University of Pennsylvania, ESE 6190

Model Predictive Control

Chapter 7: Guaranteeing Feasibility and Stability

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F. Borrelli, A. Bemporad, and M. Morari, Predictive Control for Linear and Hybrid Systems, Cambridge University Press, 2017. [Ch. 12].

Outline

1. Receding Horizon Control and Model Predictive Control
2. Motivation
3. Challenges: Feasibility and Stability
4. Guaranteeing Feasibility and Stability
5. Extension to Nonlinear MPC

Outline

Infinite Time Constrained Optimal Control (what we would like to solve)

$$
\begin{aligned}
J_0^\star(x(0)) = \min \; & \sum_{k=0}^{\infty} q(x_k, u_k) \\
\text{subj. to } & x_{k+1} = Ax_k + Bu_k, k = 0, 1, 2, \ldots \\
& x_k \in \mathcal{X}, u_k \in \mathcal{U}, k = 0, 1, 2, \ldots \\
& x_0 = x(0)
\end{aligned}
$$

- **Stage cost** $q(x, u)$ describes "cost" of being in state x and applying input u.
- Optimizing over a trajectory provides a **tradeoff between short- and long-term benefits** of actions
- We'll see that such a control law has many beneficial properties...
  ...but we can't compute it: there are an **infinite number of variables**

Receding Horizon Control (what we can sometimes solve)

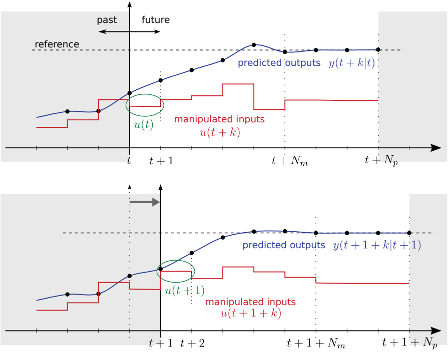
$$
\begin{aligned}
J_t^\star(x(t)) = \min_{U_t} \; & p(x_{t+N}) + \sum_{k=0}^{N-1} q(x_{t+k}, u_{t+k}) \\
\text{subj. to } & x_{t+k+1} = Ax_{t+k} + Bu_{t+k}, k = 0, \dots, N-1 \\
& x_{t+k} \in \mathcal{X}, u_{t+k} \in \mathcal{U}, k = 0, \dots, N-1 \\
& x_{t+N} \in \mathcal{X}_f \\
& x_t = x(t)
\end{aligned}
$$

where $\mathcal{U}_t = \{u_t, \dots, u_{t+N-1}\}$.

Truncate after a finite horizon:

- $p(x_{t+N})$: Approximates the 'tail' of the cost
- $\mathcal{X}_f$: Approximates the 'tail' of the constraints

On-line Receding Horizon Control



- At each sampling time, solve a **CFTOC**.
- Apply the optimal input **only during** $[t, t+1]$
- At $t+1$ solve a CFTOC over a **shifted horizon** based on new state measurements
- The resultant controller is referred to as **Receding Horizon Controller (RHC)** or **Model Predictive Controller (MPC)**.

On-line Receding Horizon Control: MPC

1. MEASURE the state $x(t)$ at time instance t
2. OBTAIN $U_t^\star(x(t))$ by solving the optimization problem
3. IF 'problem infeasible' THEN STOP
4. APPLY the first element $u_t^\star$ of $U_t^\star$ to the system
5. WAIT for the new sampling time $t+1$, GOTO 1)

Note that, we need a constrained optimization solver for step 2).

Outline

MPC: The Motivation

$$x^+ = f(x, u) \qquad\qquad (x, u) \in \mathcal{X}, \mathcal{U}$$

Design control law $u = \kappa(x)$ such that the system:

1. Satifies constraints : $\{x_i\} \subset \mathcal{X}$, $\{u_i\} \subset \mathcal{U}$
2. Is stable: $\lim_{i \to \infty} x_i = 0$
3. Optimizes "performance"
4. Maximizes the set $\{x_0 \mid \text{Conditions 1-3 are met}\}$

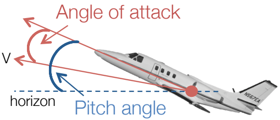
In this lecture, we will demonstrate that these objectives can be met in a predictive control framework.

Example: Cessna Citation Aircraft

Linearized continuous-time model:
(at altitude of 5000m and a speed of 128.2 m/sec)

$$\dot{x} = \begin{bmatrix} -1.2822 & 0 & 0.98 & 0 \\ 0 & 0 & 1 & 0 \\ -5.4293 & 0 & -1.8366 & 0 \\ -128.2 & 128.2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -0.3 \\ 0 \\ -17 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$$



- Input: elevator angle
- States: x_1: angle of attack, x_2: pitch angle, x_3: pitch rate, x_4: altitude
- Outputs: pitch angle and altitude
- Constraints: elevator angle ± 0.262rad ($\pm 15°$), elevator rate ± 0.524rad ($\pm 60°$), pitch angle ± 0.349 ($\pm 39°$)

Open-loop response is unstable (open-loop poles: 0, 0, $-1.5594 \pm 2.29i$)

LQR and Linear MPC with Quadratic Cost

- Quadratic cost
- Linear system dynamics
- Linear constraints on inputs and states

LQR

$$
\begin{aligned}
J_\infty(x(t)) = \min \sum_{k=0}^{\infty} x_k^T Q x_k + u_k^T R u_k \\
\text{subj. to } x_{k+1} = A x_k + B u_k \\
x_0 = x(t)
\end{aligned}
$$

MPC

$$
\begin{aligned}
J_0^*(x(t)) = \min_{U_0} \sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k \\
\text{subj. to } x_{k+1} = A x_k + B u_k \\
x_k \in \mathcal{X},\ u_k \in \mathcal{U} \\
x_0 = x(t)
\end{aligned}
$$

Assume: $Q = Q^T \succeq 0$, $R = R^T \succ 0$

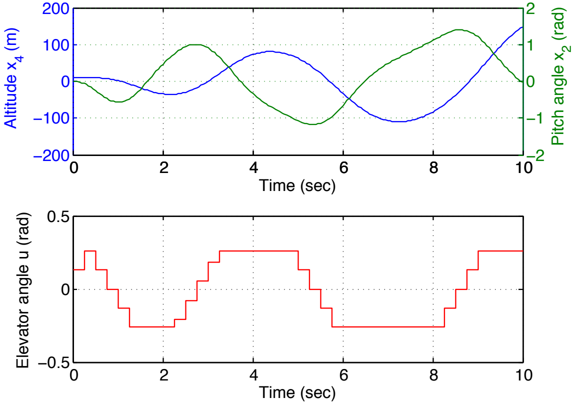
Example: LQR with saturation

Linear quadratic regulator with saturated inputs.

At time $t = 0$ the plane is flying with a deviation of $10m$ of the desired altitude, i.e. $x_0 = [0; 0; 0; 10]$

Problem parameters:

Sampling time 0.25sec, $Q = I$, $R = 10$



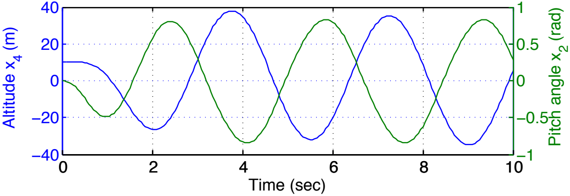
- Closed-loop system is unstable
- Applying LQR control and saturating the controller can lead to instability!

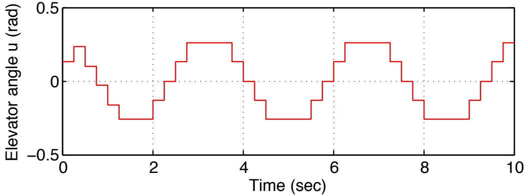
Example: MPC with Bound Constraints on Inputs

MPC controller with input constraints $|u_i| \leq 0.262$

Problem parameters:

$$\boxed{\begin{array}{l}\text{Sampling time } 0.25\text{sec}, \\ Q = I,\ R = 10,\ N = 10\end{array}}$$





The MPC controller uses the knowledge that the elevator will saturate, but it does not consider the rate constraints.

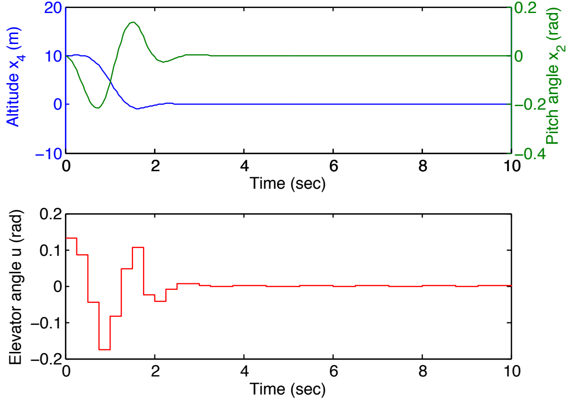
⇒ System does not converge to desired steady-state but to a limit cycle

Example: MPC with all Input Constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\, T_s$

Problem parameters:

$$\boxed{\begin{array}{l}\text{Sampling time } 0.25\text{sec},\\ Q = I,\ R = 10,\ N = 10\end{array}}$$



The MPC controller considers all constraints on the actuator

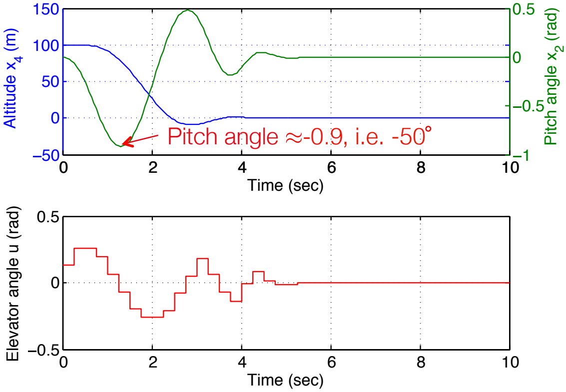
- Closed-loop system is stable
- Efficient use of the control authority

Example: Inclusion of state constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\, T_s$

Problem parameters:

$\boxed{\begin{array}{l}\text{Sampling time 0.25sec,} \\ Q = I,\ R = 10,\ N = 10\end{array}}$



Increase step:
At time $t = 0$ the plane is flying with a deviation of 100m of the desired altitude, i.e.
$x_0 = [0; 0; 0; 100]$

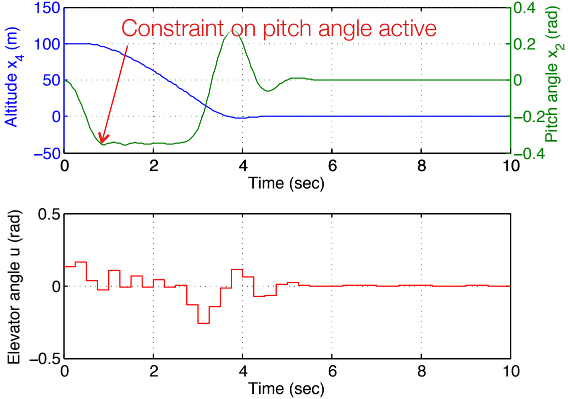
- Pitch angle too large during transient

Example: Inclusion of state constraints

MPC controller with input constraints $|u_i| \leq 0.262$
and rate constraints $|\dot{u}_i| \leq 0.349$
approximated by $|u_k - u_{k-1}| \leq 0.349\,T_s$

Problem parameters:

Sampling time 0.25sec,
$Q = I,\ R = 10,\ N = 10$



Add state constraints for passenger comfort:

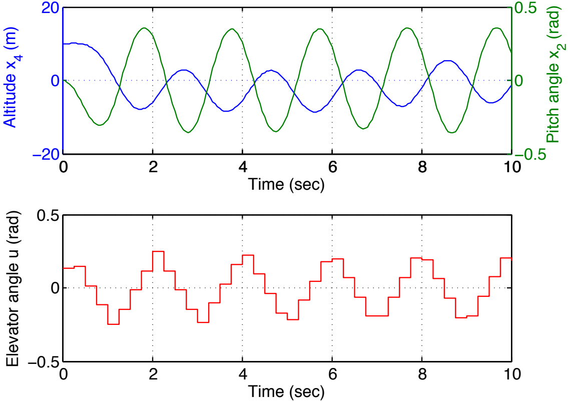
$$|x_2| \leq 0.349$$

Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\,T_s$

Problem parameters:

$\boxed{\text{Sampling time } 0.25\text{sec}, \; Q = I, \; R = 10, \; N = 4}$



Decrease in the prediction horizon causes loss of the stability properties

Outline

Loss of Feasibility and Stability

What can go wrong with "standard" MPC?

- No feasibility guarantee, i.e., the MPC problem may not have a solution
- No stability guarantee, i.e., trajectories may not converge to the origin

Example: Loss of feasibility - Double Integrator

Consider the double integrator

$$\begin{cases} x(t+1) &= \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} x(t) + \begin{bmatrix}0\\1\end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix}1 & 0\end{bmatrix} x(t) \end{cases}$$

subject to the input constraints

$$-0.5 \leq u(t) \leq 0.5$$

and the state constraints

$$\begin{bmatrix}-5\\-5\end{bmatrix} \leq x(t) \leq \begin{bmatrix}5\\5\end{bmatrix}.$$

Compute a receding horizon controller with quadratic objective with

$$N = 3,\ P = Q = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\ R = 10.$$

Example: Loss of feasibility - Double Integrator

The QP problem associated with the RHC is

$$H = \begin{bmatrix} 13.50 & -10.00 & -0.50 \\ -10.00 & 22.00 & -10.00 \\ -0.50 & -10.00 & 31.50 \end{bmatrix},\ F = \begin{bmatrix} -10.50 & 10.00 & -0.50 \\ -20.50 & 10.00 & 9.50 \end{bmatrix},\ Y = \begin{bmatrix} 14.50 & 23.50 \\ 23.50 & 54.50 \end{bmatrix}$$

$$G_0 = \begin{bmatrix} 0.50 & -1.00 & 0.50 \\ -0.50 & 1.00 & -0.50 \\ -0.50 & 0.00 & 0.50 \\ -0.50 & 0.00 & -0.50 \\ 0.50 & 0.00 & -0.50 \\ 0.50 & 0.00 & 0.50 \\ -1.00 & 0.00 & 0.00 \\ 0.00 & -1.00 & 0.00 \\ 1.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & -1.00 \\ 0.00 & 0.00 & 1.00 \\ 0.00 & 0.00 & 0.00 \\ -0.50 & 0.00 & 0.50 \\ 0.00 & 0.00 & 0.00 \\ 0.50 & 0.00 & -0.50 \\ -0.50 & 0.00 & 0.50 \\ 0.50 & 0.00 & -0.50 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \end{bmatrix},\ E_0 = \begin{bmatrix} 0.50 & 0.50 \\ -0.50 & -0.50 \\ 0.50 & 0.50 \\ -0.50 & -0.50 \\ -0.50 & -0.50 \\ 0.50 & 0.50 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 1.00 & 1.00 \\ -0.50 & -0.50 \\ -1.00 & -1.00 \\ 0.50 & 0.50 \\ -0.50 & -1.50 \\ 0.50 & 1.50 \\ 1.00 & 0.00 \\ 0.00 & 1.00 \\ -1.00 & 0.00 \\ 0.00 & -1.00 \end{bmatrix},\ w_0 = \begin{bmatrix} 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \end{bmatrix}$$

Example: Loss of feasibility - Double Integrator

1) MEASURE the state $x(t)$ at time instance t
2) OBTAIN $U_0^*(x(t))$ by solving the CFTOC
3) IF $U_0^*(x(t)) = \emptyset$ THEN 'problem infeasible' STOP
4) APPLY the first element u_0^* of U_0^* to the system
5) WAIT for the new sampling time $t+1$, GOTO 1)

Time step 0:

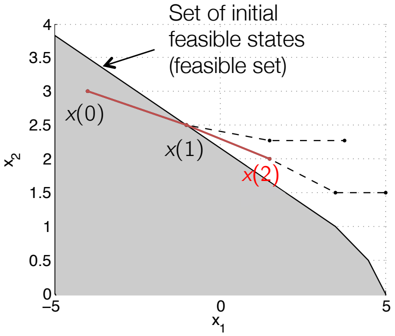
$$x_0 = [-4; 3], \quad u_0^\star(x) = -0.5$$

Time step 1:

$$x_0 = [-1; 2.5], \quad u_0^\star(x) = -0.5$$

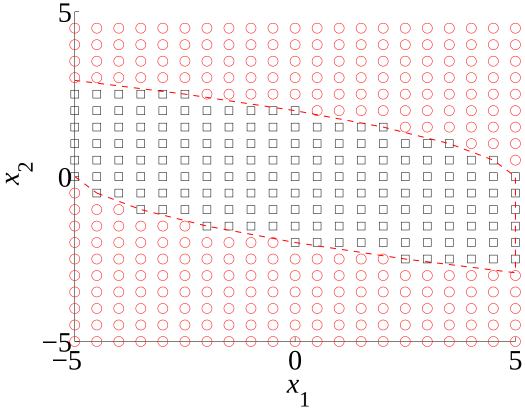
Time step 2:

$x_0 = [1.5; 2]$, Problem infeasible



Depending on initial condition, closed loop trajectory may lead to states for which optimization problem is infeasible.

Example: Loss of feasibility - Double Integrator



Boxes (**Circles**) are initial points leading (not leading) to feasible closed-loop trajectories

Example: Feasibility and stability are function of tuning

Unstable system $x(t+1) = \begin{bmatrix} 2 & 1 \\ 0 & 0.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$

Input constraints $-1 \leq u(t) \leq 1$

State constraints $\begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x(t) \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, Parameters: $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

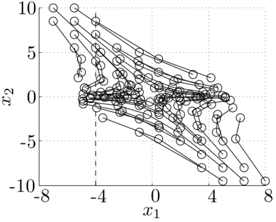
Investigate the stability properties for different horizons N and weights R by solving the finite-horizon MPC problem in a receding horizon fashion...

Example: Feasibility and stability are function of tuning

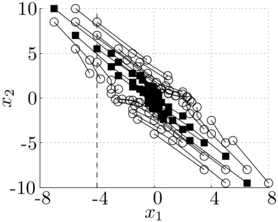
1. $R = 10$, $N = 2$: all trajectories unstable.
2. $R = 2$, $N = 3$: some trajectories stable.
3. $R = 1$, $N = 4$: more stable trajectories.

■ Initial points with convergent trajectories

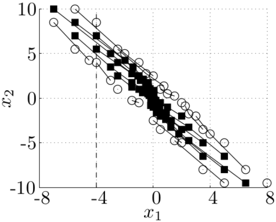
○ Initial points that diverge



Case 1



Case 2



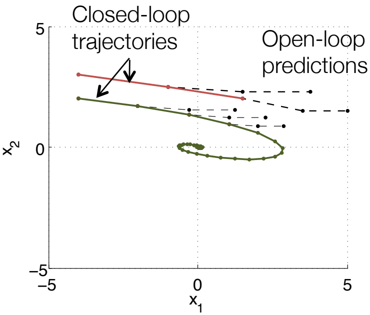
Case 3

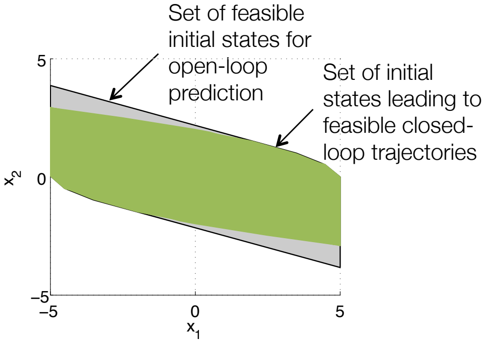
Feasible initial points depend on the horizon N but not on the cost $R \Longrightarrow$ Parameters have complex effect on trajectories.

Summary: Feasibility and Stability

Problems originate from the use of a 'short sighted' strategy

⇒ Finite horizon causes deviation between the open-loop prediction and the closed-loop system:





Ideally we would solve the MPC problem with an infinite horizon, but that is computationally intractable

⇒ Design finite horizon problem such that it approximates the infinite horizon

Summary: Feasibility and Stability

- Infinite-Horizon
  If we solve the RHC problem for $N = \infty$ (as done for LQR), then the open loop trajectories are the same as the closed loop trajectories. Hence
  - If problem is feasible, the closed loop trajectories will be always feasible
  - If the cost is finite, then states and inputs will converge asymptotically to the origin

- Finite-Horizon
  RHC is "short-sighted" strategy approximating infinite horizon controller. But
  - **Feasibility**. After some steps the finite horizon optimal control problem may become infeasible. (Infeasibility occurs without disturbances and model mismatch!)
  - **Stability**. The generated control inputs may not lead to trajectories that converge to the origin.

Feasibility and stability in MPC - Solution

Main idea: Introduce terminal cost and constraints to explicitly ensure feasibility and stability:

$$
\begin{aligned}
J_0^*(x_0) = \quad \min_{U_0} \quad & p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k) && \text{Terminal Cost} \\
\text{subj. to} \quad & \\
& x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\
& x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\
& x_N \in \mathcal{X}_f && \text{Terminal Constraint} \\
& x_0 = x(t)
\end{aligned}
$$

$p(\cdot)$ and $\mathcal{X}_f$ are chosen to mimic an infinite horizon.

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3. Challenges: Feasibility and Stability
4. Guaranteeing Feasibility and Stability
5. Extension to Nonlinear MPC

Outline

4. Guaranteeing Feasibility and Stability

    Proof for $\mathcal{X}_f = 0$

    General Terminal Sets

    Example

Feasibility and Stability of MPC: Proof

Main steps:

- Prove recursive feasibility by showing the existence of a feasible control sequence at all time instants when starting from a feasible initial point
- Prove stability by showing that the optimal cost function is a Lyapunov function

Two cases:

1. Terminal constraint at zero: $x_N = 0$
2. Terminal constraint in some (convex) set: $x_N \in \mathcal{X}_f$

General notation:

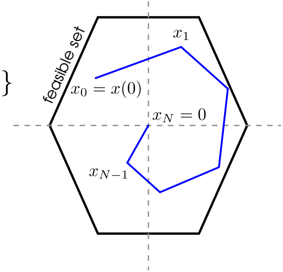
$$J_0^*(x_0) = \min_{U_0} \quad \underbrace{p(x_N)}_{\text{terminal cost}} + \sum_{i=0}^{N-1} \underbrace{q(x_i, u_i)}_{\text{stage cost}}$$

Quadratic case: $q(x_i, u_i) = x_i^T Q x_i + u_i^T R u_i,\ p(x_N) = x_N^T P x_N$

Stability of MPC - Zero terminal state constraint

Terminal constraint: $x_N \in \mathcal{X}_f = 0$

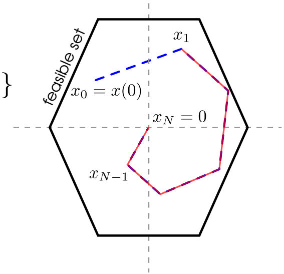
- Assume feasibility of x_0 and let $\{u_0^*,\ u_1^*,\ \dots,\ u_{N-1}^*\}$ be the optimal control sequence computed at x_0 and $\{x(0),\ x_1,\ \dots,\ x_N\}$ be the corresponding state trajectory
- Apply u_0^* and let system evolve to $x(1) = Ax_0 + Bu_0^*$



Stability of MPC - Zero terminal state constraint

Terminal constraint: $x_N \in \mathcal{X}_f = 0$

- Assume feasibility of x_0 and let $\{u_0^*,\ u_1^*,\ \dots,\ u_{N-1}^*\}$ be the optimal control sequence computed at x_0 and $\{x(0),\ x_1,\ \dots,\ x_N\}$ be the corresponding state trajectory
- Apply u_0^* and let system evolve to $x(1) = Ax_0 + Bu_0^*$
- At $x(1)$ the control sequence $\{u_1^*,\ u_2^*,\ \dots,\ u_{N-1}^*,\ 0\}$ is feasible (apply 0 control input $\Rightarrow x_{N+1} = 0$)

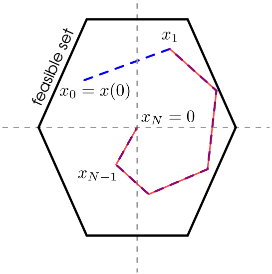


$\Rightarrow$ **Recursive feasibility ✔**

Stability of MPC - Zero terminal state constraint

Terminal constraint: $x_N \in \mathcal{X}_f = 0$

Goal: Show $J_0^*(x_1) < J_0^*(x_0) \quad \forall x_0 \neq 0$



$$J_0^*(x_0) = \underbrace{p(x_N)}_{=0} + \sum_{i=0}^{N-1} q(x_i, u_i^*)$$

$$\begin{aligned} J_0^*(x_1) \leq \tilde{J}_0(x_1) &= \sum_{i=1}^{N} q(x_i, u_i^*) \\ &= \sum_{i=0}^{N-1} q(x_i, u_i^*) - q(x_0, u_0^*) + q(x_N, u_N) \\ &= J_0^*(x_0) - \underbrace{q(x_0, u_0^*)}_{\text{Subtract cost at stage 0}} + \underbrace{q(0, 0)}_{\text{=0, Add cost for staying at 0}} \end{aligned}$$

$\Rightarrow$ $J_0^*(x)$ is a Lyapunov function $\rightarrow$ (Lyapunov) Stability ✔

Example: Impact of Horizon with Zero Terminal Constraint

System dynamics:

$$x_{k+1} = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u_k$$

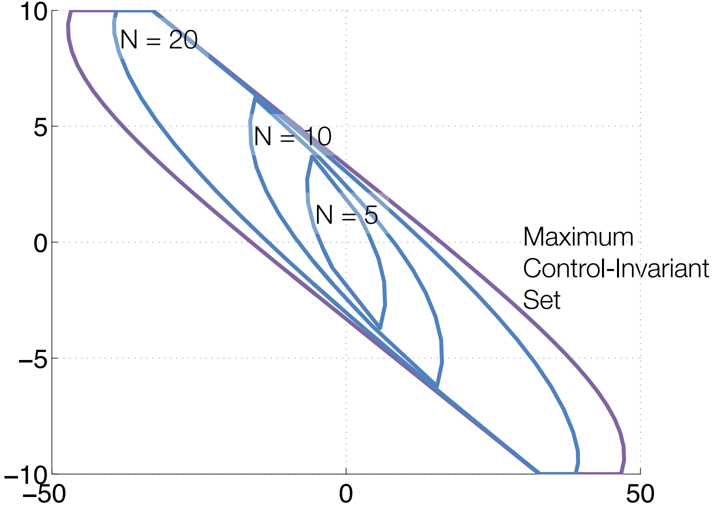
Constraints:

$$\begin{aligned} \mathcal{X} &:= \{x \,|\, -50 \le x_1 \le 50,\ -10 \le x_2 \le 10\} = \{x \,|\, A_x x \le b_x\} \\ \mathcal{U} &:= \{u \,|\, \|u\|_\infty \le 1\} = \{u \,|\, A_u u \le b_u\} \end{aligned}$$

Stage cost:

$$q(x,u) := x' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + u^\top u$$

Example: Impact of Horizon with Zero Terminal Constraint



The horizon can have a strong impact on the region of attraction.

MPC Stability and Feasibility - Summary

IF we choose the terminal constraint: $x_N \in \mathcal{X}_f = 0$, THEN

- The set of feasible initial states $\mathcal{X}_0$ is also the set of initial states which are persistently feasible (feasible at all $t \geq 0$) for the system in closed-loop with the designed MPC.
- The equilibrium point $(0, 0)$ is asymptotically stable according to Lyapunov.
- $J_0^\star(x)$ is a Lyapunov function for the closed loop system (system + MPC) defined over $\mathcal{X}_0$. Then $\mathcal{X}_0$ is the region of attraction of the equilibrium point.
- Proof works for any nonlinear system and positive definite and continuous cost.

Outline

4. Guaranteeing Feasibility and Stability

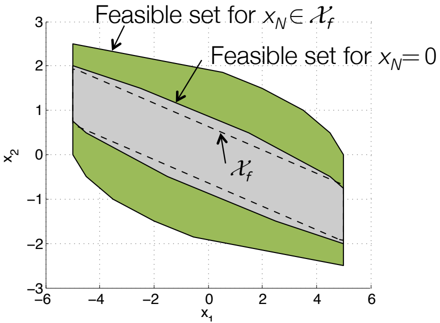
   Proof for $\mathcal{X}_f = 0$

   General Terminal Sets

   Example

Extension to More General Terminal Sets

Problem: The terminal constraint $x_N = 0$ reduces the size of the feasible set
Goal: Use convex set $\mathcal{X}_f$ to increase the region of attraction



Double integrator

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\begin{bmatrix} -5 \\ -5 \end{bmatrix} \leq x(t) \leq \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$-0.5 \leq u(t) \leq 0.5$$

$$N = 5, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = 10$$

Goal: Generalize proof to the constraint $x_N \in \mathcal{X}_f$

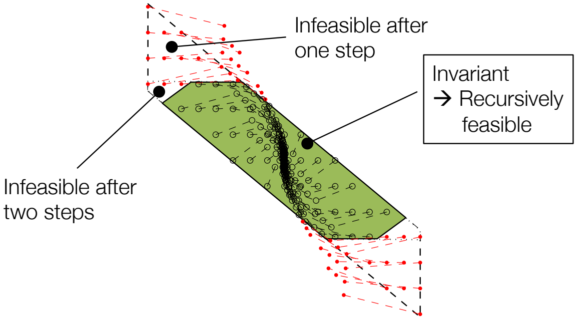
Invariant sets

Definition: Invariant set

A set $\mathcal{O}$ is called **positively invariant** for system $x(t+1) = f_{cl}(x(t))$, if

$$x(0) \in \mathcal{O} \Rightarrow x(t) \in \mathcal{O}, \quad \forall t \in \mathbb{N}_+$$

The positively invariant set that contains every closed positively invariant set is called the maximal positively invariant set $\mathcal{O}_\infty$.



Summary of important sets (1/2)

Consider the constrained system

$$\begin{aligned} x(t+1) = & g(x(t), u(t)) \\ & x(t) \in \mathcal{X}, u(t) \in \mathcal{U} \end{aligned} \qquad (\star)$$

and the MPC controller: $u(t) = \text{MPC}(x(t), g, Q, R, P, N, \mathcal{X}, \mathcal{U}, \mathcal{X}_f)$, compactly rewritten as $u(t) = \text{MPC}(x(t), par)$.

Below are the important set:

$\mathcal{X}$: State constraints: we want the system state to be in $\mathcal{X}$ at all time instants.

$\mathcal{X}_0$: Set of $\bar{x}$ such that $\text{MPC}(\bar{x}, par)$ is feasible. A control input U_0 can only be found if $x(0) \in \mathcal{X}_0$. The set $\mathcal{X}_0$ depends on $\mathcal{X}$ and $\mathcal{U}$, on the controller horizon N and on the controller terminal set $\mathcal{X}_f$. It does not depend on the objective function.

Summary of important sets (2/2)

Below are the important sets:

$\mathcal{O}_\infty$: The maximum positive invariant set for system ($\star$) in closed loop with $u(t) = \text{MPC}(x(t), par)$. It depends on the MPC controller and as such on all parameters affecting the controller, i.e. $\mathcal{X}, \mathcal{U}, N, \mathcal{X}_f$ and the objective function with its parameters P, Q, and R. Clearly $\mathcal{O}_\infty \subseteq \mathcal{X}_0$ because if it were not there would be points in $\mathcal{O}_\infty$ for which the control problem is not feasible. Because of invariance, the closed-loop is persistently feasible for all states $x(0) \in \mathcal{O}_\infty$.

$\mathcal{C}_\infty$: The maximal control invariant set $\mathcal{C}_\infty$ for system ($\star$). It is only affected by the sets $\mathcal{X}$ and $\mathcal{U}$, the contraints on states and inputs. It is the largest set over which we can expect any controller to work. $\mathcal{X}_0$ has generally no relation with $\mathcal{C}_\infty$ (it can be larger, smaller, etc). Clearly, $\mathcal{O}_\infty \subseteq \mathcal{C}_\infty$.

Stability of MPC - Main Result

Assumptions

1. Stage cost is positive definite, i.e. it is strictly positive and only zero at the origin
2. Terminal set is **invariant** under the local control law $v(x_k)$:

   $$x_{k+1} = Ax_k + Bv(x_k) \in \mathcal{X}_f, \quad \text{for all } x_k \in \mathcal{X}_f$$

   All state and input **constraints are satisfied** in $\mathcal{X}_f$:

   $$\mathcal{X}_f \subseteq \mathcal{X},\ v(x_k) \in \mathcal{U}, \quad \text{for all } x_k \in \mathcal{X}_f$$

3. Terminal cost is a continuous **Lyapunov function** in the terminal set $\mathcal{X}_f$ and satisfies:

   $$p(x_{k+1}) - p(x_k) \leq -q(x_k, v(x_k)), \quad \text{for all } x_k \in \mathcal{X}_f$$

Under those 3 assumptions:

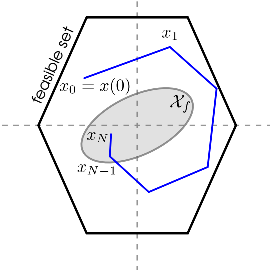
Theorem

The closed-loop system under the MPC control law $u_0^*(x)$ is asymptotically stable and the set $\mathcal{X}_f$ is positive invariant for the system

$$x(k+1) = Ax + Bu_0^*(x).$$

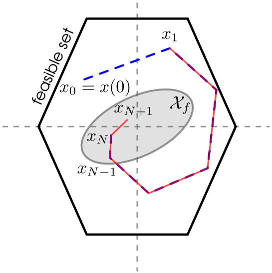
Stability of MPC - Outline of the Proof

- Assume feasibility of $x(0)$ and let $\{u_0^*,\ u_1^*,\ \ldots,\ u_{N-1}^*\}$ be the optimal control sequence computed at $x(0)$ and $\{x(0),\ x_1,\ \ldots,\ x_N\}$ the corresponding state trajectory



Stability of MPC - Outline of the Proof

- Assume feasibility of $x(0)$ and let $\{u_0^*,\ u_1^*,\ \ldots,\ u_{N-1}^*\}$ be the optimal control sequence computed at $x(0)$ and $\{x(0),\ x_1,\ \ldots,\ x_N\}$ the corresponding state trajectory



- At $x(1)$, $\{u_1^*,\ u_2^*,\ \ldots,\ v(x_N)\}$ is feasible:

$$x_N \text{ is in } \mathcal{X}_f \rightarrow v(x_N) \text{ is feasible}$$

$$\text{and } x_{N+1} = Ax_N + Bv(x_N) \text{ in } \mathcal{X}_f$$

$\Rightarrow$ **Terminal constraint provides recursive feasibility**

Asymptotic Stability of MPC - Outline of the Proof

$$J_0^*(x_0) = \sum_{i=0}^{N-1} q(x_i, u_i^*) + p(x_N)$$

Feasible, sub-optimal sequence for $x_1 : \{u_1^*,\ u_2^*,\ \dots,\ v(x_N)\}$

$$\begin{aligned}
J_0^*(x_1) &\leq \sum_{i=1}^{N} q(x_i, u_i^*) + p(Ax_N + Bv(x_N)) \\
&= \sum_{i=0}^{N-1} q(x_i, u_i^*) + p(x_N) - q(x_0, u_0^*) + p(Ax_N + Bv(x_N)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - p(x_N) + q(x_N, v(x_N)) \\
&= J_0^*(x_0) - q(x_0, u_0^*) + \underbrace{p(Ax_N + Bv(x_N)) - p(x_N) + q(x_N, v(x_N))}_{p(x) \leq 0} \\
&\Longrightarrow J_0^*(x_1) - J_0^*(x_0) \leq -q(x_0, u_0^*), \quad q > 0
\end{aligned}$$

$J_0^*(x)$ **is a Lyapunov function decreasing along the closed loop trajectories**
$\Rightarrow$ The closed-loop system under the MPC control law is asymptotically stable

MPC Stability and Feasibility - Summary

IF we choose: $\mathcal{X}_f$ to be an invariant set (Assumption 2) and the terminal cost $p(x)$ to be a Lyapunov function with the decrease described in Assumption 3, THEN

- The set of feasible initial states $\mathcal{X}_0$ is also the set of initial states which are persistently feasible (feasible for all $t \geq 0$) for the system in closed-loop with the designed MPC.
- The equilibrium point $(0, 0)$ is asymptotically stable according to Lyapunov.
- $J_0^\star(x)$ is a Lyapunov function for the closed loop system (system + MPC) defined over $\mathcal{X}_0$. Then $\mathcal{X}_0$ is the region of attraction of the equilibrium point.
- Proof works for any nonlinear system and positive definite and continuous stage cost.

Choice of Terminal Sets and Cost - Linear System, Quadratic Cost

$$
\begin{array}{llll}
J_0^*(x_0) = & \min\limits_{U_0} & x_N' P x_N + \sum\limits_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k & \text{Terminal Cost} \\
& \text{subj. to} & & \\
& & x_{k+1} = A x_k + B u_k,\ k = 0, \ldots, N-1 & \\
& & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 & \\
& & x_N \in \mathcal{X}_f & \text{Terminal Constraint} \\
& & x_0 = x(t) &
\end{array}
$$

Choice of Terminal Sets and Cost - Linear System, Quadratic Cost

- Design unconstrained LQR control law

$$F_\infty = -(B' P_\infty B + R)^{-1} B' P_\infty A$$

where P_∞ is the solution to the discrete-time algebraic Riccati equation:

$$P_\infty = A' P_\infty A + Q - A' P_\infty B (B' P_\infty B + R)^{-1} B' P_\infty A$$

- Choose the terminal weight $P = P_\infty$
- Choose the terminal set $\mathcal{X}_f$ to be the maximum invariant set for the closed-loop system $x_{k+1} = (A + BF_\infty)x_k$:

$$x_{k+1} = Ax_k + BF_\infty(x_k) \in \mathcal{X}_f, \quad \text{for all } x_k \in \mathcal{X}_f$$

All state and input **constraints are satisfied** in $\mathcal{X}_f$:

$$\mathcal{X}_f \subseteq \mathcal{X},\ F_\infty x_k \in \mathcal{U}, \quad \text{for all } x_k \in \mathcal{X}_f$$

Choice of Terminal Sets and Cost - Linear System, Quadratic Cost

1. The stage cost is a positive definite function

2. By construction the terminal set is **invariant** under the local control law $v = F_\infty x$

3. Terminal cost is a continuous **Lyapunov function** in the terminal set $\mathcal{X}_f$ and satisfies:

$$
\begin{aligned}
&x_{k+1}' P x_{k+1} - x_k' P x_k \\
&= x_k'(-P_\infty + A' P_\infty A - A' P_\infty B (B' P_\infty B + R)^{-1} B' P_\infty A - F_\infty' R F_\infty) x_k \\
&= -x_k' Q x_k - v_k' R v_k
\end{aligned}
$$

$\boxed{\text{All the Assumptions of the Feasibility and Stability Theorem are verified.}}$

Example: Unstable Linear System

System dynamics:

$$x_{k+1} = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u_k$$

Constraints:

$$\begin{aligned} \mathcal{X} &:= \{x \mid -50 \le x_1 \le 50,\ -10 \le x_2 \le 10\} = \{x \mid A_x x \le b_x\} \\ \mathcal{U} &:= \{u \mid \|u\|_\infty \le 1\} = \{u \mid A_u u \le b_u\} \end{aligned}$$

Stage cost:

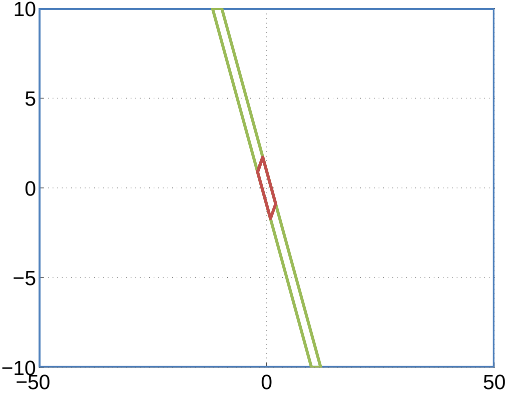
$$q(x, u) := x' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + u^\top u$$

Horizon: $N = 10$

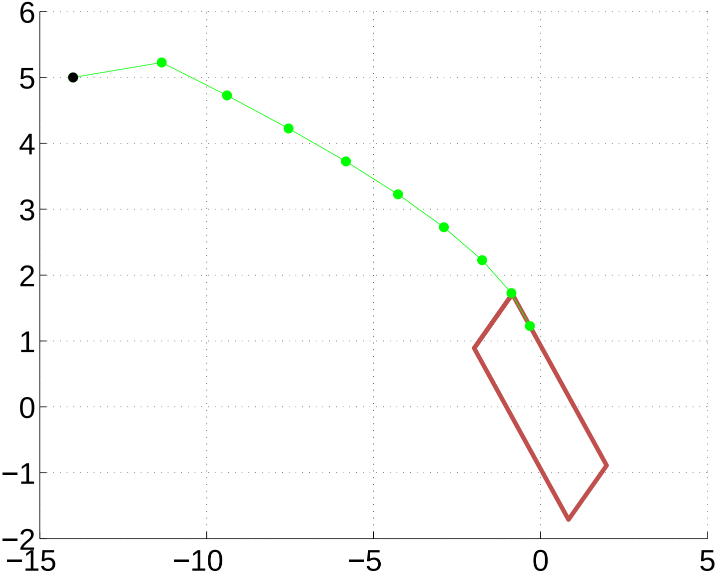
Example: Designing MPC Problem

1. Compute the optimal LQR controller and cost matrices: F_∞, P_∞
2. Compute the maximal invariant set $\mathcal{X}_f$ for the closed-loop linear system $x_{k+1} = (A + BF_\infty)x_k$ subject to the constraints

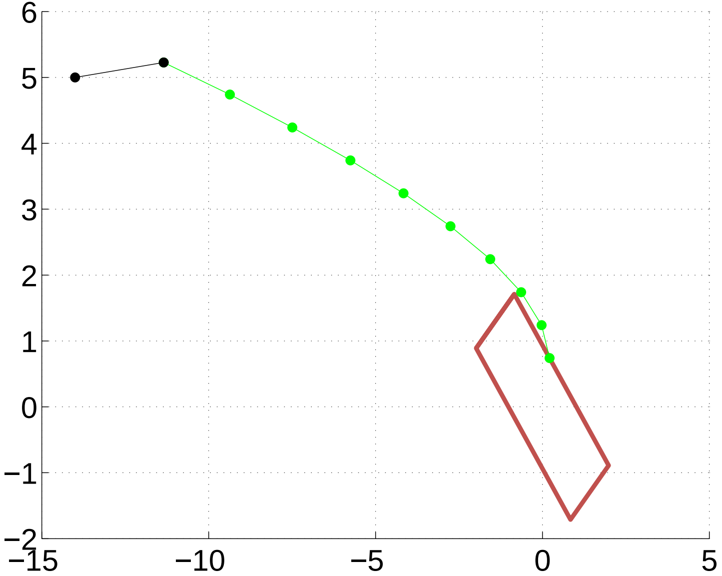
$$\mathcal{X}_{\text{cl}} := \left\{ x \,\middle|\, \begin{bmatrix} A_x \\ A_u F_\infty \end{bmatrix} x \leq \begin{bmatrix} b_x \\ b_u \end{bmatrix} \right\}$$



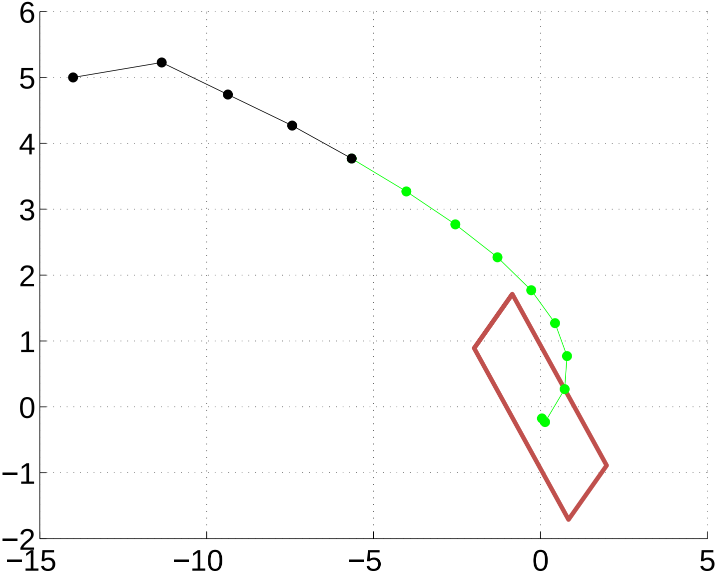
Example: Closed-loop behaviour



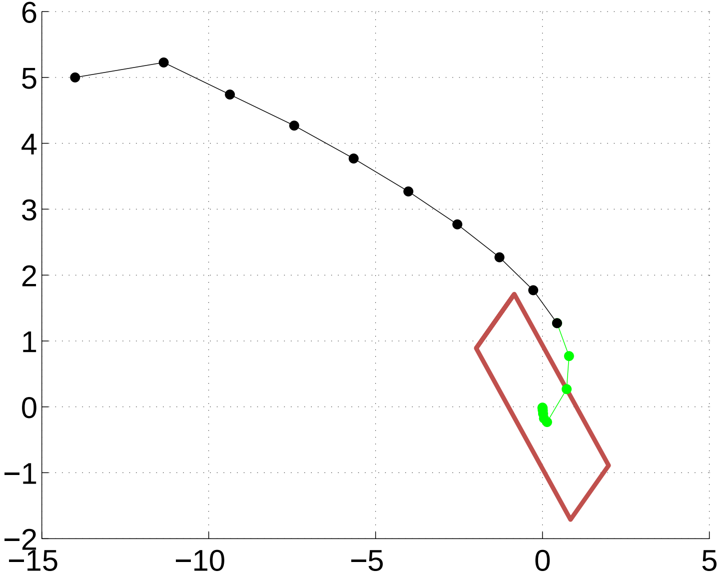
Example: Closed-loop behaviour



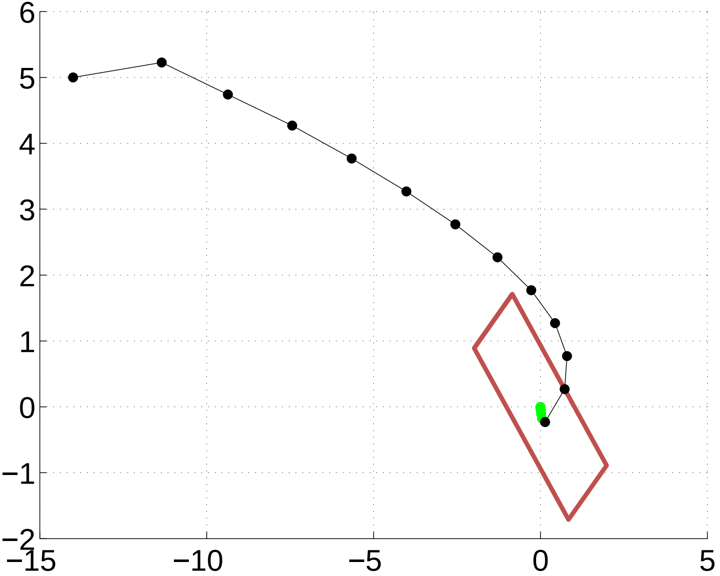
Example: Closed-loop behaviour



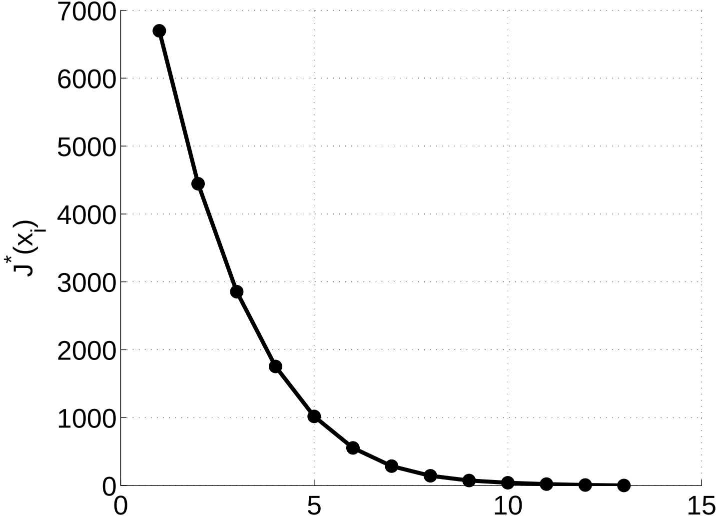
Example: Closed-loop behaviour



Example: Closed-loop behaviour



Example: Lyapunov Decrease of Optimal Cost



Stability of MPC - Remarks

- The terminal set $\mathcal{X}_f$ and the terminal cost ensure recursive feasibility and stability of the closed-loop system.
  But: the terminal constraint reduces the region of attraction.
  (Can extend the horizon to a sufficiently large value to increase the region)

Are terminal sets used in practice?

- Generally not...
  - Not well understood by practitioners
  - Requires advanced tools to compute (polyhedral computation or LMI)
- Reduces region of attraction
  - A 'real' controller must provide *some* input in *every* circumstance
- Often unnecessary
  - Stable system, long horizon $\rightarrow$ will be stable and feasible in a (large) neighbourhood of the origin

Choice of Terminal Set and Cost: Summary

- Terminal constraint provides a sufficient condition for stability
- Region of attraction without terminal constraint may be larger than for MPC with terminal constraint but characterization of region of attraction extremely difficult
- $\mathcal{X}_f = 0$ simplest choice but small region of attaction for small N
- Solution for linear systems with quadratic cost
- In practice: Enlarge horizon and check stability by sampling
- With larger horizon length N, region of attraction approaches maximum control invariant set

Outline

4. Guaranteeing Feasibility and Stability

    Proof for $\mathcal{X}_f = 0$

    General Terminal Sets

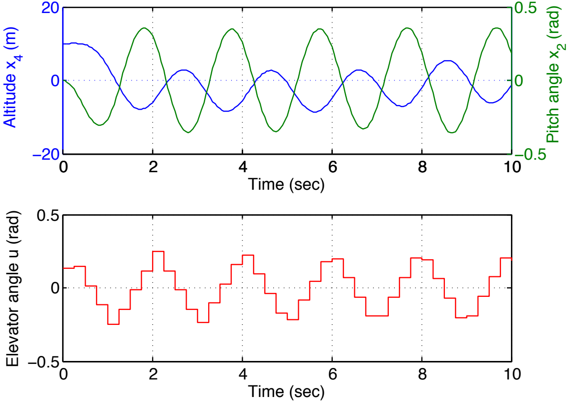
    Example

Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\, T_s$

Problem parameters:

Sampling time 0.25sec, $Q = I$, $R = 10$, $N = 4$



Decrease in the prediction horizon causes loss of the stability properties

Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\,T_s$

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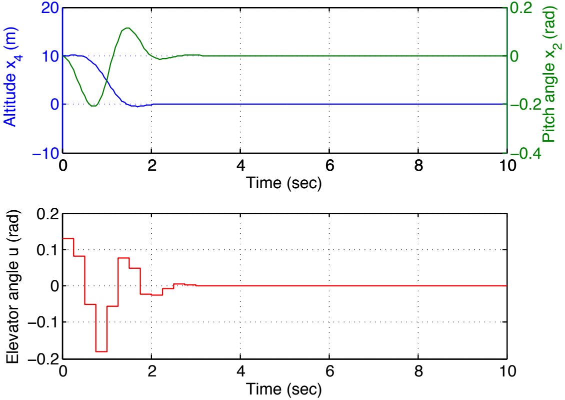

Inclusion of terminal cost and constraint provides stability

Outline

Extension to Nonlinear MPC

Consider the nonlinear system dynamics: $x(t+1) = g(x(t), u(t))$

$$
\begin{array}{lll}
J_0^*(x(t)) = & \min\limits_{U_0} & p(x_N) + \sum\limits_{k=0}^{N-1} q(x_k, u_k) \\
& \text{subj. to} & x_{k+1} = g(x_k, u_k),\ \ k = 0, \ldots, N-1 \\
& & x_k \in \mathcal{X},\ \ u_k \in \mathcal{U},\ \ k = 0, \ldots, N-1 \\
& & x_N \in \mathcal{X}_f \\
& & x_0 = x(t)
\end{array}
$$

- Presented assumptions on the terminal set and cost did not rely on linearity
- Lyapunov stability is a general framework to analyze stability of nonlinear dynamic systems

$\rightarrow$ Results can be directly extended to nonlinear systems.

However, computing the sets $\mathcal{X}_f$ and function p can be very difficult!

Summary

Finite-horizon MPC may not be stable!

Finite-horizon MPC may not satisfy constraints for all time!

- An infinite-horizon provides stability and invariance.
- We 'fake' infinite-horizon by forcing the final state to be in an invariant set for which there exists an invariance-inducing controller, whose infinite-horizon cost can be expressed in closed-form.
- These ideas extend to non-linear systems, but the sets are difficult to compute.