⊕ $X \sim Exp(\lambda)$

$$L(\vec{x}, \lambda) = \prod_{j=1}^{n} \lambda e^{-\lambda x_j} = \lambda^n e^{-\lambda \sum_{j=1}^{n} x_j}$$

$$\frac{\partial}{\partial \lambda} \ln(L) = \frac{n}{\lambda} - \sum_{j=1}^{n} x_j \Rightarrow \hat{\lambda} = \frac{n}{\sum_{j=1}^{n} x_j}$$

$$\left.\begin{aligned} &\mathbb{E}[X] = \frac{1}{\lambda} \\ &\bar{X}_n^{(1)} = \frac{1}{n}\sum_{j=1}^{n} x_j \end{aligned}\right\} \Rightarrow \hat{\lambda} = \frac{n}{\sum_{j=1}^{n} x_j}$$

$$\hat{\lambda} = \frac{n}{\sum_{j=1}^{n} x_j} \xrightarrow[n\to\infty]{п.с.} \frac{1}{\mathbb{E}[X_1]} = \lambda$$ – състоятелна оценка

$$Y,\ f_Y = \beta \cdot y^{\beta - 1} \cdot 1\{y \in [0,1]\} \cdot 1\{\beta > 0\}$$

$$L(\vec{y}, \beta) = \prod_{j=1}^{n} \beta \cdot y_j^{\beta-1} = \beta^n \prod_{j=1}^{n} y_j^{\beta-1}$$

$$\cancel{\frac{\partial}{\partial \beta} \ln(L) = \frac{n}{\beta} + (\beta - 1)\sum_{j=1}^{n} \ln(y_j) = 0 \Leftrightarrow \frac{n}{\beta} = (1-\beta)\sum_{j=1}^{n} \ln(y_j)}$$

$$\cancel{q=1\ \ \beta(1-\beta) = \frac{n}{\sum_{j=1}^{n} \ln(y_j)}}$$

$$\frac{\partial}{\partial \beta} \ln(L) = \frac{n}{\beta} + \sum_{j=1}^{n} \ln(y_j) = 0 \Rightarrow \frac{n}{\beta} = -\sum_{j=1}^{n} \ln(y_j) \Rightarrow \hat{\beta} = \frac{-n}{\sum_{j=1}^{n} \ln(y_j)}$$

$$\hat{\beta} = \frac{-n}{\sum_{j=1}^{n} \ln(y_j)} \xrightarrow[n\to\infty]{п.с.} \frac{-1}{\mathbb{E}[\ln(Y_1)]}$$ (УЗГЧ) → състоятелна

(128) $X \sim N(\mu, \sigma^2)$. м.п.о. за:

1. μ като σ е известно / неизвестно

$$\hat{\mu} = \frac{1}{n}\sum_{j=1}^{n} X_j$$

2. σ^2 като μ е известно

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n}(X_j - \mu)^2$$

3. σ^2 като μ е неизвестно

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n}(X_j - \hat{\mu})^2$$

Проверка за неизместеност:

1. $\mathbb{E}[\hat{\mu}] = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[X_j] = \mathbb{E}[X_1] = \mu$ ✓

2. $\mathbb{E}[\hat{\sigma}^2] = \frac{1}{n}\sum_{j=1}^{n}\mathbb{D}[X_j] = \mathbb{D}[X_1] = \sigma^2$ ✓

3. $\mathbb{E}[\hat{\sigma}^2] = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[(X_j - \overline{X}_n^{(1)})^2] = \mathbb{E}[\overline{X}_n^{(2)} - (\overline{X}_n^{(1)})^2] = \overset{\mathbb{E}}{\left[\frac{1}{n}\sum_{j=1}^{n}X_j^2 - \frac{1}{n^2}\left(\sum_{j=1}^{n}X_j\right)^2\right]} =$

$= \mathbb{E}[X_1^2] - \frac{n}{n^2}\mathbb{E}[X_1^2] - \frac{2(n-1)n(\mathbb{E}[X_1])^2}{2n^2} =$

$= \frac{n-1}{n}\mathbb{E}[X_1^2] - \frac{n-1}{n}(\mathbb{E}[X_1])^2 = \frac{n-1}{n}\sigma^2 \neq \sigma^2$ - асимптотично неизместима

Проверка за състоятелност:

1. $\hat{\mu} = \frac{1}{n}\sum_{j=1}^{n}X_j \xrightarrow[n\to\infty]{п.с} \mathbb{E}[X_1] = \mu$ (УЗГЧ) ✓

2. $\hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n}(X_j - \mu)^2 \xrightarrow[n\to\infty]{п.с} \mathbb{E}[(X_1 - \mu)^2] = \sigma^2$ (УЗГЧ) ✓

3. $\hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n}(X_j - \hat{\mu})^2 \xrightarrow[n\to\infty]{п.с} \mathbb{E}[(X_1 - \hat{\mu})^2] = \sigma^2$ (УЗГЧ) ✓

129 Нека $X \sim Ber(\theta)$. Намерете ефективна оценка за θ

$$X = \begin{cases} 1, & \theta \\ 0, & 1-\theta \end{cases}$$

$$\mathbb{D}[X] = \theta(1-\theta)$$

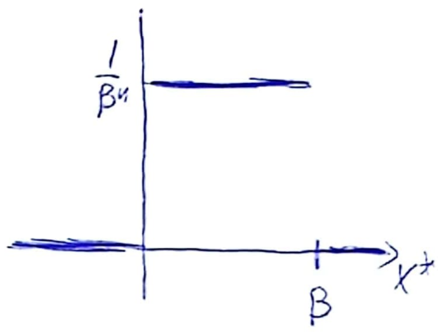
$$\mathbb{D}[X(\hat{\theta})] = \min_{\theta \in \Theta} \mathbb{D}[X(\theta)]$$

$\min \mathbb{D}[X] = 0$, при $\theta \in \{0, 1\}$

(131) $X \sim Unif(0,\beta)$

$f_X(x) = \frac{1}{\beta} \cdot 1\{x \in (0,\beta)\}$ Нека $X^* = \max_{j \in [1,n]} X_j$

$L(\vec{X},\beta) = \prod_{j=1}^{n} \frac{1}{\beta} \cdot 1\{x \in (0,\beta)\} = \frac{1}{\beta^n} \cdot 1\{x^* \in (0,\beta)\}$



$L(\vec{X},\hat{\beta}) = \sup_{\beta \in \Theta} L(\vec{X},\beta)$

$\Rightarrow \max L = \begin{cases} \frac{1}{\beta^n}, & x^* \le \beta \\ 0, & x^* \ge \beta \end{cases} \Rightarrow \sup L = \frac{1}{\beta^n}, \beta \ge x^* \Rightarrow$ избираме $\hat{\beta} = X^*$

Нека $Y = X^*$, $\mathbb{E}[Y] = ?$

$\mathbb{P}(Y \le x) = \mathbb{P}(\max X_j \le x) = \mathbb{P}(X_1 \le x)^n = \left[\frac{x}{\beta}\right]^n$

$f_Y(x) = \frac{d}{dx} \frac{x^n}{\beta^n} = \frac{n x^{n-1}}{\beta^n}$

$\Rightarrow \mathbb{E}[Y] = \int_0^\beta x \frac{n x^{n-1}}{\beta^n} dx = \frac{n}{\beta^n} \int_0^\beta x^n dx = \frac{n}{\beta^n} \frac{\beta^{n+1}}{n+1} = \frac{n}{n+1}\beta$

$\Rightarrow \mathbb{E}[\hat{\beta}] = \frac{n}{n+1}\beta \neq \beta \rightarrow$ изместена, но асимптотично неизместена

$\hat{\beta} = \max\{X_j\} \xrightarrow[n\to\infty]{п.с} \beta$ (УЗГЧ)

$\mathbb{E}[2\bar{X}] = \frac{2}{n}\sum_{j=1}^{n} \mathbb{E}[X_j] = \frac{2}{n} \cdot n\, \mathbb{E}[X_1] = 2 \cdot \frac{\beta}{2} = \beta$ – неизместена

$\hat{\beta} = 2\bar{X} = 2 \frac{\sum_{j=1}^{n} X_j}{n} \xrightarrow[n\to\infty]{п.с} 2\mathbb{E}[X_1] = \beta$ – състоятелна

133 $X \sim Ge(p)$ м.м.о. за p

$$\mathbb{E}[X] = \frac{q}{p}$$

$$M_1(X) = \mathbb{E}[X] = \frac{q}{p}$$

$$\overline{X_n}^{(1)} = \frac{1}{n}\sum_{j=1}^{n} X_j$$

$$\overline{X_n}^{(1)} = M_1(X) \Rightarrow \frac{q}{p} = \overline{X_n}^{(1)} \Rightarrow p = \frac{q}{\overline{X_n}^{(1)}}$$

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134 $X \sim Unif(\theta_1, \theta_2)$

$$M_1(X) = \mathbb{E}[X] = \boxed{\frac{\theta_1+\theta_2}{2}}$$

$$M_2(X) = \mathbb{E}[X^2] = D[X] + (\mathbb{E}[X])^2 = \frac{(\theta_2-\theta_1)^2}{12} + \frac{(\theta_1+\theta_2)^2}{4} = \frac{\theta_2^2 - 2\theta_1\theta_2 + \theta_1^2 + 3\theta_1^2 + 6\theta_1\theta_2 + 3\theta_2^2}{12}$$

$$= \frac{4(\theta_2^2 + \theta_1\theta_2 + \theta_1^2)}{12} = \boxed{\frac{\theta_1^2+\theta_1\theta_2+\theta_2^2}{3}}$$

$$\begin{cases} M_1(x) = \overline{X_n}^{(1)} \\ M_2(x) = \overline{X_n}^{(2)} \end{cases} \Leftrightarrow \begin{cases} \frac{\theta_1+\theta_2}{2} = \frac{1}{n}\sum_{j=1}^{n} X_j \\ \frac{\theta_1^2+\theta_1\theta_2+\theta_2^2}{3} = \frac{1}{n}\sum_{j=1}^{n} X_j^2 \end{cases} \Leftrightarrow \begin{cases} (\theta_1+\theta_2)^2 = 4(\overline{X_n}^{(1)})^2 \\ (\theta_1+\theta_2)^2 - \theta_1\theta_2 = 3\overline{X_n}^{(2)} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \theta_1\theta_2 = 4(\overline{X_n}^{(1)})^2 - 3\overline{X_n}^{(2)} \\ \theta_1+\theta_2 = 2\overline{X_n}^{(1)} \end{cases} \Leftrightarrow \begin{cases} \theta_1 = 2\overline{X_n}^{(1)} - \theta_2 \\ \theta_2 = \frac{4(\overline{X_n}^{(1)})^2 - 3\overline{X_n}^{(2)}}{2\overline{X_n}^{(1)} - \theta_2} \end{cases} \ldots\ldots$$

7) $X,\ f_X(x;\theta) = C(\theta)e^{-\theta x^2},\ x>0,\ \theta>0$

м.п.о. за θ от n набл., където $C(\theta) = k\theta^{\frac{1}{2}}$

$$L(\vec{X},\theta) = \prod_{j=1}^{n} k\theta^{\frac{1}{2}} \cdot e^{-\theta \sum_{j=1}^{n} x_j^2} = k^n \theta^{\frac{n}{2}} \cdot e^{-\theta \sum_{j=1}^{n} x_j^2}$$

$$\frac{\partial}{\partial\theta} \ln(L) = \frac{\partial}{\partial\theta}\left[n\ln(k) + \frac{n}{2}\ln(\theta) - \theta \sum_{j=1}^{n} x_j^2\right] = \frac{n}{2\theta} - \sum_{j=1}^{n} x_j^2 = 0 \Rightarrow \hat{\theta} = \frac{n}{2\sum_{j=1}^{n} x_j^2}$$

Проверка за състоятелност:

$$\hat{\theta} = \frac{n}{2\sum_{j=1}^{n} X_j^2} \xrightarrow[n\to\infty]{п.с.} \frac{1}{2\mathbb{E}[X_1^2]}$$

$$\mathbb{E}[X_1^2] = \int_0^\infty k\theta^{\frac{1}{2}} x^2 e^{-\theta x^2} dx = k\theta^{\frac{1}{2}} \int_0^\infty x^2 e^{-\theta x^2} dx \quad \cancel{\frac{y=\theta x}{dx}}$$

$$\cancel{= k\theta^{\frac{1}{2}} \int_0^\infty \frac{x^2}{2\theta x} de^{-\theta x^2} = -\frac{k}{2}\theta^{-\frac{1}{2}} \int_0^\infty x^2 de^{-\theta x^2} = -\frac{k}{2}\theta^{-\frac{1}{2}}\left[x^2 e^{-\theta x^2}\Big|_0^\infty - \int_0^\infty e^{-\theta x^2} dx^2\right]}$$

$$\cancel{= \frac{k}{2}\theta^{-\frac{1}{2}} \frac{1}{\theta} \int_0^\infty e^{-\theta x^2} d\theta x^2 = \frac{k}{2}\theta^{-\frac{3}{2}}\left[-e^{-\theta x^2}\Big|_0^\infty\right] = \frac{k}{2}\theta^{-\frac{3}{2}}}$$

$$= k\theta^{\frac{1}{2}} \int_0^\infty x^2 e^{-\theta x^2} dx = \frac{k\theta^{\frac{1}{2}}}{\theta^{\frac{3}{2}}} \int_0^\infty \theta x^2 e^{-\theta x^2} d\sqrt{\theta}x \overset{y^2=\theta x^2}{=} \frac{k}{\theta} \underbrace{\int_0^\infty y^2 e^{-y^2} dy}_{\frac{\sqrt{\pi}}{2}} =$$

$$= \frac{k\sqrt{\pi}}{2} \cdot \frac{1}{\theta} \to \text{състоятелност}$$

$X,\ f_X(x,\theta) = C_\theta x^2,\ x \in (-\theta, 2\theta)$

а) $\int_{-\theta}^{2\theta} C_\theta . x^2 dx = C_\theta . \left[\frac{x^3}{3}\right]_{-\theta}^{2\theta} = C_\theta \left[\frac{8\theta^3}{3} + \frac{\theta^3}{3}\right] = C_\theta . 3\theta^3 = 1 \Rightarrow C_\theta = \frac{1}{3\theta^3}$

б) $\vec{X} = (X_1, X_2, \dots, X_n)$ м.м.о за θ

$\overline{X_n}^{(1)} = \frac{1}{n}\sum_{j=1}^{n} X_j$

$M_1(X) = \mathbb{E}[X] = \int_{-\infty}^{\infty} x . \frac{1}{3\theta^3} . x^2 dx = \frac{1}{3\theta^3}\left[\frac{x^4}{4}\right]_{-\theta}^{2\theta} = \frac{1}{3\theta^3}\left[\frac{16\theta^4 - \theta^4}{4}\right] = \frac{15}{12}\theta$

$\Rightarrow \frac{15}{12}\theta = \overline{X_n}^{(1)} \Rightarrow \hat{\theta} = \frac{12}{15}\overline{X_n}^{(1)}$

в) $\hat{\theta} = \frac{12}{15} . \frac{1}{n}\sum_{j=1}^{n} X_j$

$\lim_{n\to\infty} \hat{\theta} = \lim_{n\to\infty} \mathbb{E}\left[\frac{12}{15} . \frac{1}{n}\sum_{j=1}^{n} X_j\right] = \frac{12}{15} . \frac{15}{12}\theta\ \checkmark \to$ неизм.

$\hat{\theta} = \frac{12}{15}\overline{X_n}^{(1)} = \frac{12}{15} . \frac{1}{n}\sum_{j=1}^{n} X_j \xrightarrow[n\to\infty]{п.с.} \frac{12}{15}\mathbb{E}[X_1] = \theta$ (УЗГЧ) $\to$ асстоят.

г) м.м.п за θ

$L(\vec{X}, \theta) = \prod_{j=1}^{n} \frac{x^2}{3\theta^3} = \frac{x^{2n}}{3^n\theta^{3n}}$

$\ln(L) = 2n\ln(x) - n\ln(3) - 3n\ln(\theta)$

$\frac{\partial}{\partial\theta}\ln(L) = \frac{3n}{\theta} = 0 \Rightarrow ?$