
# Lecture 4

# Systems of Linear Equations

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# Systems of Linear Equations

## Motivation: from Mechanics

**The vertical displacement of the beam is represented by a function $y(x)$, where $0 \le x \le L$ along the beam of length $L$.**

$$EIy'''' = f(x)$$

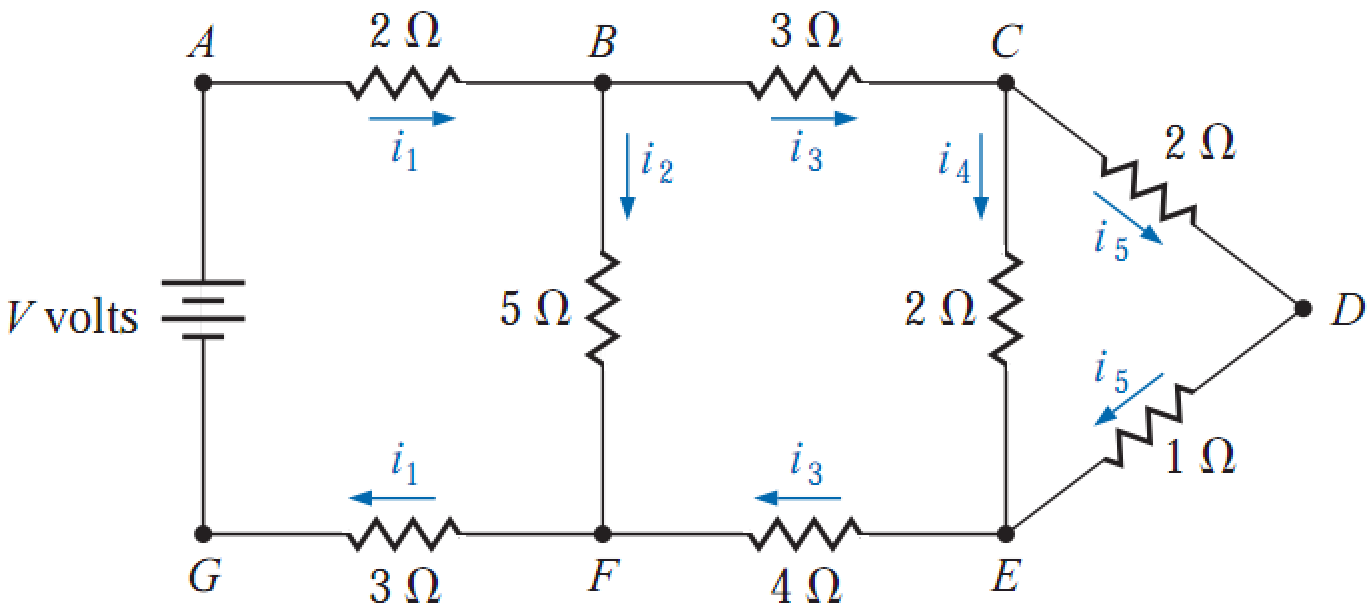
$$y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2} = \frac{h^4}{EI} f(x_i)$$

$$y(0) = y'(0) = y''(L) = y'''(L) = 0$$

$$\begin{bmatrix} 16 & -9 & \frac{8}{3} & -\frac{1}{4} & & & & & \\ -4 & 6 & -4 & 1 & & & & & \\ 1 & -4 & 6 & -4 & 1 & & & & \\ & 1 & -4 & 6 & -4 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & -4 & 6 & -4 & 1 & \\ & & & & 1 & -4 & 6 & -4 & 1 \\ & & & & & \frac{16}{17} & -\frac{60}{17} & \frac{72}{17} & -\frac{28}{17} \\ & & & & & -\frac{12}{17} & \frac{96}{17} & -\frac{156}{17} & \frac{72}{17} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \frac{h^4}{EI} \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ \vdots \\ \vdots \\ f(x_{n-1}) \\ f(x_n) \end{bmatrix}$$

# Systems of Linear Equations

## Motivation: from Electrical Circuits




**Kirchhoff's laws of electrical circuits:**

$$
\begin{aligned}
5i_1 + 5i_2 &= V, \\
i_3 - i_4 - i_5 &= 0, \\
2i_4 - 3i_5 &= 0, \\
i_1 - i_2 - i_3 &= 0, \\
5i_2 - 7i_3 - 2i_4 &= 0.
\end{aligned}
$$

# Systems of Linear Equations

- **References**

**[1] Timothy Sauer, Numerical analysis (2nd ed.), Pearson Education, 2012. Chapter 2**

**[2] Cleve Moler, Numerical Computing with MATLAB, Society for Industrial and Applied Mathematics, 2004. Chapter 2**

**[2] Richard L. Burden, J. Douglas Faires, Numerical analysis (9th ed.), Brooks/Cole, 2011. Chapter 6**

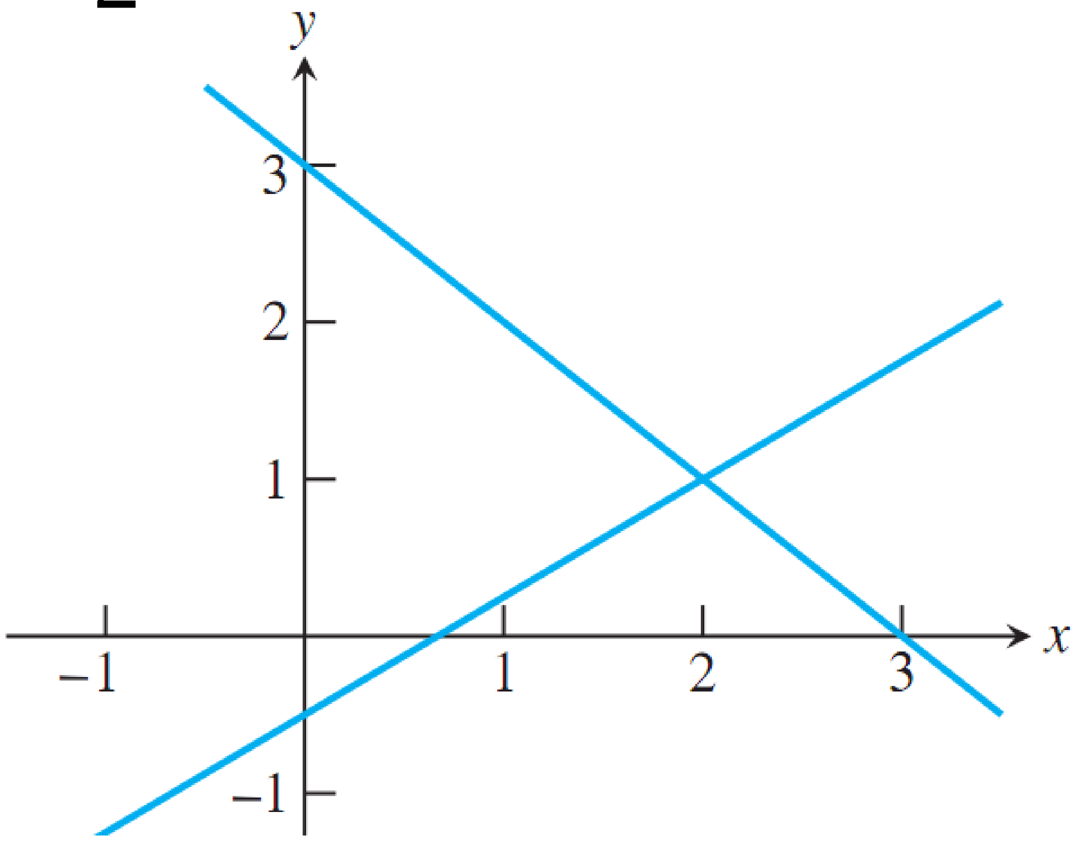
# Systems of Linear Equations

- **Direct Methods**
  - Gaussian Elimination
  - The LU Factorization
  - Sources of Error
  - The PA= LU Factorization
- **Iterative Methods**
  - Jacobi Method
  - Gauss–Seidel Method

# Direct Methods

## Gaussian Elimination: Example 1

$$\begin{cases} x + y = 3 \\ 3x - 4y = 2 \end{cases}$$



# Direct Methods

## Gaussian Elimination: Example 1

$$\begin{cases} x + y = 3 \\ 3x - 4y = 2 \end{cases}$$

**Subtracting 3 · [x + y = 3] from the second equation**

$$\begin{cases} x + y = 3 \\ -7y = -7 \end{cases}$$

$$-7y = -7 \longrightarrow y = 1$$

$$x + y = 3 \longrightarrow x + (1) = 3 \longrightarrow x = 2$$

# Direct Methods

- **Gaussian Elimination: Example 1**

$$\begin{cases} x + y = 3 \\ 3x - 4y = 2 \end{cases}$$

**The same elimination work can be done in the absence of variables by writing the system in tableau form:**

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & -4 & 2 \end{array}\right] \longrightarrow \begin{array}{c}\text{subtract } 3 \times \text{ row } 1 \\ \text{from row } 2\end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -7 & -7 \end{array}\right]$$

# Direct Methods

## Gaussian Elimination: Example 2

$$\begin{aligned} x + 2y - z &= 3 \\ 2x + y - 2z &= 3 \\ -3x + y + z &= -6 \end{aligned} \quad \Longleftrightarrow \quad \left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 2 & 1 & -2 & 3 \\ -3 & 1 & 1 & -6 \end{array}\right]$$

**To eliminate column 1:**

$$\longrightarrow \begin{array}{c}\text{subtract } 2 \times \text{row } 1 \\ \text{from row } 2\end{array} \longrightarrow \left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ -3 & 1 & 1 & -6 \end{array}\right]$$

$$\longrightarrow \begin{array}{c}\text{subtract } -3 \times \text{row } 1 \\ \text{from row } 3\end{array} \longrightarrow \left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ 0 & 7 & -2 & 3 \end{array}\right]$$

# Direct Methods

## Gaussian Elimination: Example 2

$$\longrightarrow \begin{array}{c}\text{subtract } -3 \times \text{ row } 1\\ \text{from row } 3\end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ 0 & 7 & -2 & 3 \end{array}\right]$$

**To eliminate column 2:**

$$\longrightarrow \begin{array}{c}\text{subtract } -\frac{7}{3} \times \text{ row } 2\\ \text{from row } 3\end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

$$\begin{aligned} x + 2y - z &= 3 \\ -3y &= -3 \\ -2z &= -4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= 3 - 2y + z \\ -3y &= -3 \\ -2z &= -4 \end{aligned}$$

# Direct Methods

## Gaussian Elimination: Operation Counts

**For any positive integer *n*,**

$$1 + 2 + 3 + 4 + \cdots + n = n(n+1)/2$$

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = n(n+1)(2n+1)/6$$

# Direct Methods

## Gaussian Elimination: Operation Counts

The general form of the tableau for *n* equations in *n* unknowns is

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right]$$

To carry out the elimination step:

```
for j = 1 : n-1
  for i = j+1 : n
    eliminate entry a(i,j)
  end
end
```

# Direct Methods

## Gaussian Elimination: Operation Counts

$$\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \end{array}$$

**to eliminate the $a_{21}$ entry**

$$\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & \dots & a_{2n} - \frac{a_{21}}{a_{11}} a_{1n} & b_2 - \frac{a_{21}}{a_{11}} b_1 \end{array}$$

**this requires one division (to find the multiplier $a_{21}/a_{11}$), plus $n$ multiplications and $n$ additions**

**Eliminating each entry $a_{i1}$ in the first column uses $2n + 1$ operations.**

# Direct Methods

## Gaussian Elimination: Operation Counts

**The row operation used to eliminate entry $a_{ij}$ is**

$$
\begin{array}{ccccccc}
0 & 0 & a_{jj} & a_{j,j+1} & \dots & a_{jn} & | & b_j \\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & | & \vdots \\
0 & 0 & 0 & a_{i,j+1} - \dfrac{a_{ij}}{a_{jj}} a_{j,j+1} & \dots & a_{in} - \dfrac{a_{ij}}{a_{jj}} a_{jn} & | & b_i - \dfrac{a_{ij}}{a_{jj}} b_j
\end{array}
$$

**This requires one division, $n - j + 1$ multiplications, and $n - j + 1$ addition/subtractions.**

# Direct Methods

## Gaussian Elimination: Operation Counts

**The elimination of each $a_{ij}$ requires the following number of operations, including divisions, multiplication, and addition/subtractions:**

$$\begin{bmatrix}
0 & & & & & & \\
2n+1 & 0 & & & & & \\
2n+1 & 2(n-1)+1 & 0 & & & & \\
2n+1 & 2(n-1)+1 & 2(n-2)+1 & 0 & & & \\
\vdots & \vdots & \vdots & \ddots & \ddots & & \\
\vdots & \vdots & \vdots & & & & \\
2n+1 & 2(n-1)+1 & 2(n-2)+1 & \cdots & 2(3)+1 & 0 & \\
2n+1 & 2(n-1)+1 & 2(n-2)+1 & \cdots & 2(3)+1 & 2(2)+1 & 0
\end{bmatrix}$$

# Direct Methods

## Gaussian Elimination: Operation Counts

**Starting on the right, we total up the operations as:**

$$\sum_{j=1}^{n-1}\sum_{i=1}^{j} 2(j+1)+1 = \sum_{j=1}^{n-1} 2j(j+1)+j$$

$$= 2\sum_{j=1}^{n-1} j^2 + 3\sum_{j=1}^{n-1} j = 2\frac{(n-1)n(2n-1)}{6} + 3\frac{(n-1)n}{2}$$

$$= (n-1)n\left[\frac{2n-1}{3}+\frac{3}{2}\right] = \frac{n(n-1)(4n+7)}{6}$$

$$= \boxed{\frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n}$$

# Direct Methods

## Gaussian Elimination: Operation Counts

**After the elimination is completed, the tableau is:**

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & | & b_1 \\ 0 & a_{22} & \dots & a_{2n} & | & b_2 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ 0 & 0 & \dots & a_{nn} & | & b_n \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{nn}x_n &= b_n \end{aligned}$$

***where the $a_{ij}$ refer to the revised, not original, entries.***

# Direct Methods

- **Gaussian Elimination: Operation Counts**

**To carry out the back-substitution step:**

$$x_1 = \frac{b_1 - a_{12}x_2 - \cdots - a_{1n}x_n}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{23}x_3 - \cdots - a_{2n}x_n}{a_{22}}$$

$$\vdots$$

$$x_n = \frac{b_n}{a_{nn}}.$$

# Direct Methods

## Gaussian Elimination: Operation Counts

We start at the bottom and work our way up to the top equation.

Counting operations yields:

$$1 + 3 + 5 + \cdots + (2n - 1) = \sum_{i=1}^{n} 2i - 1$$

$$= 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = 2\frac{n(n+1)}{2} - n = n^2$$

# Direct Methods

## Gaussian Elimination: Operation Counts

**For a system of *n* equations in *n* variables**

- **Operation count for the elimination step of Gaussian elimination:**

$$\frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n$$

- **Operation count for the back-substitution step of Gaussian elimination:**

$$n^2$$

# Direct Methods

## Gaussian Elimination: Operation Counts

**Estimate the time required to carry out back substitution on a system of 500 equations in 500 unknowns, on a computer where elimination takes 1 second.**

$$\frac{(500)^2}{2(500)^3/3} = \frac{3}{2(500)} = 0.003 \text{ sec}$$

***Smaller powers of n in operation counts can often be safely neglected.***

# Direct Methods

- **LU Factorization: Example 1 revisited**

$$\begin{cases} x + y = 3 \\ 3x - 4y = 2 \end{cases}$$

⇕ ***in matrix form***

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

# Direct Methods

## LU Factorization: Example 1 revisited

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} \longrightarrow \boxed{\begin{array}{c}\text{subtract } 3 \times \text{row } 1 \\ \text{from row } 2\end{array}} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix} = U$$

**To store the multiplier 3 used in the elimination step:**

$$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

# Direct Methods

- **LU Factorization: Example 1 revisited**

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix}$$

$$LU = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} = A$$

# Direct Methods

## Gaussian Elimination: Example 2 revisited

$$\begin{aligned} x + 2y - z &= 3 \\ 2x + y - 2z &= 3 \\ -3x + y + z &= -6 \end{aligned} \iff \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 1 & -2 & 3 \\ -3 & 1 & 1 & -6 \end{array}\right]$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix} \xrightarrow{\substack{\text{subtract } 2 \times \text{row 1} \\ \text{from row 2}}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{\substack{\text{subtract } -3 \times \text{row 1} \\ \text{from row 3}}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 7 & -2 \end{bmatrix}$$

$$\xrightarrow{\substack{\text{subtract } -\frac{7}{3} \times \text{row 2} \\ \text{from row 3}}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = U$$

# Direct Methods

- **Gaussian Elimination: Example 2 revisited**

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -\frac{7}{3} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -\frac{7}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix}$$

# Direct Methods

- **LU Factorization**

**An $m \times n$ matrix $L$ is lower triangular if its entries satisfy $l_{ij} = 0$ for $i < j$ .**

**An $m \times n$ matrix $U$ is upper triangular if its entries satisfy $u_{ij} = 0$ for $i > j$ .**

# Direct Methods

## LU Factorization

**Fact #1**: Let $L_{ij}(-c)$ denote the lower triangular matrix whose only nonzero entries are 1's on the main diagonal and $-c$ in the $(i, j)$ position. Then $A \to L_{ij}(-c)A$ represents the row operation "subtracting c times row j from row i ."

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - ca_{11} & a_{22} - ca_{12} & a_{23} - ca_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

# Direct Methods

- **LU Factorization**

**Fact #2:** $L_{ij}(-c)^{-1} = L_{ij}(c)$

$$\begin{bmatrix} 1 & 0 & 0 \\ -c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Direct Methods

- **LU Factorization: Example 1 revisited**

**Using Facts #1 & 2:**

**The elimination step can be represented by**

$$L_{21}(-3)A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix}$$

**To multiply both sides on the left by** $L_{21}(-3)^{-1}$

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} = \underline{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix}}$$

***the LU factorization of A***

# Direct Methods

## LU Factorization

**Fact #3: The following matrix product equation holds**

$$\begin{bmatrix} 1 & & \\ c_1 & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ c_2 & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ & c_3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ c_1 & 1 & \\ c_2 & c_3 & 1 \end{bmatrix}$$

# Direct Methods

- **LU Factorization: Example 2 revisited**

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & \frac{7}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ 3 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = U$$

$$A = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -3 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & -\frac{7}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ 2 & 1 & \\ -3 & -\frac{7}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = LU$$

# Direct Methods

## LU Factorization: Application

**Once $L$ and $U$ are known, the problem $Ax = b$ can be written as $LUx = b$.**

**Define a new "auxiliary" vector $c = Ux$. Then back substitution is a two-step procedure:**

(a) Solve $Lc = b$ for $c$.
(b) Solve $Ux = c$ for $x$.

# Direct Methods

- **LU Factorization: Example 1 revisited**

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow c_1 = 3, c_2 = -7$$

$$\begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix} \Rightarrow x_2 = 1, x_1 = 2.$$

# Direct Methods

- **LU Factorization: Complexity**

**Suppose we need to solve a number of different problems with the same *A* and different *b*.**

$$Ax = b_1$$
$$Ax = b_2$$
$$\vdots$$
$$Ax = b_k$$

- **Classical Gaussian elimination:**

  **Approximately** $2kn^3/3$ **operations**

- **The LU approach:**

  **Approximately** $2n^3/3 + 2kn^2$ **operations**

# Direct Methods

- **Sources of Error: Error Magnification**

**The infinity norm, or maximum norm, of the vector $x = (x_1, \ldots, x_n)$ is**

$$||x||_\infty = \max |x_i|, i = 1, \ldots, n$$

**Let $x_a$ be an approximate solution of the linear system $Ax = b$. The residual is the vector**

$$r = b - Ax_a$$

**The backward error:** $||b - Ax_a||_\infty$

**The forward error:** $||x - x_a||_\infty$

# Direct Methods

## Sources of Error: Example 3

Find the forward and backward errors for the approximate solution [−1,3.0001] of the system

$$x_1 + x_2 = 2$$
$$1.0001x_1 + x_2 = 2.0001$$

The backward error is the infinity norm of the vector

$$b - Ax_a = \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3.0001 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 2.0001 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.0001 \\ 0.0001 \end{bmatrix}$$

which is 0.0001

# Direct Methods

## Sources of Error: Example 3

**Find the forward and backward errors for the approximate solution [−1,3.0001] of the system**

$$x_1 + x_2 = 2$$
$$1.0001x_1 + x_2 = 2.0001$$

**The forward error is the infinity norm of the difference**

$$x - x_a = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 3.0001 \end{bmatrix} = \begin{bmatrix} 2 \\ -2.0001 \end{bmatrix}$$

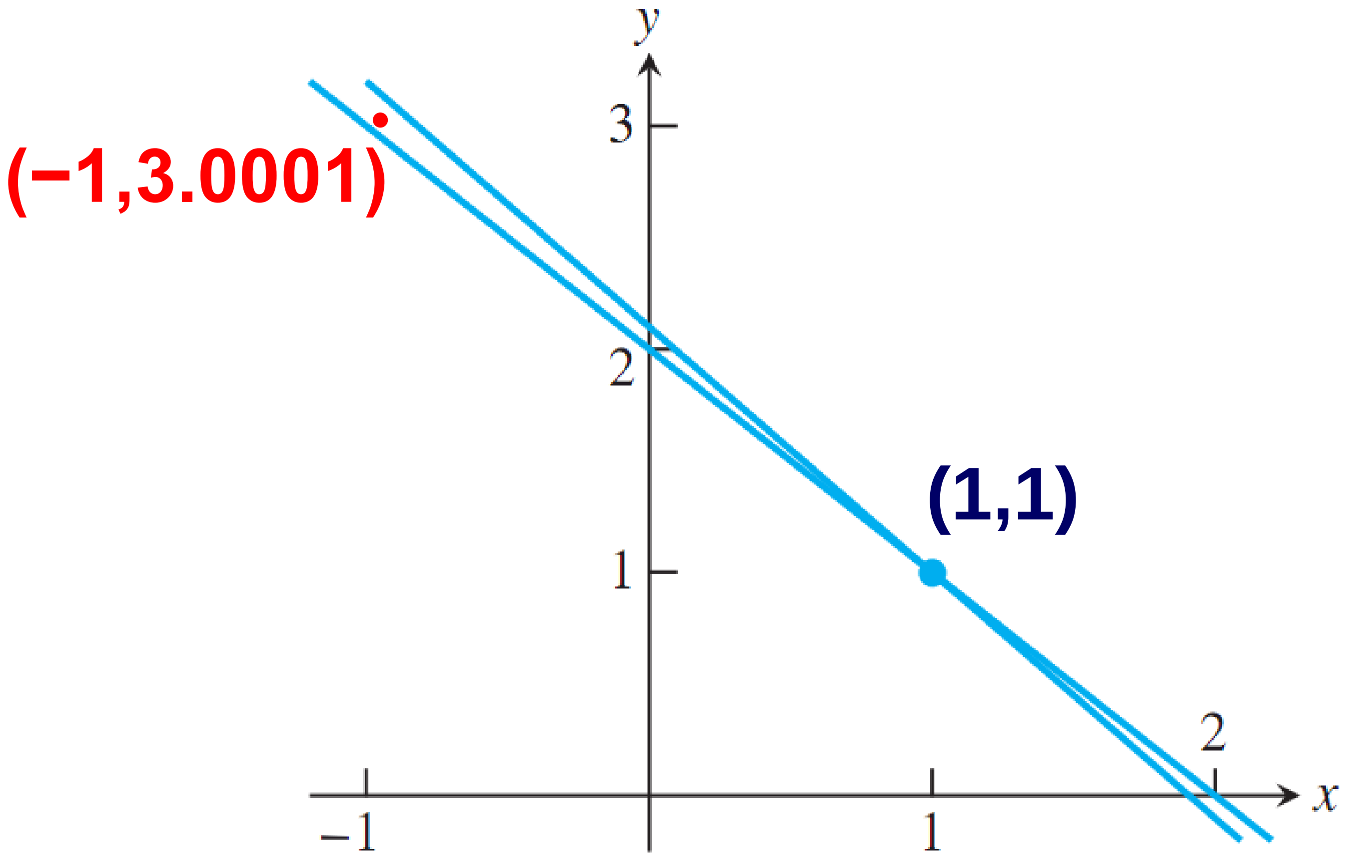
**which is 2.0001**

# Direct Methods

- **Sources of Error: Example 3**

$$x_1 + x_2 = 2$$

$$1.0001x_1 + x_2 = 2.0001$$



# Direct Methods

## Sources of Error: Error Magnification

Denote the residual by $r = b - Ax_a$.

The **relative backward error** of system $Ax = b$ is defined to be

$$\frac{||r||_\infty}{||b||_\infty}$$

and **the relative forward error** is

$$\frac{||x - x_a||_\infty}{||x||_\infty}$$

The **error magnification factor** for $Ax = b$ is

$$\frac{\text{relative forward error}}{\text{relative backward error}} = \frac{\dfrac{||x - x_a||_\infty}{||x||_\infty}}{\dfrac{||r||_\infty}{||b||_\infty}}$$

# Direct Methods

## Sources of Error: Example 3 revisited

$$x_1 + x_2 = 2$$

$$1.0001x_1 + x_2 = 2.0001$$

The relative forward error is

$$\frac{2.0001}{1} = 2.0001$$

The relative backward error is

$$\frac{0.0001}{2.0001} \approx 0.00005$$

The error magnification factor is

$$2.0001/(0.0001/2.0001) = 40004.0001$$

# Direct Methods

- **Sources of Error: Error Magnification**

**The condition number of a square matrix A, cond(A), is the maximum possible error magnification factor for solving Ax = b, over all right-hand sides b.**

**The condition number of the n × n matrix A is**

$$\mathrm{cond}(A) = ||A|| \cdot ||A^{-1}||$$

# Direct Methods

## Sources of Error: Matrix Norm

**Analogous to the norm of a vector, the matrix norm of an n × n matrix A is defined as**

$$||A||_\infty = \text{maximum absolute row sum}$$

**that is, total the absolute values of each row, and assign the maximum of these n numbers to be the norm of A.**

# Direct Methods

- **Sources of Error: Example 3 revisited**

$$x_1 + x_2 = 2$$

$$1.0001x_1 + x_2 = 2.0001$$

$$A = \begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix} \qquad ||A|| = 2.0001$$

$$A^{-1} = \begin{bmatrix} -10000 & 10000 \\ 10001 & -10000 \end{bmatrix} \qquad ||A^{-1}|| = 20001$$

$$\text{cond}(A) = (2.0001)(20001) = 40004.0001$$

***This is exactly the error magnification***

# Direct Methods

## Sources of Error: Example 4

$$10^{-20}x_1 + x_2 = 1$$
$$x_1 + 2x_2 = 4$$

**Exact solution**

$$\left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 1 & 2 & 4 \end{array}\right] \xrightarrow{\substack{\text{subtract } 10^{20}\times \text{ row 1} \\ \text{from row 2}}} \left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 0 & 2-10^{20} & 4-10^{20} \end{array}\right]$$

$$(2-10^{20})x_2 = 4-10^{20} \longrightarrow x_2 = \frac{4-10^{20}}{2-10^{20}}$$

$$10^{-20}x_1 + \frac{4-10^{20}}{2-10^{20}} = 1$$

$$x_1 = 10^{20}\left(1 - \frac{4-10^{20}}{2-10^{20}}\right) = \frac{-2\times 10^{20}}{2-10^{20}}.$$

# Direct Methods

## Sources of Error: Example 4

**The computer version of Gaussian elimination**

$$\left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 1 & 2 & 4 \end{array}\right] \longrightarrow \begin{array}{c}\text{subtract } 10^{20} \times \text{ row 1} \\ \text{from row 2}\end{array} \longrightarrow \left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 0 & 2-10^{20} & 4-10^{20} \end{array}\right]$$

$$-10^{20}x_2 = -10^{20} \longrightarrow x_2 = 1$$

**The machine arithmetic version of the top equation becomes**

$$10^{-20}x_1 + 1 = 1 \longrightarrow x_1 = 0$$

# Direct Methods

## Sources of Error: Example 4

**Repeat the computer version of Gaussian elimination, after changing the order of the two equations**

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 10^{-20} & 1 & 1 \end{array}\right] \longrightarrow \begin{array}{c}\text{subtract } 10^{-20} \times \text{ row 1} \\ \text{from row 2}\end{array}$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 - 2 \times 10^{-20} & 1 - 4 \times 10^{-20} \end{array}\right]$$

$$\longrightarrow \begin{aligned} x_1 + 2x_2 &= 4 \\ x_2 &= 1 \end{aligned}$$

**yield the computed solution $x_1$ = 2 and $x_2$ = 1.**

# Direct Methods

## The PA= LU Factorization

The key to this improvement is an efficient protocol for **exchanging rows of the coefficient matrix**, called partial pivoting.

The partial pivoting protocol consists of comparing numbers before carrying out each elimination step.

# Direct Methods

## The PA= LU Factorization

**At the start of Gaussian elimination, partial pivoting asks that we select the *p*th row, where**

$$|a_{p1}| \geq |a_{i1}|$$

**for all $1 \leq i \leq n$, and exchange rows 1 and *p*. The multiplier used to eliminate $a_{i1}$ will be**

$$m_{i1} = \frac{a_{i1}}{a_{11}} \qquad |m_{i1}| \leq 1$$

***The same check is applied to every choice of pivot during the algorithm.***

# Direct Methods

- **LU Factorization: Example 1 revisited**

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

**Since $|a_{21}| > |a_{11}|$, we exchange rows 1 and 2**

$$\left[\begin{array}{cc|c} 3 & -4 & 2 \\ 1 & 1 & 3 \end{array}\right] \longrightarrow \boxed{\begin{array}{c}\text{subtract } \frac{1}{3} \times \text{ row 1} \\ \text{from row 2}\end{array}} \longrightarrow \left[\begin{array}{cc|c} 3 & -4 & 2 \\ 0 & \frac{7}{3} & \frac{7}{3} \end{array}\right]$$

***When we solved this system the first time, the multiplier was 3***

# Direct Methods

## The PA= LU Factorization

**A permutation matrix is an $n \times n$ matrix consisting of all zeros, except for a single 1 in every row and column.**

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

# Direct Methods

## The PA= LU Factorization

**Let P be the n × n permutation matrix formed by a particular set of row exchanges applied to the identity matrix. Then, for any n × n matrix A, PA is the matrix obtained by applying exactly the same set of row exchanges to A.**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

# Direct Methods

## The PA= LU Factorization

**The PA = LU factorization is simply the LU factorization of a row-exchanged version of A.**

**Under partial pivoting, the rows that need exchanging are not known at the outset, so we must be careful about fitting the row exchange information into the factorization.**

# Direct Methods

- **The PA= LU Factorization: Example 5**

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} \longrightarrow \text{exchange rows 1 and 2} \longrightarrow \begin{bmatrix} 4 & 4 & -4 \\ 2 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix}$$

# Direct Methods

- **The PA= LU Factorization: Example 5**

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} \longrightarrow \text{exchange rows 1 and 2} \longrightarrow \begin{bmatrix} 4 & 4 & -4 \\ 2 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{array}{c} \text{subtract } \frac{1}{2} \times \text{ row 1} \\ \text{from row 2} \end{array} \longrightarrow \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{array}{c} \text{subtract } \frac{1}{4} \times \text{ row 1} \\ \text{from row 3} \end{array} \longrightarrow \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ \frac{1}{4} & 2 & 2 \end{bmatrix}$$

# Direct Methods

## The PA= LU Factorization: Example 5

$$\longrightarrow \begin{array}{c}\text{subtract } \frac{1}{4} \times \text{ row } 1\\ \text{from row } 3\end{array} \longrightarrow \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ \frac{1}{4} & 2 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \text{exchange rows 2 and 3} \longrightarrow \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -1 & 7 \end{bmatrix}$$

$$\longrightarrow \begin{array}{c}\text{subtract } -\frac{1}{2} \times \text{ row } 2\\ \text{from row } 3\end{array} \longrightarrow \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -\frac{1}{2} & 8 \end{bmatrix}$$

# Direct Methods

- **The PA= LU Factorization: Example 5**

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\underset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix}}$$

# Direct Methods

## The PA= LU Factorization: Application

**Once $P$, $L$ and $U$ are known, the problem $Ax = b$ can be written as $LUx = Pb$.**

**Define a new "auxiliary" vector $c = Ux$. Then back substitution is a two-step procedure:**

1. $Lc = Pb$ for $c$.
2. $Ux = c$ for $x$.

# Direct Methods

## The PA= LU Factorization: Example 5

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$$

```
>> A = [2,1,5;4,4,-4;1,3,1];
>> [L,U,P] = lu(A)
```

# Systems of Linear Equations

- Direct Methods
  - Gaussian Elimination
  - The LU Factorization
  - Sources of Error
  - The PA= LU Factorization
- Iterative Methods
  - Jacobi Method
  - Gauss–Seidel Method

# Iterative Methods

## Jacobi Method: Basic Idea

**The Jacobi Method is a form of fixed-point iteration for a system of equations**

**Solve the *i* th equation for the *i* th unknown. Then, iterate as in Fixed-Point Iteration, starting with an initial guess.**

# Iterative Methods

## Jacobi Method: Example 6

$$\begin{cases} 3u + v = 5 \\ u + 2v = 5 \end{cases}$$

**Begin by solving the first equation for *u* and the second equation for *v*.**

$$u = \frac{5 - v}{3}$$

$$v = \frac{5 - u}{2}$$

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

# Iterative Methods

## Jacobi Method: Example 6

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{5-v_0}{3} \\ \frac{5-u_0}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-0}{3} \\ \frac{5-0}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{5-v_1}{3} \\ \frac{5-u_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-5/2}{3} \\ \frac{5-5/3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ \frac{5}{3} \end{bmatrix}$$

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{5-5/3}{3} \\ \frac{5-5/6}{2} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{25}{12} \end{bmatrix}.$$

# Iterative Methods

## Jacobi Method: Example 7

$$\begin{cases} u + 2v = 5 \\ 3u + v = 5 \end{cases} \qquad \begin{cases} 3u + v = 5 \\ u + 2v = 5 \end{cases}$$

**Begin by solving the first equation for *u* and the second equation for *v*.**

$$u = 5 - 2v$$
$$v = 5 - 3u$$

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

# Iterative Methods

## Jacobi Method: Example 7

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 5 - 2v_0 \\ 5 - 3u_0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 5 - 2v_1 \\ 5 - 3u_1 \end{bmatrix} = \begin{bmatrix} -5 \\ -10 \end{bmatrix}$$

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 5 - 2(-10) \\ 5 - 3(-5) \end{bmatrix} = \begin{bmatrix} 25 \\ 20 \end{bmatrix}$$

***the iteration diverges***

# Iterative Methods

## Jacobi Method

The $n \times n$ matrix $A = (a_{ij})$ is **strictly diagonally dominant** if, for each $1 \le i \le n$,

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

Each main diagonal entry dominates its row in the sense that it is greater in magnitude than the sum of magnitudes of the remainder of the entries in its row.

# Iterative Methods

## Jacobi Method: Theory

**THEOREM 2.10**

**If the $n \times n$ matrix $A$ is strictly diagonally dominant, then**

**(1) $A$ is a nonsingular matrix**

**(2) for every vector $b$ and every starting guess, the Jacobi Method applied to $Ax = b$ converges to the (unique) solution.**

*Cf. P.107 of Ref. [1]*

# Iterative Methods

## Jacobi Method: Algorithm

**Let D denote the main diagonal of A, L denote the lower triangle of A (entries below the main diagonal), and U denote the upper triangle (entries above the main diagonal). Then A = L + D + U**

$$
\begin{aligned}
Ax &= b \\
(D + L + U)x &= b \\
Dx &= b - (L + U)x \\
x &= D^{-1}(b - (L + U)x)
\end{aligned}
$$

$$
\begin{aligned}
x_0 &= \text{ initial vector} \\
x_{k+1} &= D^{-1}(b - (L + U)x_k) \quad \textbf{for} \quad k = 0, 1, 2, \ldots .
\end{aligned}
$$

# Iterative Methods

## Jacobi Method: Example 6 revisited

$$\begin{cases} 3u + v = 5 \\ u + 2v = 5 \end{cases} \qquad \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} u_{k+1} \\ v_{k+1} \end{bmatrix} &= D^{-1}(b - (L + U)x_k) \\ &= \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} \left( \begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix} \right) \\ &= \begin{bmatrix} (5 - v_k)/3 \\ (5 - u_k)/2 \end{bmatrix} \end{aligned}$$

# Iterative Methods

- **Gauss–Seidel Method: Basic Idea**

**The only difference between Gauss–Seidel and Jacobi is that in the former, the most recently updated values of the unknowns are used at each step, even if the updating occurs in the current step.**

# Iterative Methods

- **Gauss–Seidel Method: Example 6 revisited**

$$\begin{cases} 3u + v = 5 \\ u + 2v = 5 \end{cases}$$

**Jacobi Method:** $\begin{bmatrix} u_{k+1} \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} (5 - v_k)/3 \\ (5 - u_k)/2 \end{bmatrix}$

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{5-v_0}{3} \\ \frac{5-u_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-0}{3} \\ \frac{5-5/3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{3} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{5-v_1}{3} \\ \frac{5-u_2}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-5/3}{3} \\ \frac{5-10/9}{2} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{35}{18} \end{bmatrix}$$

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{5-v_2}{3} \\ \frac{5-u_3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-35/18}{3} \\ \frac{5-55/54}{2} \end{bmatrix} = \begin{bmatrix} \frac{55}{54} \\ \frac{215}{108} \end{bmatrix}$$

# Iterative Methods

## Gauss–Seidel Method: Algorithm

**Let D denote the main diagonal of A, L denote the lower triangle of A (entries below the main diagonal), and U denote the upper triangle (entries above the main diagonal). Then A = L + D + U**

$$Ax = b$$

$$(D + L + U)x = b$$

$$(L + D)x_{k+1} = -Ux_k + b$$

$$x_0 = \text{initial vector}$$

$$x_{k+1} = D^{-1}(b - Ux_k - Lx_{k+1}) \quad \textbf{for} \ \ k = 0, 1, 2, \ldots$$

# Iterative Methods

## Gauss–Seidel Method: Example 8

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 4 & 1 \\ -1 & 2 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

Solution: [2,−1,1]

$$u_{k+1} = \frac{4 - v_k + w_k}{3}$$

$$v_{k+1} = \frac{1 - 2u_{k+1} - w_k}{4}$$

$$w_{k+1} = \frac{1 + u_{k+1} - 2v_{k+1}}{5}$$

**Starting with**

$$[u_0, v_0, w_0] = [0, 0, 0]$$

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} \frac{4-0-0}{3} = \frac{4}{3} \\ \frac{1-8/3-0}{4} = -\frac{5}{12} \\ \frac{1+4/3+5/6}{5} = \frac{19}{30} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} \approx \begin{bmatrix} 1.6833 \\ -0.7500 \\ 0.8367 \end{bmatrix}$$

# Summary

- **Direct Methods**
  - ✓ Gaussian Elimination
  - ✓ The LU Factorization
  - ✓ Sources of Error
  - ✓ The PA= LU Factorization
- **Iterative Methods**
  - ✓ Jacobi Method
  - ✓ Gauss–Seidel Method

# Thank You !