1 Deformation Modules and LDDMM

Incorporation of a deformation prior in image reconstruction

Barbara Gris 2018

[?]. deformation modules, image reconstruction, deformation prior

The goal is the reconstruction of images of poor data by using a template image I_0 and a deformation prior. Possible applications could be the case when an image at a certain time point is given and the data (obtained by the forward operator T) for the following timepoints is poor, while it is known that the desired images transform from the template in a known way. The framework is similar to the LDDMM framework but allows only a type of 'intuitive' deformations (depending on the image and application). The large deformation is modelled as the flow of vectorfields specified by deformation modules. The deformtation φ^v evolves from the vectorfield v via the flow equation. The space of v is a subspace of all possible vectorfields, denoted by V. The space V depends on a geometrical descriptor $o \in \mathcal{O}$. An element of $\mathcal{O}$ is specified by the control variable $h \in H$ in a Hilbert space. The intuition behind this are 'base motions' associated with a 'geometrical state' of the subject, defining a family of vector fields (exactly those in V_q). From q and h a vectorfield is generated by the field generator $\zeta\colon \mathcal{O} \times H \longrightarrow V_q, (q,h) \mapsto \zeta_q(h)$. The cost $c\colon \mathcal{O} \times H \to \mathbb{R}$ is a metric and is used as a regularitzation term for the deformation in the energy functional. The energy functional

$$J(o,h) = \int_0^1 c_{o_t}(h_t)dt + D(T(\varphi_{t=1}^{\zeta_0(h)} \cdot I_0), d) \qquad (1)$$

is to be minimized. The similarity of the images is measured in the data space. In the article well-posedness of the cost as a regularization term is shown.

The deformation Module is defined as $\mathcal{M} = (\mathcal{O}, H, \zeta, c)$. Examples for local translations, contracting-dilating fields and constrained translations generator (CTG) deformation modules are given. It is shown how modules can be compound and can be linearly combined.

As in LDDMM, the shooting equations and momentum are used to find the geodesic of the trajectories in $\mathcal{O}$ minimizing the energy (1).

Experiments are shown with simulated data. For this framework, the deformation modules have to be known in advance, which is not the case for real data. Also the size of the Gaussian kernel defining the RKHS V has to be known. The algorithm is not yet satisfying when grey-scale values of the template image are not correct, which could be solved by expanding it to a metamorphosis framework.

Computing large deformation diffeomorphic metric mappings via geodesic flows of diffeomorphisms

Beg, Miller, Trouv, Younes 2005
[?]. LDDMM, Euler-Lagrange equation

The model is relying on the large deformation approach of Christensen 1996. Smoothness is enforced by the norm $\|\cdot\|_V = \|L\cdot\|_{L^2}$ as a regularizer, where L is a differential operator. It is shown that the length of the shortest path in the space of vector fields defines a metric. The Euler-Lagrange equation for the solution of the matching problem is derived. Details of the algorithm are explained: a semi-lagrangian method for integration of the flow equation; the choice of the differential operator $L = -\alpha\nabla^2 + \gamma\mathrm{Id}$ (Cauchy-Navier), where $L^\dagger L f = g$. With periodic boundary conditions L is self adjoint. In comparison to Christensen's algorithm it is shown that the vector fields vary much less over time.

Shape deformation analysis from the optimal control viewpoint

Arguillre, Trlat, Trouv, Younes 2015
[?]

First approach to constrained deformations. Good Introduction to mathematical model of shape space. The shape deformation analysis problem is specified as an infinite dimensional control problem with state and control constraints. The states are diffeomorphisms and the controls are vectorfields that both are subject to some constraints. Weel-posedness of the problem is proved in inf. dim. As an extension of the Pontyagin Maximum Principle optimal solutions are characterized by geodesics. An analysis of the infinite dimensional shape space problem with constraints as well as its finite dimensional approximation is given. Algorithms (Gradient descent, Augmented Lagrangian Method) are described.

2 Image Registration

Deformable templates using large deformation kinematics

Christensen 1996
[?]. large deformation approach

Model of φ as the flow of a time-dependent vectorfield. locally optimal solution, no geodesic

Variational Problems on Flows of Diffeomorphisms for Image Matching

Dupuis, Grenander, Miller 1997
[?]. optimal control, bayesian estimation,

Under the condition $\int_0^\tau \|v(\cdot,t)\|_L^2 dt < \infty$ (L differential operator), the solution is a diffeomorphism.

3 Mathematical Fundamentals

Shapes and Diffeomorphisms

Younes 2010
[?]

Introduction to Riemannian and Sub-Riemannian Geometry

Agrachev, Barilari, Boscain 2015
[?]

A Riemannian space is a smooth manifold whose tangent apaces are endowed with Euclidean structures; each tangent space is equipped with its own Euclidean structure that depends smoothly on the point where the tangent space is attached.

A sub-Riemannian space is a smooth manifold with a fixed admissible subspace in any tangent space where admissible subspaces are equipped with Euclidean structures.

Admissible paths are those curves whose velocities are admissible. The distance between points is the infimum of the length of admissible paths connecting the points.

In a control theory spirit: admissible paths are solutions to the time-varying ODE

$$\dot{q}(t) = \sum_{i=1}^{k} u_i(t) f_i(q(t)), \tag{2}$$

with control functions u_i, initial points $q(0)$ and generators f_i of the space of admissible fields.

The Hamiltonian flow on T^*M associated to the Hamiltonian H is the sub-Riemannian geodesic flow (normal geodesic). Abnormal geodesics belong to the closure of the space of normal geodesics.

Theory of Reproducing Kernels

Aronszajn 1950
[?]