Model Predictive Control
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ESE6190, Spring 2023
Due: Feb 13

Exercise sheet 2
Optimization

Instructions: You are not allowed to use a calculator / computer unless specified.

Exercise 1 **Properties of Sets and Functions**

Let x_1, $x_2 \in \mathbb{R}^n$ be in the feasible set X of an optimization problem, i.e.

$$x_1,\, x_2 \in X = \{x \in \mathbb{R}^n \,|\, g_i(x) \le 0,\ i = 1, \dots m,\ h_j(x) = 0,\ j = 1, \dots q\} \ .$$

1. Can you find simple conditions on functions $g_i(x)$, $i = 1, \dots m$, and $h_j(x)$, $j = 1, \dots q$, such that the feasible set X is convex? [2 pts]
2. Let $z = \theta x_1 + (1-\theta)x_2$, $\theta \in [0, 1]$, be any point on the line segment between points x_1 and x_2. Can you find conditions on function $f(x)$ such that *'z is better than the worst of x_1 and x_2'*, i.e.

$$f(z) \le \max\{f(x_1), f(x_2)\} \quad ?$$

   If yes, can it happen that point z is better than both of them? Try to sketch such a situation. [5 pts]
3. Which property of the feasible set X would ensure that point $y = x_1 + x_2$ is feasible, given x_1, $x_2 \in X$? [3 pts]

Exercise 2 **Checking Convexity of Sets**

Which of the following sets are convex? Give reasons for your answers.

1. A slab, i.e. the set $\{x \in \mathbb{R}^n \,|\, \alpha \le a^T x \le \beta\}$ where $a \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$?
   Hint: Try to sketch this set for $n = 2$! [3 pts]
2. Let $s : \mathbb{R}^n \to \mathbb{R}$ be any function with domain $\operatorname{dom} s \subseteq \mathbb{R}^n$. Let set S be any subset of $\operatorname{dom} s$, though not necessarily a convex one. Is the set M defined as

$$M := \{x \,|\, \|x - y\| \le s(y) \text{ for all } y \in S\}$$

   a convex set? Note that $\|.\|$ denotes *any* norm on $\mathbb{R}^n$. [4 pts]
3. Consider s and S as above, again $\|.\|$ denotes *any* norm on $\mathbb{R}^n$. We consider the set $\tilde{M}$ defined as

$$\tilde{M} := \{x \,|\, \exists y \in S \text{ s.t. } \|x - y\| \le s(y)\}$$

   Is $\tilde{M}$ a convex set, assuming that s is a convex function? [4 pts]
4. The set of points closer to a given point x_0 than to a given set $Q \subseteq \mathbb{R}^n$, i.e.

$$\{x \,|\, \|x - x_0\|_2 \le \|x - y\|_2 \text{ for all } y \in Q\} \ ,$$

   where $\|.\|_2$ denotes the standard 2-norm (also called *Euclidean* norm)? [4 pts]

Exercise 3 1-Norm, ∞-Norm

Instead of the standard 2-norm, the 1-norm or the ∞-norm are sometimes used in the MPC cost function. In this exercise you should show that both a minimization problem with a 1-norm objective and an ∞-norm objective

$$\min_x \ \|Ax\|_p\,, \qquad p \in \{1, \infty\}$$

can be recast as a linear program (LP)

$$\begin{aligned} &\min_y \ b^T y \\ &\text{s.t. } \ Fy \le g \ , \end{aligned}$$

with vectors b, g and matrix F defined appropriately. [10 pts]

Note that we assume matrix $A \in \mathbb{R}^{N\times n}$ where its N row vectors are denoted as a_i^T, $i = 1, \ldots N$, i.e.

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_N^T \end{bmatrix} .$$

Exercise 4 Linear Regression with $1-$ and $\infty-$Norms

We now use the ideas from Exercise 3 to solve a linear regression problem: Use the command `linprog` from MATLAB to solve the LP's stemming from a reformulation of the linear regression problems

$$\min_x \ \|Ax - b\|_p\,, \qquad p \in \{1, \infty\} \ ,$$

where A is a random matrix and b is a random vector, e.g. created in MATLAB with the commands `A=rand(10,5)` and `b=rand(10,1)`. [10 pts]

Note: What we are actually doing here is to solve the overdetermined linear equation system $Ax = b$. The solution for the case $p = 2$ is called the linear least square solution and is given directly in MATLAB by `A\b`.

Optional: Try to install e.g. Yalmip (http://users.isy.liu.se/johanl/yalmip/) which helps you to model optimization problems in MATLAB in a more convenient way than before. Also, have a look at different LP solvers (under 'Solvers' on the latter website) that are more powerful than MATLAB's `linprog`.

Exercise 5 Quadratic Program

Consider the optimization problem:

$$\begin{aligned} \min \quad & \frac{1}{2}(x_1^2 + x_2^2 + 0.1x_3^2) + 0.55\, x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 1 \\ & x_1 \ge 0 \\ & x_2 \ge 0 \\ & x_3 \ge 0 \end{aligned}$$

1. Show that $x^* = (0.5, 0.5, 0)$ is a local minimum. [5 pts]
2. Is x^* also a global minimum? Explain why, or why not. [5 pts]