Extra Credit

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Chapter 3

Def 3.1.1

Let f be a fn. w/ doman $D \subseteq \mathbb{R}$. Then f has a limit as x approaches infinity iff $\exists L \in \mathbb{R}$ s.t. for every $\mathcal{E} > 0, \exists M \in \mathbb{R}^+$ s.t. $|f(x) - L| < \mathcal{E}$, if $x \geq M$ and $x \in D$. If such an L exists, then L is called the limit of the fn f as x tends to infinity and we write $\lim_{x\to\infty} f(x) = L$

Def 3.1.2

If $\lim_{x\to\infty} f(x) = L$, then the line $y = L$ is called a horizontal asymptote for the function f.

Thm 3.1.6

Suppose that $D \subseteq \mathbb{R}$ is an unbounded above domain of the function f ; that is, D contains arbitrarily large values. Then, $\lim_{x\to\infty} f(x) = L$ iff for every sequence $\{x_n\}$ in D that diverges to plus infinity, that is, $\lim_{n\to\infty} x_n = \infty$, the sequence $\{f(x_n)\}$ converges to L.

Thm 3.1.7

Suppose that the functions f, g, and h are defined on $D \subseteq \mathbb{R}$, which is unbounded above, with $\lim_{x\to\infty} f(x) = A, \lim_{x\to\infty} g(x) = B$, and $\lim_{x\to\infty} h(x) = C$. Then

(a) $\lim_{x\to\infty} f(x)$ is unique

(b) f must be eventually bounded above and below

(c) $\lim_{x\to\infty}[f(x) - A] = 0$

(d) $\lim_{x\to\infty} |f(x)| = |\lim_{x\to\infty} f(x)| = |A|$

(e) $\lim_{x\to\infty}(f \pm g)(x) = \lim_{x\to\infty} f(x) \pm \lim_{x\to\infty} g(x) = A \pm B$

(f) $\lim_{x\to\infty}(fg)(x) = [\lim_{x\to\infty} f(x)][\lim_{x\to\infty} g(x)] = AB$

(g) $\lim_{x\to\infty}[f(x)]^n = [\lim_{x\to\infty} f(x)]^n = A^n, \forall n \in \mathbb{N}$

(h) $\lim_{x\to\infty} (f/g)(x) = \dfrac{A}{B}\ if B \neq 0$

(i) $\lim_{x\to\infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to\infty} f(x)} = \sqrt[n]{A}$ if $A \geq 0$ & $f(x) \geq 0\ \forall x \in D$, with $n \in \mathbb{N}$

(j) $A \leq B\ if\ f(x) \leq g(x)$ eventually for $x \in D$

(k) $A \leq B \leq C$ if $f(x) \leq g(x) \leq h(x)$ eventually for $x \in D$. This property is called the sandwich (or squeeze) theorem

Thm 3.1.8

If the function f is defined on an unbounded above domain $D \subseteq \mathbb{R}$ and is eventually monotone and eventually bounded, then $\lim_{x\to\infty} f(x)$ is finite.

Def 3.1.9

Let f be a function with domain $D \subseteq \mathbb{R}$, which contains arbitrarily large values. We say that f tends to plus infinity as x tends to $+\infty$ iff for any real $K > 0$, there exists a real number $M > 0$ such that $f(x) > K$ provided that $x \geq M$ and $x \in D$. Whenever this is the case, we write $\lim_{x\to\infty} f(x) = +\infty$

Def 3.1.10

Let f be a function with domain $D \subseteq \mathbb{R}$, which contains arbitrarily large negative values. Then $\lim_{x\to-\infty} f(x) = L$ iff for every $\epsilon > 0$ there exists a real number $M > 0$ such that $|f(x) - L| < \epsilon$ if $x \leq M$ and $x \in D$

Def 3.2.1

Suppose that a function $f : D \to \mathbb{R}$, and suppose that a is an accumulation point of D. The function f has a limit as x approaches (or as x tends to) a iff there exists a real number L such that for every $\epsilon > 0$ there exists a real number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ provided that } 0 < |x - a| < \delta \text{ and } x \in D$$

write $\lim_{x \to a} f(x) = L$

Thm 3.2.5

Suppose that functions $f, g, h : D \to \mathbb{R}$, with $D \subseteq \mathbb{R}$, a is an accumulation point of D,$\lim_{x \to a} f(x) = A$,$\lim_{x \to a} g(x) = B$, and $\lim_{x \to a} h(x) = C$. Then all of the conclusions for Theorem 3.1.7 are true with ∞ replaced by a and with "eventually" replaced by "near $x = a$."

Thm 3.2.6

Let the function f be defined on some deleted neighborhood D of the real number a. The following two statements are equivalent.

(a) $\lim_{x \to a} f(x) = L$

(b) For every sequence $\{x_n\}$ converging to $x = a$, with $x_n \in D$ and $x_n \neq a$ eventually, the sequence $\{f(x_n)\}$ converges to L

Def 3.2.12

Suppose that the function $f : D \to \mathbb{R}$ with D a subset of $\mathbb{R}$ and a an accumulation point of D. Then the function f tends to plus infinity as x approaches, tends to, a iff for any given real number $K > 0$, there exists $\delta > 0$ such that $f(x) > K$, provided that $0 < |x - a| < \delta$ and $x \in D$. Write $\lim_{x \to a} f(x) = +\infty$

Thm 3.2.14

Let the functions f and g be defined on some deleted neighborhood of $x = a$. If $\lim_{x \to a} f(x) = L > 0$ and $\lim_{x \to a} g(x) = +\infty$, then $\lim_{x \to a}(fg)(x) = +\infty$.

Def 3.3.1

Suppose that the function $f : D \to \mathbb{R}$, with D a subset of $\mathbb{R}$ and a an accumulation point of the set $D \cap (a, \infty) = \{x \in D | x > a\}$ Then the function f has a right-hand limit (limit from the right) as x approaches, tends to, a iff there exists a real number L such that for every $\epsilon > 0$ there exists a positive real number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ provided that } 0 < x - a < \delta \text{ and } x \in D$$

we write $\lim_{x \to a^+} f(x) = L$

Def 3.3.2

Suppose that the function $f : D \to \mathbb{R}$, with D a subset of and a an accumulation point of $D \cap (a, \infty)$. Then the function f tends to infinity as x approaches, tends to, a from the right iff for any given real number $K > 0$, there exists a positive $\delta > 0$ such that $f(x) > K$, provided that $0 < x - a < \delta$ and $x \in D$. We write $\lim_{x \to a^+} f(x) = +\infty$

Def 3.3.4

If the limit from the right or from the left at $x = a$ of a function f is infinite, meaning $+\infty$or $-\infty$, then the line $x = a$ is called a vertical asymptote.

Thm 3.3.7

Let a function f be defined for $x \in (0, a)$, with $a > 0$ a real number. If

$$\lim_{x \to 0^+} f(x) \text{ or } \lim_{t \to \infty} f\left(\frac{1}{t}\right)$$

Chapter 4

Def 4.1.1

(Local) Suppose that a function $f : D \to \mathbb{R}$, with D a subset of $\mathbb{R}$. Then f is continuous at $a \in D$ iff for any given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$, provided that $|x - a| < \delta$ and $x \in D$.

Def 4.1.2

(Global) A function $f : D \to \mathbb{R}$ is continuous on a set $E \subseteq D$ iff f is continuous at each point (value of x) in E. If f is continuous at every point in its domain, D, we simply say that f is continuous.

Sequential Criterion for Continuity Thm

Let $f : D \to \mathbb{R}$. f is continuous at A in domain D iff for every seq $\{a_n\} \subseteq D, \lim_{n\to\infty} a_n = A$, then seq $\lim_{n\to\infty} f(A_n) = f(A)$

Def 4.1.6

Suppose a function $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$. Then f is right continuous at a, meaning that f is continuous from the right at a iff for any given $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$, provided that $0 < x - a < \delta$ and $x \in D$.

Thm 4.1.7

Suppose that D is the domain of f

(a) If f is continuous at a, then there exists $\delta > 0$ such that f is bounded on the set $(a - \delta, a + \delta) \cap D$.

(b) If f is right continuous at a, then there exists $\delta > 0$ such that f is bounded on the set $[a, a + \delta) \cap D$.

(c) If f is left continuous at a, then there exists $\delta > 0$ such that f is bounded on the set $(a - \delta, a] \cap D$.

(d) If f is continuous at a, and $f(a) > 0$, then there exists $\delta > 0$ such that $f(x) > \frac{1}{2}f(a)\forall x \in (a - \delta, a + \delta) \cap D$, or $f(a) < 0 f(x) < \frac{1}{2}f(a)$.

(e) Suppose that $D = (a, b)$, f is continuous at $c \in D$, and $f(c) > 0$. Then there exists a neighborhood N_ϵ of c such that $f(x) > 0\ \forall x \in N_\epsilon \cap (a, b)$.

Thm 4.1.8

Suppose that functions $f, g : D \to \mathbb{R}$ with $D \subset \mathbb{R}$ are continuous at a. Then,

(a) $f \pm g$ are continuous at a

(b) fg is continuous at a

(c) $\dfrac{f}{g}$ is continuous at a, provided that $g(a) \neq 0$

Thm 4.1.9

Consider functions $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ with $A, B \subseteq \mathbb{R}$ such that $f(A) \subseteq B$. If f is continuous at some $x = a \in A$ and g is continuous at $b = f(a) \in B$, then the function $g \circ f$ is continuous at $x = a$

Def 4.2.1

A function $g : E \to \mathbb{R}$ with $E \subseteq R$ is an extension of the function $f : D \to \mathbb{R}$ provided that $D \subset E$ and $f(x) = g(x)\ \forall x \in D$. If g is continuous, then g is called a continuous extension of f.

Def 4.2.3

A function $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$ and a an accumulation point of D has a removable discontinuity at $x = a$ if either

(a) $a \notin D$ and $\lim_{x\to a} f(x)$ is finite, or

(b) $a \notin D$ and $\lim_{x\to a} f(x) = L \neq f(a)$

Def 4.2.5

Suppose that fo ra function $f : D \to \mathbb{R}$, one of the following three conditions is satisfied

(a) $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = M$

(b) a is not an accumulation point of $D \cap (a, \infty), a \in D$ and $\lim_{x \to a^-} f(x) = M$. In this case, L will denote the value of $f(a)$

(c) a is not an accumulation point of $D \cap (-\infty, a), a \in D$ and $\lim_{x \to a^+} f(x) = L$. In this case, M will denote the value of $f(a)$

Def 4.2.7

The function f is piecewise continuous on $D \subseteq \mathbb{R}$ iff there exists finitely many points $x_1, x_2, \ldots, x_n$, such that

(a) f is continuous on D except at $x_1, x_2, \ldots, x_n$, and

(b) f has simple discontinuities at $x_1, x_2, \ldots, x_n$

Def 4.3.1

A set $E \subseteq \mathbb{R}$ is said to be closed iff every accumulation point of E is in E.

Def 4.3.2

A set $E \subseteq \mathbb{R}$ is said to be open iff for each $x \in E$ there exists a neighborhood I of x such that I is entirely contained in E.

Thm 4.3.3

A set $E \subseteq \mathbb{R}$ is closed iff $\mathbb{R} \backslash E$ is open.

Thm 4.3.4

If a function f is continuous on a closed and bounded interval $[a, b]$, then f is bounded on $[a, b]$.

Thm 4.3.5

(Extreme Value Theorem) If f is a continuous function on an interval $[a, b]$, then f attains its maximum and minimum values on $[a, b]$.

Thm 4.3.6

(Bolzano's Intermediate Value Theorem) If a function f is continuous on $[a, b]$ and if k is a real number between $f(a)$ and $f(b)$, then there exists a real number $c \in (a, b)$ such that $f(c) = k$.

Def 4.3.7

A function $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$ satisfies the intermediate value property on D iff for every $x_1, x_2 \in D$ with $x_1 < x_2$ and any real constant k between $f(x_1)$ and $f(x_2)$ there exists at least one constant $c \in (x_1, x_2)$ such that $f(c) = k$.

Cor 4.3.8

Any polynomial of odd degree has at least one real root.

Cor 4.3.9

If a function $f : [a, b] \to \mathbb{R}$ is nonconstant and continuous, then the range of f is an interval $[c, d]$ with $c, d \in \mathbb{R}$.

Thm 4.3.10

(Brouwer's Fixed-Point Theorem) If a function $f : [a, b] \to [a, b]$, is continuous, then f has at least one fixed point; that is, there exists at least one real number $p \in [a, b]$ such that $f(p) = p$.

Thm 4.3.11

If a function $f : D \to \mathbb{R}$ is a continuous injection and $D = [a, b]$, then $f^{-1} : R_f \to D$ is continuous.

Def 4.4.2

A fn $f : D \to \mathbb{R}$ is continuous on a set $E \subseteq D$ iff f is continuous @ each pt. in E

Def 4.4.1

f is continuous @ $a \in D$, iff given $\epsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(a)| < \epsilon$ provided that $|x - a| < \delta$ & $x \in D$

Def 4.4.3

A fn $f : D \to \mathbb{R}$ is uniformly continuous on a set $E \subseteq D$ iff given $\epsilon > 0, \exists \delta > 0$ s.t. on $|f(x) - f(t)| < \epsilon \ \forall x, t \in E$ satisfying $|x - t| < \delta$

Thm 4.4.6

If $D \subseteq \mathbb{R}$ is a closed & a bounded set, & a fn $f : D \to \mathbb{R}$ is continuous, then f is uniformly continuous.

Def 4.4.6

A fn $f : D \to \mathbb{R}$ w $F \subseteq \mathbb{R}$ is a Lipschitz fn iff $\exists L > 0$ s.t.

$$|f(x) - f(t)| \leq L|x - t| \ \forall x, t \in D$$

Thm 4.4.11

If a fn is a Lipschitz fn, then it is uniformly continuous

Thm : Uniform Continuous to Cauchy

If $f : D \to \mathbb{R}$ is uniformly continuous & if $\{x_n\}$ is a cauchy sequence in D, then $\{f(x_n)\}$ is a cauchy sequence in $\mathbb{R}$