c}).\vec{b} - (\vec{b}.\vec{c}).\vec{a}] = \vec{0}$

Детерминанта на Грам

$\Gamma(\vec{a}) = \vec{a}^2$

$$\Gamma(\vec{a}, \vec{b}) = (\vec{a} \times \vec{b})^2 = \begin{vmatrix} \vec{a}^2 & \vec{a}.\vec{b} \\ \vec{b}.\vec{a} & \vec{b}^2 \end{vmatrix}$$

$$\Gamma(\vec{a}, \vec{b}, \vec{c}) = (\vec{a}\ \vec{b}\ \vec{c})^2 = \begin{vmatrix} \vec{a}^2 & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}^2 & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}^2 \end{vmatrix}$$

Зад 4 Нека $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$

$|\vec{a}| = 3$, $|\vec{b}| = 2$, $|\vec{c}| = 4$

$$\measuredangle(\vec{a}, \vec{c})_e = \measuredangle(\vec{a}, \vec{b})_e = \measuredangle(\vec{b}, \vec{c})_e = \frac{\pi}{3}$$

а) $V_{OABC} = ?$ б) $S_{OAB} = ?$

Реш:

а) $$V_{OABC} = \frac{1}{6} \left| (\vec{OA}\ \vec{OB}\ \vec{OC}) \right| = \frac{1}{6} \left| (\vec{a}\ \vec{b}\ \vec{c}) \right|$$

$$(\vec{a}\ \vec{b}\ \vec{c})^2 = \begin{vmatrix} \vec{a}^2 & \vec{a}\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}^2 & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}^2 \end{vmatrix}$$

$\vec{a}^2 = 9$ $\quad \vec{a}.\vec{b} = 3.2.\frac{1}{2} = 3$

$\vec{b}^2 = 4$ $\quad \vec{a}.\vec{c} = 3.4.\frac{1}{2} = 6$

$\vec{c}^2 = 16$ $\quad \vec{b}.\vec{c} = 2.4.\frac{1}{2} = 4$

$$= \begin{vmatrix} 9 & 3 & 6 \\ 3 & 4 & 4 \\ 6 & 4 & 16 \end{vmatrix} = 9.4.16 + \cancel{3.4.6} + \cancel{3.4.6} - \cancel{6.4.6} - 4.4.9 - 3.3.16$$

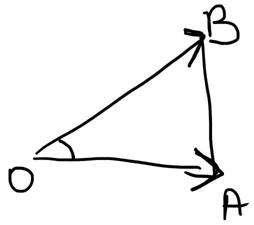
$$= 36.16 - 4.36 - 4.36$$

$$= 8.36 = 288$$

$$(\vec{a}\ \vec{b}\ \vec{c})^2 = 288 \Rightarrow (\vec{a}\ \vec{b}\ \vec{c}) = \pm\sqrt{288}$$

$$V_{OABC} = \frac{1}{6} \left| (\vec{a}\ \vec{b}\ \vec{c}) \right| = \frac{1}{6} \left| \pm\sqrt{288} \right| = \frac{1}{6} . 12\sqrt{2} = 2\sqrt{2}$$

б) $S_{OAB} = ?$



$$S = \frac{|\vec{OA} \times \vec{OB}|}{2} = \frac{|\vec{a} \times \vec{b}|}{2}$$

$$= \frac{|\vec{a}||\vec{b}|.\sin\measuredangle(\vec{a}, \vec{b})_e}{2} = \frac{3.2.\frac{\sqrt{3}}{2}}{2} = \frac{3\sqrt{3}}{2}$$

Зад 5 Дадени са векторите $\vec{a}, \vec{b}$, като $|\vec{a}| = |\vec{b}| = 1$, $\measuredangle(\vec{a}, \vec{b})_e = \frac{\pi}{3}$. Нека

$$\vec{OA} = \vec{a}, \quad \vec{OB} = \vec{a} \times \vec{b}, \quad \vec{OC} = \vec{b} \times (\vec{a} \times \vec{b}).$$

Да се докаже, че $\vec{OA}, \vec{OB}, \vec{OC}$ са ЛНЗ. Да се намери $V_{OABC} = \frac{1}{6} \left| (\vec{OA}\ \vec{OB}\ \vec{OC}) \right|$

Реш: $\vec{OA}, \vec{OB}, \vec{OC}$ са ЛНЗ $\iff (\vec{OA}\ \vec{OB}\ \vec{OC}) \neq 0$

$$(\vec{OA}\ \vec{OB}\ \vec{OC}) = (\vec{OA} \times \vec{OB}) . \vec{OC} = (\vec{a} \times (\vec{a} \times \vec{b})) . (\vec{b} \times (\vec{a} \times \vec{b}))$$

$$= (-(\vec{a} \times \vec{b}) \times \vec{a}) . (-(\vec{a} \times \vec{b}) \times \vec{b}) = ((\underset{1}{\vec{a}} \times \underset{2}{\vec{b}}) \times \underset{3}{\vec{a}}) . ((\underset{1}{\vec{a}} \times \underset{2}{\vec{b}}) \times \underset{3}{\vec{b}})$$

$$= ((\vec{a} . \vec{a}) . \vec{b} - (\vec{b} . \vec{a}) . \vec{a}) . ((\vec{a} . \vec{b}) . \vec{b} - (\vec{b} . \vec{b}) . \vec{a})$$

$$= \left(1 . \vec{b} - \left(1 . 1 . \frac{1}{2}\right) . \vec{a}\right) . \left(\frac{1}{2} . \vec{b} - 1 . \vec{a}\right)$$

$$= \left(\vec{b} - \frac{1}{2}\vec{a}\right) . \left(\frac{1}{2}\vec{b} - \vec{a}\right) = \frac{1}{2} . \vec{b}^2 - \vec{b} . \vec{a} - \frac{1}{4}\vec{a} . \vec{b} + \frac{1}{2}\vec{a}^2$$

$$= \cancel{\frac{1}{2} . 1} - \cancel{\frac{1}{2}} - \frac{1}{4} . \frac{1}{2} + \frac{1}{2} . 1 = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8} \neq 0 \Rightarrow \vec{OA}, \vec{OB}, \vec{OC}$$ са ЛНЗ

$$V = \frac{1}{6} \left| (\vec{OA}\ \vec{OB}\ \vec{OC}) \right| = \frac{1}{6} \left| \frac{3}{8} \right| = \frac{1}{16}$$

Зад. $\vec{a}, \vec{b}$ – единични

$\vec{OA} = \vec{b}$

$\vec{OB} = (\vec{a} \times \vec{b}) \times (\vec{a} + \vec{b})$

Да се намери ъгълът $\sphericalangle(\vec{a}, \vec{b})$ е така, че медианата през върха B на $\triangle ABO$ да бъде колинеарна с вектора $\vec{a}$.

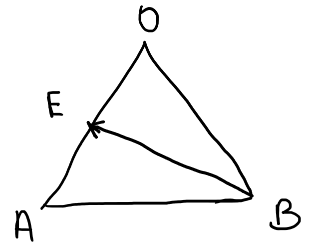
Ако $\vec{OC} = [(\vec{a} \times \vec{b}) \times \vec{a}] \times [(\vec{a} \times \vec{b}) \times \vec{b}]$, то да се намери $V_{OABC} = ?$

Реш:

Нека т. E – среда на OA

$\vec{BE} \parallel \vec{a} \rightarrow \vec{BE} = \lambda \vec{a}$

Реш.:



Нека т. E - среда на OA

$\overrightarrow{BE} \parallel \vec{a}$ $\quad \overrightarrow{BE} = \underset{\text{число}}{\lambda} . \vec{a}$

$\overrightarrow{BE} = \overrightarrow{OE} - \overrightarrow{OB} = \frac{1}{2}\overrightarrow{OA} - \overrightarrow{OB}$

$$\overrightarrow{BE} = \frac{1}{2}\vec{b} - (\vec{a}\times\vec{b})\times(\vec{a}+\vec{b}) = \frac{1}{2}\vec{b} - (\underset{1}{\vec{a}}\times\underset{2}{\vec{b}})\times\underset{3}{\vec{a}} - (\underset{1}{\vec{a}}\times\underset{2}{\vec{b}})\times\underset{3}{\vec{b}}$$

$$= \frac{1}{2}\vec{b} - \left((\vec{a}.\vec{a}).\vec{b} - \underbrace{(\vec{b}.\vec{a})}.\vec{a}\right) - \left((\vec{a}.\vec{b}).\vec{b} - (\vec{b}.\vec{b}).\vec{a}\right)$$

$$= \frac{1}{2}\vec{b} - \left(1.\vec{b} - (1.1.\cos\sphericalangle(\vec{b},\vec{a})_e).\vec{a}\right) - \left((\cos\sphericalangle(\vec{b},\vec{a})_e)\cdot\vec{b} - 1.\vec{a}\right)$$

$$= \frac{1}{2}\vec{b} - \vec{b} + \cos\sphericalangle(\vec{a},\vec{b})_e.\vec{a} - \cos\sphericalangle(\vec{a},\vec{b})_e\cdot\vec{b} + \vec{a}$$

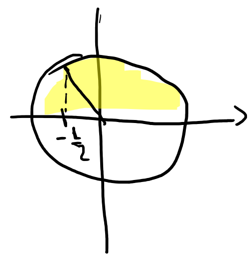
$$\overrightarrow{BE} = (\cos\sphericalangle(\vec{a},\vec{b})_e + 1)\vec{a} + \underbrace{\left(-\frac{1}{2} - \cos\sphericalangle(\vec{a},\vec{b})_e\right)}_{0}\vec{b}$$

$$\overrightarrow{BE} = \lambda \cdot \vec{a} + 0 \cdot \vec{b}$$

$$-\frac{1}{2} - \cos\sphericalangle(\vec{a},\vec{b})_e = 0$$

$$\cos\sphericalangle(\vec{a},\vec{b})_e = -\frac{1}{2}$$

$$\sphericalangle(\vec{a},\vec{b})_e = \frac{2\pi}{3}$$



$$V_{OABC} = \frac{1}{6}\left|(\overrightarrow{OA}\ \overrightarrow{OB}\ \overrightarrow{OC})\right|$$

$\overrightarrow{OA} = \vec{b}$

$$\overrightarrow{OB} = (\vec{a}\times\vec{b})\times(\vec{a}+\vec{b}) = (\vec{a}\times\vec{b})\times\vec{a} + (\vec{a}\times\vec{b})\times\vec{b}$$

$$= (\vec{a}.\vec{a}).\vec{b} - (\vec{b}.\vec{a}).\vec{a} + (\vec{a}.\vec{b}).\vec{b} - (\vec{b}.\vec{b})\vec{a}$$

$$= 1.\vec{b} - \left(1.1.\left(-\frac{1}{2}\right)\right).\vec{a} + \left(-\frac{1}{2}\right).\vec{b} - 1.\vec{a}$$

$$= \vec{b} + \frac{1}{2}\vec{a} - \frac{1}{2}\vec{b} - \vec{a} = -\frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}$$

$$\overrightarrow{OC} = \left[(\vec{a}\times\vec{b})\times\vec{a}\right]\times\left[(\vec{a}\times\vec{b})\times\vec{b}\right]$$

$|\vec{b}\times\vec{b}| = |\vec{b}||\vec{b}|\underbrace{\sin\sphericalangle(\vec{b},\vec{b})}_{0} = 0$

$$= \left[\vec{b} + \frac{1}{2}\vec{a}\right]\times\left[-\frac{1}{2}\vec{b} - \vec{a}\right] = -\frac{1}{2}\underbrace{\vec{b}\times\vec{b}}_{\vec{0}} - \vec{b}\times\vec{a} +$$

$$= [\vec{b} + \frac{1}{2}\vec{a}] \times [-\frac{1}{2}\vec{b} - \vec{a}] = -\frac{1}{2}\underbrace{\vec{b}\times\vec{b}}_{\vec{0}} \ldots$$

$$+ (-\frac{1}{4})\vec{a}\times\vec{b} - \frac{1}{2}\underbrace{\vec{a}\times\vec{a}}_{\vec{0}}$$

$$\vec{OC} = \underbrace{-\vec{b}\times\vec{a}} - \frac{1}{4}\vec{a}\times\vec{b} = \vec{a}\times\vec{b} - \frac{1}{4}\vec{a}\times\vec{b} = \frac{3}{4}\vec{a}\times\vec{b}$$

$$V_{OABC} = \frac{1}{6}|(\vec{OA}\ \vec{OB}\ \vec{OC})| = \frac{1}{6}|(\vec{OA}\times\vec{OB})\cdot\vec{OC}|$$

$$= \frac{1}{6}|(\vec{b}\times(-\frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}))\cdot(\frac{3}{4}\vec{a}\times\vec{b})|$$

$$= \frac{1}{6}|(-\frac{1}{2}\vec{b}\times\vec{a} + \frac{1}{2}\underbrace{\vec{b}\times\vec{b}}_{\vec{0}})\cdot(\frac{3}{4}\vec{a}\times\vec{b})|$$

$$= \frac{1}{6}|(\frac{1}{2}\vec{a}\times\vec{b})\cdot(\frac{3}{4}\vec{a}\times\vec{b})| = \frac{1}{{}_2\cancel{6}}\cdot\frac{1}{2}\cdot\frac{\cancel{3}}{4}\cdot|(\vec{a}\times\vec{b})^2|$$

$$= \frac{1}{16}|\vec{a}^2\cdot\vec{b}^2 - (\vec{a}\vec{b})^2| = \frac{1}{16}|1\cdot1 - (1\cdot1\cdot\cos\frac{2\pi}{3})^2|$$

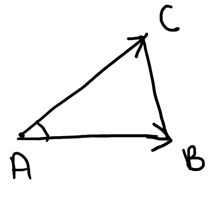
$$= \frac{1}{16}|1 - (-\frac{1}{2})^2| = \frac{1}{16}|\frac{3}{4}| = \frac{3}{64}$$

ОКС

Зад. $A(1,-1,2)$; $B(2,1,-1)$; $C(-1,2,1)$ – $\triangle ABC$

$S_{\triangle ABC} = ?$

Реш:



$$S_{\triangle ABC} = \frac{|\vec{AB}\times\vec{AC}|}{2}$$

$\vec{AB}(1, 2, -3)$

$\vec{AC}(-2, 3, -1)$

$$\vec{AB}\times\vec{AC}\left(\begin{vmatrix} 2 & -3 \\ 3 & -1 \end{vmatrix}, \begin{vmatrix} -3 & 1 \\ -1 & -2 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix}\right)$$

II III, III I, I II

$\overrightarrow{AB} \times \overrightarrow{AC} \; (7, 7, 7)$

$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{7^2 + 7^2 + 7^2} = 7\sqrt{3}$

$S_{\triangle ABC} = \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{2} = \frac{7\sqrt{3}}{2}$