Model Predictive Control
Prof. Manfred Morari

ESE 6190, Spring 2023
Due: April 3

Exercise sheet 5
MPC Theory II

Instructions:

1. Apart from your solutions, you are also required to upload your code to Gradescope. Make sure the code is well commented and add a `readme.m` file describing how to run your code. You could use MATLAB publisher to print out your solutions. This allows the comments and the plots to appear inline as they are in the MATLAB script.
2. You will need the Multi-Parametric Toolbox in Exercise 1. Follow the installation instructions here: `https://www.mpt3.org/Main/Installation`

Exercise 1 **Linear MPC Design**

Consider the following system

$$x(k+1) = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \tag{1}$$

The state and input constraints are

$$\mathcal{U} : -1 \leq u(k) \leq 1; \tag{2a}$$

$$\mathcal{X} : \begin{bmatrix} -15 \\ -15 \end{bmatrix} \leq x(k) \leq \begin{bmatrix} 15 \\ 15 \end{bmatrix}; \tag{2b}$$

Design a 2-norm MPC with $Q = eye(2)$, $R = 1$ and $N = 3$.

Approach 1

1. **[4 pts]** Choose $\mathcal{X}_f = 0$, and a terminal weight P so that asymptotic stability is guaranteed for all $x_0 \in \mathcal{X}_0$.
2. **[4 pts]** Let $x(0) = [2, -1]^T$ and plot the closed-loop state trajectory (plot $x(2)$ vs $x(1)$ on one figure) as well as the open-loop trajectories predicted by the MPC for 25 simulation steps.
3. **[4 pts]** For the same $x(0) = [2, -1]^T$, analyze the mismatch between predicted vs closed-loop trajectories for $N = 10$ in the MPC design.

Approach 2

4. **[3 pts]** Now choose P to be the solution to the discrete-time algebraic Riccati equation (DARE), which is the infinite LQR solution. Then, set $\mathcal{X}_f$ to be the maximal invariant set O_∞ defined by the closed loop solution $x(k+1) = (A + BF_\infty)x(k)$. To compute and plot O_∞ using MPT3 tools, use the following code:

```
system = LTISystem('A',Acl);
Xtilde = Polyhedron('A',[eye(2);-eye(2);Finf;-Finf],'b',[15;15;15;15;1;1]);
Oinf = system.invariantSet('X',Xtilde)
figure
plot(Oinf)
```

where `Acl` is the closed loop matrix $A + BF_\infty$ and `Finf` is the LQR control law gain. Note that the set `Xtilde` is just the intersection of $\mathcal{X}$ and the set of x such that $F_\infty x \in \mathcal{U}$. The set O_∞ is then the set $\{x | (\texttt{Oinf.A})x \leq \texttt{Oinf.b}\}$ where `Oinf.A` and `Oinf.B` are extracted from the computed O_∞.

5. **[4 pts]** Let $x(0) = [2, -1]^T$ and plot the closed-loop state trajectory as well as the open-loop trajectories predicted by the MPC with these new terminal cost and terminal constraint chosen in part 4 for $N = 3$ for 25 simulation steps. Comment on any differences.

Approach 3

6. **[4 pts]** Instead of using the unconstrained LQR solution to design our MPC, now use a different stabilizing controller $u = -Kx$ where K is*:

$$K = \begin{bmatrix} 1.595 & 2.35 \end{bmatrix}$$

What is the maximal positive invariant set O_∞ under this control law? **Make sure to edit the code in part 4 to take the new feedback law into account**. How does it compare to O_∞ computed in part 4? Find a new terminal cost such that it is a Lyapunov Function for $x \in \mathcal{X}_f = O_\infty$. You should be able to design a quadratic terminal cost with weight P by simply using the function `dlyap`.

* Aside: the gain matrix K was chosen by pole placement Matlab code `place(A,B,[0.1,−0.25])`.

7. **[4 pts]** Let $x(0) = [2, -1]^T$ and plot the closed-loop state trajectory as well as the open-loop trajectories predicted by the MPC with these new terminal cost and terminal constraint from part 6 for $N = 3$ for 25 simulation steps. How does this compare to the other solutions?

Persistent Feasibility

8. **[4 pts]** Use now $N = 1$. Can you choose $\mathcal{X}_f$ so that persistent feasibility is guaranteed for all x_0 belonging to the maximal control invariant set C_∞? Compute and plot C_∞ using MPT3.

9. **[4 pts]** With the design in the previous part (8) you have a very short horizon and persistent feasibility in $\mathcal{X}_0 = C_\infty$. It seems a great MPC design with just $N = 1$! What are we missing compared to an infinite horizon constrained optimal control? Let $x(0) = [2, -1]^T$ and plot the closed-loop state trajectory as well as the open-loop trajectories predicted by the MPC with $\mathcal{X}_f = C_\infty$ and the terminal cost given by the infinite LQR solution.

Exercise 2 **Terminal Invariant Set**

Consider the linear system

$$x^+ = \begin{bmatrix} 0.5 & 0 \\ 4 & 0.8 \end{bmatrix} x + \begin{bmatrix} 0.3 & 0.2 \\ -0.6 & 0.9 \end{bmatrix} u$$

with constraints on the input $\|u\|_\infty \leq 1$.

1. **[3 pts]** What is the maximum control invariant set for this system? Justify your answer.

2. **[5 pts]** Consider the following standard MPC optimization problem, and let $\pi(x)$ be the resulting receding-horizon control law.

$$
\begin{aligned}
\min \quad & \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i + x_N^\top Q_f x_N \\
\text{s.t.} \quad & x_{i+1} = A x_i + B u_i \\
& u_i \in U \quad i \in \{0, \dots, N-1\} \\
& x_N \in X_f \\
& x_0 = x
\end{aligned}
$$

Describe how to choose a terminal control law, K_f, terminal weight Q_f and terminal set X_f so that the closed-loop system $x^+ = Ax + B\pi(x)$ has a maximal invariant set equal to that given in part 1 and is asymptotically stable.

Exercise 3 **Terminal Invariant Set**

The following 1-dimensional system

$$x_{k+1} = 1.5 x_k + u_k$$

with constraints

$$
\begin{aligned}
-1 \le x_k \le 1, \qquad & k = 1, 2, \dots \\
-0.1 \le u_k \le 0.1, \qquad & k = 0, 1, \dots
\end{aligned}
$$

is given. In this exercise, we consider linear state feedback controllers of the form $u_k = K x_k$, which result in the following closed loop systems:

$$x_{k+1} = (A + BK) x_k \, .$$

For the above system we shall study its maximal positively invariant set $\mathcal{O} \subseteq \mathcal{X}_{\text{cl}}$, where

$$\mathcal{X}_{\text{cl}} := \{ x \in \mathbb{R} \mid -1 \le x \le 1, \; -0.1 \le Kx \le 0.1 \} \, .$$

1. **[2 pts]** Show that for $K = -2$, the maximal positively invariant set $\mathcal{O}$ is symmetric, i.e. it takes the form $\mathcal{O} = [-a, a]$, for some $a \in \mathbb{R}_+$.

2. **[6 pts]** Find the K that maximizes the parameter a, which defines the invariant set $\mathcal{O} = [-a, a]$.

   Hint: You may assume that the set $\mathcal{O}$ is symmetric for any K.

3. **[4 pts]** Does there exist $\mathcal{X}_f$, Q, R and P such that the control law of the MPC problem defined in (3) guarantees asymptotic stability? Justify your answer!

$$
\begin{aligned}
\min_U \quad & \sum_{k=0}^{N-1} (x_k^\top Q x_k + u_k^\top R u_k) + x_N^\top P x_N \\
\text{s.t.} \quad & x_0 = x(0), \\
& x_{k+1} = 1.5 x_k + u_k, \qquad k = 0, 1, \dots, N-1, \\
& -1 \le x_k \le 1, \qquad k = 0, 1, \dots, N-1, \\
& -0.1 \le u_k \le 0.1, \qquad k = 0, 1, \dots, N-1, \\
& x_N \in \mathcal{X}_f.
\end{aligned}
\tag{3}
$$

   Hint 1: It is not necessary to explicitly compute Q, R and P.

   Hint 2: If you were not able to solve part 2, use $K = -1$ and $\mathcal{O} = [-0.1, 0.1]$