University of Pennsylvania, ESE 6190

Model Predictive Control

Chapter 9: Reachable Sets and Invariant Sets

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F. Borrelli, A. Bemporad, and M. Morari, Predictive Control for Linear and Hybrid Systems, Cambridge University Press, 2017. [Ch. 4, 10].

Outline

1. Polyhedra and Polytopes

2. Reachable Sets

3. Invariant Sets

4. Reachability and Controllability – Robust Case

Outline

1. Polyhedra and Polytopes

2. Reachable Sets

3. Invariant Sets

4. Reachability and Controllability – Robust Case

Outline

Definitions: Polyhedra and polytopes

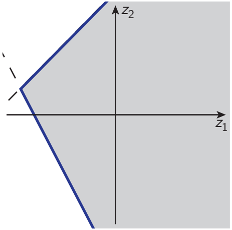
A polyhedron is the intersection of a **finite** number of closed halfspaces:

$$\begin{aligned} Z &= \{z \mid a_1^\top z \leq b_1,\ a_2^\top z \leq b_2, \ldots, a_m^\top z \leq b_m\} \\ &= \{z \mid Az \leq b\} \end{aligned}$$

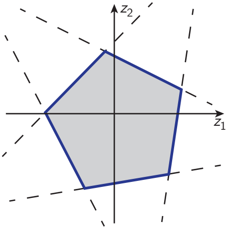
where $A := [a_1, a_2, \ldots, a_m]^\top$ and $b := [b_1, b_2, \ldots, b_m]^\top$.

A polytope is a **bounded** polyhedron.

Polyhedra and polytopes are always convex.



An (unbounded) polyhedron



A polytope

General Set Definitions and Operations

- An *n*-**dimensional ball** $B(x_0, \rho)$ is the set $B(x_0, \rho) = \{x \in \mathbb{R}^n | \sqrt{\|x - x_0\|_2} \leq \rho\}$. x_0 and ρ are the center and the radius of the ball, respectively.
- The **convex combination** of $x_1, \ldots, x_k$ is defined as the point $\lambda_1 x_1 + \ldots + \lambda_k x_k$ where $\sum_{i=1}^{k} \lambda_i = 1$ and $\lambda_i \geq 0$, $i = 1, \ldots, k$.
- The **convex hull** of a set $K \subseteq \mathbb{R}^n$ is the set of all convex combinations of points in K and it is denoted as $\mathrm{conv}(K)$:

$$\mathrm{conv}(K) := \{\lambda_1 x_1 + \ldots + \lambda_k x_k \mid x_i \in K,\ \lambda_i \geq 0,\ i = 1, \ldots, k, \\ \sum_{i=1}^{k} \lambda_i = 1\}.$$

General Set Definitions and Operations

- A **cone** spanned by a finite set of points $K = \{x_1, \ldots, x_k\}$ is defined as

$$\mathrm{cone}(K) = \{\sum_{i=1}^{k} \lambda_i x_i,\ \ \lambda_i \geq 0, i = 1, \ldots, k\}.$$

- The **Minkowski sum** of two sets $P, Q \subseteq \mathbb{R}^n$ is defined as

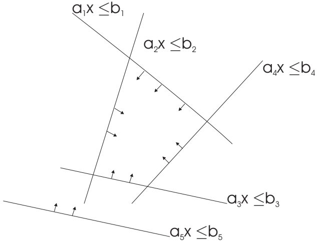
$$P \oplus Q := \{x + y | x \in P,\ \ y \in Q\}.$$

Polyhedra Representations

An $\mathcal{H}$-**polyhedron** $\mathcal{P}$ in $\mathbb{R}^n$ denotes an intersection of a finite set of closed halfspaces in $\mathbb{R}^n$:

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

In Matlab: P = Polytope(A,b)
A two-dimensional $\mathcal{H}$-polyhedron



Inequalities which can be removed without changing the polyhedron are called **redundant**. The representation of an $\mathcal{H}$-polyhedron is **minimal** if it does not contain redundant inequalities.

Polyhedra Representations

- A $\mathcal{V}$-**polyhedron** $\mathcal{P}$ in $\mathbb{R}^n$ denotes the Minkowski sum:

$$\mathcal{P} = \mathrm{conv}(V) \oplus \mathrm{cone}(Y)$$

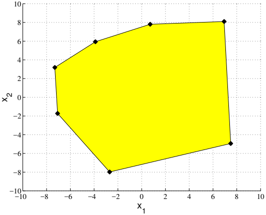
  for some $V = [V_1, \ldots, V_k] \in \mathbb{R}^{n \times k}$, $Y = [y_1, \ldots, y_{k'}] \in \mathbb{R}^{n \times k'}$.
- Any $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron.
- An $\mathcal{H}$-**polytope** ($\mathcal{V}$-**polytope**) is a bounded $\mathcal{H}$-polyhedron ($\mathcal{V}$-polyhedron). Any $\mathcal{H}$-polytope is a $\mathcal{V}$-polytope
- The **dimension of a polytope (polyhedron)** $\mathcal{P}$ is the dimension of its affine hull and is denoted by $\dim(\mathcal{P})$.
- A polytope $\mathcal{P} \subset \mathbb{R}^n$, is **full-dimensional** if it is possible to fit a non-empty n-dimensional ball in $\mathcal{P}$
- If $\|A_i\|_2 = 1$, where A_i denotes the i-th row of a matrix A, we say that the polytope $\mathcal{P}$ is **normalized**.

Polyhedra Representations

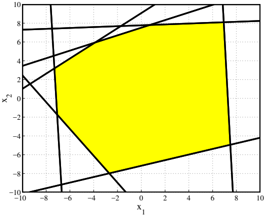
- A linear inequality $cz \leq c_0$ is said to be **valid** for $\mathcal{P}$ if it is satisfied for all points $z \in \mathcal{P}$.
- A **face** of $\mathcal{P}$ is any nonempty set of the form

$$\mathcal{F} = \mathcal{P} \cap \{z \in \mathbb{R}^s | \; cz = c_0\}$$

  where $cz \leq c_0$ is a **valid** inequality for $\mathcal{P}$.
- The faces of dimension 0,1, dim($\mathcal{P}$)-2 and dim($\mathcal{P}$)-1 are called **vertices**, **edges**, **ridges**, and **facets**, respectively.



(a) $\mathcal{V}$-representation.



(b) $\mathcal{H}$-representation.

Polytopal Complexes

A set $\mathcal{C} \subseteq \mathbb{R}^n$ is called a **P-collection** (in $\mathbb{R}^n$) if it is a collection of a finite number of n-dimensional polytopes, i.e.

$$\mathcal{C} = \{\mathcal{C}_i\}_{i=1}^{N_C},$$

where $\mathcal{C}_i := \{x \in \mathbb{R}^n \mid \;\; C_i^x x \leq C_i^c\}$, $\dim(\mathcal{C}_i) = n$, $i = 1, \ldots, N_C$, with $N_C < \infty$.

The **underlying set** of a P-collection $\mathcal{C} = \{\mathcal{C}_i\}_{i=1}^{N_C}$ is the point set

$$\underline{\mathcal{C}} := \bigcup_{\mathcal{P} \in \mathcal{C}} \mathcal{P} = \bigcup_{i=1}^{N_C} \mathcal{C}_i.$$

In Matlab: Q = [P1, P2, P3], R = [P4, Q, [P5, P6], P7]

Special Polytopal Complexes

- A collection of sets $\{\mathcal{C}_i\}_{i=1}^{N_C}$ is a **strict partition** of a set $\mathcal{C}$ if **(i)** $\bigcup_{i=1}^{N_C} \mathcal{C}_i = \mathcal{C}$ and **(ii)** $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$, $\forall i \neq j$.
- $\{\mathcal{C}_i\}_{i=1}^{N_C}$ is a **strict polyhedral partition** of a polyhedral set $\mathcal{C}$ if $\{\mathcal{C}_i\}_{i=1}^{N_C}$ is a strict partition of $\mathcal{C}$ and $\bar{\mathcal{C}}_i$ is a polyhedron for all i, where $\bar{\mathcal{C}}_i$ denotes the closure of the set $\mathcal{C}_i$
- A collection of sets $\{\mathcal{C}_i\}_{i=1}^{N_C}$ is a **partition** of a set $\mathcal{C}$ if **(i)** $\bigcup_{i=1}^{N_C} \mathcal{C}_i = \mathcal{C}$ and **(ii)** $(\mathcal{C}_i \backslash \partial \mathcal{C}_i) \cap (\mathcal{C}_j \backslash \partial \mathcal{C}_j) = \emptyset$, $\forall i \neq j$.

Functions on Polytopal Complexes

- A function $h(\theta) : \Theta \to \mathbb{R}^k$, where $\Theta \subseteq \mathbb{R}^s$, is **piecewise affine (PWA)** if there exists a strict partition $R_1,\ldots,R_N$ of Θ and $h(\theta) = H^i\theta + k^i$, $\forall\theta \in R_i$, $i = 1, \ldots, N$.
- A function $h(\theta) : \Theta \to \mathbb{R}^k$, where $\Theta \subseteq \mathbb{R}^s$, is **piecewise affine on polyhedra** (PPWA) if there exists a strict polyhedral partition $R_1,\ldots,R_N$ of Θ and $h(\theta) = H^i\theta + k^i$, $\forall\theta \in R_i$, $i = 1, \ldots, N$.
- A function $h(\theta) : \Theta \to \mathbb{R}$, where $\Theta \subseteq \mathbb{R}^s$, is **piecewise quadratic (PWQ)** if there exists a strict partition $R_1,\ldots,R_N$ of Θ and $h(\theta) = \theta' H^i\theta + k^i\theta + l^i$, $\forall\theta \in R_i$, $i = 1, \ldots, N$.
- A function $h(\theta) : \Theta \to \mathbb{R}$, where $\Theta \subseteq \mathbb{R}^s$, is **piecewise quadratic on polyhedra** (PPWQ) if there exists a strict polyhedral partition $R_1,\ldots,R_N$ of Θ and $h(\theta) = \theta' H^i\theta + k^i\theta + l^i$, $\forall\theta \in R_i$, $i = 1, \ldots, N$.

Outline

Basic Operations on Polytopes

- **Convex Hull** of a set of points $V = \{V_i\}_{i=1}^{N_V}$, with $V_i \in \mathbb{R}^n$,

$$\text{conv}\,(V) = \{x \in \mathbb{R}^n \mid x = \sum_{i=1}^{N_V} \alpha_i V_i,\ 0 \le \alpha_i \le 1,\ \sum_{i=1}^{N_V} \alpha_i = 1\}. \qquad (1)$$

  In Matlab: P=hull(V), V matrix containing vertices of the polytope P

- **Vertex Enumeration** of a polytope $\mathcal{P}$ given in $\mathcal{H}$-representation. (dual of the convex hull operation)
  In Matlab: V=extreme(P)

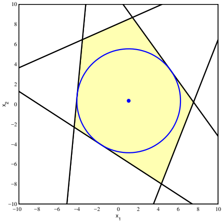
Used to switch from a $\mathcal{V}$-representation of a polytope to an $\mathcal{H}$-representation.

Basic Operations on Polytopes

- **Polytope reduction** is the computation of the minimal representation of a polytope. A polytope $\mathcal{P} \subset \mathbb{R}^n$, $\mathcal{P} = \{x \in \mathbb{R}^n \; : \; Hx \leq k\}$ is in a **minimal representation** if the removal of any row in $Hx \leq k$ would change it (i.e., if there are no redundant constraints).

  In Matlab: P = Polytope(A,b,normal,minrep), minrep=1
- The **Chebychev Ball** of a polytope $\mathcal{P}$ corresponds to the largest radius ball $\mathcal{B}(x_c, R)$ with center x_c, such that $\mathcal{B}(x_c, R) \subset \mathcal{P}$.

  In Matlab: P.xCheb, P.rCheb

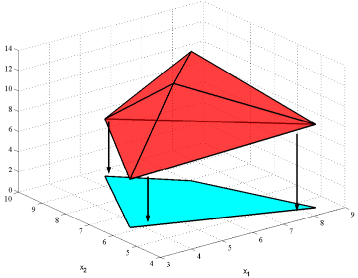


Basic Operations on Polytopes

- **Projection** Given a polytope $\mathcal{P} = \{[x'y']' \in \mathbb{R}^{n+m} \ : \ H^x x + H^y y \leq k\} \subset \mathbb{R}^{n+m}$ the projection onto the x-space $\mathbb{R}^n$ is defined as

$$\text{proj}_x(\mathcal{P}) := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m : H^x x + H^y y \leq k\}.$$

In Matlab: Q = projection(P,dim)



Basic Operations on Polytopes

- **Set-Difference** The set-difference of two polytopes $\mathcal{Y}$ and $\mathcal{R}_0$

$$\mathcal{R} = \mathcal{Y} \setminus \mathcal{R}_0 := \{x \in \mathbb{R}^n \;:\; x \in \mathcal{Y}, x \notin \mathcal{R}_0\},$$

  in general, can be a nonconvex and disconnected set and can be described as a P-collection $\mathcal{R} = \bigcup_{i=1}^{m} \mathcal{R}_i$. The P-collection can be computed by consecutively inverting the half-spaces defining $\mathcal{R}_0$ as described next.
  In Matlab: P1 \ P2

Theorem

Let

$$\mathcal{R}_i = \left\{ x \in \mathcal{Y} \;\middle|\; \begin{array}{l} A^i x > b^i \\ A^j x \leq b^j, \forall j < i \end{array} \right\} i = 1, \ldots, m$$

Then $\{\bar{\mathcal{R}}_0, \mathcal{R}_1, \ldots, \mathcal{R}_m\}$ is a strict polyhedral partition of $\mathcal{Y}$.

Basic Operations on Polytopes

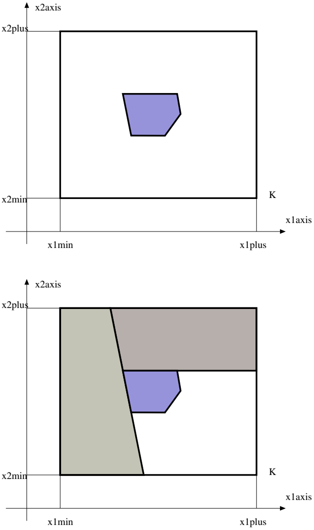


Figure: Two dimensional example: partition of the rest of the space $\mathcal{X} \backslash \mathcal{R}_0$.

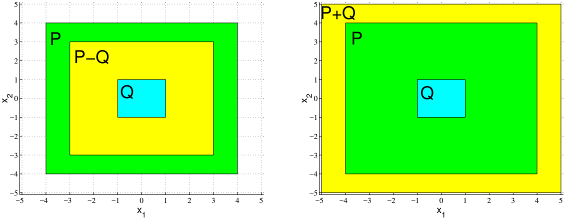
Basic Operations on Polytopes

- The **Pontryagin Difference** (also known as Minkowski difference) of two polytopes $\mathcal{P}$ and $\mathcal{Q}$ is a polytope

$$\mathcal{P} \ominus \mathcal{Q} := \{x \in \mathbb{R}^n \mid x + q \in \mathcal{P},\ \forall q \in \mathcal{Q}\}.$$

- The **Minkowski sum** of two polytopes $\mathcal{P}$ and $\mathcal{Q}$ is a polytope

$$\mathcal{P} \oplus \mathcal{Q} := \{x \in \mathbb{R}^n \mid \exists y \in \mathcal{P},\ \exists z \in \mathcal{Q},\ x = y + z\}.$$



(a) Pontryagin difference $\mathcal{P} \ominus \mathcal{Q}$.

(b) Minkowski sum $\mathcal{P} \oplus \mathcal{Q}$.

Minkowski Sum of Polytopes

- The Minkowski sum is computationally expensive.
  Consider

$$P = \{y \in \mathbb{R}^n \mid P^y y \le P^c\}, \quad \mathcal{Q} = \{z \in \mathbb{R}^n \mid Q^z z \le Q^c\},$$

it holds that

$$\begin{aligned}
W &= P \oplus Q \\
&= \Big\{x \in \mathbb{R}^n \mid \exists y\ P^y y \le P^c,\ \exists z\ Q^z z \le Q^c,\ y, z \in \mathbb{R}^n,\ x = y + z\Big\} \\
&= \Big\{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^n,\ \text{s.t. } P^y y \le P^c,\ Q^z(x - y) \le Q^c\Big\} \\
&= \left\{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^n,\ \text{s.t. } \begin{bmatrix} 0 & P^y \\ Q^z & -Q^z \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le \begin{bmatrix} P^c \\ Q^c \end{bmatrix}\right\} \\
&= \operatorname{proj}_x\left(\left\{ [x' y'] \in \mathbb{R}^{n+n} \;\middle|\; \begin{bmatrix} 0 & P^y \\ Q^z & -Q^z \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le \begin{bmatrix} P^c \\ Q^c \end{bmatrix} \right\} \right).
\end{aligned}$$

Pontryagin Difference of Polytopes

- The Pontryagin difference is "not computationally expensive".
- Consider

$$\mathcal{P} = \{y \in \mathbb{R}^n \mid , P^y y \leq P^b\}, \qquad \mathcal{Q} = \{z \in \mathbb{R}^n \mid Q^z z \leq Q^b\},$$

  Then:

$$\mathcal{P} \ominus \mathcal{Q} = \{x \in \mathbb{R}^n \mid P^y x \leq P^b - H(P^y, \mathcal{Q})\}$$

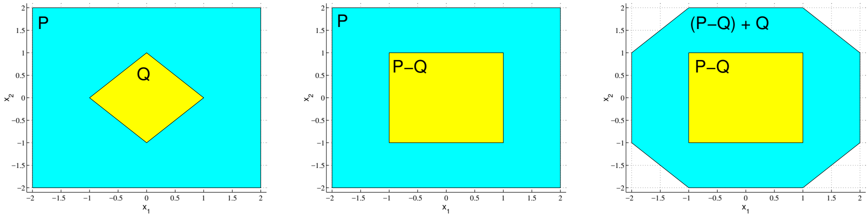
  where the i-th element of $H(P^y, \mathcal{Q})$ is

$$H_i(P^y, \mathcal{Q}) := \max_{z \in \mathcal{Q}} P_i^y z$$

  and P_i^y is the i-th row of the matrix P^y.
- For special cases (e.g. when $\mathcal{Q}$ is a hypercube), more efficient computational methods exist.

Basic Operations on Polytopes

Note that $(\mathcal{P} \ominus \mathcal{Q}) \oplus \mathcal{Q} \subseteq \mathcal{P}$.



(c) Two polytopes $\mathcal{P}$ and $\mathcal{Q}$.

(d) Polytope $\mathcal{P}$ and Pontryagin difference $\mathcal{P} \ominus \mathcal{Q}$.

(e) Polytope $\mathcal{P} \ominus \mathcal{Q}$ and the set $(\mathcal{P} \ominus \mathcal{Q}) \oplus \mathcal{Q}$.

Figure: Illustration that $(\mathcal{P} \ominus \mathcal{Q}) \oplus \mathcal{Q} \subseteq \mathcal{P}$.

Affine Mappings and Polyhedra

- Consider a polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n \mid Hx \leq k\}$, with $H \in \mathbb{R}^{n_P \times n}$ and an affine mapping $f(z)$

$$f: \; z \in \mathbb{R}^n \mapsto Az + b, \; A \in \mathbb{R}^{n \times n}, \; b \in \mathbb{R}^n$$

- Define the composition of $\mathcal{P}$ and f as the following polyhedron

$$\mathcal{P} \circ f := \{z \in \mathbb{R}^n \mid Hf(z) \leq k\} = \{z \in \mathbb{R}^n \mid HAz \leq k - Hb\}$$

- Useful for backward-reachability

Affine Mappings and Polyhedra

- Consider a polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n \mid Hx \leq k\}$, with $H \in \mathbb{R}^{n_P \times n}$ and an affine mapping $f(z)$

$$f: \; z \in \mathbb{R}^n \mapsto Az + b, \; A \in \mathbb{R}^{n \times n}, \; b \in \mathbb{R}^n$$

- Define the composition of f and $\mathcal{P}$ as the following polyhedron

$$f \circ \mathcal{P} := \{y \in \mathbb{R}^n \mid y = Ax + b \; \forall x \in \mathbb{R}^n, \; Hx \leq k\}$$

- The polyhedron $f \circ \mathcal{P}$ in can be computed as follows. Write $\mathcal{P}$ in $\mathcal{V}$-representation $\mathcal{P} = \mathrm{conv}(V)$ and map the vertices $V = \{V_1, \ldots, V_k\}$ through the transformation f. Because the transformation is affine, the set $f \circ \mathcal{P}$ is the convex hull of the transformed vertices

$$f \circ \mathcal{P} = \mathrm{conv}(F), \; F = \{AV_1 + b, \ldots, AV_k + b\}.$$

- Useful for forward-reachability

Outline

Outline

2. Reachable Sets

Set Definition

We consider the following two types of systems **autonomous systems:**

$$x(t+1) = f_a(x(t)), \tag{2}$$

and **systems subject to external inputs:**

$$x(t+1) = f(x(t), u(t)). \tag{3}$$

Both systems are subject to state and input constraints

$$x(t) \in \mathcal{X}, \;\; u(t) \in \mathcal{U}, \; \forall \; t \geq 0.$$

The sets $\mathcal{X}$ and $\mathcal{U}$ are polyhedra and contain the origin in their interior.

Reach Set Definition

For the autonomous system (2) we denote the one-step reachable set as

$$\mathsf{Reach}(\mathcal{S}) := \{x \in \mathbb{R}^n \mid \exists\, x(0) \in \mathcal{S} \text{ s.t. } x = f_a(x(0))\}$$

For the system (3) with inputs we denote the one-step reachable set as

$$\mathsf{Reach}(\mathcal{S}) := \{x \in \mathbb{R}^n \mid \exists\, x(0) \in \mathcal{S},\ \exists\, u(0) \in \mathcal{U} \text{ s.t. } x = f(x(0), u(0))\}$$

Pre Set Definition

“Pre" sets are the dual of one-step reachable sets. The set

$$\mathrm{Pre}(\mathcal{S}) := \{x \in \mathbb{R}^n \mid f_a(x) \in \mathcal{S}\}$$

defines the set of states which evolve into the target set $\mathcal{S}$ in one time step for the system (2).

Similarly, for the system (3) the set of states which can be driven into the target set $\mathcal{S}$ in one time step is defined as

$$\mathrm{Pre}(\mathcal{S}) := \{x \in \mathbb{R}^n \mid \exists u \in \mathcal{U} \text{ s.t. } f(x, u) \in \mathcal{S}\}$$

Outline

2. Reachable Sets

    Pre and Reach Sets Definition

    Pre and Reach Sets Computation

    Summary

    Controllable Sets

    N-Step Reachable Sets

Pre Set Computation - Autonomous Systems

Assume the system is linear and autonomous

$$x(t+1) = Ax(t)$$

Let

$$\mathcal{S} = \{x \mid Hx \leq h\}, \tag{4}$$

Then the set $\mathrm{Pre}(\mathcal{S})$ is

$$\mathrm{Pre}(\mathcal{S}) = \{x \mid HAx \leq h\}$$

Note that by using polyhedral notation, the set $\mathsf{Pre}(\mathcal{S})$ is simply $\mathcal{S} \circ A$.

Reach Set Computation - Autonomous Systems

The set $\mathsf{Reach}(\mathcal{S})$ is obtained by applying the map A to the set $\mathcal{S}$.
Write $\mathcal{S}$ in $\mathcal{V}$-representation

$$\mathcal{S} = \mathrm{conv}(V) \tag{5}$$

and map the set of vertices V through the transformation A.
Because the transformation is linear, the reach set is simply the convex hull of the transformed vertices

$$\mathrm{Reach}(\mathcal{S}) = A \circ \mathcal{S} = \mathrm{conv}(AV) \tag{6}$$

Pre Set Computation - Systems with Inputs

Consider the system

$$x(t+1) = Ax(t) + Bu(t)$$

Let

$$\mathcal{S} = \{x \mid Hx \le h\}, \quad \mathcal{U} = \{u \mid H_u u \le h_u\}, \tag{7}$$

The Pre set is

$$\mathrm{Pre}(\mathcal{S}) = \left\{ x \in \mathbb{R}^n \,\middle|\, \exists u \in \mathbb{R} \text{ s.t. } \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \le \begin{bmatrix} h \\ h_u \end{bmatrix} \right\}$$

Note that by using the definition of the Minkowski we can compactly write the set as:

$$\begin{aligned} \mathrm{Pre}(\mathcal{X}) = \quad & \{x \mid \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu \in \mathcal{X}\} \\ & \{x \mid \exists y \in \mathcal{X},\ \exists u \in \mathcal{U},\ Ax = y - Bu\} \\ & \{x \mid Ax = \mathcal{X} \oplus (-B) \circ \mathcal{U}\} \\ = \quad & (\mathcal{X} \oplus (-B) \circ \mathcal{U}) \circ A \end{aligned} \tag{8}$$

Reach Set Computation - Systems with Inputs

The set $\mathsf{Reach}(\mathcal{S})$ to the set $\mathcal{S}$ and then considering the effect of the input $u \in \mathcal{U}$.

Recall

$$A \circ \mathcal{S} = \mathrm{conv}(AV) \tag{9}$$

and therefore

$$\mathsf{Reach}(\mathcal{S}) = \{y + \bar{u} \mid y \in A \circ \mathcal{X},\ \bar{u} \in B \circ \mathcal{U}\}$$

and therefore

$$\mathsf{Reach}(\mathcal{X}) = (A \circ \mathcal{X}) \oplus (B \circ \mathcal{U})$$

Outline

2. Reachable Sets

Summary

In summary, the sets $\text{Pre}(\mathcal{X})$ and $\text{Reach}(\mathcal{X})$ are the results of linear operations on the polyhedra $\mathcal{X}$ and $\mathcal{U}$ and therefore are polyhedra. By using the definition of the Minkowski sum and of affine operation on polyhedra we can compactly summarize the Pre and Reach operations on linear systems as follows:

	$x(t+1) = Ax(t)$	$x(t+1) = Ax(t) + Bu(t)$
$\text{Pre}(\mathcal{X})$	$\mathcal{X} \circ A$	$(\mathcal{X} \oplus (-B \circ \mathcal{U})) \circ A$
$\text{Reach}(\mathcal{X})$	$A \circ \mathcal{X}$	$(A \circ \mathcal{X}) \oplus (B \circ \mathcal{U})$

Table: Pre and Reach operations for linear systems subject to polyhedral state and input constraints $x(t) \in \mathcal{X}$, $u(t) \in \mathcal{U}$

Outline

2. Reachable Sets

Controllable Sets

Definition: N-Step Controllable Set $\mathcal{K}_N(\mathcal{O})$

For a given target set $\mathcal{O} \subseteq \mathcal{X}$, the N-step controllable set $\mathcal{K}_N(\mathcal{O})$ is defined as:

$$\mathcal{K}_N(\mathcal{O}) := \mathrm{Pre}(\mathcal{K}_{N-1}(\mathcal{O})) \cap \mathcal{X}, \quad \mathcal{K}_0(\mathcal{O}) = \mathcal{O}, \qquad N \in \mathbb{N}^+.$$

All states $x_0 \in \mathcal{K}_N(\mathcal{O})$ can be driven,through a time-varying control law, to the target set $\mathcal{O}$ in N steps, while satisfying input and state constraints.

Definition: Maximal Controllable Set $\mathcal{K}_\infty(\mathcal{O})$

For a given target set $\mathcal{O} \subseteq \mathcal{X}$, the maximal controllable set $\mathcal{K}_\infty(\mathcal{O})$ for the system $x(t+1) = f(x(t), u(t))$ subject to the constraints $x(t) \in \mathcal{X},\ u(t) \in \mathcal{U}$ is the union of all N-step controllable sets contained in $\mathcal{X}$ ($N \in \mathbb{N}$).

Outline

2. Reachable Sets

N-Step Reachable Sets

Definition: N-Step Reachable Set $\mathcal{R}_N(\mathcal{X}_0)$

For a given initial set $\mathcal{X}_0 \subseteq \mathcal{X}$, the N-step reachable set $\mathcal{R}_N(\mathcal{X}_0)$ is

$$\mathcal{R}_{i+1}(\mathcal{X}_0) := \mathsf{Reach}(\mathcal{R}_i(\mathcal{X}_0)), \quad \mathcal{R}_0(\mathcal{X}_0) = \mathcal{X}_0, \qquad i = 0, \ldots, N-1$$

All states $x_0 \in \mathcal{X}_0$ can will evolve to the N-step reachable set $\mathcal{R}_N(\mathcal{X}_0)$ in N steps

Same definition of Maximal Reachable Set $\mathcal{R}_\infty(\mathcal{X}_0)$ can be introduced.

Outline

1. Polyhedra and Polytopes

2. Reachable Sets

3. Invariant Sets

4. Reachability and Controllability – Robust Case

Outline

3. Invariant Sets

   Invariant Sets

   Control Invariant Sets

Invariant Sets

Invariant sets

- are computed for **autonomous systems**
- for a **given** feedback controller $u = g(x)$, provide the set of initial states whose trajectory will never violate the system constraints.

Definition: Positive Invariant Set

A set $\mathcal{O} \subseteq \mathcal{X}$ is said to be a positive invariant set for the autonomous system $x(t+1) = f_a(x(t))$ subject to the constraints $x(t) \in \mathcal{X}$, if

$$x(0) \in \mathcal{O} \quad \Rightarrow \quad x(t) \in \mathcal{O}, \quad \forall t \in \mathbb{N}^+$$

Definition: Maximal Positive Invariant Set $\mathcal{O}_\infty$

The set $\mathcal{O}_\infty$ is the maximal invariant set if $\mathcal{O}_\infty$ is invariant and $\mathcal{O}_\infty$ contains all the invariant sets contained in $\mathcal{X}$.

Invariant Sets

Theorem: Geometric condition for invariance

A set $\mathcal{O}$ is a positive invariant set if and only if

$$\mathcal{O} \subseteq \text{Pre}(\mathcal{O})$$

Proof: We prove the contrapositive for both the necessary and sufficient parts.

- (necessary) If $\mathcal{O} \nsubseteq \text{Pre}(\mathcal{O})$ then $\exists \bar{x} \in \mathcal{O}$ such that $\bar{x} \notin \text{Pre}(\mathcal{O})$. From the definition of $\text{Pre}(\mathcal{O})$, $f_a(\bar{x}) \notin \mathcal{O}$ and thus $\mathcal{O}$ is not a positive invariant.
- (sufficient) If $\mathcal{O}$ is not a positive invariant set then $\exists \bar{x} \in \mathcal{O}$ such that $f_a(\bar{x}) \notin \mathcal{O}$. This implies that $\bar{x} \in \mathcal{O}$ and $\bar{x} \notin \text{Pre}(\mathcal{O})$ and thus $\mathcal{O} \nsubseteq \text{Pre}(\mathcal{O})$

Clearly

$$\mathcal{O} \subseteq \text{Pre}(\mathcal{O}) \iff \text{Pre}(\mathcal{O}) \cap \mathcal{O} = \mathcal{O} \tag{10}$$

Invariant Sets

Algorithm

Input: f_a , $\mathcal{X}$
Output: $\mathcal{O}_\infty$

$\Omega_0 := \mathcal{X}$
$\Omega_{k+1} := \mathrm{Pre}(\Omega_k) \cap \Omega_k$
if $\Omega_{k+1} = \Omega_k$ **then**
    $\mathcal{O}_\infty \leftarrow \Omega_{k+1}$
else
    GOTO 2:
end if

The algorithm generates the set sequence $\{\Omega_k\}$ satisfying $\Omega_{k+1} \subseteq \Omega_k, \forall k \in \mathbb{N}$ and it terminates when $\Omega_{k+1} = \Omega_k$ so that Ω_k is the maximal positive invariant set $\mathcal{O}_\infty$ for $x(t+1) = f_a(x(t))$.

Outline

3. Invariant Sets

   Invariant Sets

   Control Invariant Sets

Control Invariant Sets

Control invariant sets

- are computed for systems **subject to external inputs**
- provide the set of initial states for which **there exists** a controller such that the system constraints are never violated.

Definition: Control Invariant Set

A set $\mathcal{C} \subseteq \mathcal{X}$ is said to be a control invariant set if

$$x(t) \in \mathcal{C} \quad \Rightarrow \quad \exists u(t) \in \mathcal{U} \text{ such that } f(x(t), u(t)) \in \mathcal{C}, \quad \forall t \in \mathbb{N}^+$$

Definition: Maximal Control Invariant Set $\mathcal{C}_\infty$

The set $\mathcal{C}_\infty$ is said to be the maximal control invariant set for the system $x(t+1) = f(x(t), u(t))$ subject to the constraints in $x(t) \in \mathcal{X}$, $u(t) \in \mathcal{U}$, if it is control invariant and contains all control invariant sets contained in $\mathcal{X}$.

Control Invariant Sets

Same geometric condition for control invariants holds: $\mathcal{C}$ is a control invariant set if and only if

$$\mathcal{C} \subseteq \mathrm{Pre}(\mathcal{C}) \tag{11}$$

Algorithm

Input: f, $\mathcal{X}$ and $\mathcal{U}$
Output: $\mathcal{C}_\infty$

1: $\Omega_0 := \mathcal{X}$
2: $\Omega_{k+1} := \mathrm{Pre}(\Omega_k) \cap \Omega_k$
3: **if** $\Omega_{k+1} = \Omega_k$ **then**
4:     $\mathcal{C}_\infty \leftarrow \Omega_{k+1}$
5: **else**
6:     GOTO 2:
7: **end if**

The algorithm generates the set sequence $\{\Omega_k\}$ satisfying $\Omega_{k+1} \subseteq \Omega_k, \forall k \in \mathbb{N}$ and it terminates if $\Omega_{k+1} = \Omega_k$ so that Ω_k is the maximal control invariant set $\mathcal{C}_\infty$ for the constrained system.

Invariant Sets and Control Invariant Sets

- The set $\mathcal{O}_\infty$ ($\mathcal{C}_\infty$) is **finitely determined** if and only if $\exists\ i \in \mathbb{N}$ such that $\Omega_{i+1} = \Omega_i$.
- The smallest element $i \in \mathbb{N}$ such that $\Omega_{i+1} = \Omega_i$ is called the **determinedness index**.
- For linear system with linear constraints the sets $\mathcal{O}_\infty$ and $\mathcal{C}_\infty$ are polyhedra if they are finitely determined.
- For autonomous systems, if the does not terminate then $\mathcal{O}_\infty = \bigcap_{k\geq 0} \Omega_k$. If $\Omega_k = \emptyset$ for some integer k then $\mathcal{O}_\infty = \emptyset$. More complicated for non-autonomous systems.
- For all states contained in the maximal control invariant set $\mathcal{C}_\infty$ there exists a control law, such that the system constraints are never violated. This does not imply that there exists a control law which can drive the state into a user-specified target set.

Stabilizable Sets

Observe that controllable sets $\mathcal{K}_N(\mathcal{O})$ where the target $\mathcal{O}$ is a control invariant set are special sets

Definition: N-step (Maximal) Stabilizable Set
For a given control invariant set $\mathcal{O} \subseteq \mathcal{X}$, the N-step (maximal) stabilizable set is the N-step (maximal) controllable set $\mathcal{K}_N(\mathcal{O})$ ($\mathcal{K}_\infty(\mathcal{O})$).

In addition to guaranteeing that from $\mathcal{K}_N(\mathcal{O})$ we reach $\mathcal{O}$ in N steps, one can ensure that once it has reached $\mathcal{O}$, the system can stay there at all future time instants.

Set Evolution of $\mathcal{K}_N(\mathcal{X}_f)$

Theorem

Let the target set $\mathcal{X}_f$ be a control invariant subset of $\mathcal{X}$. Then,

1. The i-step controllable set $\mathcal{K}_i(\mathcal{X}_f)$, $i = 0, 1, \ldots$ is control invariant and contained within the maximal control invariant set:
$$\mathcal{K}_i(\mathcal{X}_f) \subseteq \mathcal{C}_\infty$$
2. $\mathcal{K}_i(\mathcal{X}_f) \supseteq \mathcal{K}_j(\mathcal{X}_f)$ if $i > j$.
3. The size of $\mathcal{K}_i(\mathcal{X}_f)$ set stops increasing (with increasing i) if and only if the maximal stabilizable set is finitely determined and i is larger than its determinedness index $\bar{N}$.
4. Furthermore,
$$\mathcal{K}_i(\mathcal{X}_f) = \mathcal{K}_\infty(\mathcal{X}_f) \text{ if } i \geq \bar{N}$$

Outline

1. Polyhedra and Polytopes

2. Reachable Sets

3. Invariant Sets

4. Reachability and Controllability – Robust Case

Outline

Set Definition

We consider the following two types of systems:

autonomous systems

$$x(t+1) = f_a(x(t), w(t)) \tag{12}$$

systems subject to external inputs

$$x(t+1) = f(x(t), u(t), w(t)) \tag{13}$$

Both systems are subject to disturbance $w(t)$ and to the constraints

$$x(t) \in \mathcal{X},\ \ u(t) \in \mathcal{U},\ \ w(t) \in \mathcal{W}\ \forall\ t \geq 0. \tag{14}$$

The sets $\mathcal{X}$ and $\mathcal{U}$ and $\mathcal{W}$ are polytopes and contain the origin in their interior.

Reach Set Definition

For the autonomous system (12) we denote the one-step robust reachable set

$$\text{Reach}(\mathcal{S}, \mathcal{W}) := \{x \in \mathbb{R}^n \mid \exists\ x(0) \in \mathcal{S},\ \exists\ w \in \mathcal{W} \text{ such that } x = f_a(x(0), w)\}$$

For the system (13) with inputs we denote the one-step robust reachable set as

$$\begin{aligned}\text{Reach}(\mathcal{S}, \mathcal{W}) := \quad & \{x \in \mathbb{R}^n \mid \exists\ x(0) \in \mathcal{S},\ \exists\ u \in \mathcal{U}, \\ & \ \exists\ w \in \mathcal{W},\ \text{such that } x = f(x(0), u, w)\}\end{aligned}$$

Pre Set Definition

“Pre" sets are the dual of one-step reachable sets. The set

$$\text{Pre}(\mathcal{S}, \mathcal{W}) := \{x \in \mathbb{R}^n \mid f_a(x, w) \in \mathcal{S},\ \forall w \in \mathcal{W}\}$$

defines the set of system states which evolve into the target set $\mathcal{S}$ in one time step for all possible disturbances $w \in \mathcal{W}$.

Similarly, the set of states which can be robustly driven into the target set $\mathcal{S}$ in one time step is defined as

$$\text{Pre}(\mathcal{S}, \mathcal{W}) := \{x \in \mathbb{R}^n \mid \exists u \in \mathcal{U} \text{ s.t. } f(x, u, w) \in \mathcal{S},\ \forall w \in \mathcal{W}\}. \tag{15}$$

Outline

4. Reachability and Controllability – Robust Case

   Pre and Reach Sets Definition

   Pre and Reach Sets Computation

   Summary

Pre Set Computation -Autonomous Systems

Assume the system is linear and autonomous

$$x(t+1) = Ax(t) + w(t)$$

Let

$$\mathcal{S} = \{x \mid Hx \leq h\}, \tag{16}$$

Then the set $\text{Pre}(\mathcal{S}, \mathcal{W})$ is

$$\text{Pre}(\mathcal{S}, \mathcal{W}) = \{x \mid HAx \leq h - Hw,\ \forall w \in \mathcal{W}\}.$$

which can be represented as

$$\text{Pre}(\mathcal{S}, \mathcal{W}) = \{x \in \mathbb{R}^n \mid HAx \leq \tilde{h}\}$$

with

$$\tilde{h}_i = \min_{w \in \mathcal{W}} (h_i - H_i w).$$

Note that by using polyhedral notation, the Pre set can be written as

$$\begin{aligned} \text{Pre}(\mathcal{S}, \mathcal{W}) &= \{x \in \mathbb{R}^n \mid Ax + w \in \mathcal{S},\ \forall w \in \mathcal{W}\} = \{x \in \mathbb{R}^n \mid Ax \in \mathcal{S} \ominus \mathcal{W}\} = \\ &= (\mathcal{S} \ominus \mathcal{W}) \circ A. \end{aligned}$$

Reach Set Computation - Autonomous Systems

The set

$$\text{Reach}(\mathcal{X}, \mathcal{W}) = \{y \mid \exists x \in \mathcal{X},\ \exists w \in \mathcal{W} \text{ such that } y = Ax + w\}$$

is obtained by applying the map A to the set $\mathcal{X}$ and then considering the effect of the disturbance $w \in \mathcal{W}$.
Write $\mathcal{X}$ in $\mathcal{V}$-representation

$$\mathcal{X} = \text{conv}(V) \tag{17}$$

Because the transformation is linear, the composition of the map A with the set $\mathcal{X}$, denoted as $A \circ \mathcal{X}$, is simply the convex hull of the transformed vertices

$$A \circ \mathcal{X} = \text{conv}(AV). \tag{18}$$

Rewrite the set

$$\text{Reach}(\mathcal{X}, \mathcal{W}) = \{y \in \mathbb{R}^n \mid \exists\ z \in A \circ \mathcal{X},\ \exists w \in \mathcal{W} \text{ such that } y = z + w\}.$$

We can use the definition of Minkowski sum and rewrite the Reach set as

$$\text{Reach}(\mathcal{X}, \mathcal{W}) = (A \circ \mathcal{X}) \oplus \mathcal{W}.$$

Pre Set Computation

Consider the system

$$x(t+1) = Ax(t) + Bu(t) + w(t)$$

Let

$$\mathcal{X} = \{x \mid Hx \le h\}, \quad \mathcal{U} = \{u \mid H_u u \le h_u\}, \tag{19}$$

The Pre set is

$$\text{Pre}(\mathcal{X}, \mathcal{W}) = \left\{ x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m \text{ s.t. } \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \le \begin{bmatrix} h - Hw \\ h_u \end{bmatrix},\ \forall\, w \in \mathcal{W} \right\}$$

which can be compactly written as

$$\text{Pre}(\mathcal{X}, \mathcal{W}) = \left\{ x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m \text{ s.t. } \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \le \begin{bmatrix} \tilde{h} \\ h_u \end{bmatrix} \right\}.$$

where

$$\tilde{h}_i = \min_{w \in \mathcal{W}} (h_i - H_i w).$$

Note that one can use polyhedral operations and rewrite the set as:

$$\text{Pre}(\mathcal{X}, \mathcal{W}) = ((\mathcal{X} \ominus \mathcal{W}) \oplus (-B \circ \mathcal{U})) \circ A \tag{20}$$

Reach Set Computation

The set $\mathrm{Reach}(\mathcal{X})$

$$\mathrm{Reach}(\mathcal{X}, \mathcal{W}) = \{y \mid \exists x \in \mathcal{X},\ \exists u \in \mathcal{U},\ \exists w \in \mathcal{W} \text{ s.t. } y = Ax + Bu + w\}$$

is obtained by applying the map A to the set $\mathcal{X}$ and then considering the effect of the input $u \in \mathcal{U}$ and of the disturbance $w \in \mathcal{W}$.

We can use the polyhedral operations and rewrite $\mathrm{Reach}(\mathcal{X}, \mathcal{W})$ as

$$\mathrm{Reach}(\mathcal{X}, \mathcal{W}) = (A \circ \mathcal{X}) \oplus (B \circ \mathcal{U}) \oplus \mathcal{W}.$$

Outline

4. Reachability and Controllability – Robust Case

    Pre and Reach Sets Definition

    Pre and Reach Sets Computation

    Summary

Summary

In summary, for linear systems with additive disturbances the sets Pre($\mathcal{X}, \mathcal{W}$) and Reach($\mathcal{X}, \mathcal{W}$) are the results of linear operations on the polytopes $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{W}$ and therefore are polytopes. By using the definition of Minkowski sum, Pontryagin difference and affine operation on polyhedra we obtain the following.

	$x(t+1) = Ax(t) + w(t)$	$x(k+1) = Ax(t) + Bu(t) + w(t)$
Pre($\mathcal{X}, \mathcal{W}$)	$(\mathcal{X} \ominus \mathcal{W}) \circ A$	$(\mathcal{X} \ominus \mathcal{W} \oplus -B \circ \mathcal{U}) \circ A$
Reach($\mathcal{X}$), $\mathcal{W}$	$(A \circ \mathcal{X}) \oplus \mathcal{W}$	$(A \circ \mathcal{X}) \oplus (B \circ \mathcal{U}) \oplus \mathcal{W}$

Table: **Pre** and **Reach** operations for uncertain linear systems subject to polyhedral input and state constraints $x(t) \in \mathcal{X}$, $u(t) \in \mathcal{U}$ with additive polyhedral disturbances $w(t) \in \mathcal{W}$

Note that the summary applies also to the class of systems $x(k+1) = Ax(t) + Bu(t) + E\tilde{d}(t)$ where $\tilde{d} \in \tilde{\mathcal{W}}$. This can be transformed into $x(k+1) = Ax(t) + Bu(t) + w(t)$ where $w \in \mathcal{W} := E \circ \tilde{\mathcal{W}}$.