The Shapley Value

The Shapley value gives one specific set of payments for coalition members, which are deemed fair. The problem with identifying fairness in characteristic form games is best illustrated by an example. Figure 1 shows a game for two players.

S	$v(S)$
()	0
(1)	1
(2)	3
(12)	6

Figure 1. *Example game. If the agents form coalition (12) then how much utility should each one get?*

Clearly, we should choose the coalition (12) as it has the highest value. Now we must decide how much each agent should get. The simplest solution is to divide the total of 6 evenly amongst the coalition members, so that each agent gets 3. This seems unfair to agent 2 because agent 2 could have gotten 3 by simply staying on its own coalition (2). It seems like the fair thing to do is to give each agent a payment that is proportional to the value it contributes to the coalition, that is, the amount that value increases by having the agent in the coalition. But, how do we extend this idea to cases with more than 2 agents?

Shapley was able to extend this idea by realizing that each agent should get a payment that corresponds to its marginal contribution to the final value. An agent's marginal contribution to a coalition is the difference between the value before the agent joins the coalition and after he joined. For example, if before you join Initech their annual profits are \$10M but after you are there for a year they increase to \$11M then you can claim that your marginal contribution is to Initech is \$1M assuming, of course, that everything else stays the same during that year.

The one remaining problem is that there are many different orderings in which n agents could have joined the coalition, namely, there are n!

orderings of n elements. The Shapley value simply averages over all possible orderings. That is, the Shapley value gives each agent a utility proportional to its average marginal contribution to every possible coalition, in every possible order it could have been formed.

More formally, we define the Shapley value as follows. Let $B(\pi, i)$ be the set of agents in the agent ordering π which appear before agent i. The Shapley value for agent i given A agents is given by:

$$\phi(A, i) = \frac{1}{A!} \sum_{\pi \in \Pi_A} v(B(\pi, i) \cup i) - v(B(\pi, i)),$$

where Π_A is the set of all possible orderings of the set A. Another way to express the same formula is:

$$\phi(A, i) = \sum_{S \subseteq A} \frac{(|A| - |S|)!\,(|S| - i)!}{|A|!} [v(S) - v(S - \{i\})].$$

Notice that the Shapley values are calculated for a particular coalition A in the definition above. They are not meant as a way of determining which is the best coalition structure. They can only be used to distribute the payments of a coalition once it is formed.

Let's calculate the Shapley values for the game in figure 1 and the grand coalition (12). Since there are only two agents it means that there are only two possible orderings: (12) and (21). As such we have that:

$$\begin{aligned}
\phi(\{1,2\},1) &= \frac{1}{2} \cdot (v(1) - v() + v(21) - v(2)) \\
&= \frac{1}{2} \cdot (1 - 0 + 6 - 3) = 2 \\
\phi(\{1,2\},2) &= \frac{1}{2} \cdot (v(12) - v(1) + v(2) - v()) \\
&= \frac{1}{2} \cdot (6 - 1 + 3 - 0) = 4
\end{aligned}$$

A somewhat surprising and extremely useful characteristic of the Shapley value is that it is always feasible. In our example the payments of 4 and 2 add up to 6 which is the same value we get in the grand coalition (12). Another nice feature of the Shapley value is that it always exists and is unique. Thus, we do not have to worry about coordination mechanism to choose among different payments. A final interesting result is that the Shapley value might not be in the core, even for cases where the core exists. This is a potential problem as it means that the resulting payments might not be stable and some agents might choose to leave the coalition in order to receive a higher payment on a different coalition.

Unfortunately, while the Shapley value has some very attractive theoretical properties, it does have some serious drawbacks when we try to use it for building multiagent systems. The biggest problem is computational. The Shapley value requires us to calculate at least $2^{|A|}$ orderings, this is only possible for very small sets A. It also requires that we know the value of v for every single subset 5. In many real-world applications the calculation of v is complex. For example, it might require simulating how a particular coalition of agents would work together. These complex calculations could dramatically increase the total time. Finally, the Shapley value does not give us the actual coalition structure. Thus, it only solves the second part of the coalition formation problem. We must still determine which coalition the agents will form and how they will do it.

We will illustrate the method on another game:

$$v(A) = v(B) = v(C) = 0$$
$$v(AB) = 2 \qquad v(AC) = 4 \qquad v(BC) = 6$$
$$v(ABC) = 7,$$

The calculation is as follows:

Order	Value added by A	B	C
ABC	0	2	5
ACB	0	3	4
BAC	2	0	5
BCA	1	0	6
CAB	4	3	0
CBA	1	6	0
	8	14	20

$$\varphi = \tfrac{1}{6}(8, 14, 20) = (1\tfrac{1}{3}, 2\tfrac{1}{3}, 3\tfrac{1}{3}).$$

For an example of how to arrive at the numbers in the table, consider the order BCA. The value added by each player is calculated as follows:

$$\begin{aligned} &\text{B:} \quad v(\text{B}) - v(\phi) = 0 - 0 = 0 \\ &\text{C:} \quad v(\text{BC}) - v(\text{B}) = 6 - 0 = 6 \\ &\text{A:} \quad v(\text{ABC}) - v(\text{BC}) = 7 - 6 = 1. \end{aligned}$$

We can think of the Shapley value of player *i* in a game as the average amount that player *i* contributes when the grand coalition forms, given that all orders of coalition formation are equally likely. It would seem fair to give player *i* this average contribution.

References

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