Метрични канонични уравнения на криви от втора степен

Крива от втора степен:

$k: \ a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0$

<u>Зад.</u> ОКС $K = O\vec{e_1}\vec{e_2}$

Да се намери канонично уравнение на кривата k и последователните координатни трансформации, чрез които се достига до него. Да се намерят координатите на фокусите спрямо първоначалната к.с. K.

а) $k: \ 5x^2 + \overset{2.4.xy}{8xy} + 5y^2 - 18x - 18y + 9 = 0$

$a_{11}x^2 + 2a_{12}xy + a_{22}y^2$

I ст) Намираме собствените стойности и собствените вектори на матрицата

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

$$\begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

$(5-\lambda)^2 - 4^2 = 0$

$(1-\lambda)(9-\lambda) = 0$

$\lambda_1 = 1; \ \lambda_2 = 9$

$\lambda_1 + \lambda_2 = a_{11} + a_{22}$

Търсим собствен вектор $\vec{b_1}(\alpha_1, \beta_1)$, $|\vec{b_1}| = 1$, отговарящ на $\lambda_1 = 1$

$$\left|\begin{array}{l} \begin{pmatrix} 5-1 & 4 \\ 4 & 5-1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \sqrt{\alpha_1^2 + \beta_1^2} = 1 \end{array}\right. \qquad \left|\begin{array}{l} 4\alpha_1 + 4\beta_1 = 0 \\ 4\alpha_1 + 4\beta_1 = 0 \\ \alpha_1^2 + \beta_1^2 = 1 \end{array}\right.$$

$$\left|\begin{array}{l} \alpha_1 = -\beta_1 \\ \alpha_1^2 + \beta_1^2 = 1 \end{array}\right. \qquad \left|\begin{array}{l} \alpha_1 = -\beta_1 \\ (-\beta_1)^2 + \beta_1^2 = 1 \end{array}\right. \qquad \begin{array}{l} 2\beta_1^2 = 1 \\ \beta_1^2 = \frac{1}{2} \end{array}$$

Нека $\beta_1 = \frac{-\sqrt{2}}{2} \Rightarrow \alpha_1 = \frac{\sqrt{2}}{2}$

$$\lambda_1 = 1 \longleftrightarrow \vec{b_1}\left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$$

Собствен вектор $\vec{b_2}(\alpha_2, \beta_2)$, $|\vec{b_2}| = 1$, отговарящ на $\lambda_2 = 9$

$$\left|\begin{array}{l} \begin{pmatrix} 5-9 & 4 \\ 4 & 5-9 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \sqrt{\alpha_2^2 + \beta_2^2} = 1 \end{array}\right. \qquad \left|\begin{array}{l} -4\alpha_2 + 4\beta_2 = 0 \\ 4\alpha_2 - 4\beta_2 = 0 \\ \alpha_2^2 + \beta_2^2 = 1 \end{array}\right.$$

$$\left|\begin{array}{l} \alpha_2 = \beta_2 \\ \alpha_2^2 + \beta_2^2 = 1 \end{array}\right. \qquad \beta_2^2 = \frac{1}{2}$$

Нека $\beta_2 = \frac{\sqrt{2}}{2} \Rightarrow \alpha_2 = \frac{\sqrt{2}}{2}$

$$\lambda_2 = 9 \longleftrightarrow \vec{b_2}\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

Проверка: $\vec{b_1} \cdot \vec{b_2} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} + \left(\frac{-\sqrt{2}}{2}\right) \cdot \frac{\sqrt{2}}{2} = 0$

Проверка: $\vec{b_1}\cdot\vec{b_2} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} + \left(\frac{-\sqrt{2}}{2}\right)\cdot\frac{\sqrt{2}}{2} = 0$

$\vec{b_1} \perp \vec{b_2}$

II ст) Правим смяна на ОКС

$K = O\vec{e_1}\vec{e_2} \longrightarrow K' = O\vec{b_1}\vec{b_2}$

т. $M(x, y)$ спр. K

т. $M(x', y')$ спр. K'

$$T_1: \begin{cases} x = \frac{\sqrt{2}}{2}\cdot x' + \frac{\sqrt{2}}{2}\cdot y' \\ y = -\frac{\sqrt{2}}{2}\cdot x' + \frac{\sqrt{2}}{2}\cdot y' \end{cases} \qquad \begin{array}{cc} \vec{b_1} & \vec{b_2} \end{array}$$

Заместваме x и y от T_1 в уравнението на k от условието, за да получим у-то на кривата k спрямо к.с. K'.

$$k: 5x^2 + 8xy + 5y^2 - 18x - 18y + 9 = 0 \quad \text{спр } K$$

$$k: \underbrace{5\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)^2 + 8\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)\left(-\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) + 5\left(-\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)^2}$$

$$-18\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) - 18\left(-\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) + 9 = 0$$

$$k: \lambda_1 \cdot (x')^2 + 0 \cdot x'y' + \lambda_2 \cdot (y')^2 - \cancel{9\sqrt{2}x'} - 9\sqrt{2}y'$$

$$+ \cancel{9\sqrt{2}x'} - 9\sqrt{2}y' + 9 = 0$$

$$k: 1(x')^2 + 9(y')^2 - 18\sqrt{2}y' + 9 = 0 \quad \text{спр. } K' \; (*)$$

$k: 1.(x')^2 + 9.(y')^2 - 18\sqrt{2}\, y' + 9 = 0$ спр. K' (*)

$\varepsilon: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\quad \lambda_1 . \lambda_2 > 0$

$\chi: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\quad \lambda_1 . \lambda_2 < 0$

$\pi: y^2 = 2px$ $\quad \lambda_1 . \lambda_2 = 0$

III ст. Смяна на ОКС

$K' = O\vec{b_1}\vec{b_2} \longrightarrow K'' = V\vec{b_1}\vec{b_2}$

$M(x', y')$ спр. K'

$M(x'', y'')$ спр. K''

$V(\alpha, \beta)$ – център на елипсата

$$T_2: \begin{cases} x' = x'' + \alpha \\ y' = y'' + \beta \end{cases}$$

Заместваме x' и y' от T_2 в (*), за да получим у-то на k спрямо K''

$k: (x')^2 + 9(y')^2 - 18\sqrt{2}\, y' + 9 = 0$

$k: (x'' + \alpha)^2 + 9(y'' + \beta)^2 - 18\sqrt{2}(y'' + \beta) + 9 = 0$

$k: (x'')^2 + 9.y''^2 + \underbrace{2\alpha.}_{0}x'' + \underbrace{(18\beta - 18\sqrt{2})}_{0}y''$

$\underbrace{+\alpha^2 + 9\beta^2 - 18\sqrt{2}\,\beta + 9}_{?} = 0$

$k: \frac{(x'')^2}{a^2} + \frac{(y'')^2}{b^2} = 1$

$k: \frac{(x'')^2}{a^2} + \frac{(y'')^2}{b^2} = 1$

Получаваме системата:

$$\begin{cases} 2\alpha = 0 \\ 18\beta - 18\sqrt{2} = 0 \end{cases} \Rightarrow \begin{cases} \alpha = 0 \\ \beta = \sqrt{2} \end{cases}$$

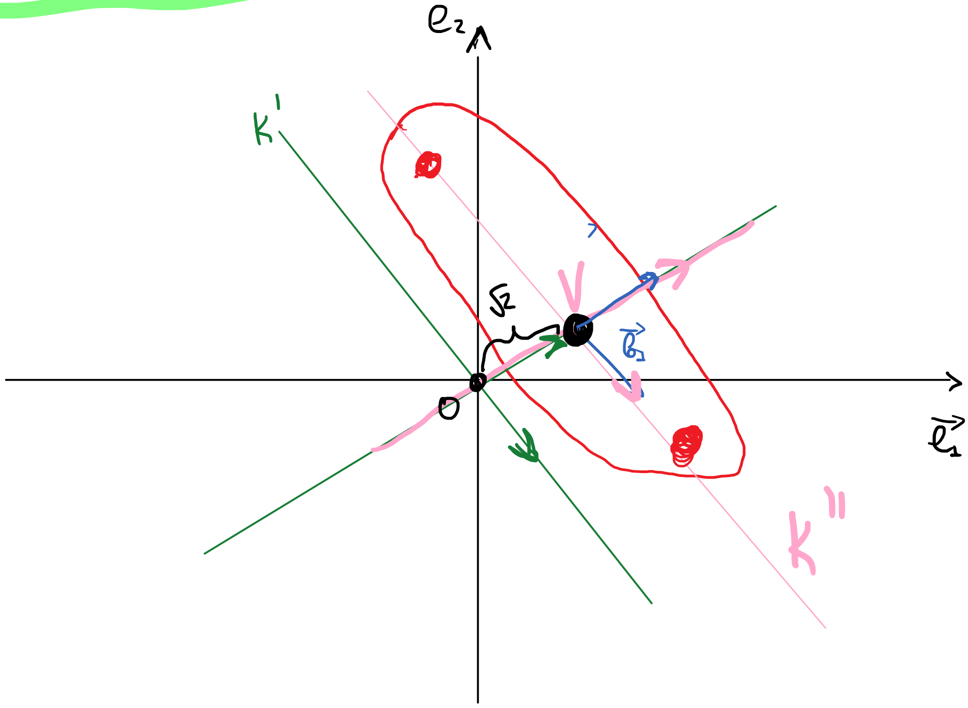
$$T_2: \begin{cases} x' = x'' + 0 \\ y' = y'' + \sqrt{2} \end{cases}$$

$$\alpha^2 + 9\beta^2 - 18\sqrt{2}\beta + 9 = 0 + 9.(\sqrt{2})^2 - 18\sqrt{2}.\sqrt{2} + 9 = -9$$

$$k: (x'')^2 + 9(y'')^2 - 9 = 0$$

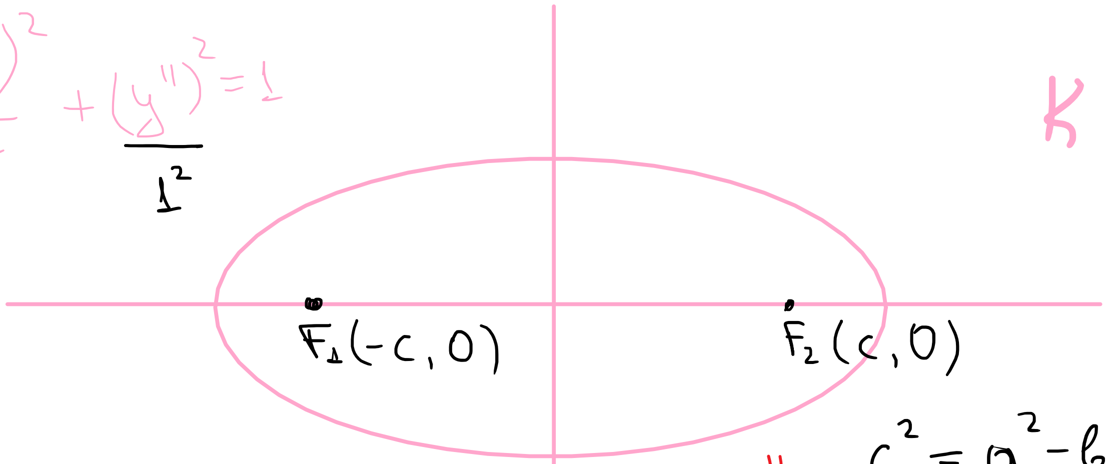
$$(x'')^2 + 9.(y'')^2 = 9 \quad / : 9$$

$$k: \frac{(x'')^2}{3^2} + (y'')^2 = 1 \quad \text{спр. } K''$$



$$\frac{(x'')^2}{3^2} + \frac{(y'')^2}{1^2} = 1$$

K''



$$c^2 = a^2 - b^2 = 9 - 1 = 8$$

$$F_1(\overbrace{-2\sqrt{2}}^{x''}; \overbrace{0}^{y''}); \quad F_2(2\sqrt{2}, 0) \text{ спр } K''$$

$$T_2 \circ T_1 \quad \begin{cases} x = \frac{\sqrt{2}}{2} x'' + \frac{\sqrt{2}}{2}(y'' + \sqrt{2}) \\ y = -\frac{\sqrt{2}}{2} x'' + \frac{\sqrt{2}}{2}(y'' + \sqrt{2}) \end{cases}$$

За F_1:

$$x = \frac{\sqrt{2}}{2} \cdot (-2\sqrt{2}) + \frac{\sqrt{2}}{2}(0 + \sqrt{2}) = -1$$

$$y = -\frac{\sqrt{2}}{2}(-2\sqrt{2}) + \frac{\sqrt{2}}{2}(0 + \sqrt{2}) = 3$$

$\Rightarrow F_1(-1, 3)$ спр K

За F_2:

$$x = \frac{\sqrt{2}}{2} 2\sqrt{2} + \frac{\sqrt{2}}{2}(0 + \sqrt{2}) = 3$$

$$y = -\frac{\sqrt{2}}{2} 2\sqrt{2} + \frac{\sqrt{2}}{2}(0 + \sqrt{2}) = -1$$

$\Rightarrow F_2(3, -1)$ спр K

$$+2(-12)xy$$

б) $k: 9x^2 - 24xy + 16y^2 - 10x - 70y + 125 = 0$

$$+2a_{12}xy$$

$I_{ст}$) Собств. ст-ти и собств. в-ри

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix}$$

$$\begin{vmatrix} 9-\lambda & -12 \\ -12 & 16-\lambda \end{vmatrix} = 0 \qquad \lambda_1 = 0 \ ; \ \lambda_2 = 25$$

Собствен в-р $\vec{e_1}(\alpha_1, \beta_1)$, $|\vec{e_1}| = 1 \iff \lambda_1 = 0$

$$\left| \begin{array}{l} \begin{pmatrix} 9-0 & -12 \\ -12 & 16-0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \sqrt{\alpha_1^2 + \beta_1^2} = 1 \end{array} \right. \qquad \left| \begin{array}{l} 9\alpha_1 - 12\beta_1 = 0 \\ -12\alpha_1 + 16\beta_1 = 0 \\ \alpha_1^2 + \beta_1^2 = 1 \end{array} \right.$$

$$\left| \begin{array}{l} \alpha_1 = \frac{4}{3}\beta_1 \\ \frac{16}{9}\beta_1^2 + \beta_1^2 = 1 \end{array} \right. \Rightarrow \frac{25}{9}\beta_1^2 = 1 \Rightarrow \beta_1 = \frac{3}{5} \Rightarrow \alpha_1 = \frac{4}{5}$$

$$\lambda_1 = 0 \iff \vec{e_1}\left(\frac{4}{5}, \frac{3}{5}\right)$$

Собствен в-р $\vec{e_2}(\alpha_2, \beta_2)$, $|\vec{e_2}| = 1 \iff \lambda_2 = 25$

$$\left| \begin{array}{l} \begin{pmatrix} 9-25 & -12 \\ -12 & 16-25 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \sqrt{\alpha_2^2 + \beta_2^2} = 1 \end{array} \right.$$

$$\lambda_2 = 25 \iff \vec{e_2}\left(-\frac{3}{5}, \frac{4}{5}\right)$$

II Смяна на ОКС $K = O\vec{e_1}\vec{e_2} \to K' = O\vec{e_1'}\vec{e_2'}$

$$T_1: \begin{cases} x = \frac{4}{5} \cdot x' + \left(-\frac{3}{5}\right) y' \\ y = \frac{3}{5} x' + \frac{4}{5} \cdot y' \end{cases}$$

$\vec{e_1}$ $\qquad$ $\vec{e_2}$

Уравнение на кривата k спрямо K'

$k: \lambda_1 (x')^2 + 0x'y' + \lambda_2 (y')^2 - 10\left(\frac{4}{5}x' - \frac{3}{5}y'\right)$

$- 70\left(\frac{3}{5}x' + \frac{4}{5}y'\right) + 125 = 0$

$k: 25(y')^2 - 8x' + 6y' - 42x' - 56y' + 125 = 0$

$k: 25(y')^2 - 50x' - 50y' + 125 = 0$

$k: (y')^2 - 2x' - 2y' + 5 = 0$ спр K' $(*)$

III. Смяна на ОКС

$K' = O\vec{e_1}\vec{e_2} \longrightarrow K'' = V\vec{e_1}\vec{e_2}$

$V(\alpha, \beta)$ – връх на параболата

$$T_2: \begin{cases} x' = x'' + \alpha \\ y' = y'' + \beta \end{cases}$$

Заместваме в $(*)$:

$k: (y'' + \beta)^2 - 2(x'' + \alpha) - 2(y'' + \beta) + 5 = 0$

$k: (y'')^2 - 2x'' + \underbrace{(2\beta - 2)}_{0} y'' +$

$+ \underbrace{\beta^2 - 2\alpha - 2\beta + 5}_{0} = 0$

$y''^2 = 2p x''$

$2\beta - 2 = 0 \Rightarrow \beta = 1$

$$\beta^2 - 2\alpha - 2\beta + 5 = 0$$
$$1 - 2\alpha - 2 + 5 = 0$$
$$\alpha = 2$$

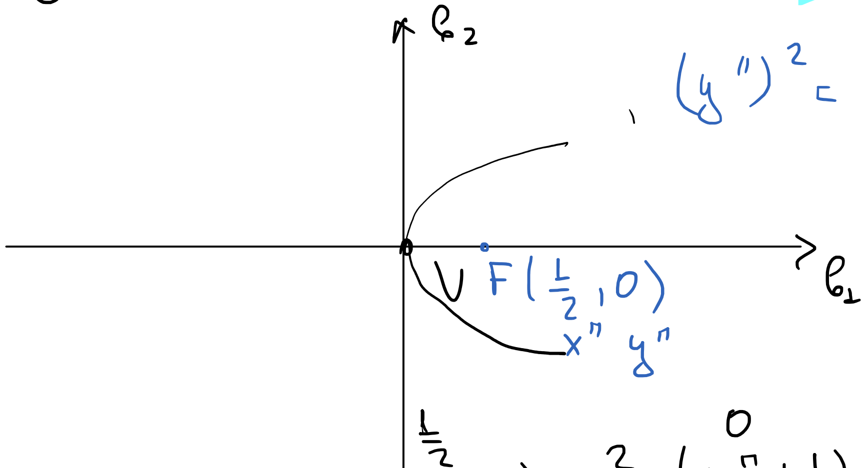
$$T_2: \begin{cases} x' = x'' + 2 \\ y' = y'' + 1 \end{cases}$$

$k: (y'')^2 - 2x'' = 0$ $\qquad$ $k: (y'')^2 = 2x''$ cup. K''

k''

$(y'')^2 = 2px''$

$p = 1$



$$T_2 \circ T_1: \begin{cases} x = \frac{4}{5}(x''+2) - \frac{2}{5}(y''+1) \\ y = \frac{3}{5}(x''+2) + \frac{4}{5}(y''+1) \end{cases}$$

$\frac{1}{2}$ $\qquad$ 0

$F\left(\frac{14}{10}, \frac{23}{10}\right)$ cup K

$0x^2 + 2\cdot\frac{3}{2}xy + 0y^2 + x - 2y = 0$

б) $k: 3xy + x - 2y = 0$

$$A = \begin{pmatrix} 0 & \frac{3}{2} \\ \frac{3}{2} & 0 \end{pmatrix}$$