Очакване и дисперсия, пораждащи функции

Дефиниция: (математическо очакване): Нека X е дискретна случайна величина. Ако $\sum_j x_j p_j$ е добре дефинирана (крайна), то очакването на X е

$$\mathbb{E}[X] := \sum_j x_j p_j = \sum_j x_j \mathbb{P}(X = x_j)$$

Лема: (свойства на очакването): Нека X, Y са дискретни случайни величини и $\mathbb{E}[X], \mathbb{E}[Y]$ съществуват.

- $X = c \Rightarrow \mathbb{E}[X] = c$
- $X = \mathbb{1}_A \Rightarrow \mathbb{E}[X] = \mathbb{P}(A)$
- $Y = cX \Rightarrow \mathbb{E}[Y] = c\mathbb{E}[X]$
- $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
- $X \perp\!\!\!\perp Y \Rightarrow \mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- $X \geq 0 \Rightarrow \mathbb{E}[X] \geq 0$
- **Доказателство:**
  - $X = c \Rightarrow \mathbb{E}[X] = c$

$$\begin{aligned} X = c \Rightarrow \mathbb{E}[X] &= \sum_j x_j \mathbb{P}(X = x_j) \\ &= \sum_j c\mathbb{P}(X = x_j) \\ &= c \sum_j \mathbb{P}(X = x_j) \\ &= c \end{aligned}$$

  - $X = \mathbb{1}_A \Rightarrow \mathbb{E}[X] = \mathbb{P}(A)$

$$\begin{aligned} X = \mathbb{1}_A \Rightarrow \mathbb{E}[X] &= \sum_j x_j \mathbb{P}(X = x_j) \\ &= 1 \cdot \mathbb{P}(X = 1) + 0 \cdot \mathbb{P}(X = 0) \\ &= \mathbb{P}(X = 1) \\ &= \mathbb{P}(A) \end{aligned}$$

- $Y = cX \Rightarrow \mathbb{E}[Y] = c\mathbb{E}[X]$

$$\begin{aligned} Y = cX \Rightarrow \mathbb{E}[Y] &= \sum_j y_j \mathbb{P}(Y = y_j) \\ &= \sum_j c x_j \mathbb{P}(X = x_j) \\ &= c \sum_j x_j \mathbb{P}(X = x_j) \\ &= c\mathbb{E}[X] \end{aligned}$$

- $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

$$\begin{aligned} \mathbb{E}[X + Y] &= \sum_{i,j} (x_i + y_j) \mathbb{P}(X = x_i, Y = y_j) \\ &= \sum_i \sum_j (x_i + y_j) \mathbb{P}(X = x_i, Y = y_j) \\ &= \sum_i \sum_j x_i \mathbb{P}(X = x_i, Y = y_j) + \sum_i \sum_j y_j \mathbb{P}(X = x_i, Y = y_j) \\ &= \sum_j x_i \sum_j \mathbb{P}(X = x_i, Y = y_j) + \sum_j y_j \sum_i \mathbb{P}(X = x_i, Y = y_j) \\ &= \sum_i x_i \mathbb{P}(X = x_i) + \sum_j y_j \mathbb{P}(Y = y_j) \\ &= \mathbb{E}[X] + \mathbb{E}[Y] \end{aligned}$$

- $X \perp\!\!\!\perp Y \Rightarrow \mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

$$\begin{aligned} X \perp\!\!\!\perp Y \Rightarrow \mathbb{E}[XY] &= \sum_{i,j} x_i y_j \mathbb{P}(X = x_i, Y = y_j) \\ &= \sum_i \sum_j x_i y_j \mathbb{P}(X = x_i) \mathbb{P}(Y = y_j) \\ &= \sum_i x_i \mathbb{P}(X = x_i) \sum_j y_j \mathbb{P}(Y = y_j) \\ &= \mathbb{E}[X]\mathbb{E}[Y] \end{aligned}$$

- $X \geq 0 \Rightarrow \mathbb{E}[X] = \sum_j x_j \mathbb{P}(X = x_j) \geq 0$, понеже $x_j \geq 0$

Дефиниция: (дисперсия): Нека X е дискретна случайна величина. Ако $\sum_j (x_j - \mathbb{E}[X])^2 p_j$ е добре дефинирана (крайна), то дисперсията на X е

$$\mathbb{D}[X] := \sum_j (x_j - \mathbb{E}[X])^2 p_j$$

- минималната квадратична грешка

Дефиниция: (стандартно отклонение): Нека X е дискретна случайна величина и $\mathbb{D}[X] < \infty$, то $\sqrt{\mathbb{D}[X]}$ е стандартното отклонение на X.

Твърдение: $\mathbb{D}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

- **Доказателство:**
  - $$\mathbb{E}[(X - \mathbb{E}[X])^2] \stackrel{def}{=} \sum_j (x_j - \mathbb{E}[X])^2 p_j \stackrel{def}{=} \mathbb{D}[X]$$
  - $$\begin{aligned} \mathbb{E}[(X - \mathbb{E}X)^2] &= \mathbb{E}[X^2 - 2X\mathbb{E}[X] + (\mathbb{E}[X])^2] \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + (\mathbb{E}[X])^2 \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \end{aligned}$$

Лема: (свойства на дисперсията): Нека X, Y са дискретни случайни величини и $\mathbb{D}[X], \mathbb{D}[Y]$ съществуват.

- $\mathbb{D}[X] \geq 0$
- $\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$
- $X = c = \text{const} \Rightarrow \mathbb{D}[X] = 0$
- $\mathbb{D}[cX] = c^2\mathbb{D}[X]$
- $X \perp\!\!\!\perp Y \Rightarrow \mathbb{D}[X + Y] = \mathbb{D}[X] + \mathbb{D}[Y]$
- **Доказателство:**
  - $\mathbb{D}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] \geq 0$, понеже $(X - \mathbb{E}[X])^2 \geq 0$
  - Следва от горното и $\mathbb{D}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
  - $X = c = \text{const} \Rightarrow \mathbb{D}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[(c - c)^2] = \mathbb{E}[0] = 0$
  - $\mathbb{D}[cX] = c^2\mathbb{D}[X]$

$$\begin{aligned} \mathbb{D}[cX] &= \mathbb{E}[(cX - \mathbb{E}[cX])^2] \\ &= \mathbb{E}[(cX - c\mathbb{E}[X])^2] \\ &= \mathbb{E}[c^2(X - \mathbb{E}[X])^2] \\ &= c^2\mathbb{E}[(X - \mathbb{E}[X])^2] \\ &= c^2\mathbb{D}[X] \end{aligned}$$

  - $X \perp\!\!\!\perp Y \Rightarrow \mathbb{D}[X + Y] = \mathbb{D}[X] + \mathbb{D}[Y]$

$$
\begin{aligned}
\mathbb{D}[X+Y] &= \mathbb{E}[(X+Y-\mathbb{E}[X+Y])^2] \\
&= \mathbb{E}[(X+Y-\mathbb{E}[X]-\mathbb{E}[Y])^2] \\
&= \mathbb{E}[(X-\mathbb{E}[X])^2] + \mathbb{E}[(Y-\mathbb{E}[Y])^2] + 2\mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] \\
&= \mathbb{D}[X] + \mathbb{D}[Y] + 2\mathbb{E}[XY] - 2\mathbb{E}[X]\mathbb{E}[Y] \\
&= \mathbb{D}[X] + \mathbb{D}[Y]
\end{aligned}
$$

като използваме: $X \perp\!\!\!\perp Y \Rightarrow \mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

Дефиниция: (пораждащи функции): Нека $X \in \mathbb{N}_0$ е дискретна случайна величина. Тогава функцията

$$
g_X(s) = \mathbb{E}[s^X] = \sum_{k=0}^{\infty} s^k \mathbb{P}(X=k)
$$

за $|s| \leq 1$ се нарича пораждаща функция на X.

Лема: (свойства на пораждащата функция):

- $\mathbb{E}[X] = g'_X(1)$
- $\mathbb{D}[X] = g''_X(1) + g'_X(1) - (g'_X(1))^2$
- $g_x^{(n)}(0) = n! \cdot \mathbb{P}(X=n)$
- $X \perp\!\!\!\perp Y \Rightarrow g_{X+Y}(s) = g_X(s) g_Y(s)$
- **Доказателства:**
  - $\mathbb{E}[X] = g'_X(1)$

$$
\frac{d}{ds} g_X(s) = \frac{d}{ds}\mathbb{E}[s^X] = \mathbb{E}[\frac{d}{ds}s^X] = \mathbb{E}[Xs^{X-1}] \Longrightarrow g'_X(1) = \mathbb{E}[X]
$$

  - $\mathbb{D}[X] = g''_X(1) + g'_X(1) - (g'_X(1))^2$

$$
\frac{d^2}{ds^2} g_X(s) = \frac{d^2}{ds^2}\mathbb{E}[s^X] = \mathbb{E}[\frac{d^2}{ds^2}s^X] = \mathbb{E}[X(X-1)s^{X-2}] \Longrightarrow \\
g''_X(1) = \mathbb{E}[X(X-1)] = \mathbb{E}[X^2] - \mathbb{E}[X]
$$

$$
\begin{aligned}
&\mathbb{E}[X] = g'_X(1) \\
&\mathbb{E}[X^2] = g''_X(1) + g'_X(1) \\
&\mathbb{D}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = g''_X(1) + g'_X(1) - (g'_X(1))^2
\end{aligned}
$$

  - $g_x^{(n)}(0) = n! \cdot \mathbb{P}(X=n)$

$$g_X^{(n)}(s) = \sum_{k=0}^{\infty}(s^k)^{(n)}\mathbb{P}(X=k) = \sum_{k=n}^{\infty}(s^k)^{(n)}\mathbb{P}(X=k) \Longrightarrow$$
$$g_X^{(n)}(0) = (s^n)^{(n)}\mathbb{P}(X=n) = n! \cdot \mathbb{P}(X=n)$$

- $X \perp\!\!\!\perp Y \Rightarrow g_{X+Y}(s) = g_X(s)g_Y(s)$

$$\begin{aligned} g_{X+Y}(s) &= \mathbb{E}[s^{X+Y}] \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} s^{i+j}\mathbb{P}(X=i, Y=j) \\ &= \sum_{i=0}^{\infty} s^i\mathbb{P}(X=i)\sum_{j=0}^{\infty} s^j\mathbb{P}(Y=j) \\ &= \mathbb{E}[s^X]\mathbb{E}[s^Y] \\ &= g_X(s)g_Y(s) \end{aligned}$$

Твърдение: $X \overset{d}{=} Y \Longleftrightarrow g_x = g_y$

Твърдение: Нека $X_1, \ldots, X_n$ са целочислени случайни величини, които са незивисими в съвкупност. Тогава:

$$Y = \sum_{j=1}^{n} X_j \Longrightarrow g_Y(s) = \prod_{j=1}^{n} g_{X_j}(s)$$

- Ако са равни по разпределение, то $g_Y(s) = (g_{X_1}(s))^n$
- **Доказателство:**
  - Ако $h_i(x)$- ограничени функции, то $\mathbb{E}[h_1(X_1)\cdots h_n(X_n)] = \prod_{j=1}^{n} h_j(X_j)$

    Нека $h_j(x) = s^x$
    Тогава

$$\begin{aligned} g_Y(s) &= \mathbb{E}[s^{X_1+\cdots+X_n}] \\ &= \mathbb{E}[s^{X_1}\cdots s^{X_n}] \\ &= \mathbb{E}[h_1(X_1)\cdots h_n(X_n)] \\ &= \prod_{j=1}^{n}\mathbb{E}[h_j(X_j)] = \prod_{j=1}^{n}\mathbb{E}[s^{X_j}] \\ &= \prod_{j=1}^{n} g_{X_j}(s) \end{aligned}$$