University of Pennsylvania, ESE 6190

Model Predictive Control

Chapter 3: Model Uncertainty and State Estimation

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F. Borrelli, A. Bemporad, and M. Morari, Predictive Control for Linear and Hybrid Systems, Cambridge University Press, 2017.

Outline

1. Uncertainty Modeling

2. State Estimation

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1. Uncertainty Modeling

   Objective Statement, Stochastic Processes

   Modeling using State Space Descriptions

   Obtaining Models from First Principles

   Obtaining Models from System Identification

Objective Statement

One of the main reasons for control is to suppress the effect of disturbances on key process outputs. A model is needed to predict the disturbances' influence on the outputs on the basis of measured signals. For unmeasured disturbances, stochastic models are used.

Objective

In this part we introduce **stochastic models** for disturbances and show how to integrate them into deterministic system models for estimation and control. We discuss how to construct models of the form:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + Fw(k) \\ y(k) &= Cx(k) + Gw(k) \end{aligned} \tag{1}$$

where $w(k)$ is a disturbance signal.

Stochastic processes (1/2)

Stochastic processes are the mathematical tool used to model uncertain signals.

- A discrete-time stochastic process is a sequence of **random variables**

$$\{w(0), w(1), w(2), ...\}$$

- The **realization** of the process is uncertain. We can model a stochastic process via its probability distribution
- In general, one must specify the joint probability distribution function (pdf) for the entire time sequence $\mathbb{P}(w(0), w(1), ...)$

Stochastic processes (2/2)

- Stochastic processes are modeled using data: Estimating the joint pdf usually is intractable
- Thus the **normal distribution assumption** is often made: only models of the *mean* and the *covariance* are needed

Further distinction of stochastic processes:

- **Stationary** stochastic processes
- **Nonstationary** stochastic processes

Informally, a stationary process with normal distribution has mean and variance that do not vary over a shifting time window.

Normal stochastic process

- Joint pdf is a normal distribution
- Completely defined by its mean and covariance function

$$\mu_w(k) := E\{w(k)\}$$
$$R_w(k,\tau) := E\{w(k+\tau)w^\top(k)\} - \mu_w(k+\tau)\mu_w(k)$$

- Stationary if $\mu_w(k) = \mu_w$ and $R_w(k,\tau) = R_w(\tau)$
- Typically data are used to estimate $\mu_w(k)$ and $R_w(k,\tau)$

Special case: Normal **white noise** stochastic process $\varepsilon(k)$

$$\mu_\varepsilon = 0$$
$$R_\varepsilon(k,\tau) = \begin{cases} R_\varepsilon & \text{if } \tau = 0 \\ 0 & \text{otherwise} \end{cases}$$

- Since jointly normally distributed and uncorrelated over time, $\varepsilon(k)$ is independent of time

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Stationary Case (1/2)

We model the stochastic process as the output $w(k)$ of a linear system driven by normal white noise $\varepsilon(k)$. It will turn out that $w(k)$ is a normal stochastic process and $\mu_w(k)$, $R_w(k,\tau)$ can be chosen through A_w,B_w,C_w. In the **stationary case** and using a **state space** description we have:

$$\begin{aligned} x_w(k+1) &= A_w x_w(k) + B_w \varepsilon(k) \\ w(k) &= C_w x_w(k) + \varepsilon(k) \end{aligned} \tag{2}$$

where:

- x_w is an additional state introduced to model the linear system's response to white noise
- all the eigenvalues of A_w lie strictly inside the unit circle.
- (2) is the standard form for many filter and control design tools. We will show how to determine A_w,B_w,C_w in practical situations.

Stationary Case (2/2)

- The output $w(k)$ of system (2) with white noise $\varepsilon(k)$ and stable A_w has the following properties:

$$\begin{aligned} E\{x_w(k)\} &= A_w E\{x_w(k-1)\} = A_w^k E\{x_w(0)\} \\ E\{x_w(k)x_w^\top(k)\} &= A_w E\{x_w(k-1)x_w^\top(k-1)\}A_w^\top + B_w R_\varepsilon B_w^\top \\ E\{x_w(k+\tau)x_w^\top(k)\} &= A_w^\tau E\{x_w(k)x_w^\top(k)\} \end{aligned} \quad (3)$$

- From this one can deduce that

$$\begin{aligned} \bar{w} = \lim_{k\to\infty} E\{w(k)\} &= 0 \\ R_w(\tau) = \lim_{k\to\infty} E\{w(k+\tau)w^\top(k)\} &= C_w A_w^\tau \bar{P}_w C_w^\top + C_w A_w^{\tau-1} B_w R_\varepsilon \end{aligned} \quad (4)$$

  where

$$\bar{P}_w = A_w \bar{P}_w A_w^\top + B_w R_\varepsilon B_w^\top, \quad (5)$$

  i.e. $\bar{P}_w$ is a positive semi-definite solution to a Lyapunov equation.
- These relations can be used in order to determine A_w, B_w, C_w matching a certain covariance $R_w(\tau)$.

Nonstationary Case (1/2)

- If a disturbance signal has "persistent" characteristics (exhibiting shifts in the mean), it is not appropriate to model it with a stationary stochastic process. For example, controller design based on stationary stochastic processes will generally lead to offset.
- In this case one can superimpose the output of a linear system driven by *integrated* white noise $\varepsilon_{\mathrm{int}}(k)$ to the stationary signal:

$$\varepsilon_{\mathrm{int}}(k+1) = \varepsilon_{\mathrm{int}}(k) + \varepsilon(k) \tag{6}$$

- The state space description is then:

$$\begin{aligned} x_w(k+1) &= A_w x_w(k) + B_w \varepsilon_{\mathrm{int}}(k) \\ w(k) &= C_w x_w(k) + \varepsilon_{\mathrm{int}}(k) \end{aligned} \tag{7}$$

Nonstationary Case (2/2)

- The state space description (7) can be rewritten, using differenced variables, as:

$$\begin{aligned} \Delta x_w(k+1) &= A_w \Delta x_w(k) + B_w \varepsilon(k-1) \\ \Delta w(k) &= C_w \Delta x_w(k) + \varepsilon(k-1) \end{aligned} \tag{8}$$

  where $\Delta w(k) := w(k) - w(k-1)$ and $\varepsilon(k)$ is a zero-mean stationary process.

- Since $\varepsilon(k)$ is a white noise signal, (8) is equivalent to:

$$\begin{aligned} \Delta x_w(k+1) &= A_w \Delta x_w(k) + B_w \varepsilon(k) \\ \Delta w(k) &= C_w \Delta x_w(k) + \varepsilon(k) \end{aligned} \tag{9}$$

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Obtaining Models from First Principles

From first principles, after linearization, one obtains an ODE of the form:

$$\begin{aligned} \dot{x}_p &= A_p^c x_p + B_p^c u + F_p^c w \\ y &= C_p x + G_p w \end{aligned} \qquad (10)$$

which can be discretized, leading to:

$$\begin{aligned} x_p(k+1) &= A_p x(k) + B_p u(k) + F_p w(k) \\ y(k) &= C_p x_p(k) + G_p w(k) \end{aligned} \qquad (11)$$

Remarks:

- Subscript p is used here to distinguish the process model matrices from the disturbance model matrices introduced before.
- If the physical disturbance variables cannot be identified, one can express the overall effect of the disturbances as a signal directly added to the output $\rightarrow$ **output disturbance**, i.e. $G_p = I$ and $F_p = 0$.

Stationary Case

We can combine the model (2) with (11) to get:

$$\begin{aligned} \begin{bmatrix} x_p(k+1) \\ x_w(k+1) \end{bmatrix} &= \begin{bmatrix} A_p & F_p C_w \\ 0 & A_w \end{bmatrix} \begin{bmatrix} x_p(k) \\ x_w(k) \end{bmatrix} + \begin{bmatrix} B_p \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} F_p \\ B_w \end{bmatrix} \varepsilon(k) \\ y(k) &= \begin{bmatrix} C_p & G_p C_w \end{bmatrix} \begin{bmatrix} x_p(k) \\ x_w(k) \end{bmatrix} + G_p \varepsilon(k) \end{aligned} \tag{12}$$

With some appropriate re-definition of system matrices, the above is in the standard state-space form of:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + \underbrace{F\varepsilon(k)}_{\varepsilon_1(k)} \\ y(k) &= Cx(k) + \underbrace{G\varepsilon(k)}_{\varepsilon_2(k)} \end{aligned} \tag{13}$$

Notice that the state is now expanded to include both the original system state x_p and the disturbance state x_w.

Nonstationary Case (1/2)

We can combine (9) with a differenced version of (11) to obtain:

$$\begin{bmatrix} \Delta x_p(k+1) \\ \Delta x_w(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A_p & F_p C_w \\ 0 & A_w \end{bmatrix}}_{A} \begin{bmatrix} \Delta x_p(k) \\ \Delta x_w(k) \end{bmatrix} + \underbrace{\begin{bmatrix} B_p \\ 0 \end{bmatrix}}_{B} \Delta u(k) + \underbrace{\begin{bmatrix} F_p \\ B_w \end{bmatrix}}_{F} \varepsilon(k) \quad (14)$$

$$\Delta y(k) = \underbrace{\begin{bmatrix} C_p & G_p C_w \end{bmatrix}}_{C} \begin{bmatrix} \Delta x_p(k) \\ \Delta x_w(k) \end{bmatrix} + \underbrace{G_p}_{G} \varepsilon(k) \quad (15)$$

where $\Delta x_p(k) := x_p(k) - x_p(k-1)$, and $\Delta x_w(k)$, $\Delta u(k)$ are defined similarly.

For estimation and control, it is further desired that the model output be y rather than Δy. This requires yet another augmentation of the state...

Nonstationary Case (2/2)

The augmented system is:

$$\begin{aligned} \begin{bmatrix} \Delta x(k+1) \\ y(k+1) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} \begin{bmatrix} \Delta x(k) \\ y(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \Delta u(k) + \begin{bmatrix} F \\ G \end{bmatrix} \varepsilon(k) \quad (16) \\ y(k) &= \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \Delta x(k) \\ y(k) \end{bmatrix} \quad (17) \end{aligned}$$

It can be brought into the standard state-space form after re-definition of the system matrices:

$$\begin{aligned} \bar{x}(k+1) &= \bar{A}\bar{x}(k) + \bar{B}\Delta u(k) + \bar{F}\varepsilon(k) \\ y(k) &= \bar{C}\bar{x}(k) \end{aligned} \quad (18)$$

except that now the system input is Δu rather than u. System (18) has n_y integrators to express the effects of the white noise disturbances and the system input Δu on the output.

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Stationary Case

Input output models obtained from an identification have the typical structure of:

$$y(z) = H_1(z)u(z) + H_2(z)\varepsilon(z) \tag{19}$$

where $H_1(z)$ and $H_2(z)$ are stable transfer matrices.

This can be brought into the form of (13) by finding a state-space realization of $\begin{bmatrix} H_1(z) & H_2(z) \end{bmatrix}$:

$$\begin{aligned} x(k+1) &= A(k) + Bu(k) + F\varepsilon(k) \\ y(k) &= Cx(k) + Du(k) + G\varepsilon(k) \end{aligned} \tag{20}$$

Remarks:

- $H_1(z)$ has relative degree of at least one
- We may assume – without loss of generality – that $H_2(0) = I$, $D = 0$ and $G = I$.

Nonstationary Case

In this case, the driving noise should be integrated white noise.

$$y(z) = H_1(z)u(z) + H_2(z)\underbrace{\frac{1}{1-z^{-1}}\varepsilon(z)}_{\varepsilon_{\mathrm{int}}(z)} \tag{21}$$

Using the fact $(1 - z^{-1})y(z) = (\Delta y)(z)$, we can rewrite the above as:

$$(\Delta y)(z) = H_1(z)\Delta u(z) + H_2(z)\varepsilon(z) \tag{22}$$

Denoting the realization of $\begin{bmatrix} H_1(z) & H_2(z) \end{bmatrix}$ as

$$\begin{array}{rcl} x(k+1) & = & Ax(k) + B\Delta u(k) + F\varepsilon(k) \\ \Delta y(k) & = & Cx(k) + \varepsilon(k) \end{array} \tag{23}$$

The state can be augmented with y as before to bring it into the form of (18).

Outline

1. Uncertainty Modeling

2. State Estimation

State Estimation

- **Motivation:** In many control applications, information about the previous, current or future states is required but it is usually not possible to measure all state variables. Hence the previous/current/future states need to be **estimated** from past input and measurement sequences

General Estimation Problem (1/2)

- Consider the following nonlinear time-varying discrete-time system subject to disturbances $w_1(\cdot)$ and $w_2(\cdot)$

$$\begin{aligned} x(k+1) &= g(x(k), u(k), w_1(k), k) \\ y(k) &= h(x(k), u(k), w_2(k), k) \end{aligned}$$

- The goal of **state estimation** at timestep k is to estimate $x(k+i)$ given $\{y(j), u(j)\}_{j=0,1,\ldots,k}$
- Estimating $x(k+i)$ with $i > 0$ is called **prediction** while the case with $i = 0$ is referred to as **filtering**
- Some applications require the estimation of $x(k+i)$ with $i < 0$ which is called **smoothing**

General Estimation Problem (2/2)

- State estimation is an integral part of MPC, which is why good knowledge of it is crucial for designing an effective model predictive control
- Disturbance modeling is very important in estimation, simply adding white noise to the equations of a deterministic model can give very poor estimation results
- We will denote by $\hat{x}_{i|j}$ an estimate of $x(i)$ using all information available up to and at timestep j

State Estimation of Linear Systems

- From here on we focus on the estimation for linear systems
- Consider the following model with white process and measurement noise sequences $\{\varepsilon_1(i)\}_{i=0,1,\ldots}$ and $\{\varepsilon_2(i)\}_{i=0,1,\ldots}$

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + \varepsilon_1(k) \\ y(k) &= Cx(k) + \varepsilon_2(k) \end{aligned} \tag{24}$$

- The (white) noise sequences are assumed to have zero mean, i.e. $\mathrm{E}\{\varepsilon_1(k)\} = 0$ and $\mathrm{E}\{\varepsilon_2(k)\} = 0$ $\forall k \geq 0$ and to have covariance

$$\mathrm{E}\left\{ \begin{bmatrix} \varepsilon_1(i) \\ \varepsilon_2(i) \end{bmatrix} \begin{bmatrix} \varepsilon_1(j) \\ \varepsilon_2(j) \end{bmatrix}^\top \right\} = \begin{cases} \begin{bmatrix} R_1 & R_{1,2} \\ R_{1,2}^\top & R_2 \end{bmatrix} & \text{if i=j} \\ 0 & \text{otherwise} \end{cases}$$

- Additionally, it is assumed that the noise sequences are independent of the initial state estimate $\hat{x}_{0|0}$

Outline

2. State Estimation

    Linear State Estimation

    State Observer

    Kalman Filter

Linear Estimator Structure

- We assume here that we want to design a linear estimator with the following structure
  - The **prediction step** propagates the last estimate $\hat{x}_{k-1|k-1}$ using the nominal (assume $\varepsilon_1 = 0$) model to generate the **a priori** estimate $\hat{x}_{k|k-1}$

$$\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1} + Bu(k-1)$$

  - The **update step** corrects the a priori estimate based on the prediction error $y(k) - C\hat{x}_{k|k-1}$ multiplied with the filter gain K_k

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y(k) - C\hat{x}_{k|k-1})$$

Estimation Error (1/3)

- Define the state estimation error $x^e_{i|j}$ as the difference between the true value $x(i)$ and the estimate $\hat{x}_{i|j}$, i.e.

$$x^e_{i|j} := x(i) - \hat{x}_{i|j}$$

- We focus on estimating $x(k)$ using information up to time k: $\{y(i)\}_{i=0,1,\dots,k}$, $\{u(j)\}_{j=0,1,\dots,k}$, i.e. on calculating $\hat{x}_{k|k}$
- We want to choose the filter gain K_k in order to minimize $x^e_{k|k}$ in some meaningful way

Estimation Error (2/3)

- Recall the model and the postulated filter equations

$$x(k) = Ax(k-1) + Bu(k-1) + \varepsilon_1(k-1) \tag{25}$$
$$y(k) = Cx(k) + \varepsilon_2(k) \tag{26}$$

$$\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1} + Bu(k-1) \tag{27}$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y(k) - C\hat{x}_{k|k-1}) \tag{28}$$

- Substituting $y(k)$ in (28) using (26)

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \left(C[x(k) - \hat{x}_{k|k-1}] + \varepsilon_2(k) \right)$$

- In a next step we obtain

$$\begin{aligned} x^e_{k|k} &= x(k) - \hat{x}_{k|k} \\ &= x(k) - \hat{x}_{k|k-1} - K_k \left(C[x(k) - \hat{x}_{k|k-1}] + \varepsilon_2(k) \right) \\ &= (I - K_k C)x^e_{k|k-1} - K_k \varepsilon_2(k) \end{aligned} \tag{29}$$

Estimation Error (3/3)

- Subtracting (27) from (25) we get

$$x^e_{k|k-1} = Ax^e_{k-1|k-1} + \varepsilon_1(k-1) \tag{30}$$

- Substituting finally $x^e_{k|k-1}$ from (30) in (29) yields the dynamics of the estimation error $x^e_{k|k}$ which we will analyze next

$$x^e_{k|k} = (A - K_k CA)x^e_{k-1|k-1} + (I - K_k C)\varepsilon_1(k-1) - K_k\varepsilon_2(k) \tag{31}$$

Outline

State Observer (1/2)

- The theory of state observers is based on the behavior of the estimation error in a deterministic setting ($\varepsilon_1 \equiv 0$, $\varepsilon_2 \equiv 0$) with a fixed $K_k = K$
- The error dynamics in the deterministic setting with fixed K are obtained by setting ε_1 and ε_2 to zero and fixing K_k in (31)

$$x^e_{k|k} = (A - KCA)x^e_{k-1|k-1} \tag{32}$$

- An estimator for system (24) is said to be **observer-stable** if $x^e_{k|k} \to 0$ as $k \to \infty$ for any $x^e_{0|0}$ when $\varepsilon_1 \equiv 0$ and $\varepsilon_2 \equiv 0$
- From (32) it is clear that for observer stability the eigenvalues of $A - KCA$ must lie strictly inside the unit circle

State Observer (2/2)

- Iff (CA, A) is observable, the eigenvalues of $A - KCA$ can be placed arbitrarily by choosing an appropriate K
- If A has full rank then observability of (CA, A) is equivalent to observability of (C, A)
- A slightly different ($\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + K(y(k) - C\hat{x}_{k|k-1}) + Bu(k)$) observer structure yields an error dynamics matrix $A - KC$ where observability of (C, A) is necessary and sufficient to arbitrarily place the eigenvalues through K
- Determining K such that the eigenvalues of $A - KCA$ are at desired locations is called observer pole placement
- Observer pole placement is not frequently used since
  1. In general, pole placement does not determine the observer gain K uniquely and it is unclear how to best use the remaining degrees of freedom
  2. Although the eigenvalues can be freely chosen, the eigenvectors cannot and the observer may recover only very slowly from certain types of errors

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    State Observer

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Kalman Filter Derivation (1/9)

- It is intuitively clear that incorporating statistical information on how the disturbances affect the state can improve the quality of the estimation
- The **Kalman filter** accounts for this information in an optimal (to be defined) way
- Denote the covariance of the estimation error $x^e_{i|j}$ by

$$P_{i|j} := \text{Cov}\{x^e_{i|j}\}$$

- The optimality criterion of the Kalman filter is the error covariance $P_{k|k}$, i.e. we choose K_k such that $\alpha^\top P_{k|k} \alpha$ is minimized for any α of appropriate dimension
- For simplicity it is assumed that, i.e. $\text{E}\{\varepsilon_1(i)\varepsilon_2(i)^\top\} = R_{1,2} = 0$ and $R_2 \succ 0$.

Kalman Filter Derivation (2/9)

Unbiasedness of $x^e_{k|k}$

- Recall the error update equations

$$x^e_{k|k} = (A - K_k CA)x^e_{k-1|k-1} + (I - K_k C)\varepsilon_1(k-1) - K_k\varepsilon_2(k)$$

- $x^e_{k|k}$ can be rewritten in explicit form

$$\begin{aligned} x^e_{k|k} = & \left[\prod_{i=1}^{k}(A - K_i CA)\right] x^e_{0|0} \\ & + \sum_{i=1}^{k}\left(\left[\prod_{j=i+1}^{k}(A - K_j CA)\right](I - K_i C)\varepsilon_1(i-1)\right) \\ & - \sum_{i=1}^{k}\left(\left[\prod_{j=i+1}^{k}(A - K_j CA)\right]K_i\varepsilon_2(i)\right) \end{aligned} \tag{33}$$

Kalman Filter Derivation (3/9)

Unbiasedness of $x^e_{k|k}$

- Since by assumption $E\{\varepsilon_1(i)\} = 0$ and $E\{\varepsilon_2(i)\} = 0$, we have from the previous equation

$$E\{x^e_{k|k}\} = \left[\prod_{i=1}^{k}(A - K_i CA)\right] E\{x^e_{0|0}\} \tag{34}$$

- If $K_k \to K_\infty$ as $k \to \infty$ and if the eigenvalues of $(A - K_\infty CA)$ are stable, the filter is asymptotically unbiased, i.e.

$$\lim_{k\to\infty} E\{x^e_{k|k}\} = 0 \tag{35}$$

Kalman Filter Derivation (4/9)

In the following we will use the fact that

- $x^e_{k-1|k-1}$ and $\varepsilon_1(k-1)$ are uncorrelated

  This is obvious since $\varepsilon_1(k-1)$ only influences $x(k)$, which itself is not part of $x^e_{k-1|k-1}$

- $x^e_{k|k-1}$ and $\varepsilon_2(k)$ are uncorrelated

  This is obvious since $\varepsilon_2(k)$ influences $y(k)$, which itself is not part of $x^e_{k|k-1}$

- $P_{i|j} = P_{i|j}^\top$ and $P_{i|j} \succeq 0$

  Any covariance matrix is symmetric positive-semidefinite by definition

- $S_k := CP_{k|k-1}C^\top + R_2 \succ 0$

  This follows from $P_{k|k-1} \succeq 0$ and $R_2 \succ 0$

Kalman Filter Derivation (5/9)

- In the next slides, we will compute the expressions for $P_{k|k-1}$ and $P_{k|k}$
- Recall the estimation error equations

$$\begin{aligned} x^e_{k|k-1} &= Ax^e_{k-1|k-1} + \varepsilon_1(k-1) \\ x^e_{k|k} &= (I - K_k C)x^e_{k|k-1} - K_k \varepsilon_2(k) \end{aligned}$$

- Using that $x^e_{k-1|k-1}$ and $\varepsilon_1(k-1)$ are uncorrelated we can compute $P_{k|k-1}$ as

$$\begin{aligned} P_{k|k-1} =& \text{Cov}\{Ax^e_{k-1|k-1} + \varepsilon_1(k-1)\} \\ =& \text{Cov}\{Ax^e_{k-1|k-1}\} + \text{Cov}\{\varepsilon_1(k-1)\} \\ =& A\text{Cov}\{x^e_{k-1|k-1}\}A^\top + R_1 \\ =& AP_{k-1|k-1}A^\top + R_1 \end{aligned} \tag{36}$$

Kalman Filter Derivation (6/9)

- Similarly for $P_{k|k}$ using that $x^e_{k|k-1}$ and $\varepsilon_2(k)$ are uncorrelated we get

$$\begin{aligned}
P_{k|k} =& \mathrm{Cov}\{(I - K_k C)x^e_{k|k-1} - K_k \varepsilon_2(k)\} \\
=& \mathrm{Cov}\{(I - K_k C)x^e_{k|k-1}\} + \mathrm{Cov}\{-K_k \varepsilon_2(k)\} \\
=& (I - K_k C)\mathrm{Cov}\{x^e_{k|k-1}\}(I - K_k C)^\top \\
& + K_k \mathrm{Cov}\{\varepsilon_2(k)\} K_k^\top \\
=& (I - K_k C)P_{k|k-1}(I - K_k C)^\top + K_k R_2 K_k^\top
\end{aligned}$$

Kalman Filter Derivation (7/9)

- Rewrite $P_{k|k}$

$$
\begin{aligned}
P_{k|k} =& (I - K_k C) P_{k|k-1} (I - K_k C)^\top + K_k R_2 K_k^\top \\
=& K_k \underbrace{\left(C P_{k|k-1} C^\top + R_2\right)}_{:= S_k} K_k^\top \\
& - K_k C P_{k|k-1} - (K_k C P_{k|k-1})^\top + P_{k|k-1} \\
=& (K_k - P_{k|k-1} C^\top S_k^{-1}) S_k (K_k - P_{k|k-1} C^\top S_k^{-1})^\top \\
& + P_{k|k-1} - P_{k|k-1} C^\top S_k^{-1} C P_{k|k-1}
\end{aligned} \tag{37}
$$

- Since $S_k \succ 0$ it always can be inverted and $\alpha^\top P_{k|k} \alpha$ is minimized for any α if we choose $K_k = P_{k|k-1} C^\top S_k^{-1}$. (37) can be rewritten as

$$P_{k|k} = (I - K_k C) P_{k|k-1}$$

Kalman Filter Derivation (8/9)

Kalman Filter Algorithm

- Initialize $\hat{x}_{0|0}$ and $P_{0|0}$
- The filter gain K_k and the error covariance matrix $P_{k|k}$ can be computed online or in advance

$$\begin{aligned} P_{k|k-1} &= AP_{k-1|k-1}A^\top + R_1 \\ K_k &= P_{k|k-1}C^\top(CP_{k|k-1}C^\top + R_2)^{-1} \\ P_{k|k} &= (I - K_kC)P_{k|k-1} \end{aligned}$$

- At time k
  1. Compute the a priori estimate
  $$\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1} + Bu(k-1) \tag{38}$$
  2. Get measurement $y(k)$
  3. Compute new estimate
  $$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y(k) - C\hat{x}_{k|k-1}) \tag{39}$$
- $k \to k+1$

Kalman Filter Derivation (9/9)

Remarks

- The derivation of the Kalman filter with $R_{1,2} \neq 0$ follows the same lines but requires a bit more algebra
- Generalization of the Kalman filter equations for a system where A, B, C are time-varying is straightforward. It is possible to replace A by $A(k)$, B by $B(k)$ and C by $C(k)$ in all equations (except the steady-state equations in the next slides) without changing the arguments used for the derivation

Kalman Filter Analysis (1/3)

Stability of the Kalman filter

- If $P_{k|k-1}$ converges to a steady-state solution P_∞ and the corresponding steady-state Kalman gain K_∞ satisfies the condition that $A - K_\infty CA$ has all eigenvalues strictly inside the unit circle, then the expected estimation error goes asymptotically to zero. Together with the bounded covariance, it is said that the Kalman filter is stable
- In the following slides, we investigate under which conditions the Kalman gain converges to K_∞ and the asymptotic filter gain is stabilizing, i.e. the eigenvalues of $A - K_\infty CA$ lie inside the unit circle
- Note that if $P_{k|k-1} \to P_\infty$ as $k \to \infty$ then also $P_{k|k} \to P_\infty^\top$

Kalman Filter Analysis (2/3)

- Substituting the Kalman gain into (37) yields

$$P_{k|k} = P_{k|k-1} - P_{k|k-1} C^\top S_k^{-1} C P_{k|k-1} \tag{40}$$

- Substitute $P_{k-1|k-1}$ using the above in (36) we can rewrite $P_{k|k-1}$ as

$$\begin{aligned} P_{k|k-1} =& A P_{k-1|k-1} A^\top + R_1 \\ =& A P_{k-1|k-2} A^\top - A P_{k-1|k-2} C^\top S_{k-1}^{-1} C P_{k-1|k-2} A^\top + R_1 \end{aligned}$$

- From the last equation together with $S_{k-1} = C P_{k-1|k-2} C^\top + R_2$ it can be seen that if $P_{k|k-1}$ converges it must satisfy the **Algebraic Riccati Equation**

$$P_\infty = A P_\infty A^\top - A P_\infty C^\top (C P_\infty C^\top + R_2)^{-1} C P_\infty A^\top + R_1$$

Kalman Filter Analysis (3/3)

- P_∞ is called a **stabilizing** solution if all eigenvalues of $A - K_\infty CA$ are strictly inside the unit circle
- We have the following properties:
  - If (C, A) is detectable, then $P_{k|k-1}$ converges regardless of $P_{0|0}$
  - If (C, A) is detectable, $(A, R_1^{1/2})$ is stabilizable and $P_{0|0} \succeq 0$, then $P_{k|k-1}$ converges to P_∞, where P_∞ is a stabilizing solution to the ARE