Chapter 1

Logan Rosentreter

October 4, 2021

PROBABILITY

Intro

Probability is a measure of one's belief in the occurrence of future events

e.g.

a policy holder making a claim in the next year

How to assign probabilities?

1. subjectively
2. relative frequency
   - repeat the study condition (identical) for M times
   - record times event occurred (m)
   - probability = relative frequency = m / M where $\lim_{M\to\infty} m/M = P(A)$
3. Axiomatic Model Based
   - use mathematical theory to assign probabilities (what we will learn)

Notation

Def : A random experiment is an experiment that produces outcomes that can't be predicted w/ certainty

Def : A performance of such experiment : trial

Def : A result : Outcome

Def : Sample Space : set of all possible outcomes of an experiment

- Denote S

Ex : Toss a coin 3 times.

$$S = \begin{Bmatrix} \text{HHH HHT HTH HTT} \\ \text{THH THT TTH TTT} \end{Bmatrix}$$

Ex : number of positive tests during a week. $S = \{0, 1, 2, \ldots\}$

Ex : highest temp. on labor day. $S = \{40 < \omega < 120\}$

Ex : An event is a subset of sample space, S, as $(A, B, C \ldots) \subseteq S$

Def : ω : "omega" an outcome occurs

- if A contains ω, we say A occurred

- if A does not contain ω, A did not occur

i.e. $\omega \in A, A$ occurred, $\omega \notin A, A$ did not occur

e.g. Toss coin 3 times. $A =$ getting @ least 2 heads

$A = \{HHH, HHT, HTH, THH\}$. If I toss the coin 3 times and get HTH, then A occurred if I get TTH, A did not occur

Set Theory

- A is a subset of S if $\omega \in A \implies \omega \in S$

- Union : $A \cup B = \{\omega : \omega \in A \text{ or } \omega \in B\}$

- intersection : $A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}$

- complement : $\overline{A} = \{\omega \in S : \omega \notin A\}$

Notate : $A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^n A_i$

Notate : $A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^n A_i$

Notate : $\emptyset$ = null event (empty set)

Notate : S = sure event (always occur)

Def : An event is called elementary (simple) event if it contains only 1 outcome

Def : mutually exclusive (disjoint)

- $A \cap B = \emptyset \implies$ no common outcome

$A_1, A_2, \ldots, A_k$ are all mutually exclusive if they are pairwise mutually exclusive i.e. $A_i \cap A_j = \emptyset$ for all $i \neq j$

Ex : Toss a coin 3 times

A : at least 2 heads

$$A = \{HHH, HHT, HTH, THH\}$$

B : 1st toss = tails

$$B = \{THH, THT, TTH, TTT\}$$

$$A \cap B = \{THH\} \implies \text{not mutually exclusive}$$

$$A \cup B = \{HHH, HHT, HTH, THH, THT, TTH, TTT\}$$

$$\overline{A \cup B} = \{HTT\}$$

$$\overline{A} = \{HTT\}$$

$$\overline{B} = \{HTT, HHH, HHT, HTH\}$$

DeMorgan's Law

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Definition of Probability

S Sample Space - all outcomes of an experiment

A event, $A \subseteq S$

B = set of all possible events

P = probability set function if

$P : \mathcal{B} \to [0 : 1]$

$\uparrow$

domain is sets

range = $[0, 1]$

Kolmogrov's axioms

1. $P(A) \geq 0$ for all $A \in B$

2. $P(S) = 1$

3. if $A_1, A_2, \ldots \leftarrow \mathcal{B}$ are mutually exclusive, then

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} A_i$$

Remark

$P(A)$ is the probability of A

Result

$P(\emptyset) = 0$

Proof

$$S \cap \emptyset = \emptyset$$
$$P(S \cap \emptyset) = P(S) + P(\emptyset)$$
$$P(S) = P(S) + P(\emptyset)$$
$$P(\emptyset) = 0$$

■

Discrete Sample Space

Assign probability to single outcomes (elementary / simple event) i.e.

$$P(\{e_i\}) = P_i$$
$$P_i \geq 0 \text{ for all } i$$
$$\sum_{i=1}^{\infty} P_i = 1 = P(S)$$
$$P(A) = \sum_{e_i \in A} P(\{e_i\})$$

e.g. Toss a coin 3 times

$$P(\{HHH\}) = P(\{HHT\}) = \ldots = 1/8$$
$$A: \text{ At least 2 heads}$$
$$A = \{HHH, HHT, HTH, THH\}$$
$$P(A) = P(\{HHH\}) + P(\{HHT\}) + \ldots = 1/8 + 1/8 + \ldots + 1/8 = 1/2$$

Equal Probability Model (Classical) <Discrete Space>

Discrete Sample Space S contain N outcomes ($N < \infty$) each outcome is equally likely

i.e.

$$P_1 = P_2 = \ldots = P_n$$
$$P_i > 0$$
$$\sum P_i = 1, \text{ then}$$
$$P_i = 1/N \text{ for all } i$$
$$P(A) = \sum_{i=1}^{n(A)} 1/N = \frac{n(A)}{N} \text{ where } n(A) = \# \text{ of outcomes in } A$$

Remark

When outcomes are equally likely, finding $P(A)$ reduces to "counting" problems

Ex : Select 2 students from a group of 5 (3 male 2 female)

A : At least 1 female student is selected

$P(A) =?$

$$S = \{(M_1, M_2), (M_1, M_3), (M_2, M_3), (M_1, F_1), \ldots\}$$

$$N = 10 = \binom{5}{2}$$

$$A = \{(M_1, F_1), (M_2, F_1), (M_3, F_1), (M_1, F_2), (M_2, F_2), (M_3, F_2)\}$$

$n(A) = 7$

$P(A) = 7/10 = 0.7$

1.4 Properties of Problem Rules