Model Predictive Control | ESE 6190, Spring 2023
Prof. Manfred Morari | Due: April 10

Exercise sheet 6
Explicit MPC

Instructions:

You could use MATLAB publisher to print out your solutions. This allows the comments and the plots to appear inline as they are in the MATLAB script.

Exercise 1 Using the Multi-parametric Toolbox

[0 pt] This exercise will give you an insight into parametric optimization, and explicit and hybrid model predictive control. You will familiarize yourself with two toolboxes for MATLAB, YALMIP and MPT. YALMIP is a toolbox, that simplifies the process of formulating and solving optimization problems of varying form, using various numerical solvers. This includes dealing with optimization problems that are parametric, which means they depend on parameters and we would like to know the solution as a function of those parameters. You will also familiarize yourself with MPT, a toolbox that helps you generate model predictive controllers, explicit MPC controllers and also allows you to visualize and run closed-loop simulations with little effort.

Details of the installation process can be found in `https://www.mpt3.org/Main/Installation`. Familiarize yourself with the toolbox by invoking one of the demos. Try for example:

```
» mpt_demo1
» mpt_demo2
» mpt_demo_lti1
» mpt_demo_lti4
```

Refer to MPT wiki: `https://www.mpt3.org` for more details.

Exercise 2 Explicit MPC

In this problem we will first derive the parametric solution of the given optimization problem by hand. Then we will verify the solution using the MPT3 toolbox.

Consider the following problem:

$$\begin{aligned} J^*(x_1, x_2) = \min_{z \in \mathbb{R}} \quad & \frac{1}{2}z^2 + 2x_1 z + x_2^2 \\ \text{subj. to} \quad & z \leq 1 + x_1 \\ & -z \leq 1 - x_2 \end{aligned} \tag{mpQP}$$

where $x_1, x_2 \in \mathbb{R}$ are the parameters.

1. **[4 pt]** Write out the Lagrangian and the KKT conditions for this problem.
2. **[4 pt]** Determine the region of the parameter space for which mpQP has a non-empty feasible set.
3. **[4 pt]** From the primal and dual feasibility conditions, write out the complementary cases that can occur.
4. **[4 pt]** Solve for $z^*(x_1, x_2)$ and $J(x_1, x_2)$ for each case.

5. **[4 pt]** Draw the critical regions on the parameter space.

   Now we will use YALMIP to solve mpQP and compare the solution. YALMIP was installed as part of Exercise 1, to get more info type `help yalmip` in the MATLAB command window and visit the website. To solve mpQP with YALMIP, follow the following steps.

6. **[0 pt]** First, declare the decision variable and parameter variables as symbolic variables:

```
z = sdpvar(1, 1);
x = sdpvar(2, 1);
```

7. **[4 pt]** Use the symbolic variables to define the objective and constraints of mpQP:

```
J = 0.5 * z(1)^2 + ...;
C = [ z(1) <= 1 + x(1), ...];
```

   Include $-5 <= \mathtt{x} <= 5$ in `C` to restrict the plots to this region.

8. **[0 pt]** Now convert the problem into the format of the MPT3 toolbox and call the routine to solve the multi-parametric program. Type `help Opt` and `help Opt.solve` to get more information on these two commands.

```
mpQP = Opt(C, J, x, z);
solution = mpQP.solve();
```

9. **[0 pt]** To visualize the solution and compare with the analytical results derived above, plot the critical regions, the $z^*(x_1, x_2)$, and $J(x_1, x_2)$. Type `help PolyUnion.fplot` and `help PolyUnion.plot` to get more information on these two plotting commands.

```
figure; solution.xopt.plot();
figure; solution.xopt.fplot('primal');
figure; solution.xopt.fplot('obj');
```

10. **[4 pt]** Identify the features of $z^*(x_1, x_2)$, and $J(x_1, x_2)$ - piece-wise linear/quadratic? Discountinuous, continous or continously differentiable?

Exercise 3 **Multiparametric Programming**

Familiarize yourself with the MPT `Opt` class for solving parametric linear and quadratic programs `https://www.mpt3.org/ParOpt/ParOpt`.

A: mpQP and mpLP

[7 pt] Consider a system with the state space representation $x_{k+1} = 0.5x_k + u_k$. For this dynamics, solve the QP (1) and the LP (2) parametrically by taking x_0 as a parameter. Plot the optimal $u_0^*(x_0)$ and the optimal cost $J^*(x_0)$.

$$\begin{aligned} \min_{x,u} \quad & \frac{1}{2}(x_1^2 + x_2^2 + u_0^2 + u_1^2) \\ \text{subject to} \quad & 2.5 \le x_1 \le 5 \\ & -1 \le x_2 \le 1 \\ & -2 \le u_0 \le 2 \\ & -2 \le u_1 \le 2 \end{aligned} \tag{1}$$

$$\begin{aligned} \min_{x,u} \quad & |x_1| + 0.5|x_2| + 0.5|u_0| + |u_1| \\ \text{subject to} \quad & 2.5 \le x_1 \le 5 \\ & -1 \le x_2 \le 1 \\ & -2 \le u_0 \le 2 \\ & -2 \le u_1 \le 2 \end{aligned} \tag{2}$$

B: Constrained Optimal Control, Multiparametric Programming and Dynamic Programming

Consider the discrete-time system model

$$\begin{cases} x_{k+1} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \\ y_k &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_k. \end{cases} \tag{3}$$

Define the following cost function

$$\left\| \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_4 \right\|_\infty + \sum_{k=0}^{3} \left(\left\| \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k \right\|_\infty + |0.8 u_k| \right) \tag{4}$$

and assume the constraints are

$$-1 \le u_k \le 1 \quad k = 0, 1, \cdots, 3, \tag{5a}$$

$$\begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x_k \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \quad k = 0, 1, \cdots, 4. \tag{5b}$$

1. **[4 pt]** Compute $u_0^\star(x_0)$ for $x_0 = [-5, 0]$ by solving a Linear Program.
2. **[8 pt]** Compute the state feedback solution $u_0^\star(x_0)$, $u_1^\star(x_1)$, $\cdots$, $u_3^\star(x_3)$ by using the batch approach and mpLPs. Check the solution obtained at the previous point for the same x_0.
3. **[3 pt]** Describe in words how MPT can be used to compute the state feedback solution $u_0^\star(x_0)$, $u_1^\star(x_1)$, $\cdots$, $u_3^\star(x_3)$ using dynamic programming and mpLPs.

   Hint: To extract data from the cost-to-go at each step, read the last entry of the MPT FAQ: `https://www.mpt3.org/Main/FAQ`