ESE 619 Model Predictive Control — **Homework 5: MPC Theory II**

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The solutions of this homework are entirely my own. I have discussed these problems with several classmates, they are: Yifan Xue and Shiming Liang

1. **Linear MPC Design**
Consider the following system

$$x(k+1) = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \tag{1}$$

The state and input constraints are

$$\mathcal{U} : -1 \le u(k) \le 1 \tag{2a}$$

$$\mathcal{X} : \begin{bmatrix} -15 \\ -15 \end{bmatrix} \le x(k) \le \begin{bmatrix} 15 \\ 15 \end{bmatrix} \tag{2b}$$

Design a 2-norm MPC with with $Q = I_2, R = 1$ and $N = 3$

Approach 1

1. **[4 pts]** Choose $\mathcal{X}_f = 0$ and a terminal weight P so that asymptotic stability is guaranteed for all $x_0 \in \mathcal{X}_f$
Solution:
Since the terminal set $\mathcal{X}_f = 0$. Any positive semidefinite P matrices would be fine for the system. In my implementation, I choose the $P = Q = I_2$.

2. **[4 pts]** Let $x(0) = \begin{bmatrix} 2 & -1 \end{bmatrix}^{\mathrm{T}}$ and plot the closed-loop state trajectory (plot $x_2(k)$ vs. $x_1(k)$ on one figure) as well as the open-loop trajectories predicted by the MPC for 25 simulation steps.
Solution: The required figure is shown below

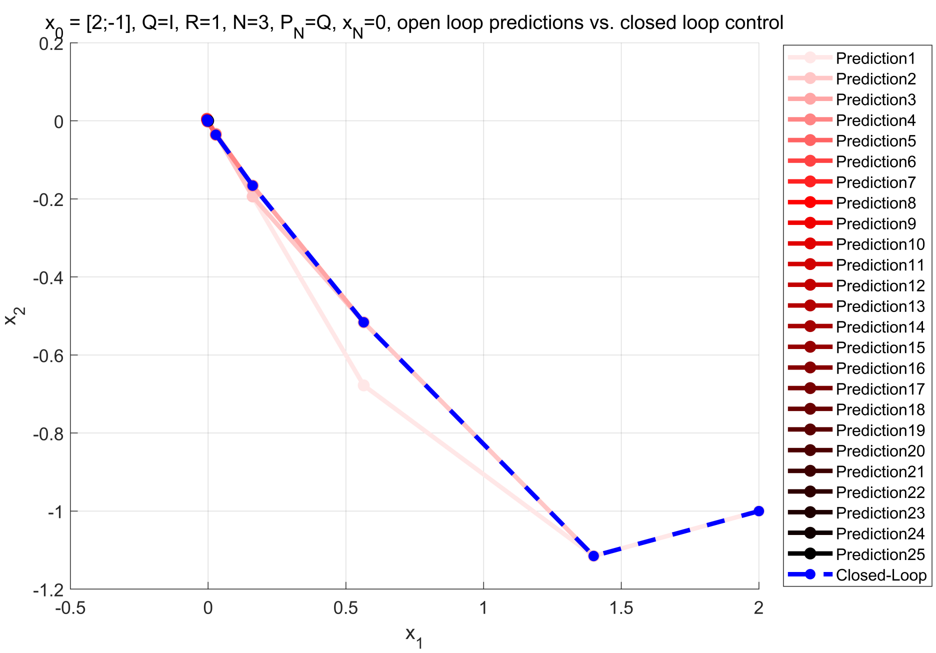

Figure 1: Open loop prediction vs. closed loop control: $Q = I_2, R = 1, N = 3, P_N = Q, \mathcal{X}_f = 0$

3. **[4 pts]** For the same x(0) $x(0) = \begin{bmatrix} 2 & -1 \end{bmatrix}^{\mathrm{T}}$, analyze the mismatch between predicted vs closed-loop trajectories for $N = 10$ in the MPC design.
Solution:

The required figure is shown below.
We can see that the closed loop trajectory overlaps with the open loop prediction. That is to say, these two matches. It is because we take a long enough horizon ($N = 10$) so that at the very first prediction can already reach the terminal set we want.

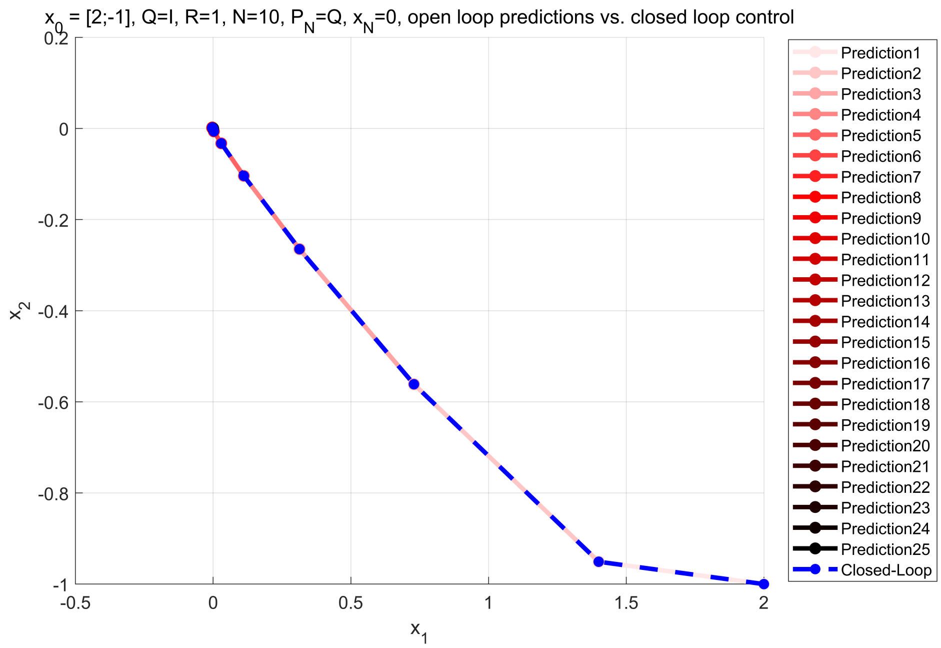

Figure 2: Open loop prediction vs. closed loop control: $Q = I_2, R = 1, N = 10, P_N = Q, \mathcal{X}_f = 0$

Approach 2

4. **[3 pts]** Now choose P to be the solution to the discrete-time algebraic Riccati equation (DARE), which is the infinite LQR solution. Then, set $\mathcal{X}_f$ to be the maximal invariant set $\mathcal{O}_\infty$ defined by the closed loop solution $x(k+1) = (A + BF_\infty)\, x(k)$. Use MPT3 tools to compute and plot $\mathcal{O}_\infty$.
Solution: The required figure is shown below

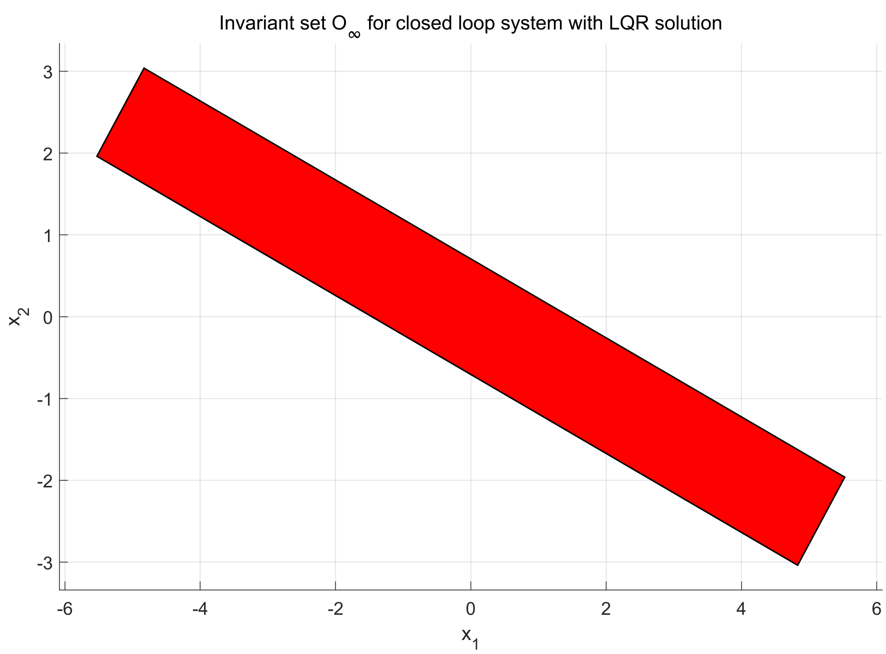

Figure 3: maximal invariant set $\mathcal{O}_\infty$ for closed loop system with LQR solution

5. **[4 pts]** Let $x(0) = \begin{bmatrix} 2 & -1 \end{bmatrix}^{\mathrm{T}}$ and plot the closed-loop state trajectory as well as the open loop trajectories predicted by the MPC with these new terminal cost and terminal constraint chosen in part 4 for $N = 3$ for 25 simulation steps. Comment on any differences.
Solution:
The required figure is shown below. We can see that with the terminal cost and terminal set given by the LQR solution, the closed loop trajectory overlaps with the open loop prediction. That is to say, these two matches. It is becuase from 1.4 we know that the initial condition $x(0) = \begin{bmatrix} 2 & -1 \end{bmatrix}^{\mathrm{T}}$ is in the maximal invariant set. So there should be no difference between the open loop and close loop results.

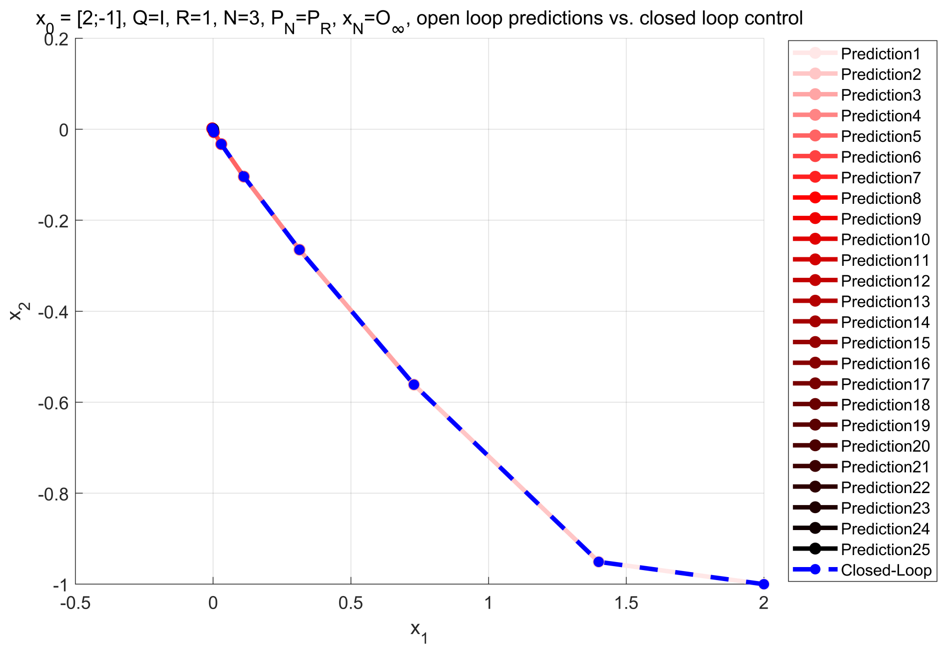

Figure 4: Open loop prediction vs. closed loop control: $Q = I_2, R = 1, N = 3, P_N = P_R, \mathcal{X}_f = \mathcal{O}_\infty$

Approach 3

6. **[4 pts]** Instead of using the unconstrained LQR solution to design our MPC, now use a different stabilizing controller $u = -Kx$ where K is*:

$$K = \begin{bmatrix} 1.595 & 2.35 \end{bmatrix}$$

What is the maximal positive invariant set $\mathcal{O}_\infty$ under this control law? How does it compare to $\mathcal{O}_\infty$ computed in part 4? Find a new terminal cost such that it is a Lyapunov Function for $x \in \mathcal{X}_f = \mathcal{O}_\infty$. You should be able to design a quadratic terminal cost with weight P by simply using the function `dlyap`

*Aside: the gain matrix K was chosen by pole placement Matlab code `place(A,B,[0.1, 0.25])`
Solution:
The required figure is shown below. We can see that with this gain derived by pole placement, the maximal invariant set becomes smaller.
The terminal cost P in this case can be derived by solving the Lyapunov equation for the closed loop system with the given control gain. i.e. define the autonomous close loop system $A_{\mathrm{cl}} \triangleq (A - BK)$ and get P by `P_K = dlyap(Acl, Q+K'*R*K)`

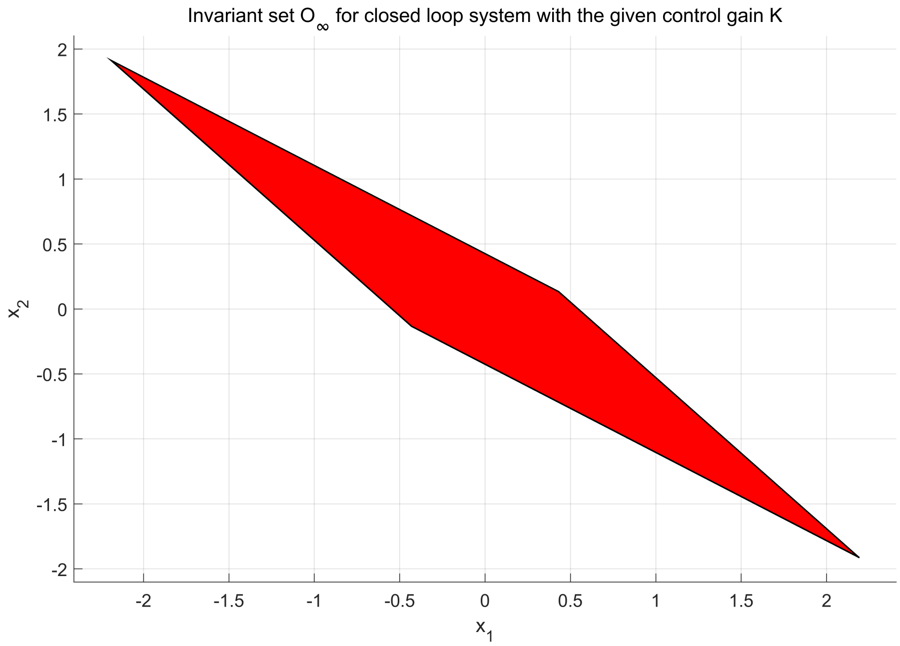

Figure 5: maximal invariant set $\mathcal{O}_\infty$ for closed loop system with $K = \begin{bmatrix} 1.595 & 2.35 \end{bmatrix}$

7. Let $x(0) = \begin{bmatrix} 2 & -1 \end{bmatrix}^{\mathrm{T}}$ and plot the closed-loop state trajectory as well as the open-loop trajectories predicted by the MPC with these new terminal cost and terminal constraint from part 6 for $N = 3$ for 25 simulation steps. How does this compare to the other solutions?

   Solution:

   The required figure is shown below. We can see now the there are some deviations from the open loop predictions and closed loop trajectory. It is because in 1.6 we can see that the terimnal set for this case is smaller and our initial condition does not lie in the terminal set. Only when the state goes into the terminal set after many steps will the two trajectory matches.

   Compared to other solutions, I think the LQR solution is better since its invariant set is larger and the LQR solution makes more accurate prediction and better closed loop trajectory.

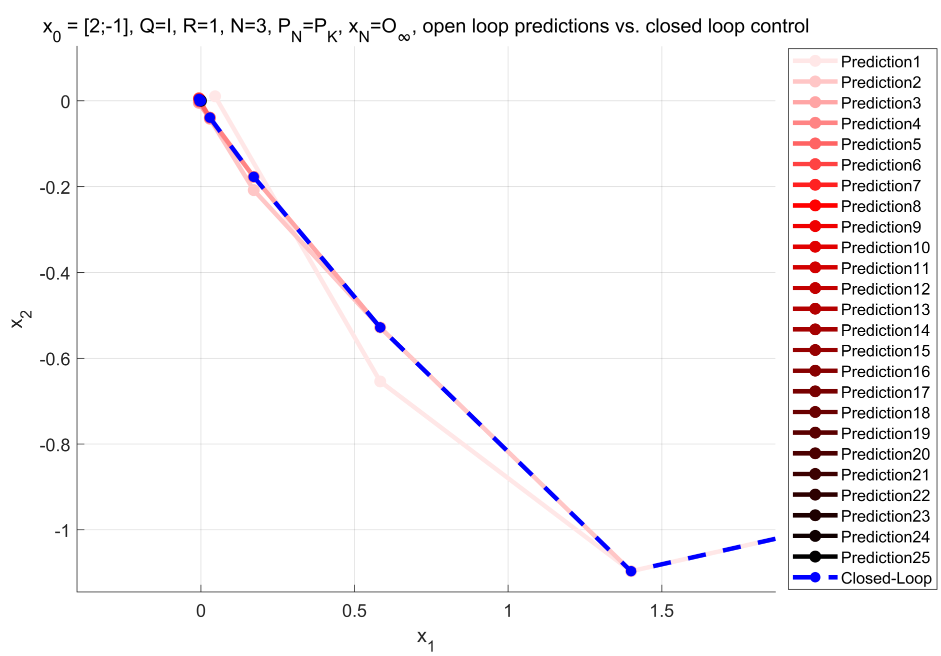

Figure 6: Open loop prediction vs. closed loop control: $Q = I_2, R = 1, N = 3, P_N = P_K, \mathcal{X}_f = \mathcal{O}_\infty$

Persistent Feasibility

8. [**4 pts**] Use now $N = 1$. Can you choose $\mathcal{X}_f$ so that persistent feasibility is guaranteed for all x_0 belonging to the maximal control invariant set $\mathcal{C}_\infty$ Compute and plot $\mathcal{C}_\infty$ using MPT3.
Solution:
The required figure is shown below.

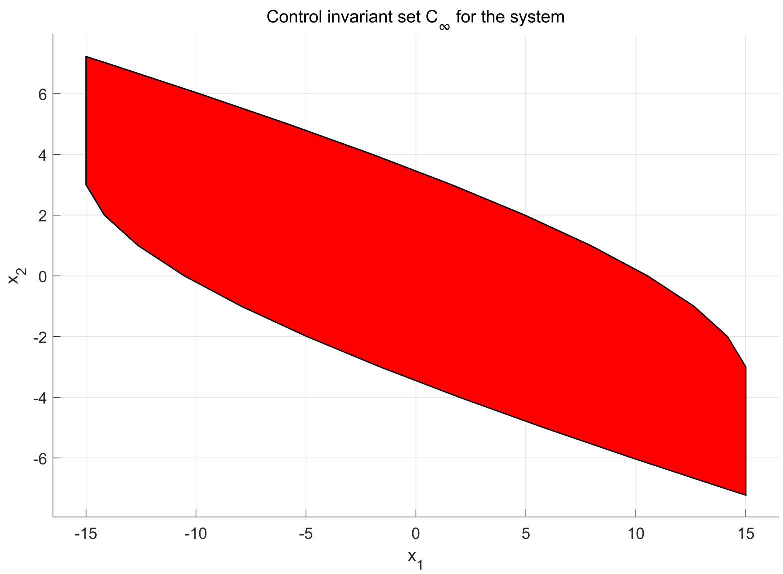

Figure 7: maximal control invariant set $\mathcal{C}_\infty$ for the system

9. [**4 pts**] With the design in the previous part 8 you have a very short horizon and persistent feasibility in $\mathcal{X}_0 = \mathcal{C}_\infty$ It seems a great MPC design with just $N = 1$! What are we missing compared to an infinite horizon constrained optimal control? Let $x(0) = \begin{bmatrix} 2 & -1 \end{bmatrix}^{\mathrm{T}}$ and plot the closed-loop state trajectory as well as the open-loop trajectories predicted by the MPC with $\mathcal{X}_f = \mathcal{C}_\infty$ and the terminal cost given by the infinite LQR solution.
Solution:
The required figure is shown below.

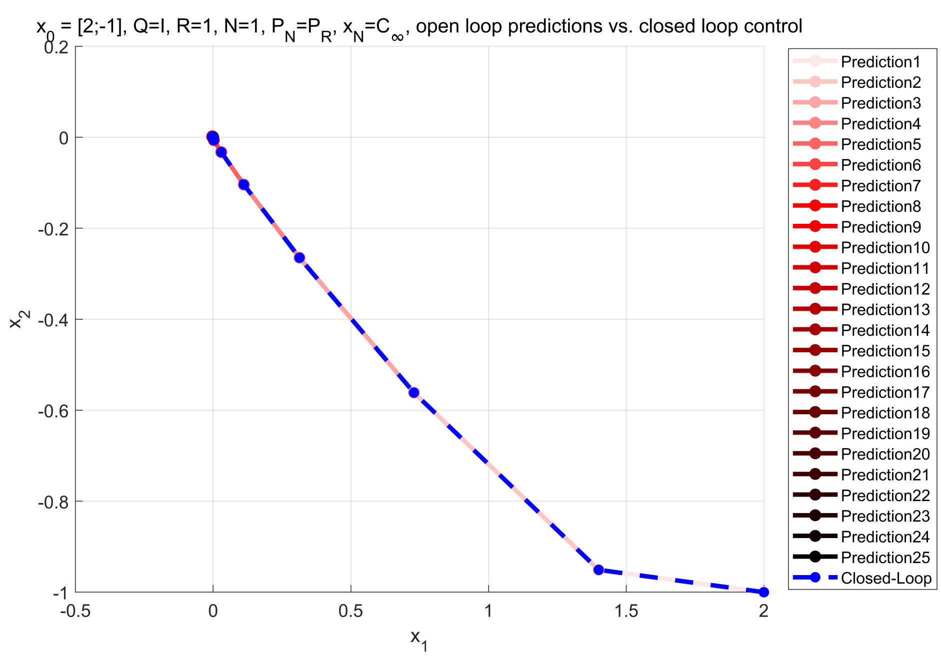

Figure 8: Open loop prediction vs. closed loop control: $Q = I_2, R = 1, N = 3, P_N = P_R, \mathcal{X}_f = \mathcal{C}_\infty$

Since the initial condition lies in the maximal control invariant set, the feasibility is guaranteed at any time. With persistent feasibility the MPC could work even with very short horizon $N = 1$. Compared with infinite horizion constrained optimal control, while the feasibility is guaranteed all the time, we actually lose the stability guarantee becuase the terminal cost might not be the Lyapunov function in the whole terminal set and the condition of $p(x_{k+1}) - p(x_k) \leq -q(x_k, v(x_k))$ might not hold.

Therefore, we need to carefully select the terminal cost, in the plot above we should select the LQR solution to be the cost, the stabilizing region is smaller than the control invariant set, but our initial condition lie in the scope of stabilizing region, so the result is almost the same as the LQR case.

2. **Terminal Invariant Set**

Consider the linear system

$$x^+ = \begin{bmatrix} 0.5 & 0 \\ 4 & 0.8 \end{bmatrix} x + \begin{bmatrix} 0.3 & 0.2 \\ -0.6 & 0.9 \end{bmatrix} u$$

with constraints on the input $\|u\|_\infty \le 1$

1. **[3 pts]** What is the maximum control invariant set for this system? Justify your answer.
   Solution:
   It is obvious that there are no state constraints. Therefore, the maximum control invariant set should be the whole space, i.e. $\mathbb{R}^2$. With any initial state in $\mathbb{R}^2$, any control law leads to the next state also being in $\mathbb{R}^2$.

2. **[5 pts]** Consider the following standard MPC optimization problem, and let $\pi(x)$ be the resulting receding-horizon control law.

$$\begin{aligned} \min\ & x_N^{\mathrm{T}} Q_f x_N + \sum_{i=0}^{N-1} x_i^{\mathrm{T}} Q x_i + u_i^{\mathrm{T}} R u_i \\ \text{s.t.}\ \ & x_{i+1} = A x_i + B u_i \\ & u_i \in \mathcal{U} \quad i \in \{0, \ldots, N-1\} \\ & x_N \in \mathcal{X}_f \\ & x_0 = x \end{aligned}$$

   Describe how to choose a terminal control law, K_f, terminal weight Q_f and terminal set $\mathcal{X}_f$ so that the closed-loop system $x^+ = Ax + B\pi(x)$ has a maximal invariant set equal to that given in part 1 and is asymptotically stable.
   Solution:
   From 2.1, we already know that the maximal control invariant set is $\mathbb{R}^2$. Therefore, we can select the terminal set to be $\mathbb{R}^2$, and we only should figure out how to make the system to be asymptotically stable.
   We can see that the autonomous system (without control) itself is already stable, so we can set the terminal control law $K_f = 0$. The terminal weight Q_f can be selected from solving the Lyapunov equation using A and Q

3. **Terminal Invariant Set**

The following 1-dimensional system

$$x_{k+1} = 1.5x_k + u_k$$

with constraints

$$\begin{aligned} -1 \le x_k &\le 1, \qquad k = 1, 2, \ldots \\ -0.1 \le u_k &\le 0.1, \qquad k = 0, 1, \ldots \end{aligned}$$

is given. In this exercise, we consider linear state feedback controllers of the form $u_k = Kx_k$, which result in the following closed loop systems:

$$x_{k+1} = (A + BK)x_k$$

1. **[3 pts]** Show that for $K = -2$, the maximal positively invariant set $\mathcal{O}$ is symmetric, i.e. it takes the form $\mathcal{O} = [-a, a]$, for some $a \in \mathbb{R}_+$

   Solution:

   when $K = -2$, the closed loop system is $x_{k+1} = 1.5x_k - 2x_k = -0.5x_k$ and we have:

   $$\begin{cases} -1 \le x_k \le 1 \\ -0.1 \le -2x \le 0.1 \end{cases} \Rightarrow \mathcal{X}_{\text{cl}} = [-0.05, 0.05] = \Omega_0$$
   $$\text{pre}\,(\Omega_0) = [-0.1, 0.1] \Rightarrow \text{pre}\,(\Omega_0) \cap \Omega_0 = \Omega_1 = [-0.05, 0.05] = \Omega_0$$
   $$\Rightarrow \mathcal{O}_\infty = \Omega_0 = [-0.05, 0.05]$$

   Therefore, for $K = -2$, the maximal positively invariant set $\mathcal{O}$ is symmetric and is $[-0.05, 0.05]$

2. **[6 pts]** Find the K that maximizes the parameter a, which defines the invariant set $\mathcal{O} = [-a, a]$

   Hint: You may assume that the set $\mathcal{O}$ is symmetric for any K.

   Solution:

   First of all, we need to ensure that the system does not diverge, i.e. at least marginally stable:

   $$-1 \le 1.5 + K \le 1 \Rightarrow -2.5 \le K \le -0.5$$

   Also, this means the initial feasible set Ω_0 would be the maximal positively invariant set, because the dynamics is apparently a contraction mapping (if $K \in (-2.5, -0.5)$) or marginally stable (if $K = -2.5$ or $K = -0.5$) so that $\text{pre}\,(\Omega) \cap \Omega = \Omega$ for all Ω. Then, consider the initial feasible set:

   $$\begin{cases} -1 \le x_k \le 1 \\ -0.1 \le Kx \le 0.1 \end{cases} \Rightarrow \mathcal{X}_{\text{cl}} = [-1, -1] \cap \left[\frac{-0.1}{K} \le x \le \frac{0.1}{K}\right] = \Omega_0$$

   To maximize the invariant set, it is obvious that we had better let $\left\|\frac{0.1}{K}\right\|$ as large as possible. Therefore:

   $$K = -0.5 \Rightarrow \mathcal{O} = [-0.2, 0.2]$$

3. **[4 pts]** Does there exist $\mathcal{X}_f, Q, R$ and P such that the control law of the MPC problem defined in (3) guarantees asymptotic stability? Justify your answer!

   $$\begin{aligned} \min_{\mathcal{U}} \;\; & x_N^{\mathrm{T}} P x_N + \sum_{i=0}^{N-1} x_i^{\mathrm{T}} Q x_i + u_i^{\mathrm{T}} R u_i \\ \text{s.t.} \;\; & x_0 = x(0) \\ & x_{k+1} = 1.5x_k + u_k \quad k = 0, 1, \ldots, N-1 \\ & -1 \le x_k \le 1 \quad k = 0, 1, \ldots, N-1 \\ & -0.1 \le u_k \le 0.1 \quad k = 0, 1, \ldots, N-1 \\ & x_N \in \mathcal{X}_f \end{aligned} \tag{3}$$

   Hint 1: It is not necessary to explicitly compute Q, R and P.

   Hint 2: If you were not able to solve part 2, use $K = -1$ and $\mathcal{O} = [-0.1, 0.1]$

   Solution: Yes, there exist, and we can deisgn the $\mathcal{X}_f, Q, R$ and P using the following procedure:

i. Choose a control gain K which is stabilizing, i.e. $K \in (-2.5, -0.5)$, and define $A_{\text{cl}} = A + BK$ can calculate the invariant set $\mathcal{O}_K$ for A_{cl}, let terminal set $\mathcal{X}_f = \mathcal{O}_K$

ii. Choose any $Q \succeq 0, R \succ 0$

iii. Set P to satisfy the following inequality:

$$(A + BK)^{\mathrm{T}} P (A + BK) - P \leq -\left(Q + K^{\mathrm{T}} R K\right)$$

A simple way to satisfy this relationship is to set P as a solution of Lyapunov equation contructed with A_{cl} and $Q + K^{\mathrm{T}} R K$