Lecture 9: Reachable Sets and Invariant Set

I. Polyhedra and Polytopes

We have already mentioned many of the things before, but here we again quickly go through all the concepts needed for this lecture for a unified notation, and there are also some new concepts.

- **General Set Definitions and Polytopes**

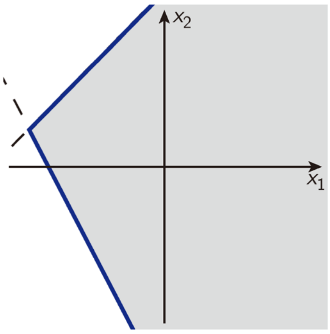
Definition (Polyhedra and polytopes):

A polyhedron is the intersection of a finite number of closed halfspaces:

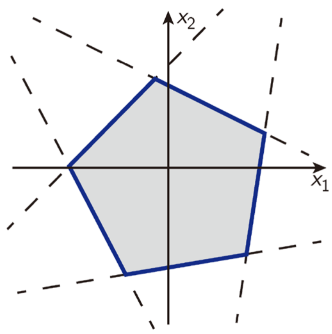
$$\mathcal{Z}=\{z \mid a_1^{\mathrm{T}} z \leq b_1,\ a_2^{\mathrm{T}} z \leq b_2, \ldots, a_m^{\mathrm{T}} z \leq b_m\}=\{z \mid Az \leq b\}$$

where $A=[a_1, a_2, \cdots, a_m]^{\mathrm{T}}$ and $b=[b_1, b_2, \ldots, b_m]^{\mathrm{T}}$.

A polytope is a bounded polyhedron. Polyhedra and polytopes **are always convex.**



An (unbounded) polyhedron



A polytope

Definition (n-dimensional ball):

An n-dimensional ball $\mathcal{B}(x_0, \rho)$ is the set $\mathcal{B}(x_0, \rho)=\left\{x \in \mathbb{R}^n \,\middle|\, \sqrt{\|x-x_0\|_2} \leq \rho\right\}$ where x_0 and ρ are the center and the radius of the ball, respectively.

Definition (convex combination):

The convex combination of $x_1, \ldots, x_k$ is defined as the point $\lambda_1 x_1+\ldots+\lambda_k x_k$ where $\sum_{i=1}^k \lambda_i=1$ and $\lambda_i \geq 0$, $i=1, \ldots, k$.

Definition (convex hull):

The convex hull of a set $K \subseteq \mathbb{R}^n$ is the set of all convex combinations of points in K and it is denoted as $\operatorname{conv}(K)$:

$$\operatorname{conv}(K)=\left\{\lambda_1 x_1+\ldots+\lambda_k x_k \,\middle|\, x_i \in K,\ \lambda_i \geq 0,\ i=1, \ldots, k,\ \sum_{i=1}^k \lambda_i=1\right\}$$

Definition (cone):

A cone spanned by a finite set of points $K=\{x_1, \ldots, x_k\}$ is defined as:

$$\operatorname{cone}(K)=\left\{\sum_{i=1}^k \lambda_i x_i,\ \lambda_i \geq 0,\ i=1, \ldots, k\right\}$$

Definition (Minkowski Sum):

The Minkowski sum of two sets $P,\ Q \subseteq \mathbb{R}^n$ is defined as:

$$P \oplus Q = \{x + y \mid x \in P, y \in Q\}$$

Definition (Polyhedra Representations of Polyhedra)

An $\mathcal{H}$-polyhedron $\mathcal{P}$ in $\mathbb{R}^n$ denotes an intersection of a finite set of closed halfspaces in $\mathbb{R}^n$:

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \le b\}$$

Inequalities that can be removed without changing the polyhedron are called **redundant**. The representation of an $\mathcal{H}$-polyhedron is **minimal** if it does not contain redundant inequalities. A linear inequality $cz \le c_0$ is said to be valid for $\mathcal{P}$ if it is satisfied for all points $z \in \mathcal{P}$.

A $\mathcal{V}$-polyhedron $\mathcal{P}$ in $\mathbb{R}^n$ denotes the Minkowski sum:

$$\mathcal{P} = \operatorname{conv}(V) \oplus \operatorname{cone}(Y)$$

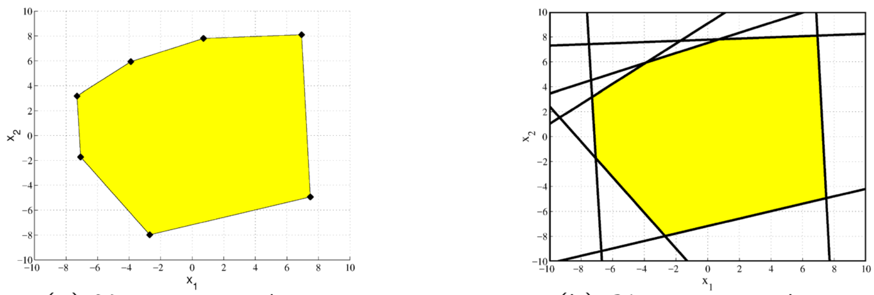
for some $V = [V_1, \ldots, V_k] \in \mathbb{R}^{n \times k}$, $Y = [y_1, \ldots, y_{k'}] \in \mathbb{R}^{n \times k'}$

Remarks on Polyhedra Representations:

1. Any $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron
2. An $\mathcal{H}$-polytope ($\mathcal{V}$-polytope) is a bounded $\mathcal{H}$-polyhedron ($\mathcal{V}$-polyhedron).

   Note: Why the definition $\mathcal{V}$-polyhedron is so complex that we need to use Minkowski sum? It is because we consider the unbounded situations: $\operatorname{conv}(V)$ is the bounded part and $\operatorname{cone}(Y)$ is the unbounded part
3. The dimension of a polytope (polyhedron) $\mathcal{P}$ is the dimension of its affine hull and is denoted by $\dim(\mathcal{P})$. A polytope (polyhedron) $\mathcal{P} \subset \mathbb{R}^n$, is full-dimensional if it is possible to fit a non-empty n-dimensional ball in $\mathcal{P}$

   Note: We care about this because many of the algorithms and operations only work for full-dimensional polyhedrons. An example of the not full-dimensional polyhedron is a 2-D polyhedron in 3-D space, whose "volume" is 0 in 3-D.
4. If $\|A_i\|_2 = 1$, where A_i denotes the i-th row of a matrix A, we say that the polytope $\mathcal{P}$ is normalized.



(a) $\mathcal{V}$-representation. (b) $\mathcal{H}$-representation.

Definition (Vertices, Edges, Ridges, and Facets):

A linear inequality $cz \le c_0$ is said to be valid for $\mathcal{P}$ if it is satisfied for all points $z \in \mathcal{P}$.

A face of $\mathcal{P}$ is any nonempty set of the form $\mathcal{F}=\mathcal{P}\cap\{z\in\mathbb{R}^s \mid cz=c_0\}$ where $cz\le c_0$ is a valid inequality for $\mathcal{P}$

The faces of dimension 0, 1, $\dim(\mathcal{P})$-2, and $\dim(\mathcal{P})$-1 are called vertices, edges, ridges, and facets, respectively. That is to say, ridges, and facets are generalizations of the vertices and edges.

Definition (Polytopal Complexes, or Partition):

A set $\mathcal{C}\subseteq\mathbb{R}^n$ is called a **P-collection** (in $\mathbb{R}^n$) if it is a collection of a finite number of n-dimensional polytopes:

$$\mathcal{C}=\{\mathcal{C}_i\}_{i=1}^{N_c}$$

Where $\mathcal{C}_i=\{x\in\mathbb{R}^n \mid C_i^x x\le C_i^c\}$, $\dim(\mathcal{C}_i=n)$, $i=1,\dots,N_c$ with $N_c<\infty$. In other words, it is a list of polytopes sets. The **underlying set** of a P-collection $\mathcal{C}=\{\mathcal{C}\}_{i=1}^{N_c}$ is the point set:

$$\underline{\mathcal{C}}=\bigcup_{\mathcal{P}\in\mathcal{C}}\mathcal{P}=\bigcup_{i=1}^{N_c}\mathcal{C}_i$$

Definition (Partition of Set):

1. A collection of sets $\{\mathcal{C}_i\}_{i=1}^{N_c}$ is a **strict partition** of a set $\mathcal{C}$ if:

   $$\text{(i) } \bigcup_{i=1}^{N_c}\mathcal{C}_i=\mathcal{C} \text{ and (ii) } \mathcal{C}_i\cap\mathcal{C}_j \quad \forall i\ne j$$

2. $\{\mathcal{C}_i\}_{i=1}^{N_c}$ is a **strict polyhedral partition** of a polyhedral set $\mathcal{C}$ if $\{\mathcal{C}_i\}_{i=1}^{N_c}$ is a strict partition of $\mathcal{C}$ and $\overline{\mathcal{C}}_i$ is a polyhedron for all i, where $\overline{\mathcal{C}}_i$ denoted the closure of the set $\mathcal{C}_i$
3. A collection of sets $\{\mathcal{C}_i\}_{i=1}^{N_c}$ is a partition of a set if:

   $$\text{(i) } \bigcup_{i=1}^{N_c}\mathcal{C}_i=\mathcal{C} \text{ and (ii) } (\mathcal{C}_i\backslash\partial\mathcal{C}_i)\cap(\mathcal{C}_j\backslash\partial\mathcal{C}_j)=\emptyset,\ \forall i\ne j$$

   Where ∂ means the boundary of a set.

Note: From the definition, we know that strict partition of a polyhedral asks each polyhedron can't even have a common edge or vertices, which would cause a big problem in real applications since we have to decide and track the belongs of the boundary carefully. Therefore, we then have the definition of a normal partition that does not care about the intersections on boundaries.

Definition (Functions on Polytopal Complexes)

A function $h(\theta):\Theta\to\mathbb{R}^k$, where $\Theta\subseteq\mathbb{R}^s$, is **piecewise affine (PWA)** if there exists a **strict partition** $R_1,\dots,R_N$ of Θ and $h(\theta)=H^i\theta+k^i,\ \forall\theta\in R_i,\ i=1,\dots,N$

A function $h(\theta):\Theta\to\mathbb{R}^k$, where $\Theta\subseteq\mathbb{R}^s$, is **piecewise affine on polyhedra (PPWA)** if there exists a **strict polyhedral partition** $R_1,\dots,R_N$ of Θ and $h(\theta)=H^i\theta+k^i,\ \forall\theta\in R_i,\ i=1,\dots,N$

A function $h(\theta):\Theta\to\mathbb{R}^k$, where $\Theta\subseteq\mathbb{R}^s$, is **piecewise quadratic (PWQ)** if there exists a **strict partition** $R_1,\dots,R_N$ of Θ and $h(\theta)=\theta^{\mathrm{T}}H^i\theta+k^i\theta+I^i,\ \forall\theta\in R_i,\ i=1,\dots,N$

A function $h(\theta):\Theta\to\mathbb{R}^k$, where $\Theta\subseteq\mathbb{R}^s$, is s **piecewise quadratic on polyhedra (PPWQ)** if there exists a **strict polyhedral partition** $R_1,\dots,R_N$ of Θ and $h(\theta)=\theta^{\mathrm{T}}H^i\theta+k^i\theta+I^i,\ \forall\theta\in R_i,$ $i=1,\dots,N$

Note: As mentioned above, though by definition we require strict partition, it is actually very hard to follow that strict partition in real applications. **Generally, if there are no discontinuities in the function, it does not matter much whether the partition is strict or not. Or in other words, we usually require continuity at the boundary.**

- **Basic Operations on Polytope**

Definition (convex hull): see the definition before

Definition (Vertex Enumeration):

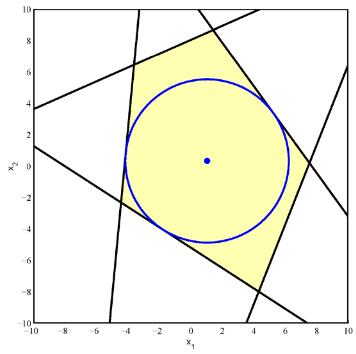
The Vertex Enumeration of a polytope $\mathcal{P}$ given in $\mathcal{H}$-representation. (dual of the convex hull operation). Used to switch from a $\mathcal{V}$-representation of a polytope to an $\mathcal{H}$-representation.

Note: This operation requires checking any two inequalities (pairs inequality) and doing the intersection, which is combinatorial and complicated. Thus, try not to switch representations frequently.

Definition (Polytope Reduction):

Polytope reduction is the computation of the minimal representation of a polytope. A polytope $\mathcal{P}\subset\mathbb{R}^n, \mathcal{P}=\{x\in\mathbb{R}^n: Hx\le k\}$ is in a minimal representation if the removal of any row in $Hx\le k$ would change it (i.e., if there are no redundant constraints).

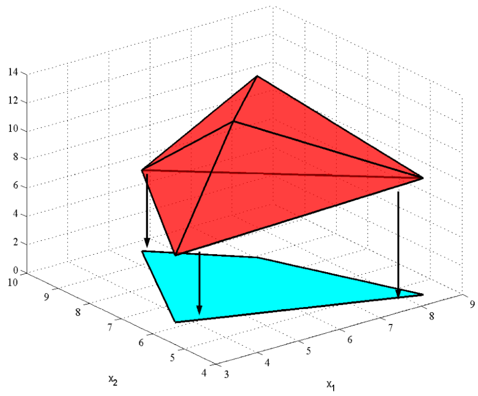
Definition (Chebyshev Ball):



The Chebyshev Ball of a polytope P corresponds to the largest radius ball $\mathcal{B}(x_c, R)$ with the center x_c, such that $\mathcal{B}(x_c, R)\subset\mathcal{P}$.

Note: The calculation of the Chebyshev Ball (largest ball inscribing the polytope) can be done efficiently. However, calculating the smallest ball surrounding the polytope is hard.

Definition (Projection):



Given a polytope:

$\mathcal{P}=\{[x^{\mathrm{T}}\ \ y^{\mathrm{T}}]^{\mathrm{T}}\in\mathbb{R}^{n+m}:\ H^x x+H^y y\le b\}\subset\mathbb{R}^{n+m}$, the projection onto the x-space $\mathbb{R}^n$ is defined as:

$$\mathrm{proj}_x(\mathcal{P})=\{x\in\mathbb{R}^n \mid\ \exists y\in\mathbb{R}^m:\ H^x x+H^y y\le b\}$$

In MATLAB with polyhedron created using MPT toolbox, we can call: Q = projection(P, dim) to get the projection.

Definition (Set-Difference):

The set-difference of two polytopes $\mathcal{Y}$ and $\mathcal{R}_0$ is;

$$\mathcal{R}=\mathcal{Y}\backslash\mathcal{R}_0=\{x\in\mathbb{R}^n \mid x\in\mathcal{Y}, x\notin\mathcal{R}_0\}$$

Generally, the set-difference can be a nonconvex and disconnected set and can be described as a P-collection $\mathcal{R}=\bigcup_{i=1}^{m}\mathcal{R}_i$. The P-collection can be computed by consecutively inverting the half-spaces defining $\mathcal{R}_0$ as described next.

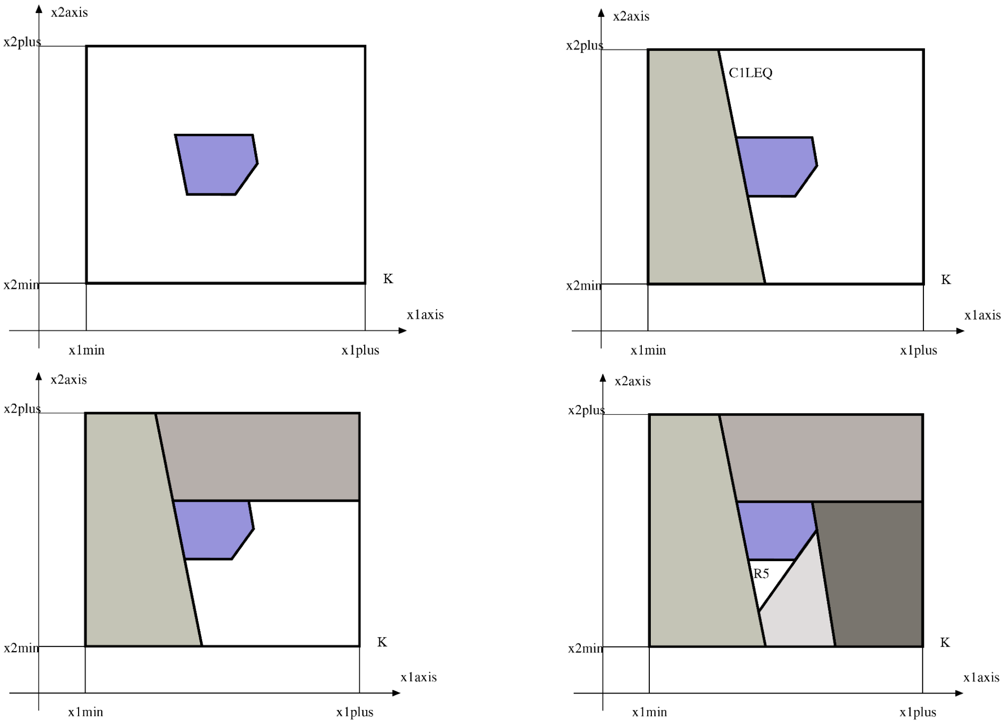
Theorem (Partition of the Space):

Let

$$\mathcal{R}_i = \left\{ x \in \mathcal{Y} \middle| \begin{array}{l} A^i x > b^i \\ A^j x \le b^i,\ \forall j < i \end{array} \right\},\ i = 1, \dots, m$$

Then $\left\{ \bar{\mathcal{R}}_0, \mathcal{R}_1, \dots, \mathcal{R}_m \right\}$ is a strict polyhedral partition of $\mathcal{Y}$

Note: The importance of this theorem is that it tells us the whole space (no matter whether convex or not) can be partitioned into a group of convex polytopes. So the set difference, which is nonconvex, can also be handled by first doing partitions, as shown below.



Definition (Pontryagin Difference, or Minkowski Difference):

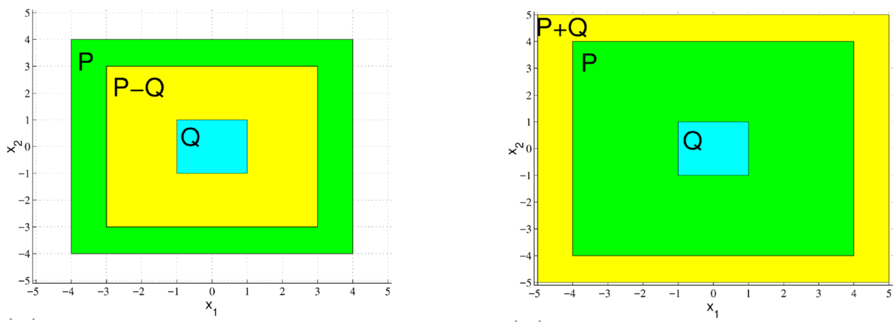
The Pontryagin Difference (also known as Minkowski difference) of two polytopes $\mathcal{P}$ and $\mathcal{Q}$ is a polytope defined as:

$$\mathcal{P} \ominus \mathcal{Q} = \{ x \in \mathbb{R}^n \mid x + q \in \mathcal{P},\ \forall q \in \mathcal{Q} \}$$

Recall our definition of Minkowski sum:

$$\mathcal{P} \oplus \mathcal{Q} = \{ x \in \mathbb{R}^n \mid\ \exists y \in \mathcal{P},\ \exists z \in \mathcal{Q},\ x = y + z \}$$

these two can be concluded as the following figure (left: Minkowski difference, right: Minkowski sum).



Theorem (Minkowski Sum of Polytopes as Projection Operation):

Consider

$$\mathcal{P}=\{y\in\mathbb{R}^n \mid P^y y\le P^c\},\ \ \mathcal{Q}=\{z\in\mathbb{R}^n \mid Q^z z\le Q^c\}$$

It holds that:

$$\begin{aligned} W &= \mathcal{P}\oplus\mathcal{Q} \\ &= \{x\in\mathbb{R}^n \mid \ \exists y\ P^y y\le P^c, \exists z\ Q^z z\le Q^c,\ y,z\in\mathbb{R}^n,\ x=y+z\} \\ &= \{x\in\mathbb{R}^n \mid \ \exists y\in\mathbb{R}^n \text{ s.t. } P^y y\le P^c,\ Q^z(x-y)\le Q^c\} \\ &= \left\{x\in\mathbb{R}^n \,\middle|\, \ \exists y\in\mathbb{R}^n \text{ s.t.} \begin{bmatrix} 0 & P^y \\ Q^z & -Q^z \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\le\begin{bmatrix} P^c \\ Q^c \end{bmatrix}\right\} \\ &= \operatorname{proj}_x\left(\left\{[x^{\mathrm{T}} \ \ y^{\mathrm{T}}]^{\mathrm{T}}\in\mathbb{R}^{n+m} \,\middle|\, \begin{bmatrix} 0 & P^y \\ Q^z & -Q^z \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\le\begin{bmatrix} P^c \\ Q^c \end{bmatrix}\right\}\right) \end{aligned}$$

Note: From this, we can see that the Minkowski sum is computationally expensive.

Theorem (Pontryagin Difference of Polytopes as Series of Linear Programs):

Consider

$$\mathcal{P}=\{y\in\mathbb{R}^n \mid P^y y\le P^c\},\ \ \mathcal{Q}=\{z\in\mathbb{R}^n \mid Q^z z\le Q^c\}$$

Then:

$$\begin{aligned} \mathcal{P}\ominus\mathcal{Q} &= \{x\in\mathbb{R}^n \mid x+z\in\mathcal{P}, \forall z\in\mathcal{Q}\} \\ &= \{x\in\mathbb{R}^n \mid P^y(x+z)\le P^c, \forall z\in\mathcal{Q}\} \\ &= \{x\in\mathbb{R}^n \mid P^y x\le P^c-P^y z,\ \forall z\in\mathcal{Q}\} \\ &= \{x\in\mathbb{R}^n \mid P^y x\le P^c-H(P^y,\mathcal{Q})\} \end{aligned}$$

where the i-th element of $H(P^y,\mathcal{Q})$ is:

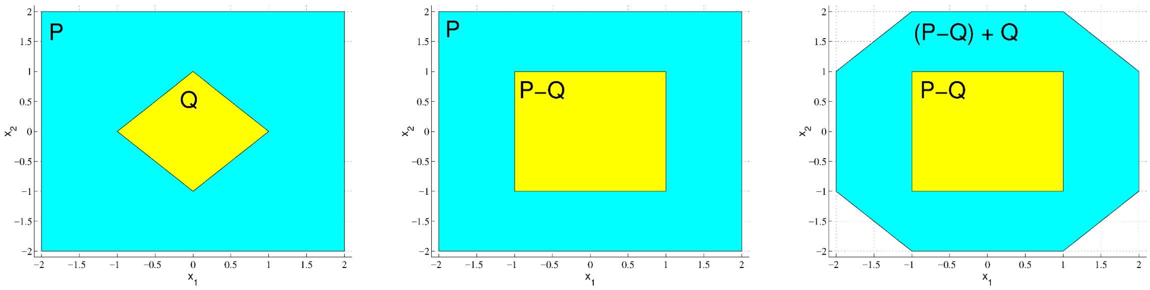
$$H_i(P^y,\mathcal{Q})=\max_{z\in\mathcal{Q}} P_i^y z$$

And P_i^y is the i-th row of the matrix P^y

Note: From this, we can see that the Pontryagin difference is "not that computationally expensive" because we only need to do several linear programs. And For special cases (e.g. when $\mathcal{Q}$ is a hypercube), more efficient computational methods exist.

Remark: Minkowski sum is not the inverse operation of Minkowski difference:

consider $(\mathcal{P}\ominus\mathcal{Q})\oplus\mathcal{Q}$: in fact $(\mathcal{P}\ominus\mathcal{Q})\oplus\mathcal{Q}\subseteq\mathcal{P}$, as shown below



Definition (Affine Mappings and Composition)

Consider a polyhedron $\mathcal{P}=\{x\in\mathbb{R}^n \mid Hx\leq k\}$, with $H\in\mathbb{R}^{n_p\times n}$ and an affine mapping $f(z)$:

$$f\colon z\in\mathbb{R}^n\mapsto Az+b,\ A\in\mathbb{R}^{n\times n}, b\in\mathbb{R}^n$$

1. Define the composition of $\mathcal{P}$ and f as the following polyhedron:

$$\mathcal{P}\circ f=\{z\in\mathbb{R}^n \mid Hf(z)\leq k\}=\{z\in\mathbb{R}^n \mid HAz\leq k-Hb\}$$

   This is useful for **backward-reachability (Pre Set calculation)**

2. Meanwhile, the composition of f and $\mathcal{P}$ as the following polyhedron

$$f\circ\mathcal{P}=\{y\in\mathbb{R}^n \mid y=Ax+b, \forall x\in\mathbb{R}^n, Hx\leq k\}$$

   This is useful for **forward-reachability (Reach Set calculation)**

   Note: The polyhedron $f\circ\mathcal{P}$ can be computed as follows: Write $\mathcal{P}$ in $\mathcal{V}$-representation, i.e., $\mathcal{P}=\text{conv}(V)$ and map the vertices $V=\{V_1,V_2,\dots,V_k\}$ through the transformation f. Because the transformation is affine, the set $f\circ\mathcal{P}$ is the convex hull of the transformed vertices:

$$f\circ\mathcal{P}=\text{conv}(F), F=\{AV_1+b,\ \dots, AV_k+b\}$$

   This property is useful for hand-calculating the reach set and helps in invariant set checking. See Homework examples like HW4 Ex 1.12. Also note that doing $f\circ\mathcal{P}$ in $\mathcal{H}$-representation is hard.

II. Reach Sets

Still, we consider two systems: the autonomous system $x(k+1)=f_a(x(k))$ and the controlled system subject to external inputs $x(k+1)=f(x(k),u(k))$. Both systems are subject to state and input constraints $x(t)\in\mathcal{X},\ u(t)\in\mathcal{U},\ \forall t\geq 0$. Where the sets $\mathcal{X}$ and $\mathcal{U}$ are polyhedra and contain the origin in their interior.

- **Pre and Reach Sets Definition**

Definition (Reach Set):

For the autonomous system $x(k+1)=f_a(x(k))$, we denote the one-step reachable set as:

$$\text{Reach}(\mathcal{X})=\{x\in\mathbb{R}^n \mid \exists x(0)\in\mathcal{X},\ \text{s.t. } x=f_a(x(0))\}$$

For the controlled system $x(k+1)=f(x(k),u(k))$, we denote the one-step reachable set as:

$$\text{Reach}(\mathcal{X})=\{x\in\mathbb{R}^n \mid \exists x(0)\in\mathcal{X},\ \exists u(0)\in\mathcal{U},\ \text{s.t. } x=f(x(0),u(0))\}$$

Definition (Pre Set):

Pre sets are the dual of one-step reachable sets.

For the autonomous system $x(k+1)=f_a(x(k))$, we denote the one-step pre set as:

$$\text{Pre}(\mathcal{X})=\{x\in\mathbb{R}^n \mid f_a(x)\in\mathcal{X}\}$$

For the controlled system $x(k+1)=f(x(k),u(k))$, we denote the one-step pre set as:

$$\text{Pre}(\mathcal{X})=\{x\in\mathbb{R}^n \mid \exists u\in\mathcal{U},\ \text{s.t. } f(x,u)\in\mathcal{X}\}$$

- **Pre and Reach Sets Computation**

Pre and Reach Set Computation – Autonomous Systems

Assume the system is linear and autonomous

$$x(t+1)=Ax(t)$$

Let the set of the feasible state be defined as

$$\mathcal{X}=(x|\ Hx\leq h)$$

then the set $\text{Pre}(\mathcal{X})$ is:

$$\text{Pre}(\mathcal{X})=\{x|\ HAx\leq h\}$$

Note that by using the polyhedral notation (operation), the set $\text{Pre}(\mathcal{X})$ is simply $\mathcal{X}\circ A$

Similarly, the set $\text{Reach}(\mathcal{X})$ is obtained by applying the map A to the set $\mathcal{X}$. Write the set $\mathcal{X}$ in $\mathcal{V}$-representation:

$$\mathcal{X}=\text{conv}(V)$$

and map the set of vertices V through the transformation A. Because the transformation is linear, the reach set is simply the convex hull of the transformed vertices. Using the polyhedral notation:

$$\text{Reach}(\mathcal{X})=A\circ\mathcal{X}=\text{conv}(AV)$$

Pre and Reach Set Computation – Controlled Systems

Now consider the system with control input:

$$x(t+1)=Ax(t)+Bu(t)$$

Let the set of the feasible state and input be defined as

$$\mathcal{X}=\{x|\ Hx\leq h\},\ \mathcal{U}=\{u|H_u u\leq h_u\}$$

then the set $\text{Pre}(\mathcal{X})$ is:

$$\text{Pre}(\mathcal{X})=\left\{x\in\mathbb{R}^n\ \middle|\ \exists u\in\mathbb{R},\ \text{s.t.}\begin{bmatrix}HA & HB\\ 0 & H_u\end{bmatrix}\begin{bmatrix}x\\ u\end{bmatrix}\leq\begin{bmatrix}h\\ h_u\end{bmatrix}\right\}$$

Note that by using the definition of the Minkowski sum we can compactly write the set as:

$$\begin{aligned}\text{Pre}(\mathcal{X})&=\{x|\ \exists u\in\mathcal{U}\ \text{s.t.}\ Ax+Bu\in\mathcal{X}\}\\ &=\{x|\ \exists y\in\mathcal{X},\exists u\in\mathcal{U}\ \text{s.t.}\ Ax=y-Bu\}\\ &=\{x|\ Ax=\mathcal{X}\oplus(-B)\circ\mathcal{U}\}\\ &=(\mathcal{X}\oplus(-B)\circ\mathcal{U})\circ A\end{aligned}$$

Similarly, the set $\text{Reach}(\mathcal{X})$ considering the effect of the input $u\in\mathcal{U}$ is:

$$\begin{aligned}\text{Reach}(\mathcal{X})&=\{y+\bar{u}|\ y\in A\circ\mathcal{X},\ \bar{u}\in B\circ\mathcal{U}\}\\ &=(A\circ\mathcal{X})\oplus(B\circ\mathcal{U})\end{aligned}$$

- **Summary**

In summary, for the feasible set of the state $\mathcal{X}$ and the feasible set of the control input $\mathcal{U}$, the sets $\text{Pre}(\mathcal{X})$ and $\text{Reach}(\mathcal{X})$ are the results of linear operations on the polyhedra $\mathcal{X}$ and $\mathcal{U}$ and therefore are polyhedra. By using the definition of the Minkowski sum and the definition affine operation on polyhedra, we can compactly summarize the $\text{Pre}(\cdot)$ and $\text{Reach}(\cdot)$ operations on linear systems as follows:

	Autonomous: $x(t+1)=Ax(t)$	Controlled: $x(t+1)=Ax(t)+Bu(t)$
$\mathrm{Pre}(\mathcal{X})$	$\mathcal{X}\circ\mathcal{A}$	$(\mathcal{X}\oplus(-B)\circ\mathcal{U})\circ A$
$\mathrm{Reach}(\mathcal{X})$	$\mathcal{A}\circ\mathcal{X}$	$(A\circ\mathcal{X})\oplus(B\circ\mathcal{U})$

- **Controllable Sets**

Definition (N-Step Controllable Set $\mathcal{K}_N(\mathcal{O})$):

For a given target set $\mathcal{O}\subseteq\mathcal{X}$, the N-step controllable set $\mathcal{K}_N(\mathcal{O})$ is defined as:

$$\mathcal{K}_N(\mathcal{O})=\mathrm{Pre}(\mathcal{K}_{N-1}(\mathcal{O}))\cap\mathcal{X},\ \mathcal{K}_0(\mathcal{O})=\mathcal{O},\quad N\in\mathbb{N}^+$$

All states $x_0\in\mathcal{K}_N(\mathcal{O})$ can be driven, through a time-varying control law, to the target set $\mathcal{O}$ in N steps, while satisfying input and state constraints.

Definition (Maximal Controllable Set $\mathcal{K}_\infty(\mathcal{O})$):

For a given target set $\mathcal{O}\subseteq\mathcal{X}$, the maximal controllable set $\mathcal{K}_\infty(\mathcal{O})$ for the controlled system $x(t+1)=f(x(t),u(t))$ subject to the constraints $x(t)\in\mathcal{X},\ u(t)\in\mathcal{U},\ \forall t\geq 0$ is the union of all N-step controllable set contained in $\mathcal{X}$ ($N\in\mathbb{N}$)

- **N-Step Reachable Set**

Definition (N-Step Reachable Set $\mathcal{R}_N(\mathcal{X}_0)$)

For a given initial set $\mathcal{X}_0\subseteq\mathcal{X}$, the N-step reachable set $\mathcal{R}_N(\mathcal{X}_0)$ is:

$$\mathcal{R}_{i+1}(\mathcal{X}_0)=\mathrm{Reach}(\mathcal{R}_i(\mathcal{X}_0)),\ \mathcal{R}_0(\mathcal{X}_0)=\mathcal{X}_0,\ i=0,\ldots,N-1$$

All states $x_0\in\mathcal{X}_0$ will evolve to the N-step reachable set $\mathcal{R}_N(\mathcal{X}_0)$ in N steps.

The same definition of Maximal Reachable Set $\mathcal{R}_\infty(\mathcal{X}_0)$ can be introduced.

III. Invariant Sets

This section is also a recall section since we have discussed most of the concepts in previous lectures.

- **Invariant Sets**

Invariant sets

1. are computed for autonomous systems
2. for a given feedback controller $u=g(x)$ (i.e., we can form a closed loop autonomous system), provide the set of initial states whose trajectory will never violate the system constraints.

Definition (Positive Invariant Set):

A set $\mathcal{O}\subseteq\mathcal{X}$ is said to be a positive invariant set for the autonomous system $x(t+1)=f_a(x(t))$, subject to the constraints $x(t)\in\mathcal{X}$, if:

$$x(0)\in\mathcal{O}\ \Rightarrow\ x(t)\in\mathcal{O},\ \forall t\in\mathbb{N}^+$$

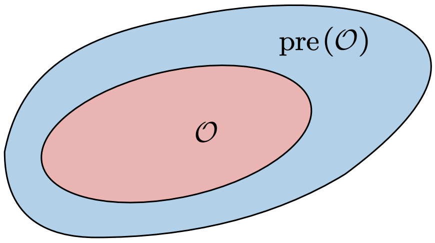
Definition (Maximal Positive Invariant Set):

The set $\mathcal{O}_\infty\subset\mathcal{X}$ is the maximal invariant set with respect to $\mathcal{X}$ if $0\in\mathcal{O}_\infty$, $\mathcal{O}_\infty$ is invariant and $\mathcal{O}_\infty$ contains all invariant sets that contain the origin.

Theorem (Geometric condition for invariance):

A set $\mathcal{O}$ is a positive invariant set if and only if

$$\mathcal{O} \subseteq \text{pre}(\mathcal{O})$$



We can prove from contradiction for both the necessary and sufficient conditions:

We can prove from contradiction for both the necessary and sufficient conditions:

(Necessity $\Rightarrow$):

If $\mathcal{O} \nsubseteq \text{pre}(\mathcal{O})$, then $\exists \bar{x} \in \mathcal{O}$ such that $\bar{x} \notin \mathcal{O}$. From the definition of $\text{pre}(\mathcal{O})$, $f(\bar{x}) \notin \mathcal{O}$ and thus $\mathcal{O}$ is not a positive invariant set.

(Sufficiency $\Leftarrow$):

If $\mathcal{O}$ is not a positive invariant set, then $\exists \bar{x} \in \mathcal{O}$ such that $f(\bar{x}) \notin \mathcal{O}$. This implies that $\bar{x} \in \mathcal{O}$ and $\bar{x} \notin \text{pre}(\mathcal{O})$ and thus $\mathcal{O} \nsubseteq \text{pre}(\mathcal{O})$.

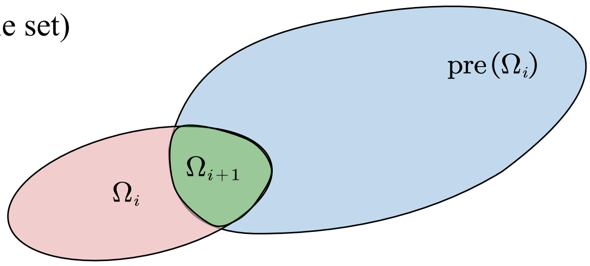
Also note that $\mathcal{O} \subseteq \text{pre}(\mathcal{O}) \Leftrightarrow \text{pre}(\mathcal{O}) \cap \mathcal{O} = \mathcal{O}$

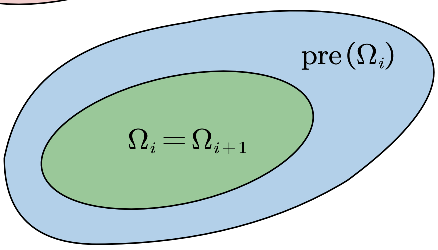
Algorithm (Conceptual Algorithm to Compute Invariant Set):

Input: $f_a, \mathcal{X}$ (system dynamics and feasible set)

Output: $\mathcal{O}_\infty$ (maximum invariant set)

$\Omega_0 \leftarrow \mathcal{X}$
loop
  $\Omega_{i+1} \leftarrow \text{pre}(\Omega_i) \cap \Omega_i$
  if $\Omega_{i+1} = \Omega_i$ **then**
    return $\mathcal{O}_\infty = \Omega_i$
    end if
  end loop





The algorithm generates the set sequence $\{\Omega_i\}$ satisfying $\Omega_{i+1} \subseteq \Omega_i$ for all $i \in \mathbb{N}$ and it terminates when $\Omega_{i+1} = \Omega_i$ so that Ω_i is the maximal positive invariant set $\mathcal{O}_\infty$ for $x(t+1) = f_a(x(t))$.

- **Control Invariant Sets**

Control invariant sets

1. are computed for systems subject to external inputs
2. provide the set of initial states for which **there exists** a controller such that the system constraints are never violated.

Definition (Control Invariant Set):

A set $\mathcal{C} \subseteq \mathcal{X}$ is said to be a control invariant set if for controlled system $x(k+1) = f(x(k), u(k))$:

$$x(t) \in \mathcal{C} \Rightarrow \exists u \in \mathcal{U} \text{ such that } f(x(t), u(t)) \in \mathcal{C}, \quad \forall t \in \mathbb{N}^+$$

Definition (Maximal Control Invariant Set):

The set $\mathcal{C}_\infty$ is said to be the maximal control invariant set for the system $x(t+1)=f(x(t),u(t))$ subject to the constraints $(x(t),u(t))\in\mathcal{X}\times\mathcal{U}$ if it is control invariant and contains all control invariant sets contained in $\mathcal{X}$

Theorem (Geometric Condition for Control Invariance):

A set $\mathcal{C}$ is a control invariant set if and only if $\mathcal{C}\subseteq \mathrm{pre}(\mathcal{C})$

Algorithm (Conceptual Algorithm to Compute Control Invariant Set):

$\Omega_0 \leftarrow \mathcal{X}$
loop
  $\Omega_{i+1} \leftarrow \mathrm{pre}(\Omega_i)\cap\Omega_i$
  if $\Omega_{i+1}=\Omega_i$ **then**
    return $\mathcal{C}_\infty=\Omega_i$
    end if
  end loop

Input: f, $\mathcal{X}$, and $\mathcal{U}$ (system dynamics and feasible set)

Output: $\mathcal{C}_\infty$ (maximum control invariant set)

Note: Everything generalized smoothly and we can prove them using the same scheme. The only drawback and biggest challenge in real practice are that the calculation of pre-set is much more complicated.

- **Invariant Sets and Control Invariant Sets: Determinedness**

The set $\mathcal{O}_\infty$ ($\mathcal{C}_\infty$) is **finitely determined** if and only if $\exists i\in\mathbb{N}$ such that $\Omega_{i+1}=\Omega_i$. The smallest element $i\in\mathbb{N}$ such that $\Omega_{i+1}=\Omega_i$ is called the determinedness index.

Remarks:

1. For the linear system with linear constraints, the sets $\mathcal{O}_\infty$ and $\mathcal{C}_\infty$ are polyhedra if they are finitely determined.
2. For autonomous systems, if the algorithm does not terminate, then $\mathcal{O}_\infty=\bigcap_{k\geq 0}\Omega_k$. If $\Omega_k=\emptyset$ for some integer k, then $\mathcal{O}_\infty=\emptyset$. More complicated for non-autonomous systems.
3. For all states contained in the maximal control invariant set $\mathcal{C}_\infty$ there exists a control law, such that the system constraints are never violated. This does not imply that there exists a control law that can drive the state into a user-specified target set.

- **Stabilizable Sets**

Observe that controllable sets $\mathcal{K}_N(\mathcal{O})$ where the target $\mathcal{O}$ is a control invariant set are special sets, which leads to our definition of the stabilizable sets.

Definition (N-step (Maximal) Stabilizable Set):

For a given control invariant set $\mathcal{O}\subseteq\mathcal{X}$, the N-step (Maximal) Stabilizable Set is the N-step (maximal) controllable set $\mathcal{K}_N(\mathcal{O})(\mathcal{K}_\infty(\mathcal{O}))$.

In addition to guaranteeing that from $\mathcal{K}_N(\mathcal{O})$ we reach $\mathcal{O}$ in N steps, one can ensure that once it has reached $\mathcal{O}$, the system can stay there at all future time instants.

Theorem (Set Evolution of $\mathcal{K}_N(\mathcal{X}_f)$)

Let the target set $\mathcal{X}_f$ be a control invariant subset of $\mathcal{X}$. Then:

1. The i-step controllable set $\mathcal{K}_i(\mathcal{X}_f)$, is control invariant and contained within the maximal control invariant set:
$$\mathcal{K}_i(\mathcal{X}_f) \subseteq \mathcal{C}_\infty$$
2. $\mathcal{K}_i(\mathcal{X}_f) \supseteq \mathcal{K}_j(\mathcal{X}_f)$ if $i > j$
3. The size $\mathcal{K}_i(\mathcal{X}_f)$ set stops increasing (with increasing i) if and only if the maximal stabilizable set is finitely determined and i is larger than its determinedness index $\bar{N}$
4. Furthermore:
$$\mathcal{K}_i(\mathcal{X}_f) = \mathcal{K}_\infty(\mathcal{X}_f) \text{ if } i \geq \bar{N}$$

IV. Reachability and Controllability – Robust Case

Now, we add disturbance to the system we considered, i.e. we now consider autonomous systems:
$$x(t+1) = f_a(x(t), w(t))$$
And the controlled system is subject to external inputs:
$$x(t+1) = f(x(t), u(t), w(t))$$
Both systems are subject to disturbance $w(t)$ and to the constraints:
$$x(t) \in \mathcal{X},\ u(t) \in \mathcal{U},\ w(t) \in \mathcal{W},\ \forall t \geq 0$$
The sets $\mathcal{X}$ and $\mathcal{U}$ and $\mathcal{W}$ are polytopes and contain the origin in their interior.

- **Robust Pre and Reach Sets Definition**

Definition (Robust Reach Sets):

For the autonomous system we denote the one-step robust reachable set:
$$\text{Reach}(\mathcal{X},\mathcal{W}) = \{x \in \mathbb{R}^n \mid\ \exists x(0) \in \mathcal{X}\ \exists w \in \mathcal{W} \text{ s.t. } x = f_a(x(0), w)\ \}$$
For the controlled system with inputs we denote the one-step robust reachable set as:
$$\text{Reach}(\mathcal{X},\mathcal{W}) = \{x \in \mathbb{R}^n \mid\ \exists x(0) \in \mathcal{X},\ \exists u \in \mathcal{U},\ \exists w \in \mathcal{W} \text{ s.t. } x = f(x(0), u, w)\ \}$$

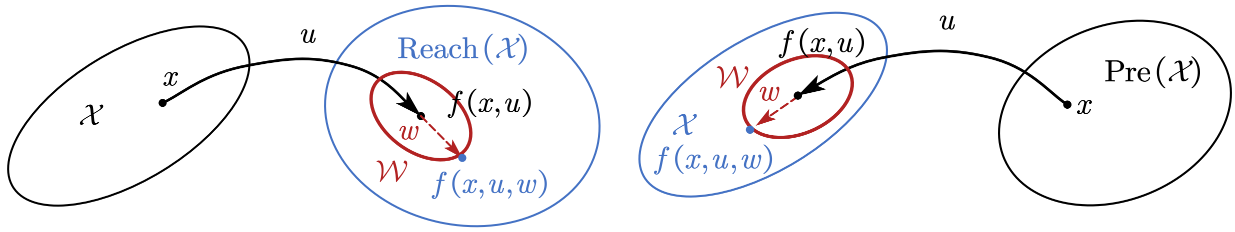
Definition (Robust Pre Sets):

Pre sets are the dual of one-step reachable sets. Therefore:

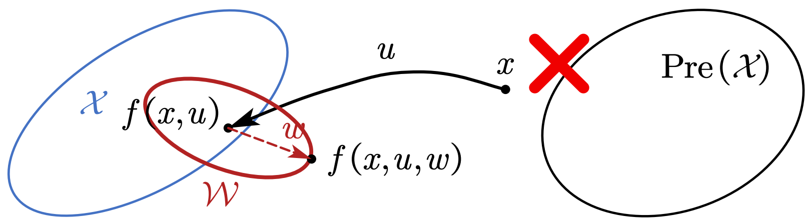
For the autonomous system the set:
$$\text{Pre}(\mathcal{X},\mathcal{W}) = \{x \in \mathbb{R}^n \mid f_a(x, w) \in \mathcal{X},\ \forall w \in \mathcal{W}\}$$
defines the set of system states that evolve into the target set $\mathcal{X}$ in one time step **for all possible disturbances** $w \in \mathcal{W}$

For the autonomous system, the set of states which can be robustly driven into (i.e. controlled into) the target set $\mathcal{X}$ in one time step is defined as:
$$\text{Pre}(\mathcal{X},\mathcal{W}) = \{x \in \mathbb{R}^n \mid\ \exists \mathcal{U} \text{ s.t. } f(x, u, w) \in \mathcal{X},\ \forall w \in \mathcal{W}\}$$
The concepts for robust reach set and pre set can be illustrated as below, note that we will consider the

case of linear systems and the disturbance that can be added directly using the superposition property of the linear systems:



One thing to note is if a point is mapped to $\mathcal{X}$, but with uncertainty disturbance, it runs out of $\mathcal{X}$, then it can not be considered as the element of pre set:



- **Robust Pre and Reach Sets Computation**

Pre Set Computation - Autonomous Systems

Assume the system is linear and autonomous:

$$x(t+1)=Ax(t)+w(t)$$

Let

$$\mathcal{X}=\{x|\ Hx\leq h\}$$

Then the set $\text{Pre}(\mathcal{S},\mathcal{W})$ is:

$$\text{Pre}(\mathcal{X},\mathcal{W})=\{x|\ HAx\leq h-Hw,\ \forall w\in\mathcal{W}\}$$

Note that it is for all $w\in\mathcal{W}$, which means we want the tightest bound for the term on the right, which is equivalent to saying:

$$\text{Pre}(\mathcal{X},\mathcal{W})=\left\{x\middle|\ HAx\leq\tilde{h}\right\}\ \text{ with }\ \tilde{h}_i=\min_{w\in\mathcal{W}}(h_i-H_iw)$$

Note that by using polyhedral notation, the robust pre set can be written as:

$$\begin{aligned}\text{Pre}(\mathcal{X},\mathcal{W})&=\{x\in\mathbb{R}^n|\ Ax+w\in\mathcal{X},\ \forall w\in\mathcal{W}\}=\{x\in\mathbb{R}^n|\ Ax\in\mathcal{X}\ominus\mathcal{W}\}\\&=(\mathcal{X}\ominus\mathcal{W})\circ A\end{aligned}$$

Reach Set Computation - Autonomous System

The set:

$$\text{Reach}(\mathcal{X},\mathcal{W})=\{y\in\mathbb{R}^n|\ \exists x\in\mathcal{X},\ \exists w\in\mathcal{W}\ \text{ s.t. } y=Ax+w\}$$

is obtained by applying the map A to the set $\mathcal{X}$ and then considering the effect of the disturbance $w\in\mathcal{W}$.

Write $\mathcal{X}$ in $\mathcal{V}$-representation

$$\mathcal{X}=\text{conv}(V)$$

Since the transformation is linear, the composition of the map A with the set $\mathcal{X}$, denoted as $A \circ \mathcal{X}$, is simply the convex hull of the transformed vertices:

$$A \circ \mathcal{X} = \text{conv}(AV)$$

Rewrite the set:

$$\text{Reach}(\mathcal{X}, \mathcal{W}) = \{y \in \mathbb{R}^n \mid \exists z \in A \circ \mathcal{X},\ \exists w \in \mathcal{W},\ \text{s.t. } y = z + w\}$$

We can use the definition of Minkowski sum and rewrite the Reach set as:

$$\text{Reach}(\mathcal{X}, \mathcal{W}) = (A \circ \mathcal{X}) \oplus \mathcal{W}$$

Pre Set Computation - Controlled Systems

Assume now the system is linear and controlled:

$$x(t+1) = Ax(t) + Bu(t) + w(t)$$

Let

$$\mathcal{X} = \{x \mid H_x x \leq h_x\},\ \mathcal{U} = \{u \mid H_u u \leq h_u\}$$

The pre set is:

$$\text{Pre}(\mathcal{X}, \mathcal{W}) = \left\{x \in \mathbb{R}^n \,\middle|\, \exists u \in \mathbb{R}^m \text{ s.t. } \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} h - Hw \\ h_u \end{bmatrix}, \forall w \in \mathcal{W} \right\}$$

Similarly, this can be compactly written as:

$$\text{Pre}(\mathcal{X}, \mathcal{W}) = \left\{x \in \mathbb{R}^n \,\middle|\, \exists u \in \mathbb{R}^m,\ \text{s.t. } \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} \tilde{h} \\ h_u \end{bmatrix} \right\} \text{ with } \tilde{h}_i = \min_{w \in \mathcal{W}} (h_i - H_i w)$$

Note that one can use polyhedral operations and rewrite the set as:

$$\text{Pre}(\mathcal{X}, \mathcal{W}) = ((\mathcal{X} \ominus \mathcal{W}) \oplus (-B \circ \mathcal{U})) \circ A$$

Reach Set Computation - Controlled System

For the controlled system, the reach set is:

$$\text{Reach}(\mathcal{X}, \mathcal{W}) = \{y \in \mathbb{R}^n \mid \exists x \in \mathcal{X},\ \exists u \in \mathcal{U},\ \exists w \in \mathcal{W},\ \text{s.t. } y = Ax + Bu + w\}$$

is obtained by applying the map A to the set $\mathcal{X}$ and then considering the effect of the input $u \in \mathcal{U}$ and of the disturbance $w \in \mathcal{W}$

We can use the polyhedral operations and rewrite $\text{Reach}(\mathcal{X}, \mathcal{W})$ as:

$$\text{Reach}(\mathcal{X}, \mathcal{W}) = (A \circ \mathcal{X}) \oplus (B \circ \mathcal{U}) \oplus \mathcal{W}$$

- **Summary for Robust Case**

In summary, for **linear systems with additive disturbances,** the robust pre set and reach set are the results of linear operations on the polytopes, and therefore are polytopes. By using the definition of Minkowski sum, Pontryagin difference, and affine operation on polyhedra, we obtain the following:

	$x(t+1) = Ax(t) + w(t)$	$x(t+1) = Ax(t) + Bu(t) + w(t)$
$\text{Pre}(\mathcal{X}, \mathcal{W})$	$(\mathcal{X} \ominus \mathcal{W}) \circ \mathcal{A}$	$((\mathcal{X} \ominus \mathcal{W}) \oplus (-B \circ \mathcal{U})) \circ A$
$\text{Reach}(\mathcal{X}, \mathcal{W})$	$(\mathcal{A} \circ \mathcal{X}) \oplus \mathcal{W}$	$(A \circ \mathcal{X}) \oplus (B \circ \mathcal{U}) \oplus \mathcal{W}$

Note:

The above summary also applies to the class of systems $x(t+1) = Ax(t) + Bu(t) + E\tilde{d}(t)$ where $\tilde{d}(t) \in \tilde{\mathcal{W}}$. This can be transformed into $x(t+1) = Ax(t) + Bu(t) + w(t)$ where we define $w \in \mathcal{W} = E \circ \tilde{\mathcal{W}}$