Model Predictive Control
Prof. Manfred Morari

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Exercise sheet 1
Dynamical Systems

Instructions: You are not allowed to use a calculator/computer unless specified. Multiple answers are possible in multiple choice questions, unless specified.

Exercise 1 **Analysis of LTI Discrete-Time Systems**

A: Consider the discrete-time dynamic system with the following state space representation:

$$
\begin{aligned}
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} &= \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \alpha & 0 \\ 0 & \frac{1}{2} & -\frac{5}{4} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 4 \\ 0 \end{bmatrix} u(k) \\
y(k) &= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}
\end{aligned} \tag{1}
$$

1) Let $\alpha = 0$.

   a) Is the system stable? [2 pts]
   b) Which states of the system belong to the controllable subsystem? [2 pts]
   c) Which states of the system belong to the observable subsystem? [2 pts]

2) The reachable subspace is defined as the set of states the system can reach starting from the origin. Use your knowledge of controllability to compute the reachable subspace as a function of the parameter α. [4 pts]

3) Now let $\alpha = \frac{1}{2}$. Is it possible to design a stabilizing controller for the system (1)? [4 pts]
   Hint: Use a coordinate transformation T such that $\tilde{A} = TAT^{-1}$ is diagonal.

B: Tick the correct answers:

1) Consider a SISO system with exactly one uncontrollable mode. [2 pts]

   ☐ The system can be stabilized using feedback if the uncontrollable mode is observable.
   ☐ The system can be stabilized using feedback if the uncontrollable mode is stable.
   ☐ The controlled closed-loop system is asymptotically stable if all its eigenvalues lie in the closed unit disc.
   ☐ A system that is not fully controllable can never be stabilized using feedback.

2) Consider a linear system of the form

$$x(k+1) = Ax(k) + Bu(k), \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and the system matrix A is diagonal:

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{bmatrix}. \tag{3}$$

The system is controllable if and only if [4 pts]

- ☐ all elements of B are non-zero.
- ☐ all eigenvalues of A are non-zero, and all elements of B are non-zero.
- ☐ all eigenvalues of A are distinct, and all elements of B are non-zero.

Exercise 2 **Discretization of a LTI continuous-time state-space model**

Note: You should use MATLAB or similar for this exercise.

Consider the following continuous-time dynamic system:

$$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{bmatrix} = \begin{bmatrix} -5 & 2.7 \\ -3.1 & 1.5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 4 & 2.1 \\ 1.1 & 3 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Discretize the system with $T_s = 1$ using the formulas in the slides and check the result with the Matlab function `c2d`. Compare the outputs of the continuous and the discretized model in a dynamic simulation, starting from the same initial state and applying the same input. [5 pts]

Hint: The following Matlab commands may be useful to solve the exercise: `expm` and `ode45`.

Exercise 3 **Sum of Lyapunov functions**

Let $V_i(x) := x^T P_i x$ be a Lyapunov function for the system $x^+ = Ax$ for $i = 1, 2$, with a rate of decrease of $x^T \Gamma x$, i.e.:

$$V_i(x^+) - V_i(x) \leq -x^T \Gamma x \ .$$

Show that $V(x) = \alpha V_1(x) + (1-\alpha) V_2(x)$ is also a Lyapunov function with a rate of decrease of $x^T \Gamma x$ for any $\alpha \in [0, 1]$. [5 pts]

Exercise 4 **Controllable, Observable, etc. and optimal control**

Consider the following discrete-time system with linear dynamics

$$x_{k+1} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{A(\alpha)} x_k + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u_k \,, \tag{4a}$$

where $\alpha \in \mathbb{R}$ is a parameter. The output of the system is given as

$$y_k = h(x_k) = \sum_{l=1}^{3} l \cdot \tanh(x_{k,l}) \,, \tag{4b}$$

where $x_{k,l}$ denotes the l-th state at time k.

a) In this part, the task is to analyze system (4).

i) For which values of α is the system controllable? [2 pts]

ii) Linearize the output mapping $h(x_k)$ around $\bar{x} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$ and $\bar{u} = 0$. Determine C, D in the linearized output mapping $y_k = Cx_k + Du_k$. [3 pts]
Hint: $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

For the remainder of this question you can use

$$C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad D = 0 . \tag{5}$$

iii) For which values of α is the linearized system $x_{k+1} = A(\alpha)x_k + Bu_k$, $y_k = Cx_k + Du_k$, with (C, D) as in (5), observable? For which values of α is it detectable? [5 pts]

b) Consider system (4a) with $\alpha = 1$.

i) For the unconstrained system (4a), consider the following control policy

$$u_k = -\underbrace{\begin{bmatrix} 0 & 1 & 2 \end{bmatrix}}_{K} x_k , \tag{6}$$

that has been designed such that the closed-loop (unconstrained) system is asymptotically stable. You are now given the function $V : \mathbb{R}^3 \to \mathbb{R}$, with

$$V(x) = x^\top \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix}}_{P} x .$$

Verify that $V(x)$ is a Lyapunov function for the closed-loop system (4a) with $\alpha = 1$ and u_k as in (6). [5 pts]

We have developed a steady-state Kalman filter with Kalman gain K_∞ for system (4) with $\alpha = 1$, based on the linearized output $y_k = Cx_k + Du_k$, with C, D as in (5). The observer error dynamics are given as

$$e_{k+1} := \hat{x}_{k+1} - x_{k+1} = (A - K_\infty CA)e_k .$$

ii) Instead of the actual state x_k, we will use the estimated state $\hat{x}_k$ to design our state-feedback controller, i.e. (6) is changed to

$$u_k = -K\hat{x}_k .$$

Determine the state transition matrix A_a for the augmented closed-loop system, as a function of A, B, C, D, K and K_∞:

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = A_a \begin{bmatrix} x_k \\ e_k \end{bmatrix} .$$

[2 pts]

iii) Does there exist a Lyapunov function $V_\infty(x)$ for the augmented closed-loop system in ii)? Justify your answer. [3 pts]

Exercise 5 **Stability, controllability, observability and observer design**

a) Consider the following discrete time LTI system:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k. \end{aligned} \tag{7}$$

i) If $A = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & a \end{bmatrix}$, find the interval in which the parameter a must lie for the system to be asymptotically stable. [1 pt]

ii) Assume that A is as in part a.i) with $a = \frac{2}{3}$ and that one can choose between three different actuators that result in the following three forms of the matrix B:

$$B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; B_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Which of the actuators lead to a system that is controllable? [3 pts]

iii) Assume that A is as in part a.i) with $a = \frac{2}{3}$ and that C has the following structure $C = \begin{bmatrix} 1 & c \end{bmatrix}$. For which values of c is the system observable? [3 pts]

iv) Let a, B and c be such that the system is asymptotically stable, observable but not controllable. Is it possible to design a state observer and a state feedback controller such that $\lim_{k\to\infty} y_k = 0, \forall x_0$? Justify your answer. [2 pts]

v) If the system configuration is such that A is as in part a.i) with $a = \frac{2}{3}$ and $B = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, is it possible to design a state feedback controller $u_k = -Kx_k$, $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, such that all the poles of the closed loop system have absolute values no larger than $\frac{1}{2}$? If yes, find the range in which k_1 and k_2 should lie to satisfy this requirement. [4 pts]

b) We consider the system in (9) and we want to design a state observer and a state feedback controller for it. The state observer should have the following structure:

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k + Bu_k + L(y_k - \hat{y}_k) \\ \hat{y}_k &= C\hat{x}_k, \end{aligned}$$

where $\hat{x}_k$ is the state estimate and $\hat{y}_k$ is the output estimate. The state feedback controller should have the following structure:

$$u_k = -K\hat{x}_k.$$

The closed loop system with the state observer and the controller can be described by the following difference equation:

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = F \begin{bmatrix} x_k \\ e_k \end{bmatrix}, \tag{8}$$

where $e_k = \hat{x}_k - x_k$ is the state estimation error.

i) Derive the matrix F in (8). [6 pts]

ii) Based on the result in part b.i), derive the eigenvalues of the matrix F in terms of the eigenvalues of $A - BK$ and $A - LC$. [3 pts]

Hint: You may use $\det\left(\begin{bmatrix} X_1 & X_2 \\ 0 & X_3 \end{bmatrix}\right) = \det(X_1)\det(X_3)$.

iii) If the controller gain K is selected such that the closed loop system with state measurements is stable and if the observer gain L is selected such that the dynamics of the state error e_k are stable for the open loop system, is the closed loop system (8) with both the observer and the controller stable in general? Justify your answer. [3 pts]

Exercise 6 **Stability, Lyapunov functions**

1. Consider the following piecewise affine system:

$$x^+ = \begin{cases} -x-2 & \text{if } x < -2 \\ 0.9x & \text{if } -2 \leq x \leq 2 \\ -x+2 & \text{if } x > 2 \end{cases}$$

   Is this system globally stable? [5 pts]

   ◯ Yes ◯ No

2. Find a Lyapunov function for the system $x^+ = \frac{1}{2}x\cos(x)$ if it exists. [5 pts]

Exercise 7 **Stability, Observability**

Consider the following discrete time LTI system:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k, \end{aligned} \tag{9}$$

where

$$A = \begin{bmatrix} -0.4 & -1.1 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 0.9 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}.$$

For the following statements say whether they are true or false. Justify your answers.

1.) System (9) is stable. [1 pt]
☐ true ☐ false

2.) System (9) is both controllable and stabilizable. [1 pt]
☐ true ☐ false

3.) System (9) is not controllable, but it is stabilizable. [1 pt]
☐ true ☐ false

4.) System (9) is observable, but not detectable. [1 pt]
☐ true ☐ false

5.) System (9) is not observable, but it is detectable. [1 pt]
☐ true ☐ false