CLASS101 HOMEWORK SET #1

JEREMY BEJARANO

Exercise 1. Suppose we choose any feasible capital policy, that is, any function g_0 satisfying $0 \leq g_0(k) \leq f(k)$, for all $k \geq 0$. The lifetime utility yielded by this policy, as a function of the initial capital stock k_0, is

$$w_0 = \sum_{t=0}^{\infty} \beta^t U(f(k_t) - g_0(k_t)),$$

where

$$k_{t+1} = g_0(k^t),\ t = 0, 1, 2, ...$$

Show that $w_0 = U(f(k) - g_0(k)) + \beta w_o(g_0(k))$, for all $k \geq 0$. (From *Recursive Methods in Economic Dynamics* by Stokey, Lucas, and Prescott with a solution by Irigoyen, Rossi-Hansberg, and Wright.)

Proof. Given $k_0 = k$, the lifetime utility given by the sequence $\{k_1\}_{t=1}^{\infty}$ in which $k_{t+1} = g_0(k_t)$ is

$$\begin{aligned} w_0(k) &= \sum_{t=0}^{\infty} \beta^t u(f(k_t) - g_0(k_t)) \\ &= u(f(k) - g_0(k) + \beta \sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - g_0(k_t)). \end{aligned}$$

But

$$\begin{aligned} \sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - g_0(k_t)) &= \sum_{t=0}^{\infty} \beta u(f(_{t+1}) - g_0(k_{t+1})) \\ &= w_0(k_1) \\ &= w_0(g_0(k)). \end{aligned}$$

Hence

$$w_o = u(f(k) - g_0(k)) + \beta w_0(g_0(k))$$

for all $k \geq 0$.

$\square$

Exercise 2. Let G be any group. Prove that $\phi : G \to G$ where $g \mapsto g^{-1}$ is a homomorphism if and only if G is abelian.

Proof. Suppose that ϕ is a homomorphism. So, if $a, b \in G$, then

$$\begin{aligned} \phi(ab) &= \phi(a)\phi(b) \\ (ab)^{-1} &= a^{-1} \cdot b^{-1} \\ b^{-1}a^{-1} &= a^{-1} \cdot b^{-1}. \end{aligned}$$

Taking the inverse of both sides (valid because G is a group), we get

$$ab = ba.$$

Thus, G is abelian.

Conversely, suppose that G is abelian. By definition, given $a, b \in G$, we have $ab = ba$. This implies that

$$\begin{aligned} ab &= ba \\ (ab)^{-1} &= (ba)^{-1} \\ \phi(ab) &= a^{-1}b^{-1} \\ &= \phi(a)\phi(b) \end{aligned}$$

This satisfies the definition and, thus, ϕ is a homomorphism. (Note that in this proof we inverted the product such that $(ab)^{-1} = b^{-1}a^{-1}$. This is a basic property of all groups.)

□

Exercise 3. Setup but do not solve the following problem. Marie recieves 1 large cake for her birthday and must decide how much of it to eat today (period 1), versus how much to eat tomorrow (period 2). She has a discount factor of $\delta = 0.7$, and a period utility function of $\ln b$, where b is the fraction of the cake that is consumed that period. She can store the cake in the fridge overnight, but half of whatever is stored mysteriously disappears (her husband is likely to blame).

Proof. Setup the problem as follows

$$\max_{b_1, b_2, s} \ln(b_1) + 0.6 \ln(b_2) \text{ s.t. } b_1 + s \leq 1 \text{ and } b_2 \leq \frac{1}{2}s.$$

□

Exercise 4.

(a) Order of each element...

Order of each element in $\mathbb{Z}_{12}$.

Element in $\mathbb{Z}_{12}$	Order of Element
element 0	order 1
element 1	order 12
element 2	order 6
element 3	order 4
element 4	order 3
element 5	order 12
element 6	order 2
element 7	order 12
element 8	order 3
element 9	order 4
element 10	order 6
element 11	order 12

Order of each element in $\mathbb{Z}_{15}$.

Element in $\mathbb{Z}_{15}$	Order of Element
element 0	order 1
element 1	order 15
element 2	order 15
element 3	order 5
element 4	order 15
element 5	order 3
element 6	order 5
element 7	order 15
element 8	order 15
element 9	order 5
element 10	order 3
element 11	order 15
element 12	order 5
element 13	order 15
element 14	order 15

(b) Find the order of each element...

Order of each element in U_{12}.

Element in U_{12}	Order of Element
element 1	order 1
element 5	order 5
element 7	order 7
element 11	order 11

Order of each element in U_{15}.

Element in U_{15}	Order of Element
element 1	order 1
element 2	order 8
element 4	order 4
element 7	order 13
element 8	order 2
element 11	order 11
element 13	order 7
element 14	order 14

(c) Compute the order of ...

Note that the order of a permutation in S_n is given by the least common multiple of the lengths of the disjoint cycles (see section 7.9, exercise 13).

Because (1 12 8 10 4)(2 13)(5 11 7)(6 9) is comprised of disjoint cycles, we can calculate the order by calculating the least common multiple,

$$lcm(5, 2, 3) = 30$$

.

(d) Compute ...

(1 3 5 7 9)(2 4 6)(1 3 6 9)(2 4 6 8)(1 4 8)(2 5 7 9) = (5 9 6 8)(2 7 3)