University of Pennsylvania, ESE 6190

Model Predictive Control

Chapter 6: Constrained Finite Time Optimal Control

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F. Borrelli, A. Bemporad, and M. Morari, Predictive Control for Linear and Hybrid Systems, Cambridge University Press, 2017. [Ch. 11].

Outline

1. Constrained Optimal Control
2. Basic Ideas of Predictive Control
3. Constrained Linear Optimal Control
4. Constrained Optimal Control: 2-Norm
5. Constrained Optimal Control: 1-Norm and ∞-Norm
6. Receding Horizon Control Notation

Outline

1. Constrained Optimal Control

2. Basic Ideas of Predictive Control

3. Constrained Linear Optimal Control

4. Constrained Optimal Control: 2-Norm

5. Constrained Optimal Control: 1-Norm and ∞-Norm

6. Receding Horizon Control Notation

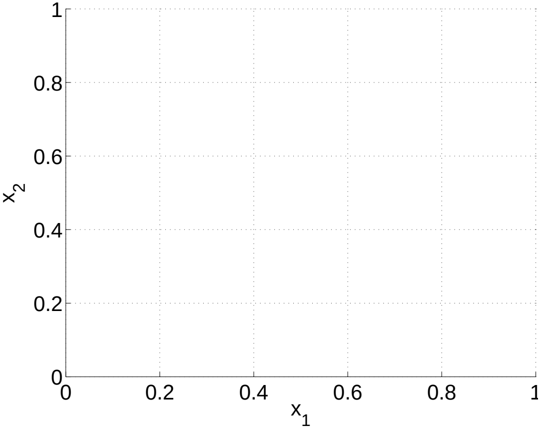
Objectives of Constrained Optimal Control

$$x^+ = f(x,u) \qquad\qquad (x,u) \in \mathcal{X}, \mathcal{U}$$

Design control law $u = \kappa(x)$ such that the system:

1. Satifies constraints : $\{x_i\} \subset \mathcal{X}$, $\{u_i\} \subset \mathcal{U}$
2. Is asymptotically stable: $\lim_{i\to\infty} x_i = 0$
3. Optimizes "performance"
4. Maximizes the set $\{x_0 \mid \text{Conditions 1-3 are met}\}$

Limitations of Linear Controllers



Does linear control work?

System:

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u$$

Constraints:

$$\mathcal{X} := \{x \mid \|x\|_\infty \leq 5\}$$
$$\mathcal{U} := \{u \mid \|u\|_\infty \leq 1\}$$

Consider an LQR controller, with $Q = I$, $R = 1$.

$\Rightarrow K = \begin{bmatrix} 0.52 & 0.94 \end{bmatrix}$

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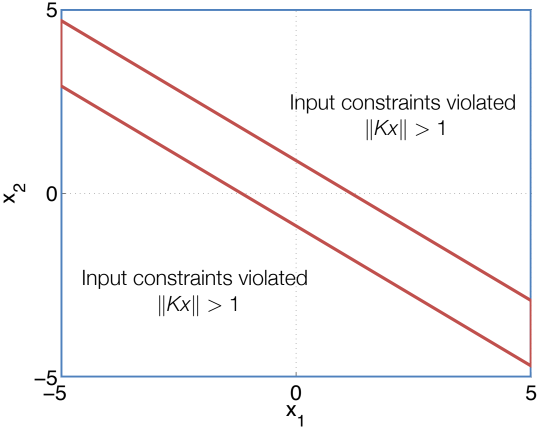

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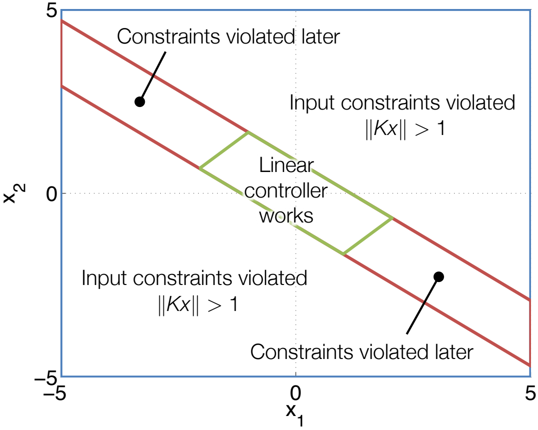

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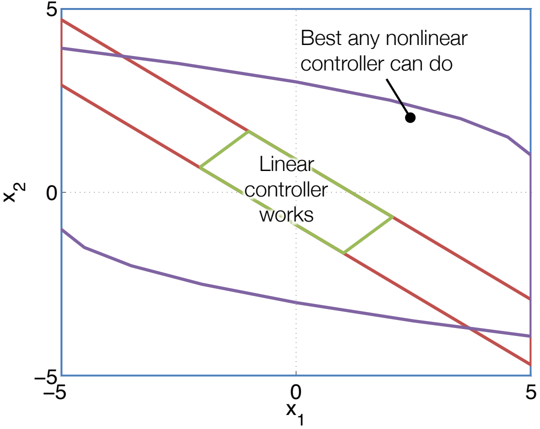
Does linear control work?

Yes, but the region where it works is very small

Limitations of Linear Controllers



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Consider an LQR controller, with $Q = I$, $R = 1$.

$\Rightarrow K = \begin{bmatrix} 0.52 & 0.94 \end{bmatrix}$

Does linear control work?

Yes, but the region where it works is very small

Use nonlinear control (MPC) to increase the region of attraction

Outline

Constrained Infinite Time Optimal Control (what we would like to solve)

$$
\begin{aligned}
J_0^*(x(0)) = \min \ & \sum_{k=0}^{\infty} q(x_k, u_k) \\
\text{s.t. } & x_{k+1} = Ax_k + Bu_k, k = 0, \dots, N-1 \\
& x_k \in \mathcal{X}, u_k \in \mathcal{U}, k = 0, \dots, N-1 \\
& x_0 = x(0)
\end{aligned}
$$

- **Stage cost** $q(x, u)$: “cost” of being in state x and applying input u
- Optimizing over a trajectory provides a **tradeoff between short- and long-term benefits** of actions
- We’ll see that such a control law has many beneficial properties...
  ... but we can’t compute it: there are an **infinite number of variables**

Constrained Finite Time Optimal Control (what we can sometimes solve)

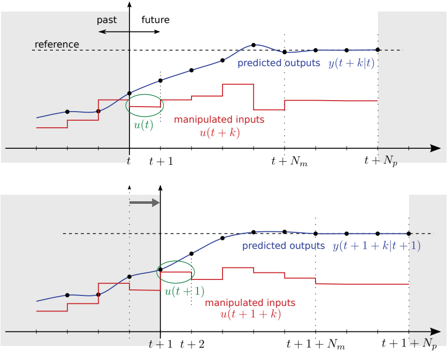
$$
\begin{aligned}
J_t^*(x(t)) = \quad \min_{U_t} \quad & p(x_{t+N}) + \sum_{k=0}^{N-1} q(x_{t+k}, u_{t+k}) \\
\text{subj. to} \quad & x_{t+k+1} = Ax_{t+k} + Bu_{t+k},\ k = 0, \ldots, N-1 \\
& x_{t+k} \in \mathcal{X},\ u_{t+k} \in \mathcal{U},\ k = 0, \ldots, N-1 \\
& x_{t+N} \in \mathcal{X}_f \\
& x_t = x(t)
\end{aligned} \tag{1}
$$

where $\mathcal{U}_t = \{u_t, \ldots, u_{t+N-1}\}$.

Truncate after a finite horizon:

- $p(x_{t+N})$: Approximates the 'tail' of the cost
- $\mathcal{X}_f$: Approximates the 'tail' of the constraints

On-line Receding Horizon Control



1. At each sampling time, solve a **CFTOC**.
2. Apply the optimal input **only during** $[t, t+1]$
3. At $t+1$ solve a CFTOC over a **shifted horizon** based on new state measurements
4. The resulting controller is referred to as **Receding Horizon Controller (RHC)** or **Model Predictive Controller (MPC)**.

On-line Receding Horizon Control

1) MEASURE the state $x(t)$ at time instance t
2) OBTAIN $U_t^*(x(t))$ by solving the optimization problem in (1)
3) IF $U_t^*(x(t)) = \emptyset$ THEN 'problem infeasible' STOP
4) APPLY the first element u_t^* of U_t^* to the system
5) WAIT for the new sampling time $t+1$, GOTO 1)

Note that we need a constrained optimization solver for step 2).

MPC Features

Pros

- Any model:
  - linear
  - nonlinear
  - single/multivariable
  - time delays
  - constraints
- Any objective:
  - sum of squared errors
  - sum of absolute errors (i.e., integral)
  - worst error over time
  - economic objective

Cons

- Computationally demanding in the general case
- May or may not be stable
- May or may not be feasible

Outline

Outline

3. Constrained Linear Optimal Control

    Problem formulation

    Feasible Sets

    Unconstrained Solution

Constrained Linear Optimal Control

Cost function

$$J_0(x(0), U_0) = p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k)$$

- $U_0 \triangleq [u'_0, \ldots, u'_{N-1}]'$
- Squared Euclidian norm: $p(x_N) = x'_N P x_N$ and $q(x_k, u_k) = x'_k Q x_k + u'_k R u_k$.
- $p = 1$ or $p = \infty$: $p(x_N) = \|P x_N\|_p$ and $q(x_k, u_k) = \|Q x_k\|_p + \|R u_k\|_p$.

Constrained Finite Time Optimal Control problem (CFTOC)

$$\begin{array}{rll} J_0^*(x(0)) = & \min_{U_0} & J_0(x(0), U_0) \\ & \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\ & & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(0) \end{array} \tag{2}$$

N is the time horizon and $\mathcal{X}$, $\mathcal{U}$, $\mathcal{X}_f$ are polyhedral regions.

Outline

3. Constrained Linear Optimal Control

   Problem formulation

   Feasible Sets

   Unconstrained Solution

Feasible Sets

Set of initial states $x(0)$ for which the optimal control problem (2) is feasible:

$$\begin{aligned}\mathcal{X}_0 = \; & \{x_0 \in \mathbb{R}^n |\; \exists (u_0, \dots, u_{N-1}) \text{ such that } x_k \in \mathcal{X},\; u_k \in \mathcal{U}, \\ & k = 0, \dots, N-1,\; x_N \in \mathcal{X}_f,\; \text{where } x_{k+1} = Ax_k + Bu_k\}\end{aligned}$$

In general $\mathcal{X}_i$ is the set of states x_i at time i for which (2) is feasible:

$$\begin{aligned}\mathcal{X}_i = \; & \{x_i \in \mathbb{R}^n |\; \exists (u_i, \dots, u_{N-1}) \text{ such that } x_k \in \mathcal{X},\; u_k \in \mathcal{U}, \\ & k = i, \dots, N-1,\; x_N \in \mathcal{X}_f,\; \text{where } x_{k+1} = Ax_k + Bu_k\},\end{aligned}$$

The sets $\mathcal{X}_i$ for $i = 0, \dots, N$ play an important role in the solution of the CFTOC problem. They are independent of the cost.

Outline

3. Constrained Linear Optimal Control

    Problem formulation

    Feasible Sets

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Unconstrained Solution

For quadratic cost (squared Euclidian norm) and **no state and input constraints**:

$$\{x \in \mathcal{X},\ u \in \mathcal{U}\} = \mathbb{R}^{n+m},\ \mathcal{X}_f = \mathbb{R}^n$$

we have the **time-varying** linear control law

$$u^*(k) = F_k x(k)\ k = 0, \ldots, N-1.$$

If $N \to \infty$, we have the **time-invariant** linear control law

$$u^*(k) = F_\infty x(k)\ k = 0, 1, \ldots$$

Next we show how to compute finite time **constrained** optimal controllers.

Outline

Outline

4. Constrained Optimal Control: 2-Norm

   Problem Formulation

   Construction of the QP with substitution

   Construction of the QP without substitution

   2-Norm State Feedback Solution

Problem Formulation

Quadratic cost function

$$J_0(x(0), U_0) = x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k \tag{3}$$

with $P \succeq 0$, $Q \succeq 0$, $R \succ 0$.
Constrained Finite Time Optimal Control problem (CFTOC).

$$\boxed{\begin{array}{lll} J_0^*(x(0)) = & \min\limits_{U_0} & J_0(x(0), U_0) \\ & \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\ & & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(0) \end{array} \qquad (4)}$$

N is the time horizon and $\mathcal{X}$, $\mathcal{U}$, $\mathcal{X}_f$ are polyhedral regions.

Outline

4. Constrained Optimal Control: 2-Norm

    Problem Formulation

    Construction of the QP with substitution

    Construction of the QP without substitution

    2-Norm State Feedback Solution

Construction of the QP with substitution

- **Step 1**: Rewrite the cost as

$$\begin{aligned} J_0(x(0), U_0) &= U_0' H U_0 + 2x(0)' F U_0 + x(0)' Y x(0) \\ &= [U_0' \; x(0)'] \left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] [U_0{}' \; x(0)']' \end{aligned}$$

  Note: $\left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] \succeq 0$ since $J_0(x(0), U_0) \geq 0$ by assumption.

- **Step 2**: Rewrite the constraints compactly as (details provided on the next slide)

$$G_0 U_0 \leq w_0 + E_0 x(0)$$

- **Step 3**: Rewrite the optimal control problem as

$$\boxed{\begin{aligned} J_0^*(x(0)) = \min_{U_0} \quad & [U_0' \; x(0)'] \left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] [U_0{}' \; x(0)']' \\ \text{subj. to} \quad & G_0 U_0 \leq w_0 + E_0 x(0) \end{aligned}}$$

Solution

$$
\begin{aligned}
J_0^*(x(0)) = \min_{U_0} \quad & [U_0' \; x(0)'] \left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] [U_0{}' \; x(0)']' \\
\text{subj. to} \quad & G_0 U_0 \leq w_0 + E_0 x(0)
\end{aligned}
$$

For a given x(0) U_0^* can be found via a QP solver.

Construction of QP constraints with substitution

If $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{X}_f$ are given by:

$$\mathcal{X} = \{x \,|\, A_x x \leq b_x\} \qquad \mathcal{U} = \{u \,|\, A_u u \leq b_u\} \qquad \mathcal{X}_f = \{x \,|\, A_f x \leq b_f\}$$

Then $G_0 U_0 \leq w_0 + E_0 x(0)$, where G_0, E_0 and w_0 are defined as follows

$$G_0 = \begin{bmatrix} A_u & 0 & \dots & 0 \\ 0 & A_u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_u \\ 0 & 0 & \dots & 0 \\ A_x B & 0 & \dots & 0 \\ A_x A B & A_x B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_f A^{N-1} B & A_f A^{N-2} B & \dots & A_f B \end{bmatrix}, E_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -A_x \\ -A_x A \\ -A_x A^2 \\ \vdots \\ -A_f A^N \end{bmatrix}, w_0 = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_f \end{bmatrix}$$

Outline

Construction of the QP without substitution

To obtain the QP problem

$$\begin{array}{rl} J_0^*(x(0)) = \min\limits_{U_0} & [U_0' \; x(0)'] \left[\begin{smallmatrix} H & F' \\ F & Q \end{smallmatrix}\right] [U_0{}' \; x(0)']' \\ \text{subj. to} & G_0 U_0 \leq w_0 + E_0 x(0) \end{array}$$

we have substituted the state equations

$$x_{k+1} = Ax_k + Bu_k$$

into the state constraints $x_k \in \mathcal{X}$.

It is often more efficient to keep the explicit equality constraints.

Construction of the QP without substitution

We transform the CFTOC problem into the QP problem

$$
\begin{aligned}
J_0^*(x(0)) = \min_z \quad & [z' \ x(0)'] \left[\begin{smallmatrix} \bar{H} & 0 \\ 0 & Q \end{smallmatrix}\right] [z' \ x(0)']' \\
\text{subj. to } \ & G_{0,\mathrm{in}} z \le w_{0,\mathrm{in}} + E_{0,\mathrm{in}} x(0) \\
& G_{0,\mathrm{eq}} z = E_{0,\mathrm{eq}} x(0)
\end{aligned}
$$

- Define variable:

$$
z = \begin{bmatrix} x_1' & \ldots & x_N' & u_0' & \ldots & u_{N-1}' \end{bmatrix}'
$$

- Equalities from system dynamics $x_{k+1} = Ax_k + Bu_k$:

$$
G_{0,\mathrm{eq}} = \left[\begin{array}{ccccc:ccccc}
I & & & & & -B & & & & \\
-A & I & & & & & -B & & & \\
 & -A & I & & & & & -B & & \\
 & & \ddots & \ddots & & & & & \ddots & \\
 & & & -A & I & & & & & -B
\end{array}\right], \quad E_{0,\mathrm{eq}} = \begin{bmatrix} A \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

Construction of the QP without substitution

If $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{X}_f$ are given by:

$$\mathcal{X} = \{x \,|\, A_x x \leq b_x\} \qquad \mathcal{U} = \{u \,|\, A_u u \leq b_u\} \qquad \mathcal{X}_f = \{x_N \,|\, A_f x_N \leq b_f\}$$

Then matrices $G_{0,\text{in}}$, $w_{0,\text{in}}$ and $E_{0,\text{in}}$ are:

$$G_{0,\text{in}} = \left[\begin{array}{ccccc|ccccc} 0 & & & & & 0 & & & & \\ & A_x & & & & & 0 & & & \\ & & \ddots & & & & & \ddots & & \\ & & & A_x & & & & & 0 & \\ & & & & A_f & & & & & 0 \\ \hline 0 & & & & & A_u & & & & \\ & 0 & & & & & A_u & & & \\ & & \ddots & & & & & \ddots & & \\ & & & 0 & & & & & A_u & \\ & & & & 0 & & & & & A_u \end{array}\right] \qquad w_{0,\text{in}} = \left[\begin{array}{c} b_x \\ b_x \\ \vdots \\ b_x \\ b_f \\ \hline b_u \\ b_u \\ \vdots \\ b_u \\ b_u \end{array}\right]$$

$$E_{0,\text{in}} = \begin{bmatrix} -A_x' & 0 & \cdots & 0 \end{bmatrix}'$$

Construction of the QP without substitution

Build cost function from MPC cost $x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k$

$$\bar{H} = \left[\begin{array}{cccc|ccc} Q & & & & & & \\ & \ddots & & & & & \\ & & Q & & & & \\ & & & P & & & \\ \hline & & & & R & & \\ & & & & & \ddots & \\ & & & & & & R \end{array}\right]$$

Matlab hint:

```
barH = blkdiag(kron(eye(N-1),Q), P, kron(eye(N),R))
```

Outline

4. Constrained Optimal Control: 2-Norm

    Problem Formulation

    Construction of the QP with substitution

    Construction of the QP without substitution

    2-Norm State Feedback Solution

2-**Norm State Feedback Solution**

Start from QP with substitution.

- **Step 1**: Define $z \triangleq U_0 + H^{-1}F'x(0)$ and transform the problem into

$$\boxed{\begin{array}{rll} \hat{J}^*(x(0)) = & \min\limits_{z} & z'Hz \\ & \text{subj. to} & G_0 z \leq w_0 + S_0 x(0), \end{array}}$$

  where $S_0 \triangleq E_0 + G_0 H^{-1} F'$, and
  $\hat{J}^*(x(0)) = J_0^*(x(0)) - x(0)'(Y - FH^{-1}F')x(0)$.

  The CFTOC problem is now a **multiparametric quadratic program (mp-QP)**.

- **Step 2**: Solve the mp-QP to get explicit solution $z^*(x(0))$

- **Step 3**: Obtain $U_0^*(x(0))$ from $z^*(x(0))$

2-**Norm State Feedback Solution**

Main Results

1. The **open loop optimal control function** can be obtained by solving the mp-QP problem and calculating $U_0^*(x(0)),\ \forall x(0) \in \mathcal{X}_0$ as $U_0{}^* = z^*(x(0)) - H^{-1}F'x(0)$.
2. The first component of the multiparametric solution has the form

$$u^*(0) = f_0(x(0)), \quad \forall x(0) \in \mathcal{X}_0,$$

   $f_0 : \mathbb{R}^n \to \mathbb{R}^m$, is continuous and PieceWise Affine on Polyhedra

$$f_0(x) = F_0^i x + g_0^i \quad \text{if} \quad x \in CR_0^i, \quad i = 1, \ldots, N_0^r$$

3. The polyhedral sets $CR_0^i = \{x \in \mathbb{R}^n | H_0^i x \leq K_0^i\}$, $i = 1, \ldots, N_0^r$ are a partition of the feasible polyhedron $\mathcal{X}_0$.
4. The value function $J_0^*(x(0))$ is convex and piecewise quadratic on polyhedra.

Example

Consider the double integrator

$$\begin{cases} x(t+1) & = & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) & = & \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

subject to constraints

$$-1 \leq u(k) \leq 1, \; k = 0, \ldots, 5$$

$$\begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x(k) \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \; k = 0, \ldots, 5$$

Compute the **state feedback** optimal controller $u^*(0)(x(0))$ solving the

CFTOC problem with $N = 6$, $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $R = 0.1$, P the solution of the ARE, $\mathcal{X}_f = \mathbb{R}^2$.

Example

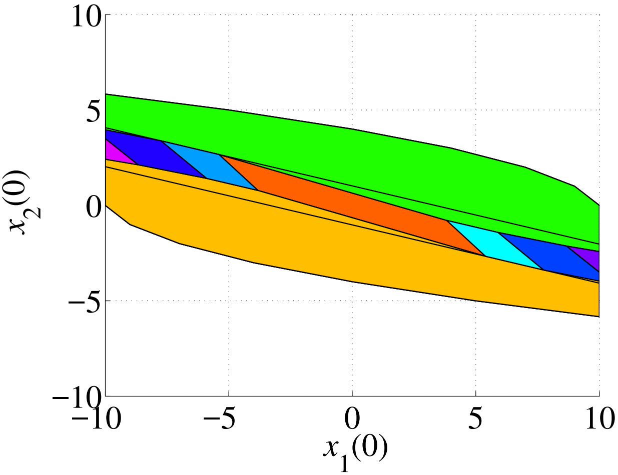

Figure: Partition of the state space for the affine control law $u^*(0)$ ($N_0^r = 13$)

Example

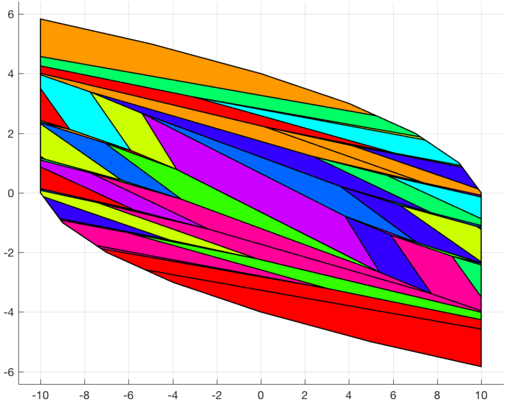

Figure: Partition of the state space for the affine control law $u^*(0)$ ($N_0^r = 61$)

Example

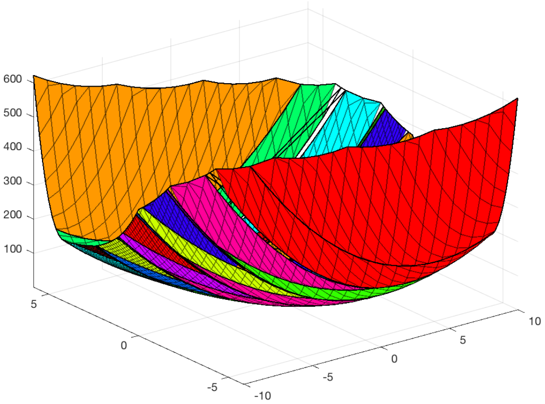

Figure: Value function for the affine control law $u^*(0)$ ($N_0^r = 61$)

Example

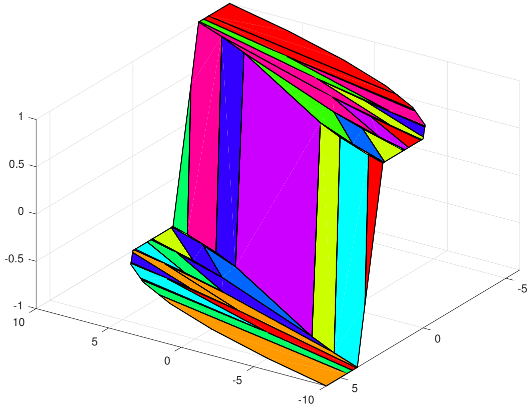

Figure: Optimal control input for the affine control law $u^*(0)$ ($N_0^r = 61$)

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Outline

5. Constrained Optimal Control: 1-Norm and ∞-Norm

   Problem Formulation

   Construction of the LP with substitution

   1- /∞-Norm State Feedback Solution

Problem Formulation

Piece-wise linear cost function

$$J_0(x(0), U_0) := \|Px_N\|_p + \sum_{k=0}^{N-1} \|Qx_k\|_p + \|Ru_k\|_p \tag{5}$$

with $p = 1$ or $p = \infty$, P, Q, R full column rank matrices

Constrained Finite Time Optimal Control Problem (CFTOC)

$$\boxed{\begin{array}{lll} J_0^*(x(0)) = & \min\limits_{U_0} & J_0(x(0), U_0) \\ & \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\ & & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(0) \end{array} \quad (6)}$$

N is the time horizon and $\mathcal{X}$, $\mathcal{U}$, $\mathcal{X}_f$ are polyhedral regions.

Outline

5. Constrained Optimal Control: 1-Norm and ∞-Norm

   Problem Formulation

   Construction of the LP with substitution

   1- /∞-Norm State Feedback Solution

Construction of the LP with substitution

Recall that the $\infty-$norm problem can be equivalently formulated as

$$
\begin{aligned}
\min_{z_0} \quad & \varepsilon_0^x + \ldots + \varepsilon_N^x + \varepsilon_0^u + \ldots + \varepsilon_{N-1}^u \\
\text{subj. to} \quad & -\mathbf{1}_n \varepsilon_k^x \leq \pm Q \left[A^k x_0 + \sum_{j=0}^{k-1} A^j B u_{k-1-j} \right], \\
& -\mathbf{1}_n \varepsilon_N^x \leq \pm P \left[A^N x_0 + \sum_{j=0}^{N-1} A^j B u_{N-1-j} \right], \\
& -\mathbf{1}_m \varepsilon_k^u \leq \pm R u_k, \\
& A^k x_0 + \sum_{j=0}^{k-1} A^j B u_{k-1-j} \in \mathcal{X}, \;\; u_k \in \mathcal{U}, \\
& A^N x_0 + \sum_{j=0}^{N-1} A^j B u_{N-1-j} \in \mathcal{X}_f, \\
& x_0 = x(0), \;\; k = 0, \ldots, N-1
\end{aligned}
$$

Construction of the LP with substitution

The problem results in the following standard LP

$$\begin{array}{ll} \min\limits_{z_0} & c_0' z_0 \\ \text{subj. to} & \bar{G}_0 z_0 \leq \bar{w}_0 + \bar{S}_0 x(0) \end{array}$$

where $z_0 := \{\varepsilon_0^x, \dots, \varepsilon_N^x, \varepsilon_0^u, \dots, \varepsilon_{N-1}^u, u_0', \dots, u_{N-1}'\} \in \mathbb{R}^s$,
$s \triangleq (m+1)N + N + 1$ and

$$\bar{G}_0 = \left[\begin{array}{cc} G_\varepsilon & 0 \\ 0 & G_0 \end{array} \right], \ \bar{S}_0 = \left[\begin{array}{c} S_\varepsilon \\ S_0 \end{array} \right], \ \bar{w}_0 = \left[\begin{array}{c} w_\varepsilon \\ w_0 \end{array} \right]$$

For a given $x(0)$ U_0^* can be obtained via an LP solver (the $1-$norm case is similar).

Outline

5. Constrained Optimal Control: 1-Norm and ∞-Norm

   Problem Formulation

   Construction of the LP with substitution

   1- /∞-Norm State Feedback Solution

1- /∞-Norm State Feedback Solution

Main Results

1. The **Open loop optimal control function** can be obtained by solving the mp-LP problem and calculating $z_0^*(x(0))$
2. The component $u_0^* = [0 \ \ldots 0 \ I_m \ 0 \ \ldots \ 0] z_0^*(x(0))$ of the multiparametric solution has the form

   $$u^*(0) = f_0(x(0)), \quad \forall x(0) \in \mathcal{X}_0,$$

   $f_0 : \mathbb{R}^n \to \mathbb{R}^m$, is continuous and PieceWise Affine on Polyhedra

   $$f_0(x) = F_0^i x + g_0^i \quad \text{if} \quad x \in CR_0^i, \quad i = 1, \ldots, N_0^r$$

3. The polyhedral sets $CR_0^i = \{x \in \mathbb{R}^n | H_0^i x \leq K_0^i\}$, $i = 1, \ldots, N_0^r$ are a partition of the feasible polyhedron $\mathcal{X}_0$.
4. In case of multiple optimizers a PieceWise Affine control law exists.
5. The value function $J_0^*(x(0))$ is convex and piecewise linear on polyhedra.

Outline

RHC Notation

$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{aligned}
$$

$$
x(t) \in \mathcal{X},\ \ u(t) \in \mathcal{U},\ \ \forall t \geq 0
$$

The CFTOC Problem

$$
\boxed{
\begin{array}{lll}
J_t^*(x(t)) = & \min\limits_{U_{t\to t+N|t}} & p(x_{t+N|t}) + \sum\limits_{k=0}^{N-1} q(x_{t+k|t}, u_{t+k|t}) \\
& & \\
& \text{subj. to} & x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t},\ \ k = 0, \ldots, N-1 \\
& & x_{t+k|t} \in \mathcal{X},\ \ u_{t+k|t} \in \mathcal{U},\ \ k = 0, \ldots, N-1 \\
& & x_{t+N|t} \in \mathcal{X}_f \\
& & x_{t|t} = x(t)
\end{array}
}
$$

with $U_{t\to t+N|t} = \{u_{t|t}, \ldots, u_{t+N-1|t}\}$.

RHC Notation

- $x(t)$ is the state of the system at time t.
- $x_{t+k|t}$ is the state of the model at time $t+k$, predicted at time t obtained by starting from the current state $x_{t|t} = x(t)$ and applying to the system model

$$x_{t+1|t} = Ax_{t|t} + Bu_{t|t}$$

  the input sequence $u_{t|t}, \ldots, u_{t+k-1|t}$.
- For instance, $x_{3|1}$ represents the predicted state at time 3 when the prediction is done at time $t = 1$ starting from the current state $x(1)$. It is different, in general, from $x_{3|2}$ which is the predicted state at time 3 when the prediction is done at time $t = 2$ starting from the current state $x(2)$.
- Similarly $u_{t+k|t}$ is read as "the input u at time $t+k$ computed at time t".

RHC Notation

- Let $U^*_{t\to t+N|t} = \{u^*_{t|t}, \ldots, u^*_{t+N-1|t}\}$ be the optimal solution. The first element of $U^*_{t\to t+N|t}$ is applied to system

$$u(t) = u^*_{t|t}(x(t)).$$

- The CFTOC problem is reformulated and solved at time $t+1$, based on the new state $x_{t+1|t+1} = x(t+1)$.

Receding horizon control law

$$f_t(x(t)) = u^*_{t|t}(x(t))$$

Closed loop system

$$x(t+1) = Ax(t) + Bf_t(x(t)) \triangleq f_{cl}(x(t)),\ \ t \geq 0$$

RHC Notation: Time-invariant Systems

As the system, the constraints and the cost function are time-invariant, the solution $f_t(x(t))$ becomes a time-invariant function of the initial state $x(t)$. Thus, we can simplify the notation as

$$
\begin{array}{llc}
J_0^*(x(t)) = & \min\limits_{U_0} & p(x_N) + \sum\limits_{k=0}^{N-1} q(x_k, u_k) \\
 & \text{subj. to} & \\
 & & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\
 & & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\
 & & x_N \in \mathcal{X}_f \\
 & & x_0 = x(t)
\end{array}
$$

where $U_0 = \{u_0, \ldots, u_{N-1}\}$.

The control law and closed loop system are **time-invariant** as well, and we write $f_0(x_0)$ for $f_t(x(t))$.