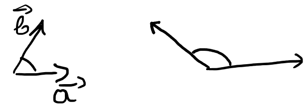
Скаларно произведение на два вектора $\rightarrow$ число

Def: $\vec{a}.\vec{b} = \begin{cases} |\vec{a}|.|\vec{b}|.\cos\sphericalangle(\vec{a},\vec{b})_e \\ 0, \text{ при } \vec{a}=\vec{0} \text{ или } \vec{b}=\vec{0} \end{cases}$

$\sphericalangle(\vec{a},\vec{b})_e$ – елементарно геометричен ъгъл $[0;\pi]$



Свойства:

1) $\vec{a}.\vec{b} = \vec{b}.\vec{a}$

2) $(\vec{a}+\vec{b}).\vec{c} = \vec{a}.\vec{c}+\vec{b}.\vec{c}$

3) $(\lambda\vec{a}).\vec{b} = \lambda.(\vec{a}.\vec{b})$

4) $\vec{a}.\vec{a} = \vec{a}^2 = |\vec{a}|^2$

5) $\vec{a}.\vec{b} = 0 \iff \vec{a}\perp\vec{b}$

6) $\cos\sphericalangle(\vec{a},\vec{b})_e = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|}$

- Координатно представяне спрямо ОКС $K = Oxyz$

$\vec{a}(a_1, a_2, a_3)$

$\vec{b}(b_1, b_2, b_3)$

$\vec{a}.\vec{b} = a_1.b_1 + a_2.b_2 + a_3.b_3$

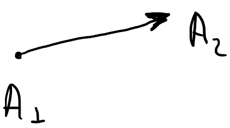
$|\vec{a}|^2 = \vec{a}^2 = a_1.a_1 + a_2.a_2 + a_3.a_3$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

- Разстояние между точки:

$A_1(x_1, y_1, z_1)$

$A_2(x_2, y_2, z_2)$



$\overrightarrow{A_1A_2}(x_2-x_1, y_2-y_1, z_2-z_1)$

$|\overrightarrow{A_1A_2}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

① Дадени са в-рите $\vec{a}, \vec{b}$ и $\vec{c}$,

$|\vec{a}| = 1, |\vec{b}| = 2, |\vec{c}| = \sqrt{2}$

$\sphericalangle(\vec{a},\vec{b})_e = \frac{\pi}{2}, \sphericalangle(\vec{b},\vec{c})_e = \frac{\pi}{2}, \sphericalangle(\vec{a},\vec{c})_e = \frac{\pi}{4}$

$\vec{p} = \vec{a}+\vec{b}-\vec{c}$

$\vec{q} = 2\vec{a}-\vec{b}+\vec{c}$

$\vec{r} = \vec{a}+\lambda.\vec{b}-\vec{c}$

Да се определят:

а) $|\vec{p}| = ?$, $|\vec{q}| = ?$

б) $\vec{p}.\vec{q} = ?$

в) $\cos\sphericalangle(\vec{p},\vec{q})_e = ?$

$\vec{a}^2 = 1$

$\vec{b}^2 = 4$

$\vec{c}^2 = (\sqrt{2})^2 = 2$

$\Rightarrow \vec{a} \ldots \frac{\pi}{2} = 0$

$r = a + \lambda . b - c$

б) $\vec{p} . \vec{q} = ?$

в) $\cos \sphericalangle(\vec{p}, \vec{q})_e = ?$

г) $\lambda = ? : \vec{p} \perp \vec{r}$

д) $\lambda = ? : |\vec{r}| = \sqrt{5}$

$\vec{c}^2 = (\sqrt{2})^2 = 2$

$\vec{a} . \vec{b} = 1 . 2 . \cos\frac{\pi}{2} = 0$

$\vec{b} . \vec{c} = 0$

$\vec{a} . \vec{c} = 1 . \sqrt{2} . \cos\frac{\pi}{4} = 1$

Реш: а) $|\vec{p}|^2 = \vec{p}^2 = (\vec{a} + \vec{b} - \vec{c})^2 = \vec{a}^2 + \vec{b}^2 + \vec{c}^2 + 2\vec{a}.\vec{b} - 2\vec{a}.\vec{c} - 2.\vec{b}.\vec{c}$

$= 1 + 4 + 2 + 2.0 - 2.1 - 2.0 = 5$

$|\vec{p}|^2 = 5 \Rightarrow |\vec{p}| = \sqrt{5}$

$|\vec{q}|^2 = \vec{q}^2 = (2\vec{a} - \vec{b} + \vec{c})^2 = 4.\vec{a}^2 + \vec{b}^2 + \vec{c}^2 - 4.\vec{a}.\vec{b} + 4.\vec{a}.\vec{c} - 2.\vec{b}.\vec{c}$

$= 4.1 + 4 + 2 - 4.0 + 4.1 - 2.0 = 14$

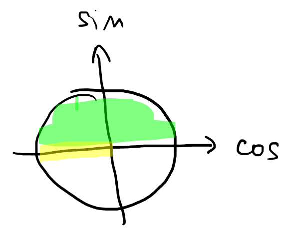
$|\vec{q}|^2 = 14 \Rightarrow |\vec{q}| = \sqrt{14}$

б) $\vec{p}.\vec{q} = (\vec{a} + \vec{b} - \vec{c}).(2\vec{a} - \vec{b} + \vec{c}) = 2\vec{a}^2 - \vec{a}.\vec{b} + \vec{a}.\vec{c} + 2\vec{b}.\vec{a} - \vec{b}^2 + \vec{b}.\vec{c} -$

$- 2\vec{c}.\vec{a} + \vec{c}.\vec{b} - \vec{c}^2 = 2.1 - 0 + 1 + 2.0 - 4 + 0 - 2.1 + 0 - 2$

$\vec{p}.\vec{q} = -5$

в) $\cos \sphericalangle(\vec{p}, \vec{q})_e = \dfrac{\vec{p}.\vec{q}}{|\vec{p}|.|\vec{q}|} = \dfrac{-5}{\sqrt{5}.\sqrt{14}} = -\dfrac{\sqrt{5}}{\sqrt{14}}$



г) $\lambda = ? : \vec{p} \perp \vec{r} \Leftrightarrow \vec{p} . \vec{r} = 0$

$(\vec{a} + \vec{b} - \vec{c}).(\vec{a} + \lambda.\vec{b} - \vec{c}) = 0$

$\vec{a}^2 + \lambda\vec{a}.\vec{b} - \vec{a}.\vec{c} + \vec{b}.\vec{a} + \lambda.\vec{b}^2 - \vec{b}.\vec{c} - \vec{c}.\vec{a} - \lambda.\vec{b}.\vec{c} + \vec{c}^2 = 0$

$1 - 1 + 4\lambda - 1 + 2 = 0$

$4\lambda + 1 = 0 \Rightarrow \lambda = -\dfrac{1}{4}$

д) $\lambda = ? : |\vec{r}| = \sqrt{5}$

$|\vec{r}|^2 = \vec{r}^2 = (\vec{a} + \lambda.\vec{b} - \vec{c})^2 = (\sqrt{5})^2$

$\vec{a}^2 + \lambda^2.\vec{b}^2 + \vec{c}^2 + 2\lambda\vec{a}.\vec{b} - 2.\vec{a}.\vec{c} - 2\lambda\vec{b}.\vec{c} = 5$

$1 + 4\lambda^2 + 2 - 2 = 5$

$$4\lambda^2 = 4$$
$$\lambda = \pm 1$$

② Да се докаже, че:
Ако един в-р в тримерно пространство ($\mathbb{R}^3$) е едновременно $\perp$ на три ЛНЗ вектора, то той е нулев

Реш: Нека $\vec{a}, \vec{b}, \vec{c}$ – ЛНЗ. $\vec{p} = \alpha.\vec{a} + \beta.\vec{b} + \gamma.\vec{c}$

$\vec{p} \perp \vec{a} \Rightarrow \vec{p}.\vec{a} = 0$

$\vec{p} \perp \vec{b} \Rightarrow \vec{p}.\vec{b} = 0$

$\vec{p} \perp \vec{c} \Rightarrow \vec{p}.\vec{c} = 0$

$|\vec{p}|^2 = \vec{p}^{\,2} = (\alpha.\vec{a} + \beta.\vec{b} + \gamma.\vec{c}).\vec{p}$

$= \alpha.\underbrace{\vec{a}.\vec{p}}_{0} + \beta\underbrace{\vec{b}.\vec{p}}_{0} + \gamma.\underbrace{\vec{c}.\vec{p}}_{0} = 0$

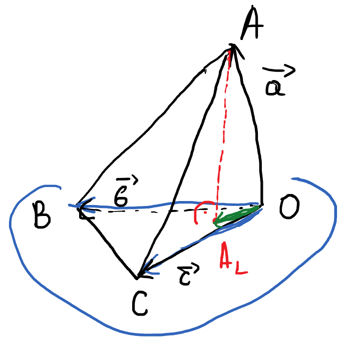
$|\vec{p}| = 0 \Rightarrow \vec{p} = \vec{0}$.

③ Дадени са ЛНЗ вектори $\vec{a}, \vec{b}$ и $\vec{c}$:
$|\vec{a}| = 2, |\vec{b}| = 1, |\vec{c}| = 3$; $\sphericalangle(\vec{a}, \vec{b})_e = \sphericalangle(\vec{a}, \vec{c})_e = \sphericalangle(\vec{b}, \vec{c})_e = \frac{\pi}{3}$

Тетраедър $OABC$: $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$

а) AA_1 - височина на тетраедъра. Да се изрази $\overrightarrow{OA_1}$ чрез $\vec{a}, \vec{b}$ и $\vec{c}$.

б) Ако $OD \perp BC$, т. $D \in BC$, да се изрази $\overrightarrow{AD}$ чрез $\vec{a}, \vec{b}$ и $\vec{c}$.



Реш: а) $\overrightarrow{AA_1} \perp \{O, B, C\}$

$\overrightarrow{AA_1} \perp \vec{b} \Rightarrow \overrightarrow{AA_1}.\vec{b} = 0$

$\overrightarrow{AA_1} \perp \vec{c} \Rightarrow \overrightarrow{AA_1}.\vec{c} = 0$.

$\overrightarrow{AA_1} = \overrightarrow{OA_1} - \overrightarrow{OA}$; $\overrightarrow{OA_1} = \alpha.\vec{b} + \beta.\vec{c}$

$\overrightarrow{AA_1} = \alpha.\vec{b} + \beta.\vec{c} - \vec{a}$

$$\begin{cases} 0 = \overrightarrow{AA_1}.\vec{b} = (\alpha.\vec{b} + \beta.\vec{c} - \vec{a}).\vec{b} = \alpha.\vec{b}^{\,2} + \beta.\vec{c}.\vec{b} - \vec{a}\vec{b} \\ 0 = \overrightarrow{AA_1}.\vec{c} = (\alpha.\vec{b} + \beta.\vec{c} - \vec{a}).\vec{c} = \alpha.\vec{b}.\vec{c} + \beta.\vec{c}^{\,2} - \vec{a}.\vec{c} \end{cases}$$

$$0 = \vec{AA_1} \cdot \vec{c} = (\alpha \cdot \vec{b} + \beta \cdot \vec{c} - \vec{a}) \cdot \vec{c} = \alpha \cdot \vec{b} \cdot \vec{c} + \beta \cdot \vec{c} \ \ \ \vec{a} \cdot \vec{c}$$

$$\left| \begin{aligned} &\alpha + \frac{3}{2}\beta - 1 = 0 \\ &\frac{3}{2}\alpha + 9\beta - 3 = 0 \end{aligned} \right. \Rightarrow \quad \begin{aligned} \alpha &= \frac{2}{3} \\ \beta &= \frac{2}{9} \end{aligned}$$

$$\left| \begin{aligned} &\vec{a}^2 = 4 \\ &\vec{b}^2 = 1 \\ &\vec{c}^2 = 9 \\ &\vec{a} \cdot \vec{b} = 2 \cdot 1 \cdot \frac{1}{2} = 1 \\ &\vec{a} \cdot \vec{c} = 2 \cdot 3 \cdot \frac{1}{2} = 3 \\ &\vec{b} \cdot \vec{c} = 1 \cdot 3 \cdot \frac{1}{2} = \frac{3}{2} \end{aligned} \right.$$

$$\vec{OA_1} = \frac{2}{3}\vec{b} + \frac{2}{9}\vec{c}$$

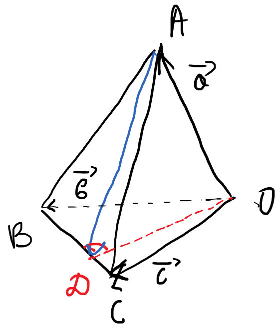
б)

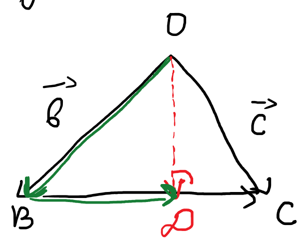
$$\vec{OD} \perp \vec{BC} \Rightarrow \vec{OD} \cdot \vec{BC} = 0$$

$$\vec{AD} = \vec{OD} - \vec{OA} = \vec{OD} - \vec{a}$$

$$\vec{OD} = \vec{OB} + \vec{BD} = \vec{OB} + \lambda \cdot \vec{BC}\ ;$$

$$\vec{BD} = \lambda \cdot \vec{BC}$$





$$\vec{BC} = \vec{OC} - \vec{OB}$$

$$\vec{BC} = \vec{c} - \vec{b}$$

$$\vec{OD} = \vec{b} + \lambda(\vec{c} - \vec{b}) = (1-\lambda)\vec{b} + \lambda \cdot \vec{c}$$

$$0 = \vec{OD} \cdot \vec{BC} = [(1-\lambda)\vec{b} + \lambda \cdot \vec{c}] \cdot (\vec{c} - \vec{b})$$

$$(1-\lambda)\vec{b} \cdot \vec{c} + \lambda \cdot \vec{c}^2 - (1-\lambda)\vec{b}^2 - \lambda \cdot \vec{c} \cdot \vec{b} = 0$$

$$(1-\lambda) \cdot \frac{3}{2} + 9\lambda - (1-\lambda)1 - \lambda \cdot \frac{3}{2} = 0$$

$$\frac{3}{2} - \frac{3}{2}\lambda + 9\lambda - 1 + \lambda - \frac{3}{2}\lambda = 0$$

$$\frac{1}{2} + 7\lambda = 0 \Rightarrow \lambda = -\frac{1}{14}$$

$$\vec{AD} = \vec{OD} - \vec{OA} = \left(1 - \left(-\frac{1}{14}\right)\right)\vec{b} - \frac{1}{14}\vec{c} - \vec{a} = -\vec{a} + \frac{15}{14}\vec{b} - \frac{1}{14}\vec{c}$$