ESE 619 Model Predictive Control
University of Pennsylvania, Spring 2023
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Homework 1: Dynamical Systems
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The solutions of this homework are entirely my own. I have discussed these problems with several classmates, they are: Yifan Xue and Shiming Liang

1. **Analysis of LTI Discrete-Time Systems**

A Consider the discrete-time dynamic system with the following state space representation:

$$\begin{aligned}\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} &= \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \alpha & 0 \\ 0 & \frac{1}{2} & -\frac{5}{4} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 4 \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}\end{aligned} \tag{1}$$

(1) Let $\alpha = 0$

(a) Is the system stable? [2 pts]

Solution: If $\alpha = 0$, the eigenvalue of system dynamics is:

$$A = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{5}{4} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{3} \end{bmatrix} \Rightarrow \lambda_A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{2} & -\frac{5}{4} \end{bmatrix}$$

$\max \left|\lambda_A^i\right| = \frac{5}{4} > 1$, hence the system is unstable

(b) Which states of the system belong to the controllable subsystem? [2 pts]

Solution: The controllability (or, reachability) matrix $\mathcal{C}$ for this system is:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 1 & -\frac{1}{2} & \frac{1}{4} \\ 4 & -6 & 8 & -\frac{41}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The controllable subspace is the range space of controllability matrix, it is obvious that:

$$\mathcal{X}_{\mathcal{C}} = \mathrm{Im}(\mathcal{C}) = \mathrm{span}\left\{\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}\right\}$$

That is, state x_2 and x_3 belong to the controllable subsystem

(c) Which states of the system belong to the observable subsystem? [2 pts]

Solution: The observability matrix $\mathcal{O}$ for this system is:

$$\mathcal{O} = \begin{bmatrix} C^{\mathrm{T}} & (CA)^{\mathrm{T}} & (CA^2)^{\mathrm{T}} & (CA^3)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{6} & -\frac{5}{12} & 0 \\ \frac{1}{18} & -\frac{7}{24} & \frac{25}{48} & 0 \\ \frac{1}{54} & \frac{13}{32} & \frac{-125}{192} & 0 \end{bmatrix}$$

The observable subspace is the range space of observability matrix, it is obvious that:

$$\mathcal{X}_{\mathcal{O}} = \mathrm{Im}(\mathcal{O}) = \mathrm{span}\left\{\begin{bmatrix} 1 & 0 & 0 & \frac{5}{24} \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} 0 & 1 & 0 & -\frac{1}{24} \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} 0 & 0 & 1 & -\frac{17}{12} \end{bmatrix}^{\mathrm{T}}\right\}$$

That is, state x_1, x_2 and x_3 belong to the observable subsystem

(2) The reachable subspace is defined as the set of states the system can reach starting from the origin. Use your knowledge of controllability to compute the reachable subspace as a function of the parameter α. [4 pts]
Solution: The reachable subspace is the range space of controllability matrix, the controllability matrix is calculted as:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 4\alpha+1 & -8\alpha-\frac{1}{2} & \frac{23\alpha}{4}+4\alpha\left(\frac{\alpha}{2}+\frac{25}{16}\right)+\frac{1}{4} \\ 4 & -6 & 2\alpha+8 & -\frac{13\alpha}{2}-\frac{41}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is already rank-deficient, but would generally have rank 2 except for the following case which would make it lose another rank:

$$\frac{-2}{4} = \frac{4\alpha+1}{-6} = \frac{-8\alpha-\frac{1}{2}}{2\alpha+8} = \frac{\frac{23\alpha}{4}+4\alpha\left(\frac{\alpha}{2}+\frac{25}{16}\right)+\frac{1}{4}}{-\frac{13\alpha}{2}-\frac{41}{4}} \Rightarrow \alpha = \frac{1}{2}$$

Therefore, the reachable subspace can be finally concluded as follows:

$$\mathcal{X}_{\mathcal{R}} = \operatorname{Im}(\mathcal{C}) = \begin{cases} \operatorname{span}\left\{\begin{bmatrix} 0 & -1 & 2 & 0 \end{bmatrix}^{\mathrm{T}}\right\} & \alpha = \frac{1}{2} \\ \operatorname{span}\left\{\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}\right\} & \text{else} \end{cases}$$

(3) Now let $\alpha = \frac{1}{2}$. Is it possible to design a stabilizing controller for the system (1)? [4 pts]
Solution: To judge the stabilizability of the system A with input B, since the system matrix is coupled, we need to use coordinate transformation to decouple the modes, first solve the eigenvalues and (generalized) eigenvector for the system matrix:

$$A = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{5}{4} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{3} \end{bmatrix} \Rightarrow \lambda_A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{3}{2} & -\frac{1}{4} \end{bmatrix}, T_A = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

The transformed modal expression is then:

$$z(k+1) = TAT^{-1}z(k) + TBu(k) = \begin{bmatrix} \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{4} \end{bmatrix} z(k) + \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \end{bmatrix} u(k) = Jz(k) + \hat{B}u(k)$$

A system is stabilizable if and only if all of its uncontrollable modes are stable, first find the controllable modes of the system:

$$\mathcal{C}_z = \begin{bmatrix} \hat{B} & J\hat{B} & J^2\hat{B} & J^3\hat{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -5 & -\frac{15}{2} & \frac{45}{4} & -\frac{135}{8} \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \operatorname{Im}(\mathcal{C}_z) = \operatorname{span}\left\{\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}\right\}$$

It can be observed that mode z_3 is in the the controllable subsystem and hence this subsystem is stabilizable, the uncontrollable modes z_1, z_2 and z_4 are inherently stable since the corrsponding eigenvalue is in the unit circle in complex plane ($\left|\frac{1}{3}\right| < 1, \left|\frac{1}{4}\right| < 1$). Therefore, if $\alpha = \frac{1}{2}$, the system is stabilizable.

B Tick the correct answers:

(1) Consider a SISO system with exactly one uncontrollable mode. [2 pts]

- ☐ The system can be stabilized using feedback if the uncontrollable mode is observable.
  The system can be stabilized using feedback if the uncontrollable mode is stable.
- ☑ The system can be stabilized using feedback if the uncontrollable mode is stable.
- ☐ The controlled closed-loop system is asymptotically stable if all its eigenvalues lie in the closed unit disc.
  The controlled closed-loop system is asymptotically stable if all its eigenvalues lie in the open unit disc.
- ☐ A system that is not fully controllable can never be stabilized using feedback.
  Even though the system is uncontrollable, it can be stabilized using feedback if the uncontrollable mode is stable.

(2) Consider a linear system of the form

$$x(k+1) = Ax(k) + Bu(k) \tag{2}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}$ and the system matrix A is diagonal:

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{bmatrix} \tag{3}$$

The system is controllable if and only if [4 pts]

- ☐ all elements of B are non-zero.
  If eigenvalues of A are repeated, obviously the controllability matrix would lose rank and thus the system is not controllable
- ☐ all eigenvalues of A are non-zero, and all elements of B are non-zero.
  If eigenvalues of A are repeated, obviously the controllability matrix would lose rank and thus the system is not controllable
- ☑ all eigenvalues of A are distinct, and all elements of B are non-zero.

2. **Discretization of a LTI continuous-time state-space model**

Consider the following continuous-time dynamic system:

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} -5 & 2.7 \\ -3.1 & 1.5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 4 & 2.1 \\ 1.1 & 3 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\
y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
\end{aligned}
$$

Discretize the system with $T_s = 1$ using the formulas in the slides and check the result with the Matlab function `c2d`. Compare the outputs of the continuous and the discretized model in a dynamic simulation, starting from the same initial state and applying the same input. [5 pts]

Solution: Firstly, apply the formulas in the slides, we have:

$$
\begin{aligned}
A &= e^{A_c T_s} = \begin{bmatrix} -3.933 & 0.6604 \\ -0.7583 & 1.1967 \end{bmatrix} \\
B &= \int_0^{T_s} e^{A^c(T_s - \tau\prime)} B^c \mathrm{d}\tau\prime = (A^c)^{-1}(A - I)B^c = \begin{bmatrix} 0.3856 & 1.4821 \\ -1.0809 & 2.3948 \end{bmatrix} \\
C &= C^c = \begin{bmatrix} 1 & 1 \end{bmatrix}
\end{aligned}
$$

Use MATLAB function `c2d`, we get the same results. Next, we need to compare the output of the continuous and discretized model, the simulation setting is the following:

Initial condition $x(t)|_{t=0} = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$. In order to satisfy the assumption of zero-th hold and for simplicity, the input control is the constant input $u = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$, the simulation time is $t = 10$ s.

The simulation result is shown below. We can see that the output of exact discretization system and continous system maetches well, showing the correctness of the discretization. The MATLAB code for simulation and plotting can be found in the appendix part

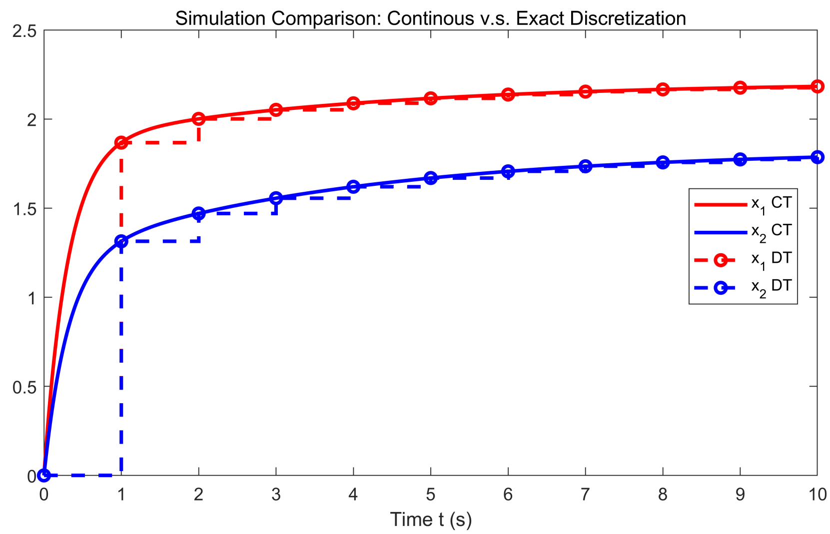

Figure 1: Simulation comparison of continuous system and exact discretized system

3. **Sum of Lyapunov functions** Let $V_i(x) = x^{\mathrm{T}} P_i x$ be a Lyapunov function for the system $x^+ = Ax$ for $i = 1, 2$, with a rate of decrease of $x^{\mathrm{T}}\Gamma x$, i.e.:

$$V_i\left(x^+\right) - V_i(x) \leq -x^{\mathrm{T}}\Gamma x$$

Show that $V(x) = \alpha V_1(x) + (1-\alpha) V_2(x)$ is also a Lyapunov function with a rate of decrease of $x^{\mathrm{T}}\Gamma x$ for any $\alpha \in [0, 1]$

Solution: Given $V_i(x)$ is Lyapunov function, aprat from the given decrease bound, we also have:

$$V_i(0) = 0, \quad \text{and} \quad V_i(x) > 0, \ \forall x \in \Omega \backslash \{0\}$$

Now we prove $V(x) = \alpha V_1(x) + (1-\alpha) V_2(x)$ is also a Lyapunov function, firstly for cases that $\alpha = 1$ or $\alpha = 0$, it is obvious that $V(x)$ is just $V_1(x)$ or $V_2(x)$ and the proposition obviously holds, for $\alpha \in (0, 1)$, we have:

$$\begin{aligned}
V(0) &= \alpha V_1(x) + (1-\alpha) V_2(x) = \alpha \times 0 + (1-\alpha) \times 0 = 0 \\
V(x) &= \underbrace{\alpha}_{>0} \underbrace{V_1(x)}_{>0} + \underbrace{(1-\alpha)}_{>0} \underbrace{V_2(x)}_{>0} > 0, \ \forall x \in \Omega \backslash \{0\} \\
V\left(x^+\right) - V(x) &= \alpha \left[V_1\left(x^+\right) - V_1(x)\right] + (1-\alpha)\left[V_2\left(x^+\right) - V_2(x)\right] \\
&\leq -\alpha x^{\mathrm{T}}\Gamma x - (1-\alpha) x^{\mathrm{T}}\Gamma x = -x^{\mathrm{T}}\Gamma x
\end{aligned}$$

Hence we prove that $V(x)$ is also a Lyapunov function with a rate of decrease of $x^{\mathrm{T}}\Gamma x$

4. **Controllable, Observable, etc. and optimal control**

Consider the following discrete-time system with linear dynamics:

$$x_{k+1} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{A(\alpha)} x_k + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u_k \tag{4a}$$

where $\alpha \in \mathbb{R}$ is a parameter. The output of the system is given as:

$$y_k = h(x_k) = \sum_{l=1}^{3} l \tanh(x_{k,l}) \tag{4b}$$

where $x_{k,l}$ denotes the l-th state at time k.

(a) In this part, the task is to analyze system (4)

(i) For which values of α is the system controllable? [2 pts]

Solution: The system is controllable if and only if the controllability matrix is full rank, the controllability matrix is calculated as:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \alpha+1 \\ 1 & 1 & 1 \end{bmatrix}$$

It is obvious that $\text{rank}(\mathcal{C}) = 2, \forall \alpha$, hence for any of the α, the system is never controllable.

(ii) Linearize the output mapping $h(x_k)$ around $\bar{x} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$ and $\bar{u} = 0$. Determine C, D in the linearized output mapping $y_k = Cx_k + Du_k$. [3 pts]

Solution: The linearization is done as below:

$$y = \underbrace{h(\bar{x}, \bar{u})}_{=\bar{y}} + \underbrace{\left.\frac{\partial h}{\partial x^{\mathrm{T}}}\right|_{x=\bar{x},u=\bar{u}}}_{=C} (x - \bar{x}) + \underbrace{\left.\frac{\partial h}{\partial u^{\mathrm{T}}}\right|_{x=\bar{x},u=\bar{u}}}_{=D} (u - \bar{u})$$

Hence we have:

$$\begin{aligned} C &= \left.\frac{\partial h}{\partial x^{\mathrm{T}}}\right|_{x=\bar{x},u=\bar{u}} = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \end{bmatrix}_{x=\bar{x},u=\bar{u}} \\ &= \begin{bmatrix} 1 - \tanh^2 x_1 & 2 - 2\tanh^2 x_2 & 3 - 3\tanh^2 x_1 \end{bmatrix}_{x=\bar{x},u=\bar{u}} \\ &= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \\ D &= \left.\frac{\partial h}{\partial u^{\mathrm{T}}}\right|_{x=\bar{x},u=\bar{u}} = 0 \end{aligned}$$

The linearized output mapping would then be:

$$\begin{aligned} y_k &= \bar{y} + Cx_k + Du_k = 0 + \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_{k,1} \\ x_{k,2} \\ x_{k,3} \end{bmatrix} + 0 \\ &= x_{k,1} + 2x_{k,2} + 3x_{k,3} \end{aligned}$$

For the remainder of this question you can use:

$$C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad D = 0 \tag{5}$$

(iii) For which values of α is the linearized system $x_{k+1} = A(\alpha)\, x_k + Bu_k, y_k = Cx_k + Du_k$ with (C, D) as in (5), observable? For which values of α is it detectable?

Solution: The system is observable if and only if the observability matrix is full rank, the observability matrix is calculated as:

$$\mathcal{O} = \begin{bmatrix} C^{\mathrm{T}} & (CA)^{\mathrm{T}} & (CA^2)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \alpha & 2 \\ 1 & \alpha^2 & \alpha+2 \end{bmatrix} \Rightarrow \det(\mathcal{O}) = 2\alpha - \alpha^2$$

Matrix $\mathcal{O}$ is rank deficient if $2\alpha - \alpha^2 = 0$, i.e. $\alpha = 0$ or $\alpha = 2$. Therefore, the linearized system is observable for any $\alpha \in \mathbb{R} \backslash \{0, 2\}$

As for detectability, if the system is observable, then it is detectable; if the system is not observable, it is detectable if and only if all of its unobservable modes are stable. Therefore, for $\alpha \in \mathbb{R} \backslash \{0, 2\}$, the system is detectable. We only need to check the cases where $\alpha = 0$ and $\alpha = 2$.

When $\alpha = 0$, we can use the Hautus Lemma (rank test) for detectability, first solve the eigenvalues of A:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda_A = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

We have one eigenvalue $\lambda_3 = 1 \in \Lambda_A^+$ lying on the unit circle, the other two are inherently stable. Therefore, we check:

$$\operatorname{rank}\left(\begin{bmatrix} A^{\mathrm{T}} - \lambda_3 I \mid C^{\mathrm{T}} \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}\right) = 3$$

Hence when $\alpha = 0$, the system is detectable

When $\alpha = 2$, we can still follow the same procedure, first solve the eigenvalues of A:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda_A = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$$

We have two eigenvalue $\lambda_2 = 1, \lambda_3 = 2 \in \Lambda_A^+$ lying on the unit circle, the other one is inherently stable. Therefore, we check:

$$\operatorname{rank}\left(\begin{bmatrix} A^{\mathrm{T}} - \lambda_2 I \mid C^{\mathrm{T}} \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}\right) = 2$$

$$\operatorname{rank}\left(\begin{bmatrix} A^{\mathrm{T}} - \lambda_3 I \mid C^{\mathrm{T}} \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}\right) = 3$$

Hence when $\alpha = 2$, the system is not detectable since for λ_2, it does not satisfy the rank condition.

(b) Consider system (4a) with $\alpha = 1$

(i) For the unconstrained system (4a), consider the following control policy

$$u_k = -\underbrace{\begin{bmatrix} 0 & 1 & 2 \end{bmatrix}}_{K} x_k \tag{6}$$

that has been designed such that the closed-loop (unconstrained) system is asymptotically stable. You are now given the function $V : \mathbb{R}^3 \to \mathbb{R}$, with:

$$V(x) = x^{\mathrm{T}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix}}_{P} x$$

Verify that $V(x)$ is a Lyapunov function for the closed-loop system (4a) with $\alpha = 1$ and u_k as in (6). [5 pts]

With $\alpha = 1$, the close-loop dynamics of the system is:

$$x(k+1) = Ax_k + Bu_k = (A - BK)\, x_k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} x_k$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} x_k \stackrel{\Delta}{=\!=} A_{\mathrm{cl}} x_k$$

To prove the given $V(x)$ is a Lyapunov function, just check:

$$V(0) = 0^{\mathrm{T}} P 0 = 0$$

$$V(x) = x^{\mathrm{T}} P x = x_1^2 + 3x_2^2 + 3x_3^2 + 4x_2x_3 = x_1^2 + 3\left(x_2 + \frac{2}{3}x_3\right)^2 + \frac{5}{3}x_3^2 > 0 \quad \forall x \in \mathbb{R}\backslash\{0\}$$

$$V(x_{k+1}) - V(x_k)$$

$$= x_k^{\mathrm{T}} \left(A_{\mathrm{cl}}^{\mathrm{T}} P A_{\mathrm{cl}} - P\right) x_k = x_k^{\mathrm{T}} \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix} \right) x_k$$

$$= -x_k^{\mathrm{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_k = -x^{\mathrm{T}} I x \quad \text{and obviously} \quad I \succ 0$$

Therefore, the given $V(x)$ is a Lyapunov function for the closed-loop system (4a) with $\alpha = 1$

We have developed a steady-state Kalman filter with Kalman gain K_∞ for system (4) with $\alpha = 1$ based on the linearized output $y_k = Cx_k + Du_k$, with C, D as in (5). The observer error dynamics are given as:

$$e_{k+1} = \hat{x}_{k+1} - x_{k+1} = (A - K_\infty CA)\, e_k$$

(ii) Instead of the actual state x_k we will use the estimated state $\hat{x}_k$ to design our state feedback controller, i.e. (6) is changed to:

$$u_k = -K\hat{x}_k$$

Determine the state transition matrix A_a, or the augmented closed-loop system, as a function of A, B, C, D, K and K_∞:

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = A_a \begin{bmatrix} x_k \\ e_k \end{bmatrix}$$

[2 pts]

Solution: According to the problem description, we have:

$$x_{k+1} = Ax_k + Bu_k = Ax_k - BK\hat{x}_k = Ax_k - BK(e_k + x_k) = (A - BK)\, x_k - BKe_k$$

Together with the given error dynamics, we can stack up the two equations and form the augmented closed-loop system as:

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = \begin{bmatrix} A - BK & -BK \\ 0 & A - K_\infty CA \end{bmatrix} \begin{bmatrix} x_k \\ e_k \end{bmatrix} \Rightarrow A_a = \begin{bmatrix} A - BK & -BK \\ 0 & A - K_\infty CA \end{bmatrix}$$

(iii) Does there exist a Lyapunov function $V_\infty(x)$ for the augmented closed-loop system in (ii)? Justify your answer. [3 pts]

Yes, there exist a Lyapunov function $V\infty(x)$ for the augmented closed-loop system in (ii). Firstly, we have proved that the state dynamics is stable in (b).(i), Secondly, since we have the converged steady state Kalman gain K_∞, it means our error dynamaics also is stable (i.e. (A, C) observable and the covariance is stabilizable). Hence according to the seperation principle (which is also proved in 5.(b).(iii)), the whole system is stable, equivalent to a Lyapunov function could be found.

5. Stability, controllability, observability and observer design

(a) Consider the following discrete time LTI system:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned} \tag{7}$$

(i) if $A = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & a \end{bmatrix}$, find the interval in which the parameter a must lie for the system to be asymptotically stable. [1 pt]

Solution: The LTI system is asymptotically stable if and only if all the eigenvalues lie in the unit circle. Therefore, we have:

$$A = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & a \end{bmatrix} \Rightarrow \lambda_A = \begin{bmatrix} \frac{1}{2} & a \end{bmatrix} \Rightarrow |\lambda_i| < 1 \Rightarrow a \in (-1, 1)$$

(ii) Assume that A is as in part (a).(i) with a $= \frac{2}{3}$ and that one can choose between three different actuators that result in the following three forms of the matrix B:

$$B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Which of the actuators lead to a system that is controllable? [3 pts]

Solution: The system is controllable if and only if the controllability matrix $\mathcal{C}$ is full rank, hence we can check:

$$\begin{aligned} \mathcal{C}_1 &= \begin{bmatrix} B_1 & AB_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -1 \end{bmatrix} \Rightarrow \operatorname{rank}(\mathcal{C}_1) = 2 \\ \mathcal{C}_2 &= \begin{bmatrix} B_2 & AB_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & \frac{2}{3} \end{bmatrix} \Rightarrow \operatorname{rank}(\mathcal{C}_2) = 1 \\ \mathcal{C}_3 &= \begin{bmatrix} B_3 & AB_3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{3} \end{bmatrix} \Rightarrow \operatorname{rank}(\mathcal{C}_3) = 2 \end{aligned}$$

Therefore, we can see that B_1 and B_3 would lead a system to be controllable

(iii) Assume that A is as in part (a).(i) with a $= \frac{2}{3}$ and that C has the following structure $C = \begin{bmatrix} 1 & c \end{bmatrix}$. For which values of c is the system observable? [3 pts]

Solution: The system is observable if and only if the observability matrix is full rank, hence we can calculate:

$$\mathcal{O} = \begin{bmatrix} C & (CA)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & c \\ \frac{1}{2} - c & \frac{2c}{3} \end{bmatrix} \Rightarrow \det(\mathcal{O}) = c^2 + \frac{c}{6}$$

The observability matrix would be rank-deficient if $c = 0$ or $c = -\frac{1}{6}$. That is to say, for any $c \in \mathbb{R} \backslash \left\{0, -\frac{1}{6}\right\}$, the system would be observable

(iv) Let a, B and c be such that the system is asymptotically stable, observable but not controllable. Is it possible to design a state observer and a state feedback controller such that $\lim_{k\to\infty} y_k = 0, \forall x_0$? Justify your answer. [2 pts]

Yes, we can define a state observer and state feedback controller such that $\lim_{k\to\infty} y_k = 0$. Firstly, the system is not controllable but asymptotically stable, therefore, it is stabilizable since all the uncontrollable modes are inherently stable. This shows that we can design a stable state feedback controller. Secondly, the system is observable, which means we can design a stable observer. According to the seperation principle, which is also proved in (b).(ii), we know that the closed loop system with both the controller and observer would be stable, hence the output would finally goes to zero.

(v) If the system configuration is such that A is as in part (a).(i) with $a = \frac{2}{3}$ and $B = \begin{bmatrix}0 & 1\end{bmatrix}^{\mathrm{T}}$, is it possible to design a state feedback controller $u_k = -Kx_k, K = \begin{bmatrix}k_1 & k_2\end{bmatrix}$, such that all the poles of the closed loop system have absolute values no larger than $\frac{1}{2}$? If yes, find the range in which k_1 and k_2 should lie to satisfy this requirement. [4 pts]

According to (a).(ii), we know that with $a = \frac{2}{3}$ and $B = \begin{bmatrix}0 & 1\end{bmatrix}^{\mathrm{T}}$, the system is uncontrollable. That means pole placement can't be done on the uncontrollable mode. And we have:

$$\lambda_A = \begin{bmatrix}\frac{1}{2} & \frac{2}{3}\end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix}0 & 0\\ 1 & \frac{2}{3}\end{bmatrix} \Rightarrow \mathrm{Im}\left(\mathcal{C}\right) = \mathrm{span}\left\{\begin{bmatrix}0 & 1\end{bmatrix}^{\mathrm{T}}\right\}$$

Therefore, the uncontrollable subspace is the state x_1 with eigenvalue $\frac{1}{2}$ which already satisfy the requirement, the only thing we need to do is to place the second pole (the one in the controllable subspace) to where we want. The calculation process can is shown below:

$$x_{k+1} = \left(A - BK\right)x_k = \left(\begin{bmatrix}\frac{1}{2} & 0\\ -1 & \frac{2}{3}\end{bmatrix} - \begin{bmatrix}0 & 0\\ k_1 & k_2\end{bmatrix}\right)x_k = \begin{bmatrix}\frac{1}{2} & 0\\ -1-k_1 & \frac{2}{3}-k_2\end{bmatrix}x_k$$

$$\Rightarrow \det\left(\lambda I - \left(A - BK\right)\right) = \frac{1}{6}\left(2\lambda - 1\right)\left(3k_2 + 3\lambda - 2\right) \Rightarrow \lambda_2 = \frac{2 - 3k_2}{3}$$

$$\left|\lambda_2\right| = \left|\frac{2 - 3k_2}{3}\right| \leq \frac{1}{2} \Rightarrow \frac{1}{6} \leq k_2 \leq \frac{7}{6}$$

Therefore, the range in which k_1 and k_2 should lie in is $k_1 \in \mathbb{R}$ and $k_2 \in \left[\frac{1}{6}, \frac{7}{6}\right]$

(b) We consider the system in (9) and we want to design a state observer and a state feedback controller for it. The state observer should have the following structure:

$$\begin{aligned}\hat{x}_{k+1} &= A\hat{x}_k + Bu_k + L\left(y_k - \hat{y}_k\right)\\ \hat{y}_k &= C\hat{x}_k\end{aligned}$$

where $\hat{x}_k$ is the state estimate and $\hat{y}_k$, is the output estimate. The state feedback controller should have the following structure:

$$u_k = -K\hat{x}_k$$

The closed loop system with the state observer and the controller can be described by the following difference equation:

$$\begin{bmatrix}x_{k+1}\\ e_{k+1}\end{bmatrix} = F\begin{bmatrix}x_k\\ e_k\end{bmatrix} \tag{8}$$

where $e_k = \hat{x}_k - x_k$ is the state estimation error.

(i) Derive the matrix F

Solution: From the given state observer and feedback controller structure, we have:

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L\left(y_k - \hat{y}_k\right) = A\hat{x}_k - BK\hat{x}_k + L\left(Cx_k - C\hat{x}_k\right) = \left(A - LC - BK\right)\hat{x}_k + LCx_k$$

$$x_{k+1} = Ax_k + Bu_k = Ax_k - BK\hat{x}_k$$

Manipulate the above equations we have:

$$\begin{aligned}e_{k+1} = \hat{x}_{k+1} - x_{k+1} &= \left(A - LC - BK\right)\hat{x}_k + LCx_k - \left(Ax_k - BK\hat{x}_k\right)\\ &= \left(A - LC\right)\hat{x}_k + \left(LC - A\right)x_k = \left(A - LC\right)\left(\hat{x}_k - x_k\right)\\ &= \left(A - LC\right)e_k\end{aligned}$$

Also, the system dynamics can be written as:

$$x_{k+1} = Ax_k + Bu_k = Ax_k - BK\hat{x}_k = Ax_k - BK\left(e_k + x_k\right) = \left(A - BK\right)x_k - BKe_k$$

Stack the state and estimation error, the above equations can be written as:

$$\begin{bmatrix}x_{k+1}\\ e_{k+1}\end{bmatrix} = \begin{bmatrix}A - KB & -KB\\ 0 & A - LC\end{bmatrix}\begin{bmatrix}x_k\\ e_k\end{bmatrix} \Rightarrow F = \begin{bmatrix}A - BK & -BK\\ 0 & A - LC\end{bmatrix}$$

(ii) Based on the result in part (b).(i), derive the eigenvalues of the matrix F in terms of the eigenvalues of $A - BK$ and $A - LC$. [3 pts]

Solution: The eigenvalues of F can be derived by solving the the characteristic polynomial problem, i.e.:

$$\det \begin{bmatrix} \lambda I - A + BK & BK \\ 0 & \lambda I - A + LC \end{bmatrix} = 0$$

It can be observed that the F matrix here is the upper triangular matrix and for upper triangular matrix, for upper triangular matrix, the determinant can be calculated directly as:

$$\det \begin{bmatrix} \lambda I - A + BK & BK \\ 0 & \lambda I - A + LC \end{bmatrix} = \det (\lambda I - A + BK) \det (\lambda I - A + LC)$$

Therefore, the eigenvalues of matrix F is the union of the eigenvalue of the two blocks, i.e.:

$$\lambda \{A - BK\} \cup \lambda \{A - LC\}$$

(iii) If the controller gain K is selected such that the closed loop system with state measurements is stable and if the observer gain L is selected such that the dynamics of the state error e_k are stable for the open loop system, is the closed loop system (8) with both the observer and the controller stable in general? Justify your answer. [3 pts]

Solution: Yes, the closed loop system (8) with both the observer and the controller would be stable in general. In (b).(ii), we prove the seperation principle, which means the observer design is decoupled from state feedback design. Therefore, if a stable observer and a stable state feedback are designed for the system, then the combined observer and feedback is stable.

6. Stability, Lyapunov Functions

   1. Consider the following piecewise affine system:

$$x^+ = \begin{cases} -x-2 & \text{if} \quad x < -2 \\ 0.9x & \text{if} \quad -2 \leq x \leq 2 \\ -x+2 & \text{if} \quad x > 2 \end{cases}$$

      Is this system globally stable? [5 pts]

      ☑ Yes
      The stability can be judged from fixed point iteration and Cobweb Diagram (Phase Diagram) and note that the absolute value of slope around origin is less than 1 ($0.9 < 1$).

      ☐ No

   2. Find a Lyapunov function for the system $x^+ = \frac{1}{2}x\cos(x)$ if it exists.
      Solution: Choose the candidate Lyapunov function to be $V(x) = x^2$, now we prove that it is the true Lyapunov function:

$$\begin{aligned} &V(x) = x^2 > 0, \forall x \in \mathbb{R} \setminus \{0\} \\ &V(0) = 0^2 = 0 \\ &V(x^+) - V(x) = \left(\frac{1}{2}x\cos(x)\right)^2 - x^2 = \frac{1}{4}x^2\left(\cos^2(x) - 4\right) \\ &\qquad\qquad\qquad\quad \leq -\frac{3}{4}x^2 < 0,\ \forall x \in \mathbb{R} \setminus \{0\} \end{aligned}$$

      Hence $V(x) = x^2$ is a Lyapunov function of the system.

7. Stability, Observability Consider the following discrete time LTI system:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned} \tag{9}$$

where

$$A = \begin{bmatrix} -0.4 & -1.1 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

For the following statements say whether they are true or false. Justify your answers.

(1) System (9) is stable. [1 pt]

☐ true ☑ false

$$A = \begin{bmatrix} -0.4 & -1.1 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 0.9 \end{bmatrix} \Rightarrow \lambda_A = \begin{bmatrix} \frac{3}{5} & \frac{9}{10} & 4 \end{bmatrix} \Rightarrow |\lambda_3| > 1$$

Hence the system is unstable

(2) System (9) is both controllable and stabilizable. [1 pt]

☐ true ☑ false

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & -\frac{11}{10} & -\frac{253}{50} \\ 1 & 5 & \frac{103}{5} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \operatorname{rank}(\mathcal{C}) = 2$$

Hence the system is uncontrollable

(3) System (9) is not controllable, but it is stabilizable. [1 pt]

☑ true ☐ false

Use Hautus Lemma to check the stabilizability:

$$\operatorname{rank}([\lambda_3 I - A \; |B]) = \operatorname{rank}\left(\begin{bmatrix} \frac{22}{5} & \frac{11}{10} & 0 & 0 \\ -4 & -1 & 0 & 1 \\ 0 & 0 & \frac{31}{10} & 0 \end{bmatrix}\right) = 3 \Rightarrow (A, B) \text{ stabilizable}$$

(4) System (9) is observable, but not detectable. [1 pt]

☐ true ☑ false

$$\mathcal{O} = \begin{bmatrix} C^{\mathrm{T}} & (CA)^{\mathrm{T}} & \left(CA^2\right)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 0 \\ \frac{18}{5} & \frac{39}{10} & 0 \\ \frac{354}{25} & \frac{777}{50} & 0 \end{bmatrix} \Rightarrow \operatorname{rank}(\mathcal{O}) = 2$$

Hence the system is unobservable. Also, it is impossible for a system to be observable but not detectable. Observability itself implies detectability.

(5) System (9) is not observable, but it is detectable. [1 pt]

☑ true ☐ false

Use Hautus Lemma to check the detectability:

$$\operatorname{rank}\left(\begin{bmatrix} A^{\mathrm{T}} - \lambda_3 I \; |C^{\mathrm{T}} \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} -\frac{22}{5} & 4 & 0 & 1 \\ -\frac{11}{10} & 1 & 0 & 1 \\ 0 & 0 & -\frac{31}{10} & 0 \end{bmatrix}\right) = 3 \Rightarrow (A, C) \text{ detectable}$$

8. Appendix: MATLAB code for simulation of exact discretization

```matlab
%% discretize using the formulas on the slides
clc
clear all
A_c = [-5, 2.7; -3.1, 1.5]
B_c = [4,   2.1;   1.1,    3]
C_c = [1, 1];
T_s = 1
% exact solution
A_d = expm(A_c.*T_s)
% fun = @(tau) expm(A_c.*(T_s-tau))*B_c
% B_d = integral(fun,0,T_s,'ArrayValued',true)
B_d = inv(A_c)*(A_d-eye(2))*B_c
C_d = C_c
% using ss first establish the state space model sys, then use c2d
D_c = zeros(1,2);
T_s = 1
sys_c = ss(A_c,B_c,C_c,D_c)
sys_d = c2d(sys_c,T_s,'zoh')

%% simulation comparison of discretized system and continuous system
% continous
t_c = 0:0.01:10;
x0 = [0; 0];
[t,x_c] = ode45(@(t,x) system_c(t,x,A_c,B_c), t_c, x0);
% discrete
t_d = 0:1:10;
x_d = zeros(2,9);
for i = 0:9
    if i == 0
        x_d(:,i+1) = A_d*x0 + B_d*[1;1];
    else
        x_d(:,i+1) = A_d*x_d(:,i) + B_d*[1;1];
    end
end
x_d = [x0, x_d]

%% plot the results
figure
plot(t_c,x_c(:,1),'-r','LineWidth',2)
hold on
plot(t_c,x_c(:,2),'-b','LineWidth',2)
stairs(t_d,x_d(1,:),'--or','LineWidth',2)
stairs(t_d,x_d(2,:),'--ob','LineWidth',2)
xlabel("Time t (s)")
legend("x_1 CT","x_2 CT","x_1 DT","x_2 DT")
title("Simulation Comparison: Continous v.s. Exact Discretization")

%% function definition for ode45
function dxdt_c = system_c(t,x,A_c,B_c)
    dxdt_c = A_c*x + B_c*[1;1];
end
```