University of Pennsylvania, ESE 6190

Model Predictive Control

Chapter 12: Hybrid MPC

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F. Borrelli, A. Bemporad, and M. Morari, Predictive Control for Linear and Hybrid Systems, Cambridge University Press, 2017. [Ch. 16, 17].

Outline

1. Modeling of Hybrid Systems

2. Optimal Control of Hybrid Systems

3. Model Predictive Control of Hybrid Systems

4. Explicit MPC of Hybrid Systems

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1. Modeling of Hybrid Systems

    Introduction

    Examples of Hybrid Systems

    Piecewise Affine (PWA) Systems

    Mixed Logical Dynamical (MLD) Hybrid Model

Introduction

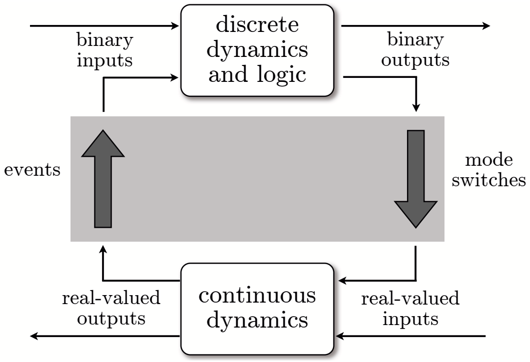
Up to this point: Discrete-time linear systems with linear constraints.

We now consider MPC for systems with

1. **Continuous dynamics**: described by one or more difference (or differential) equations; states are continuous-valued.
2. **Discrete events**: state variables assume **discrete** values, e.g.
    - binary digits $\{0, 1\}$,
    - $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}, \ldots$
    - finite set of symbols

Hybrid systems: Dynamical systems whose state evolution depends on an interaction between continuous dynamics and discrete events.

Introduction



Hybrid systems: Logic-based discrete dynamics and continuous dynamics interact through events and mode switches

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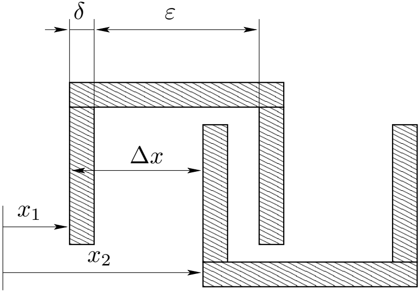
Mechanical System with Backlash

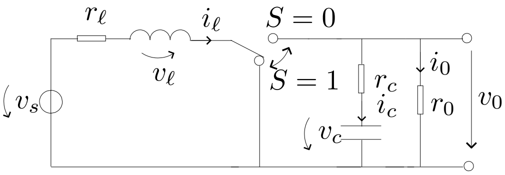


- **Continuous dynamics**: states x_1, x_2, $\dot{x}_1$, $\dot{x}_2$.
- **Discrete events**:
  a) *"contact mode"* $\Rightarrow$ mechanical parts are in contact and the force is transmitted. Condition:

$$[(\Delta x = \delta) \wedge (\dot{x}_1 > \dot{x}_2)] \ \bigvee \ [(\Delta x = \varepsilon) \wedge (\dot{x}_2 > \dot{x}_1)]$$

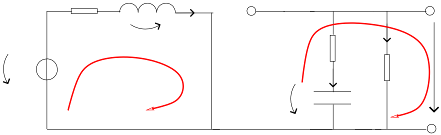
  b) *"backlash mode"* $\Rightarrow$ mechanical parts are not in contact

DCDC Converter

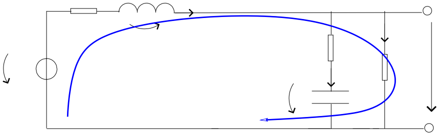


- **Continuous dynamics**: states v_ℓ, i_ℓ, v_c, i_c, v_0, i_0
- **Discrete events**: $S = 0$, $S = 1$

Mode 1 ($S = 1$)



Mode 2 ($S = 0$)



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Piecewise Affine (PWA) Systems

PWA systems are defined by:

- **affine dynamics and output** in each region:

$$\left\{\begin{array}{rcl} x(t+1) & = & A_i x(t) + B_i u(t) + f_i \\ y(t) & = & C_i x(t) + D_i u(t) + g_i \end{array}\right\} \text{ if } (x(t), u(t)) \in \mathcal{X}_{i(t)}$$

- **polyhedral partition** of the (x, u)-space:

$$\{\mathcal{X}_i\}_{i=1}^{s} := \{x, u \mid H_i x + J_i u \leq K_i\}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$

Physical constraints on $x(t)$ and $u(t)$ are defined by polyhedra $\mathcal{X}_i$

Piecewise Affine (PWA) Systems

Examples:

- *linearization* of a non-linear system at different operating point $\Rightarrow$ useful as an approximation tool
- *closed-loop MPC system* for linear constrained systems
- When the mode i is an exogenous variable, the partition disappears and we refer to the system as a **Switched Affine System** (SAS)

Definition: Well-Posedness

Let P be a PWA system and let $\mathcal{X} = \cup_{i=1}^{s} \mathcal{X}_i \subseteq \mathbb{R}^{n+m}$ be the polyhedral partition associated with it. System P is called **well-posed** if for all pairs $(x(t), u(t)) \in \mathcal{X}$ *there exists only one* index $i(t)$ satisfying the membership condition.

Binary States, Inputs, and Outputs

Remark: In the previous example, the PWA system has only continuous states and inputs.

We will formulate PWA systems including binary state and inputs by treating 0–1 binary variables as:

- **Numbers**, over which arithmetic operations are defined,
- **Boolean variables**, over which Boolean functions are defined.

We will use the notation $x = \left[\begin{smallmatrix} x_c \\ x_\ell \end{smallmatrix}\right] \in \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell}$, $n := n_c + n_\ell$; $y \in \mathbb{R}^{p_c} \times \{0,1\}^{p_\ell}$, $p := p_c + p_\ell$; $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_\ell}$, $m := m_c + m_\ell$.

Boolean Algebra: Basic Definitions and Notation

- **Boolean variable:** A variable δ is a Boolean variable if $\delta \in \{0, 1\}$, where "$\delta = 0$" means "false", "$\delta = 1$" means "true".

- **A Boolean expression** is obtained by combining Boolean variables through the logic operators $\neg$ (not), $\vee$ (or), $\wedge$ (and), $\leftarrow$ (implied by), $\rightarrow$ (implies), and $\leftrightarrow$ (iff).

- **A Boolean function** $f : \{0, 1\}^{n-1} \mapsto \{0, 1\}$ is used to define a Boolean variable δ_n as a logic function of other variables $\delta_1, \dots, \delta_{n-1}$:

$$\delta_n = f(\delta_1, \delta_2, \dots, \delta_{n-1}).$$

Example

$$
\begin{aligned}
x_c(t+1) &= 2x_c(t) + u_c(t) - 3u_\ell(t) \\
x_\ell(t+1) &= x_\ell(t) \wedge u_\ell(t)
\end{aligned}
$$

can be represented in the PWA form

$$
\begin{bmatrix} x_c(t+1) \\ x_\ell(t+1) \end{bmatrix} = \begin{cases}
\begin{bmatrix} 2x_c(t) + u_c(t) \\ 0 \end{bmatrix} & \text{if} \quad x_\ell \leq \frac{1}{2}, u_\ell \leq \frac{1}{2} \\[2ex]
\begin{bmatrix} 2x_c(t) + u_c(t) - 3 \\ 0 \end{bmatrix} & \text{if} \quad x_\ell \leq \frac{1}{2}, u_\ell \geq \frac{1}{2} + \epsilon \\[2ex]
\begin{bmatrix} 2x_c(t) + u_c(t) \\ 0 \end{bmatrix} & \text{if} \quad x_\ell \geq \frac{1}{2} + \epsilon, u_\ell \leq \frac{1}{2} \\[2ex]
\begin{bmatrix} 2x_c(t) + u_c(t) - 3 \\ 1 \end{bmatrix} & \text{if} \quad x_\ell \geq \frac{1}{2} + \epsilon, u_\ell \geq \frac{1}{2} + \epsilon.
\end{cases}
$$

by associating $x_\ell = 0$ with $x_\ell \leq \frac{1}{2}$ and $x_\ell = 1$ with $x_\ell \geq \frac{1}{2} + \epsilon$ for any $0 < \epsilon \leq \frac{1}{2}$.

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Mixed Logical Dynamical Systems

Goal: Describe hybrid system in form compatible with optimization software:

- continuous and Boolean variables
- linear equalities and inequalities

Idea: associate to each Boolean variable p_i a binary integer variable δ_i:

$$p_i \Leftrightarrow \{\delta_i = 1\}, \quad \neg p_i \Leftrightarrow \{\delta_i = 0\}$$

and embed them into a set of constraints as **linear integer inequalities**.

Two main steps:

1. Translation of Logic Rules into Linear Integer Inequalities
2. Translation continuous and logical components into Linear Mixed-Integer Relations

Final result: a compact model with linear equalities and inequalities involving real and binary variables

Boolean formulas as Linear Integer Inequalities

Goal

Given a Boolean formula $F(p_1, p_2, \dots, p_n)$ define a polyhedral set P such that a set of binary values $\{\delta_1, \delta_2, \dots, \delta_n\}$ satisfies the Boolean formula F in P

$$F(p_1, p_2, \dots, p_n) \text{ ``TRUE''} \Leftrightarrow A\delta \le B, \quad \delta \in \{0, 1\}^n$$

where: $\{\delta_i = 1\} \Leftrightarrow p_i = \text{TRUE}$.

Analytic Approach

1. Transform $F(p_1, p_2, \ldots, p_n)$ into a **Conjunctive Normal Form (CNF)**:

$$F(p_1, p_2, \ldots, p_n) = \bigwedge_j \left[\bigvee_i p_i \right]$$

2. Translation of a **CNF** into **algebraic inequalities:**

relation	Boolean	linear constraints
AND	$\delta_1 \wedge \delta_2$	$\delta_1 \geq 1$, $\delta_2 \geq 1$ or $\delta_1 + \delta_2 \geq 2$
OR	$\delta_1 \vee \delta_2$	$\delta_1 + \delta_2 \geq 1$
NOT	$\neg\delta_1$	$(1 - \delta_1) \geq 1$ or $\delta_1 = 0$
XOR	$\delta_1 \oplus \delta_2$	$\delta_1 + \delta_2 = 1$
IMPLY	$\delta_1 \rightarrow \delta_2$	$\delta_1 - \delta_2 \leq 0$
IFF	$\delta_1 \leftrightarrow \delta_2$	$\delta_1 - \delta_2 = 0$
ASSIGNMENT $\delta_3 = \delta_1 \wedge \delta_2$	$\delta_3 \leftrightarrow \delta_1 \wedge \delta_2$	$\delta_1 + (1 - \delta_3) \geq 1$ $\delta_2 + (1 - \delta_3) \geq 1$ $(1 - \delta_1) + (1 - \delta_2) + \delta_3 \geq 1$

Analytic Approach. Example

Given

$$F(p_1, p_2, p_3, p_4) \triangleq [(p_1 \wedge p_2) \Rightarrow (p_3 \wedge p_4)]$$

find the equivalent set of linear integer inequalities.

1. remove implication:

$$F(p_1, p_2, p_3, p_4) = \neg(p_1 \wedge p_2) \vee (p_3 \wedge p_4)$$

2. using DeMorgan's theorem, obtain CNF:

$$F(p_1, p_2, p_3, p_4) = (\neg p_1 \vee \neg p_2 \vee p_3) \wedge (\neg p_1 \vee \neg p_2 \vee p_4)$$

3. introduce $[\delta_i = 1] \Leftrightarrow p_i = \text{TRUE}$ and write the inequalities:

$$F(p_1, p_2, p_3, p_4) = \text{TRUE} \Leftrightarrow \left\{ \begin{array}{lcl} \delta_1 + \delta_2 - \delta_3 & \leq & 1 \\ \delta_1 + \delta_2 - \delta_4 & \leq & 1 \\ \delta_{1,2,3,4} \in \{0, 1\} & & \end{array} \right.$$

Linear Inequality As Logic Condition

Definition: Event Generator

An **event generator** is defined by function $f_{EG} : \mathcal{X}_c \times \mathcal{U}_c \times \mathbb{N}_0 \to \mathcal{D}$:

$$\delta_e(t) = f_{EG}(x_c(t), u_c(t), t)$$

Consider the Boolean expression consisting of a Boolean variable p and continuous variable $x \in \mathbb{R}^n$:

$$p \Leftrightarrow a^T x \leq b$$

where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, $x \in \mathcal{X} \subset \mathbb{R}^n$:

$$\mathcal{X} = \{x \mid a^T x - b \in [m, M]\}$$

Translated to linear inequalitites:

$$\begin{aligned} a^T x - b &\leq M(1-\delta) \\ a^T x - b &> m\delta \end{aligned}$$

Switched Affine Dynamics

Rewrite the state update functions of the SAS as

$$
\begin{aligned}
z_1(t) &= \begin{cases} A_1 x_c(t) + B_1 u_c(t) + f_1, & \text{if } (i(t) = 1), \\ 0, & \text{otherwise,} \end{cases} \\
&\;\;\vdots \\
z_s(t) &= \begin{cases} A_s x_c(t) + B_s u_c(t) + f_s, & \text{if } (i(t) = s), \\ 0, & \text{otherwise,} \end{cases} \\
x_c(t+1) &= \sum_{i=1}^{s} z_i(t),
\end{aligned}
$$

In general, use the "IF-THEN-ELSE" relations

$$
\text{IF } \delta \text{ THEN } z = a_1' x + b_1' u + f_1 \text{ ELSE } z = a_2' x + b_2' u + f_2,
$$

"IF-THEN-ELSE" Relations

$$
\begin{array}{l}
\text{IF} \quad p \qquad \text{THEN} \\
\qquad z_t = a_1^T x_t + b_1 \\
\text{ELSE} \\
\qquad z_t = a_2^T x_t + b_2
\end{array}
\quad \Longleftrightarrow \quad
\begin{array}{rcl}
(m_2 - M_1)\delta + z_t & \leq & a_2^T x_t + b_2 \\
(m_1 - M_2)\delta - z_t & \leq & -a_2^T x_t - b_2 \\
(m_1 - M_2)(1-\delta) + z_t & \leq & a_1^T x_t + b_1 \\
(m_2 - M_1)(1-\delta) - z_t & \leq & -a_1^T x_t - b_1
\end{array}
$$

where $x \in \mathcal{X}$, with

$$\sup_{x \in \mathcal{X}} \; a_i^T x + b_i \leq M_i,$$

$$\inf_{x \in \mathcal{X}} \; a_i^T x + b_i \geq m_i,$$

$$m_2 \neq M_1, \;\; m_1 \neq M_2.$$

Hybrid Modelling - An Example

Consider the following system with constraints: $|x| \leq 10$, $|u| \leq 10$

$$x_{t+1} = \begin{cases} 0.8x_t + u_t & \text{if } x_t \geq 0 \\ -0.8x_t + u_t & \text{if } x_t < 0 \end{cases}$$

1. associate $\{\delta_t = 1\} \Leftrightarrow \{x_t \geq 0\}$

$$\begin{aligned} -m\delta_t &\leq x_t - m \\ -(M+\epsilon)\delta_t &\leq -x_t - \epsilon, \end{aligned}$$

   where: $M = -m = 10$, $\epsilon > 0$.

2. state update equation:

$$x_{t+1} = 1.6\delta_t x_t - 0.8x_t + u_t$$

Hybrid Modelling - An Example

3. introduce variable: $z_t = \delta_t x_t$

$$x_{t+1} = 1.6 z_t - 0.8 x_t + u_t$$

4. constraints on z:

$$\begin{aligned} z_t &\leq M\delta_t \\ z_t &\geq m\delta_t \\ z_t &\leq x_t - m(1-\delta_t) \\ z_t &\geq x_t - M(1-\delta_t) \end{aligned}$$

MLD Hybrid Model

A DHA can be converted into the following MLD model

$$\begin{aligned} x_{t+1} &= Ax_t + B_1 u_t + B_2 \delta_t + B_3 z_t \\ y_t &= Cx_t + D_1 u_t + D_2 \delta_t + D_3 z_t \\ E_2 \delta_t + E_3 z_t &\leq E_4 x_t + E_1 u_t + E_5 \end{aligned}$$

where $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell}$, $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_\ell}$ $y \in \mathbb{R}^{p_c} \times \{0,1\}^{p_\ell}$, $\delta \in \{0,1\}^{r_\ell}$ and $z \in \mathbb{R}^{r_c}$.

Physical constraints on continuous variables:

$$\mathcal{C} = \left\{ \begin{bmatrix} x_c \\ u_c \end{bmatrix} \in \mathbb{R}^{n_c + m_c} \;\middle|\; Fx_c + Gu_c \leq H \right\}$$

MLD Hybrid Model. Well-Posedness

- **Well-Posedness:**
  for a given $x = \left[\begin{smallmatrix} x_t \\ u_t \end{smallmatrix}\right] \Rightarrow x_{t+1}$ and y_t uniquely determined

- **Complete Well-Posedness:**
  well-posedness + uniquely determined δ_t and z_t, $\forall \left[\begin{smallmatrix} x_t \\ u_t \end{smallmatrix}\right]$

- Well-posedness is sufficient for the computation of the state and output prediction

- Complete well-posedness allows transformation into equivalent hybrid models

HYbrid System DEscription Language

HYSDEL

- based on DHA
- enables description of discrete-time hybrid systems in a compact way:
  - automata and propositional logic
  - continuous dynamics
  - A/D and D/A conversion
  - definition of constraints
- automatically **generates MLD models** for MATLAB
- freely available from:

`http://control.ee.ethz.ch/~hybrid/hysdel/`

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Optimal Control for Hybrid Systems: General Formulation

Consider the CFTOC problem:

$$J^*(x(t)) = \min_{U_0} \; p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k, \delta_k, z_k),$$

$$\text{s.t.} \begin{cases} x_{k+1} = Ax_k + B_1 u_k + B_2 \delta_k + B_3 z_k \\[2ex] E_2 \delta_k + E_3 z_k \leq E_4 x_k + E_1 u_k + E_5 \\[2ex] x_N \in \mathcal{X}_f \\ x_0 = x(t) \end{cases}$$

where $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_b}$, $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_b}$, $y \in \mathbb{R}^{p_c} \times \{0,1\}^{p_b}$, $\delta \in \{0,1\}^{r_b}$ and $z \in \mathbb{R}^{r_c}$ and

$$U_0 = \{u_0, u_1, \ldots, u_{N-1}\}$$

Mixed Integer Optimization

Mixed Integer Linear Programming

Consider the following MILP:

$$\begin{array}{ll} \inf_{[z_c, z_b]} & c_c' z_c + c_b' z_b + d \\ \text{subj. to} & G_c z_c + G_b z_b \le W \\ & z_c \in \mathbb{R}^{s_c},\ z_b \in \{0,1\}^{s_b} \end{array}$$

where $z_c \in \mathbb{R}^{s_c}$, $z_b \in \{0,1\}^{s_b}$

- MILP are nonconvex, in general.
- For a fixed $\bar{z}_b$ the MILP becomes a linear program:

$$\begin{array}{ll} \inf_{[z_c, z_b]} & c_c' z_c + (c' b \bar{z}_b + d) \\ \text{subj. to} & G_c z_c \le W - G_b \bar{z}_b \\ & z_c \in \mathbb{R}^{s_c} \end{array}$$

- Brute force approach to solution: enumerating the 2^{s_b} integer values of the variable z_b and solve the corresponding LPs. By comparing the 2^{s_b} optimal costs one can find the optimizer and the optimal cost of the MILP.

Mixed Integer Quadratic Programming

Consider the following MIQP:

$$\begin{array}{ll} \inf_{[z_c, z_b]} & \frac{1}{2} z' H z + q' z + r \\ \text{subj. to} & G_c z_c + G_b z_b \leq W \\ & z_c \in \mathbb{R}^{s_c},\ z_b \in \{0,1\}^{s_b} \\ & z = [z_c, z_b], s = s_c + s_d \end{array}$$

where $H \succeq 0$, $z_c \in \mathbb{R}^{s_c}, z_b \in \{0,1\}^{s_b}$.

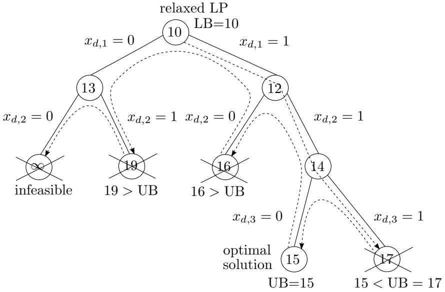
- MIQP are nonconvex, in general.
- For a fixed integer value $\bar{z}_b$ of z_b, the MIQP becomes a quadratic program:

$$\begin{array}{ll} \inf_{[z_c]} & \frac{1}{2} z_c' H_c z_c + q_c' z + k \\ \text{subj. to} & G_c z_c \leq W - G_b \bar{z}_b \\ & z_c \in \mathbb{R}^{s_c} \end{array}$$

- Brute force approach to the solution: enumerating all the 2^{s_b} integer values of the variable z_b and solve the corresponding QPs. By comparing the 2^{s_b} optimal costs one can derive the optimizer and the optimal cost of the MIQP.

Branch and Bound (B&B)

Common solution method for MIPs, based on relaxations of binaries: $\{0, 1\} \rightarrow [0, 1]$.



- Optimal cost of a solution to a modified problem where some binaries are relaxed is a **lower bound** on optimal cost.
- Any feasible solution to original problem is **upper bound** on optimal cost.
- Use bounds to rule out parts of the B&B tree systematically - much more efficient than brute force search.

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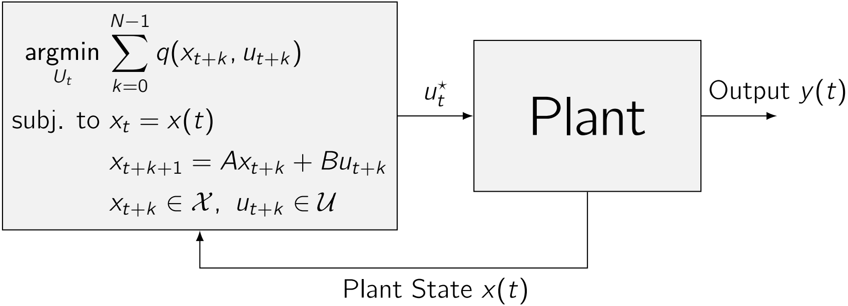
Model Predictive Control of Hybrid Systems

MPC solution: Optimization in the loop



As for linear MPC, at each sample time:

- Measure / estimate current state $x(t)$
- Find the optimal input sequence for the entire planning window N: $U_t^* = \{u_t^*, u_{t+1}^*, \ldots, u_{t+N-1}^*\}$
- Implement only the **first** control action u_t^*
- **Key difference: Requires online solution of an MILP or MIQP**

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Explicit MPC for Hybrid Systems. Quadratic Case

Theorem

The solution to the CFTOC problem based on an MLD model and with quadratic cost is **time-varying PWA feedback law** of the form:

$$u_t^*(x_t) = F_t^i x_t + G_t^i \quad \text{if } \; x_t \in \mathcal{R}_k^i$$

where $\left\{\mathcal{R}_t^i\right\}_{i=1}^{R_t}$ are regions partitioning the set of feasible states $\mathcal{X}_t^*$ and the closure $\bar{\mathcal{R}}_k^i$ of the sets $\mathcal{R}_k^i$ has the following form:

$$\bar{\mathcal{R}}_k^i := \left\{x : \; x(k)' L(j)_k^i x(k) + M(j)_k^i x(k) \leq N(j)_k^i, \; j = 1, \ldots, n_k^i\right\}, \\ k = 0, \ldots, N-1.$$

(note: quadratic and linear boundaries - not polyhedra!)

Explicit MPC for Hybrid Systems. Quadratic Case

- Denote by $\{v_i\}_{i=1}^{s^N}$ the set of all possible switching sequences over the horizon N
- Fix a certain v_i and constrain the state to switch according to sequence v_i.
- The problem becomes a **CFTOC for a linear time-varying system**. The solution is

$$u^i(x(0)) = \tilde{F}^{i,j} x(0) + \tilde{g}^{i,j}, \quad \forall x(0) \in \mathcal{T}^{i,j}, \quad j = 1, \ldots, N^{r\,i}$$

where $\mathcal{D}^i = \bigcup_{j=1}^{N^{r\,i}} \mathcal{T}^{i,j}$

- The set $\mathcal{X}_0 = \bigcup_{i=1}^{s^N} \mathcal{D}^i$ in general is **not convex**.
- The sets $\mathcal{D}^i$ can, in general, overlap. I.e., **some initial state is feasible for more than one switching sequence**.

Explicit MPC for Hybrid Systems. 1, ∞-norm Case

Theorem

The solution to the CFTOC problem based on an MLD model and with the cost based on norms $\{1, \infty\}$ is a **time-varying PPWA feedback law** of the form:

$$u_t^*(x_t) = F_t^i x_t + G_t^i \quad \text{if } \; x_t \in \mathcal{R}_k^i$$

where $\left\{\mathcal{R}_t^i\right\}_{i=1}^{R_t}$ are polyhedral regions partitioning the set of feasible states $\mathcal{X}_t^*$.

MPC for Hybrid Systems - Complexity

- The complexity strongly depends on the problem structure and the initial setup
- In general:

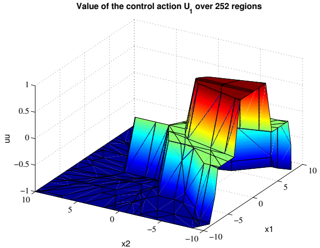
  Mixed-Integer programming is HARD

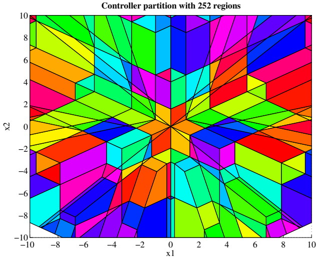
- Efficient general purpose solvers for MILP/MIQP: `CPLEX`, `XPRESS-MP` $\Rightarrow$ based on **Branch-And-Bound**, **Branch-And-Cut** methods + lots of heuristics
- On-line optimization is good for applications allowing large sampling intervals (typically **minutes**), requires expensive hardware and (even more) expensive software
- For very small problems requiring fast sampling rate
  $\Rightarrow$ **explicit solution of the MPC**

Explicit MPC for Hybrid Systems - Example

$$
\begin{cases}
x_{t+1} & = & 0.8 \begin{bmatrix} \cos\alpha_t & -\sin\alpha_t \\ \sin\alpha_t & \cos\alpha_t \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\
\alpha_t & = & \begin{cases} \pi/3 & \text{if } \begin{bmatrix} 1 & 0 \end{bmatrix} x_t \geq 0, \\ -\pi/3 & \text{if } \begin{bmatrix} 1 & 0 \end{bmatrix} x_t < 0 \end{cases} \\
x_t & \in & \begin{bmatrix} -10, & 10 \end{bmatrix} \times \begin{bmatrix} -10, & 10 \end{bmatrix}, \\
u_t & \in & \begin{bmatrix} -1, & 1 \end{bmatrix}
\end{cases}
$$

$$N = 12, P_N = Q_x = I, Q_u = 1, \infty - \text{norm}$$





Summary

- Hybrid systems: mixture of continuous and discrete dynamics
  - Many important systems fall in this class
  - Many tricks involved in modeling - automatic systems available to convert to consistent form

- Optimization problem becomes a mixed-integer linear / quadratic program
  - NP-hard (exponential time to solve)
  - Advanced commercial solvers available

- MPC theory (invariance, stability, etc) applies
  - Computing invariant sets is usually extremely difficult
  - Computing the optimal solution is extremely difficult (sub-optimal ok)