зад. 1 • Запълване на 1-во кв. $\to Ge(1)$

Запълване на 2-ро след 1-то $\to Ge(\frac{8}{9})$

$\vdots$

$$\mathbb{E}[\text{времето}] = \mathbb{E}\left[Ge(1) + \dots + Ge(\tfrac{1}{9})\right]$$

$$= 9\left(1 + \frac{1}{2} + \dots + \frac{1}{9}\right) \approx 25{,}46.$$

• $\mathbb{E}[\underbrace{\text{до първия момент с } 2 \text{ на } 1 \text{ място}}_{:= T_2}]$

$$= \mathbb{E}\left[T_2 \mathbb{1}_{\{2 \text{ в } \boxplus\}} + T_2 \mathbb{1}_{\{2 \text{ в } \boxplus\}} + \dots\right]$$

$$= 9\, \mathbb{E}\left[T_2 \mathbb{1}\{T_2 \text{ се реализира в } \boxplus\}\right]$$

$$= 9\left[2 \cdot \left(\frac{1}{9}\right)^2 + 3 \cdot \binom{8}{1} 2! \left(\frac{1}{9}\right)^3 + 4 \binom{8}{2} 3! \left(\frac{1}{9}\right)^4 + \dots\right]$$

$$= 9 \sum_{k=0}^{8} (k+2)! \left(\frac{1}{9}\right)^{k+2} \binom{8}{k} \approx 4{,}46.$$

• За 1-то $n^2\left(1 + \dots + \frac{1}{n^2}\right) \approx n^2 \log n^2 = 2n^2 \log n$

• За 2-то: $n^2 \sum_{k=0}^{n-1} (k+2)! \left(\frac{1}{n^2}\right)^{k+2} \binom{n-1}{k}$.

зад. 21. $S = X_1 + \dots + X_n = 1 + \dots + n = \frac{n(n+1)}{2}$

$\Rightarrow DS = 0,\ \mathbb{E}S = \frac{n(n+1)}{2}$

2. $D(X+Y) = \mathbb{E}(X+Y)^2 - (\mathbb{E}(X) + \mathbb{E}(Y))^2$

$= \mathbb{E}X^2 - (\mathbb{E}X)^2 + \mathbb{E}Y^2 - (\mathbb{E}Y)^2 + 2(\mathbb{E}XY - \mathbb{E}X\mathbb{E}Y)$

3. $\mathbb{E}S = \mathbb{E}(X_1 + \dots + X_n) = n \cdot \mathbb{E}X_1$ по симетрия

$\Rightarrow \mathbb{E}X_i = \frac{n+1}{2}$.

Аналог.

$\mathbb{E}(X_1^2 + \dots + X_n^2) = \mathbb{E}(1^2 + \dots + n^2) = \frac{n(n+1)(2n+1)}{6} = n\mathbb{E}X_1^2$

$\Rightarrow \mathbb{E}X_i^2 = \frac{(n+1)(2n+1)}{6}$.

$DX_i = \mathbb{E}X_i^2 - (\mathbb{E}X_i)^2 = \frac{(n+1)}{2}\left(\frac{2n+1}{3} - \frac{n+1}{2}\right)$

$= \frac{n^2-1}{12}$.

3. $0 = DS = n \cdot DX_1 + \binom{n}{2} \cdot 2\,\mathrm{Cov}(X_1, X_2)$

$= n \cdot \frac{n^2-1}{12} + n(n-1) \cdot \mathrm{Cov}(X_1, X_2)$

$\Rightarrow \boxed{\mathrm{Cov}(X_i, X_j) = \begin{cases} -\frac{n+1}{12}, & i \neq j \\ \frac{n^2-1}{12}, & i = j \end{cases}}$.

зад. 3 1. $D(M_1+\dots+M_n) = n\,DM_1$; $M_i = M_1$

$\mathbb{P}(M \le t) = \mathbb{P}(U_1 \le t)\,\mathbb{P}(V_1 \le t) = t^2$ за $t \in [0,1]$

$\Rightarrow f_M(t) = 2t\,\mathbb{1}_{\{t \in [0,1]\}}$

$\Rightarrow \mathbb{E}M = \int_0^1 2t \cdot t\,dt = \frac{2}{3}$

$\mathbb{E}M^2 = \int_0^1 t^2 \cdot 2t\,dt = \frac{1}{2} \Rightarrow DM = \frac{1}{2} - \frac{4}{9} = \boxed{\frac{1}{18}}$

$\Rightarrow D(M_1+\dots+M_n) = \frac{n}{18}$.

2. $M_1+\dots+M_n \approx N\left(n \cdot \frac{2}{3}, \frac{n}{18}\right)$ от ЦГТ.

зад. 4 $\int_1^\infty \frac{c}{x^4}\,dx = 1 = \left[-\frac{c}{3}x^{-3}\right]_1^\infty = \frac{c}{3}$

$\Rightarrow \underline{\underline{c = 3}}$

$\mathbb{E}X_1 = \int_1^\infty \frac{c}{x^3}\,dx = \left[-\frac{c}{2}x^{-2}\right]_1^\infty = \frac{c}{2} = \frac{3}{2}$

$\mathbb{E}X_1^2 = \int_1^\infty \frac{c}{x^2}\,dx = c = 3$

$\Rightarrow DX_1 = 3 - \frac{9}{4} = \underline{\underline{\frac{3}{4}}}$

$\Rightarrow$ от ЦГТ (т.к. $DX_1 < \infty$)

$\frac{S_n - n \cdot \frac{3}{2}}{\sqrt{3n/4}} \approx N(0,1)$, т.е. $a = \frac{3n}{2}$; $b = \frac{\sqrt{3n}}{2}$.

2. Нека $S := \sum_{i=1}^{40} X_i$; $\mathbb{E}S = 400 = \mathbb{D}S$

$$99\% = \mathbb{P}(S > a) = \mathbb{P}\left(\frac{S-400}{20} > \frac{a-400}{20}\right) \overset{\text{ЦГТ},\ 40>30}{\approx} \mathbb{P}\left(N(0,1) > \frac{a-400}{20}\right)$$

От табл.: $\frac{a-400}{20} \approx -2{,}33$

$$\Rightarrow a = 400 - 46{,}6 = \underline{353{,}4}.$$