4) Спрямо ОКС $K=Oxyz$ са дадени точките

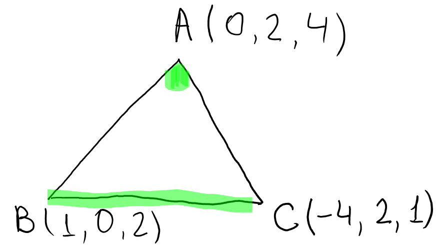
$A(0,2,4)$; $B(1,0,2)$; $C(-4,2,1)$ - върхове на $\triangle ABC$.

Да се намерят:

а) $P_{\triangle ABC}=?$

б) Да се определи вида на $\triangle ABC$ според ъглите му;

в) Координатите на петата H на височината AH на $\triangle ABC$.



Реш:

$\overrightarrow{AB}(1-0,\ 0-2,\ 2-4)$

$\overrightarrow{AB}(1,-2,-2) \Rightarrow$

$|\overrightarrow{AB}| = \sqrt{1^2+(-2)^2+(-2)^2} = 3$

$\overrightarrow{BC}(-5,2,-1) \Rightarrow |\overrightarrow{BC}| = \sqrt{(-5)^2+2^2+(-1)^2} = \sqrt{30}$

$\rightarrow \overrightarrow{CA}(4,0,3) \Rightarrow |\overrightarrow{CA}| = \sqrt{4^2+0^2+3^2} = 5$

$P_{\triangle ABC} = 3+\sqrt{30}+5 = 8+\sqrt{30}$

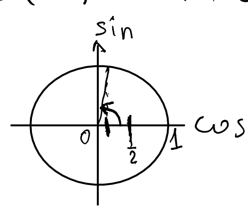
б) $\overrightarrow{AB}.\overrightarrow{AC} = |\overrightarrow{AB}|.|\overrightarrow{AC}|.\cos\sphericalangle(\overrightarrow{AB},\overrightarrow{AC})_e \rightarrow$ дефиниция за скаларно произведение

$\cos\sphericalangle(BAC) = \cos\sphericalangle(\overrightarrow{AB},\overrightarrow{AC})_e = \dfrac{\overrightarrow{AB}.\overrightarrow{AC}}{|\overrightarrow{AB}|.|\overrightarrow{AC}|}$

$\overrightarrow{AB}(1,-2,-2)$; $\overrightarrow{AC}(-4,0,-3)$

$\overrightarrow{AB}.\overrightarrow{AC} = 1.(-4)+(-2).0+(-2).(-3) = -4+6 = 2$

$\cos\sphericalangle(BAC) = \dfrac{2}{3.5} = \dfrac{2}{15}$



$\Rightarrow \triangle ABC$ е остроъгълен

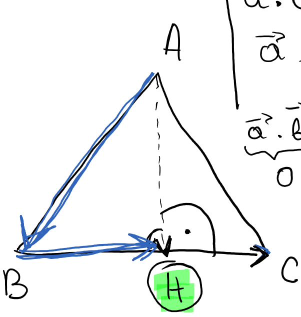
$\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$

$\underbrace{\vec{a}.\vec{b}}_{0} = \overset{\neq 0}{|\vec{a}|}.\overset{\neq 0}{|\vec{b}|}.\underbrace{\cos\sphericalangle(\vec{a},\vec{b})_e}_{0}$

$\Downarrow$

$\sphericalangle = 90^\circ$

в) $\overrightarrow{BC} \perp \overrightarrow{AH} \Rightarrow \overrightarrow{BC}.\overrightarrow{AH} = 0$



$\overrightarrow{BC}(-5,2,-1)$

Нека $\overrightarrow{BH} = \lambda.\overrightarrow{BC}$.

Нека $\overrightarrow{BH} = \alpha . \overrightarrow{BC}$.

$\sphericalangle = 90^\circ$

$$\overrightarrow{AH} = \overrightarrow{AB} + \overrightarrow{BH} = \overrightarrow{AB} + \alpha . \overrightarrow{BC}$$

$$\overrightarrow{BC} . \overrightarrow{AH} = 0 = \overrightarrow{BC} . (\overrightarrow{AB} + \alpha . \overrightarrow{BC}) = \overrightarrow{BC} . \overrightarrow{AB} + \alpha . \overrightarrow{BC}^2$$

$\overrightarrow{AB}(1, -2, -2)$ $\qquad$ $\overrightarrow{BC} . \overrightarrow{AB} = 1.(-5) + (-2).2 + (-2).(-1)$

$\overrightarrow{BC}(-5, 2, -1)$ $\qquad$ $= -5 - 4 + 2 = \cancel{-11}\ -7$

$$\overrightarrow{BC}^2 = 30$$

$$0 = -7 + \alpha . 30 \Rightarrow \alpha = \frac{7}{30}$$

$\overrightarrow{BH} = \frac{7}{30} . \overrightarrow{BC}$ $\qquad$ $H(x_H, y_H, z_H)$

$$\left| \begin{array}{l} x_H - 1 = \frac{7}{30}(-5) \\ y_H - 0 = \frac{7}{30} . 2 \\ z_H - 2 = \frac{7}{30}(-1) \end{array} \right. \Rightarrow \left| \begin{array}{l} x_H = -\frac{1}{6} \\ y_H = \frac{7}{15} \\ z_H = \frac{53}{30} \end{array} \right. \Rightarrow H\left(-\frac{1}{6}, \frac{7}{15}, \frac{53}{30}\right)$$

5) ОКС $K = Oxyz$

$A(0, 0, -2)$; $B(4, 0, -4)$, $C(2, 0, 0)$, $D(5, 3, -3)$

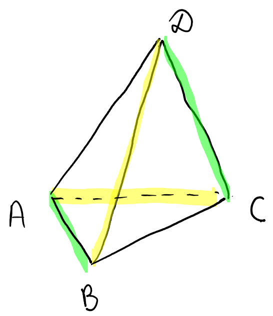
$ABCD$ - тетраедър

Да се пресметнат:

а) $\sphericalangle(AB, CD)$; $\sphericalangle(AC, BD)$

б) Коорд. на точки M и N съответно върху AB и CD такива, че $MN \perp AB$, $MN \perp CD$

Дом: в) Координатите на петата H на височината DH на $ABCD$.



Реш: а) $\sphericalangle(AB, CD) = ?$

$\overrightarrow{AB}(4, 0, -2) \Rightarrow |\overrightarrow{AB}| = \sqrt{16 + 4} = \sqrt{20}$

$\overrightarrow{CD}(3, 3, -3) \Rightarrow |\overrightarrow{CD}| = \sqrt{9 + 9 + 9} = \sqrt{27}$

$$\overrightarrow{CD}\,(3,3,-3) \Rightarrow |\overrightarrow{CD}| = \sqrt{9+9+9} = \sqrt{27}$$

B

$$\cos\measuredangle(\overrightarrow{AB},\overrightarrow{CD})_e = \frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{AB}|\cdot|\overrightarrow{CD}|} = \frac{4.3+0.3+(-2).(-3)}{\sqrt{20}\cdot\sqrt{27}} = \frac{\cancel{18}^{3}}{\cancel{2}\sqrt{5}\cdot\cancel{3}\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{5}}$$

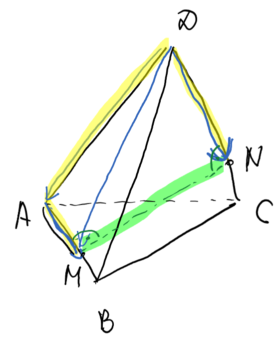
$$\overrightarrow{AC}\,(2,0,2) \Rightarrow |\overrightarrow{AC}| = \sqrt{4+0+4} = \sqrt{8}$$

$$\overrightarrow{BD}\,(1,3,1) \Rightarrow |\overrightarrow{BD}| = \sqrt{1+9+1} = \sqrt{11}$$

$$\overrightarrow{AC}\cdot\overrightarrow{BD} = 2.1+0.3+2.1 = 4$$

$$\cos\measuredangle(\overrightarrow{AC},\overrightarrow{BD})_e = \frac{\overrightarrow{AC}\cdot\overrightarrow{BD}}{|\overrightarrow{AC}|\cdot|\overrightarrow{BD}|} = \frac{4}{\sqrt{8}\cdot\sqrt{11}} = \frac{2}{\sqrt{22}}$$

б) $$\left.\begin{array}{l}\overrightarrow{MN}\perp\overrightarrow{AB} \Rightarrow \\ \overrightarrow{MN}\perp\overrightarrow{CD} \Rightarrow\end{array}\right| \begin{array}{l}\overrightarrow{MN}\cdot\overrightarrow{AB}=0\\ \overrightarrow{MN}\cdot\overrightarrow{CD}=0\end{array}$$



$$\overrightarrow{MN} = \overrightarrow{DN} - \overrightarrow{DM}$$

Нека $\overrightarrow{DN} = \lambda.\overrightarrow{DC}$; $\overrightarrow{AM} = \mu.\overrightarrow{AB}$

$$\overrightarrow{MN} = \lambda.\overrightarrow{DC} - (\overrightarrow{DA}+\overrightarrow{AM}) = \lambda.\overrightarrow{DC} - \overrightarrow{DA} - \mu.\overrightarrow{AB}$$

$$\left|\begin{array}{l}0 = \overrightarrow{MN}\cdot\overrightarrow{AB} = (\lambda.\overrightarrow{DC}-\overrightarrow{DA}-\mu.\overrightarrow{AB}).\overrightarrow{AB} = \lambda.\overrightarrow{DC}\cdot\overrightarrow{AB} - \overrightarrow{DA}\cdot\overrightarrow{AB} - \mu.\overrightarrow{AB}^2\\ 0 = \overrightarrow{MN}\cdot\overrightarrow{CD} = (\lambda.\overrightarrow{DC}-\overrightarrow{DA}-\mu.\overrightarrow{AB}).\overrightarrow{CD} = \lambda.\overrightarrow{DC}\cdot\overrightarrow{CD} - \overrightarrow{DA}\cdot\overrightarrow{CD} - \mu.\overrightarrow{AB}\cdot\overrightarrow{CD}\end{array}\right.$$

$\overrightarrow{AB}\,(4,0,-2)$

$\overrightarrow{DC}\,(-3,-3,3)$

$\overrightarrow{CD}\,(3,3,-3)$

$\overrightarrow{DA}\,(-5,-3,1)$

$\overrightarrow{DC}\cdot\overrightarrow{AB} = 4.(-3)+0.(-3)+(-2).3 = -18$

$\overrightarrow{DA}\cdot\overrightarrow{AB} = 4.(-5)+0.(-3)+(-2).1 = -22$

$\overrightarrow{AB}^2 = 20$

$\overrightarrow{DC}\cdot\overrightarrow{CD} = -27$

$\overrightarrow{DA}\cdot\overrightarrow{CD} = -27 = 3.(-5)+3.(-3)+(-3).1$

$\overrightarrow{AB}\cdot\overrightarrow{CD} = 18$

$$\left|\begin{array}{l}0 = \lambda.(-18)-(-22)-\mu.20\\ 0 = \lambda.(-27)-(-27)-\mu.18\end{array}\right. \Rightarrow \begin{array}{l}\lambda = \frac{2}{3}\\ \mu = \frac{1}{2}\end{array}$$

$$\overrightarrow{DN} = \lambda.\overrightarrow{DC} = \frac{2}{3}\cdot\overrightarrow{DC}$$

$$\left|\ x_N - 5 = \frac{2}{3}.(-3)\right. \Rightarrow \left|\ x_N = 3\right. \quad N(3, 1, -1)$$

$$\begin{cases} x_N - 5 = \frac{2}{3}\cdot(-3) \\ y_N - 3 = \frac{2}{3}\cdot(-3) \\ z_N - (-3) = \frac{2}{3}\cdot 3 \end{cases} \Rightarrow \begin{cases} x_N = 3 \\ y_N = 1 \\ z_N = -1 \end{cases} \Rightarrow N(3, 1, -1)$$

$$\vec{AM} = \mu \cdot \vec{AB} = \frac{1}{2}\vec{AB}$$

$$\begin{cases} x_M - 0 = \frac{1}{2}\cdot 4 \\ y_M - 0 = \frac{1}{2}\cdot 0 \\ z_M - (-2) = \frac{1}{2}\cdot(-2) \end{cases} \Rightarrow \begin{cases} x_M = 2 \\ y_M = 0 \\ z_M = -3 \end{cases} \Rightarrow M(2, 0, -3)$$