~~зад. 1 и 2~~ Виж зад 1 и 2 от КН контролно 2 2024

~~зад. 3~~ Виж упражнения.

~~зад. 4.~~ $X, Y \sim Exp(1)$ iid

$$\mathbb{P}(Exp(\lambda) > t) = e^{-\lambda t}$$

$$\Rightarrow \mathbb{P}\left(\sqrt{\frac{X}{Y}} > t\right) = \mathbb{P}(X > Yt^2)$$

$$= \int_0^\infty \mathbb{P}(X > yt^2) f_Y(y)\, dy$$

$$= \int_0^\infty e^{-yt^2} \cdot e^{-y}\, dy$$

$$= \frac{1}{t^2+1} \underbrace{\int_0^\infty e^{-y(t^2+1)}\, dy(t^2+1)}_{\int_0^\infty e^{-x} dx = 1}$$

$$= \frac{1}{t^2+1} \text{ за } t > 0$$

$$\Rightarrow \mathbb{E}\left[\sqrt{\frac{X}{Y}}\right] = \int_0^\infty \mathbb{P}\left(\sqrt{\frac{X}{Y}} > t\right) dt = \int_0^\infty \frac{1}{t^2+1}\, dt = \frac{\pi}{2}.$$

(аналогично може и чрез $F_{\sqrt{\frac{X}{Y}}}(t) = \frac{t^2}{1+t^2}$

$\Rightarrow f = F'$ и т.н.)

2) $\mathbb{E}\left[\sqrt{\frac{X}{Y}}\right] = \int_0^\infty \int_0^\infty \sqrt{\frac{x}{y}} f_{X,Y}(x,y)\,dx\,dy$

$$= \int_0^\infty \int_0^\infty \sqrt{\frac{x}{y}}\, e^{-x} e^{-y}\,dx\,dy$$

$$= \int_0^\infty e^{-x}\sqrt{x}\,dx \cdot \int_0^\infty e^{-y}\frac{1}{\sqrt{y}}\,dy$$

$$= \Gamma\left(\frac{3}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)$$

$\Gamma(z+1) = z\Gamma(z)$

$$= \Gamma\left(\frac{1}{2}\right)^2 \cdot \frac{1}{2}$$

$$\Rightarrow \Gamma\left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} \underset{\text{от } 1)}{=} \frac{\pi}{2} \Rightarrow \boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

~~зад. 5~~ Виж зад. 4 изпит СЕМ фев 2024

~~зад. 6~~| Това е д-во на (частен случай на) теоремата на Кохран (Cochran), разбито на отделни стъпки зад. 7 $e^{-n}\left(\frac{n^0}{0!}+\frac{n^1}{1!}+\dots+\frac{n^n}{n!}\right)$

$$=\sum_{k=0}^{n} \mathbb{P}(\mathrm{Poi}(n)=k)=\mathbb{P}(\mathrm{Poi}(n)\le n)$$

Както видяхме по-рано в курса,

Ако $X_n \sim \mathrm{Poi}(n)$, то $X_n \overset{d}{=} Y_1+\dots+Y_n$, където

$Y_i \overset{iid}{\sim} \mathrm{Poi}(1)$

$\mathbb{E}[Y_i] = D(Y_i) = 1$

$$\Rightarrow \mathbb{P}(\mathrm{Poi}(n)\le n) = \mathbb{P}(Y_1+\dots+Y_n \le n)$$

$$=\mathbb{P}\left(\frac{Y_1+\dots+Y_n-n}{\sqrt{n}} \le \frac{n-n}{\sqrt{n}}\right)$$

$$\xrightarrow[n\to\infty]{} \mathbb{P}(\mathcal{N}(0,1)\le 0)=\frac{1}{2}.$$

от ЦГТ