drawn through some fixed P_B requirement, and as k or M is increased, it is seen that the E_b/N_0 requirement is reduced. Similarly, it can be seen that the curves in Figure 9.1b, for nonorthogonal signaling such as MPSK, behave in the opposite fashion. Error-performance degrades as k or M is increased.

(b) In the case of orthogonal signaling, where error performance improves with increasing k of M, what is the cost? In terms of the orthogonal signaling we are most familiar with, MFSK, when $k = 1$ and $M = 2$ there are two tones in the signaling set. When $k = 2$ and $M = 4$, there are four tones in the set. When $k = 3$ and $M = 8$, there are eight tones, and so forth. With MFSK, only one tone is sent during each symbol time, but the available transmission bandwidth consists of the entire set of tones. Hence, as k or M is increased, it should be clear that the cost of improved error-performance is an expansion of required bandwidth.

(c) In the case of nonorthogonal signaling, such as MPSK or QAM, where error-performance degrades as k or M is increased, one might rightfully guess that the tradeoff will entail a reduction in the required bandwidth. Consider the following example. Suppose we require a data rate of $R = 9600$ bit/s. And, suppose that the modulation chosen is 8-ary PSK. Then, using Equation (9.1), we find that the symbol rate is

$$R_s = \frac{R}{\log_2 M} = \frac{9600 \text{ bit/s}}{3 \text{ bit/symbol}} = 3200 \text{ symbol/s}$$

If we decide to use 16-ary PSK for this example, the symbol rate would then be

$$R_s = = \frac{9600 \text{ bit/s}}{4 \text{ bit/symbol}} = 2400 \text{ symbol/s}$$

If we continue in this direction and use 32-ary PSK, the symbol rate becomes

$$R_s = = \frac{9600 \text{ bit/s}}{5 \text{ bit/symbol}} = 1920 \text{ symbol/s}$$

Do you see what happens as the operating point in Figure 9.1b is moved along a horizontal line from the $k = 3$ curve to the $k = 4$ curve, and finally to the $k = 5$ curve? For a given data rate and bit-error probability, each such movement allows us to signal at a slower rate. Whenever you hear the words, "slower signaling rate," that is tantamount to saying that the transmission bandwidth can be reduced. Similarly, any case of increasing the signaling rate, corresponds to a need for increasing the transmission bandwidth.

9.4 SHANNON–HARTLEY CAPACITY THEOREM

Shannon [3] showed that the system capacity C of a channel perturbed by additive white Gaussian noise (AWGN) is a function of the average received signal power S, the average noise power N, and the bandwidth W. The capacity relationship (Shannon–Hartley theorem) can be stated as

$$C = W \log_2 \left(1 + \frac{S}{N}\right) \tag{9.2}$$

When W is in hertz and the logarithm is taken to the base 2, as shown, the capacity is given in bits/s. It is theoretically possible to transmit information over such a channel at any rate R, where $R \leq C$, with an *arbitrarily small* error probability by using a sufficiently complicated coding scheme. For an information rate $R > C$, it is not possible to find a code that can achieve an arbitrarily small error probability. Shannon's work showed that the values of S, N, and W *set a limit on transmission rate, not on error probability.* Shannon [4] used Equation (9.2) to graphically exhibit a bound for the achievable performance of practical systems. This plot, shown in Figure 9.2, gives the normalized channel capacity C/W in bits/s/Hz as a function of the channel signal-to-noise ratio (SNR). A related plot, shown in Figure 9.3, indicates the normalized channel bandwidth W/C in Hz/bits/s as a function of SNR in the channel. Figure 9.3 is sometimes used to illustrate the power-bandwidth trade-off inherent in the ideal channel. However, it is not a pure trade-off [5] because the detected noise power is proportional to bandwidth:

$$N = N_0 W \tag{9.3}$$

Substituting Equation 9.3 into Equation 9.2 and rearranging terms yields

$$\frac{C}{W} = \log_2\left(1 + \frac{S}{N_0 W}\right) \tag{9.4}$$

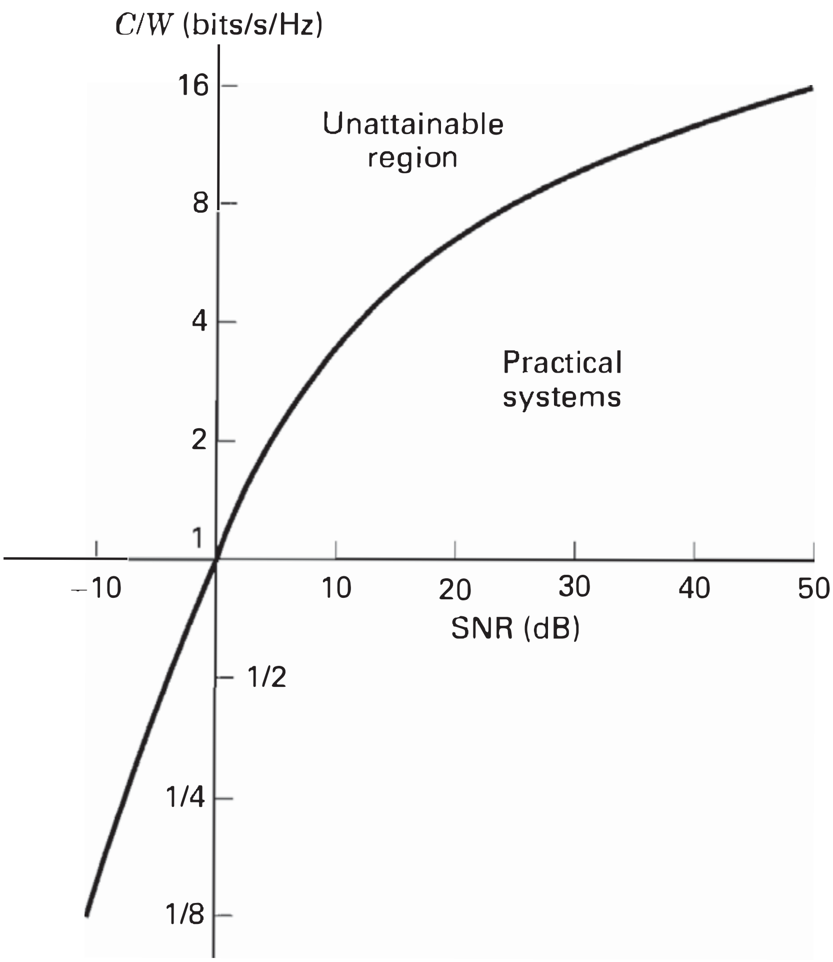

Figure 9.2 Normalized channel capacity versus channel SNR.

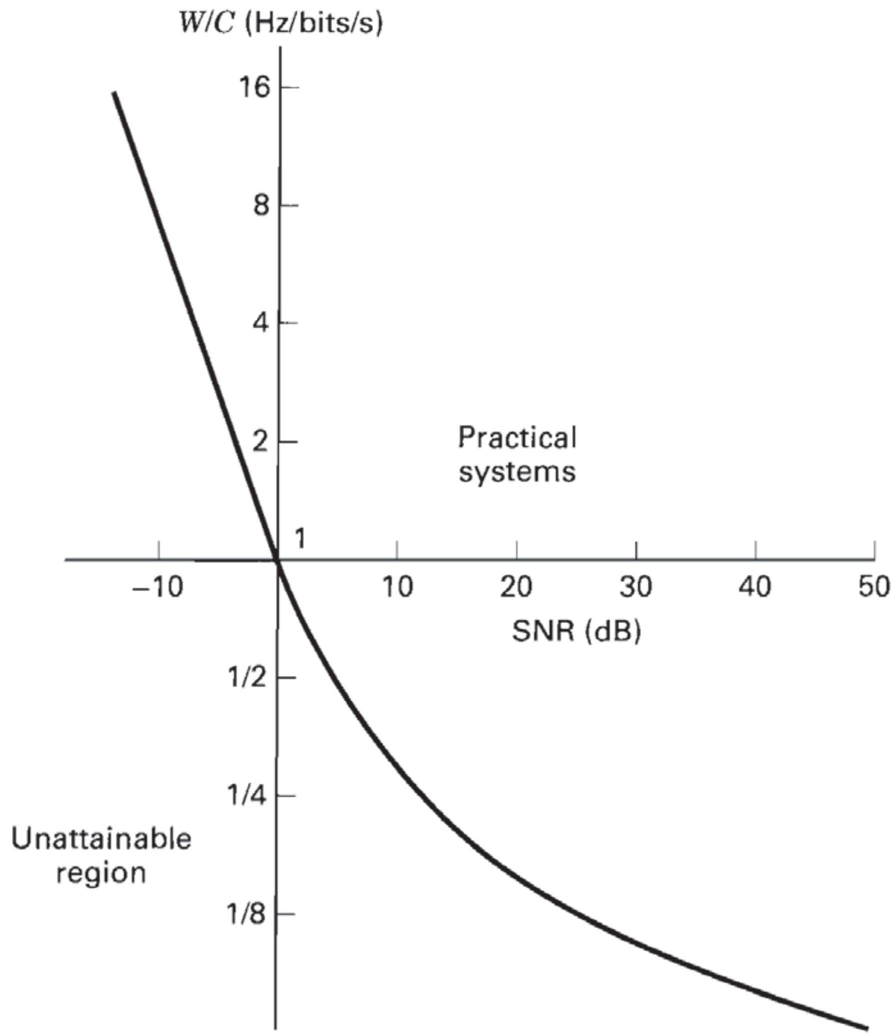

Figure 9.3 Normalized channel bandwidth versus channel SNR.

For the case where transmission bit rate is equal to channel capacity, $R = C$, we can use the identity presented in Equation 3.30 to write

$$\frac{S}{N_0 C} = \frac{E_b}{N_0} \tag{9.5}$$

Hence, we can modify Equation 9.4 as follows:

$$\frac{C}{W} = \log_2\left[1 + \frac{E_b}{N_0}\left(\frac{C}{W}\right)\right] \tag{9.6a}$$

$$2^{C/W} = 1 + \frac{E_b}{N_0}\left(\frac{C}{W}\right) \tag{9.6b}$$

$$\frac{E_b}{N_0} = \frac{W}{C}\left(2^{C/W} - 1\right) \tag{9.6c}$$

Figure 9.4 is a plot of W/C versus E_b/N_0 in accordance with Equation (9.6c). The asymptotic behavior of this curve as $C/W \to 0$ (or $W/C \to \infty$) is discussed in the next section.

9.4.1 Shannon Limit

There exists a limiting value of E_b/N_0 below which there can be no error-free communication at any information rate. Using the identity

$$\lim_{x \to 0} (1 + x)^{1/x} = e$$

we can calculate the limiting value of E_b/N_0 as follows: Let

$$x = \frac{E_b}{N_0}\left(\frac{C}{W}\right)$$

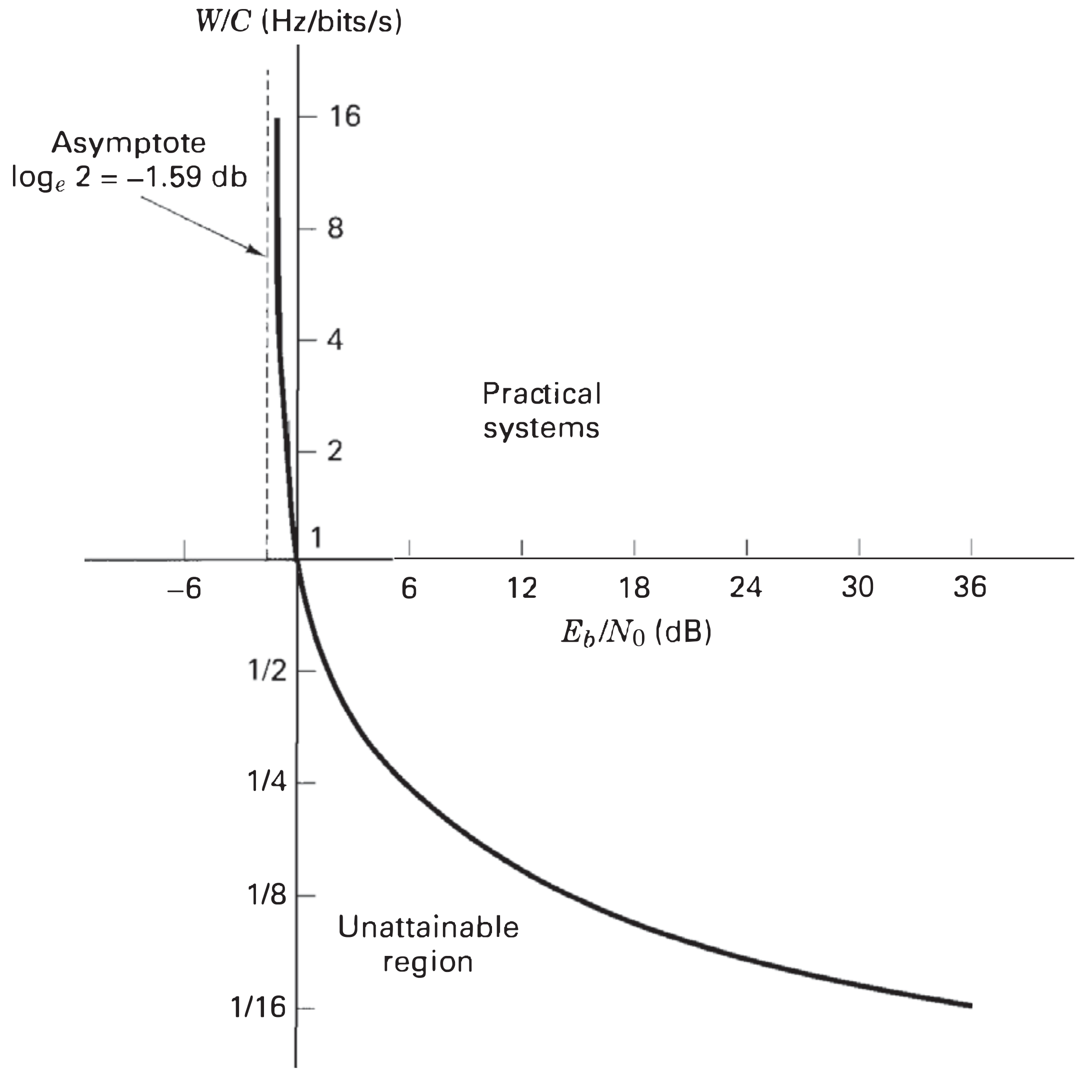

Figure 9.4 Normalized channel bandwidth versus channel E_b/N_0.

Then, from Equation (9.6a),

$$\frac{C}{W} = x \log_2 (1 + x)^{1/x}$$

and

$$1 = \frac{E_b}{N_0} \log_2 (1 + x)^{1/x}$$

In the limit, as $C/W \to 0$, we get

$$\frac{E_b}{N_0} = \frac{1}{\log_2 e} = 0.693 \tag{9.7}$$

or, in decibels,

$$\frac{E_b}{N_0} = -1.6 \text{ dB}$$

This value of E_b/N_0 is called the *Shannon limit.* On Figure 9.1a the Shannon limit is the P_B versus E_b/N_0 curve corresponding to $k \to \infty$. The curve is discontinuous, going from a value of $P_B = \frac{1}{2}$ to $P_B = 0$ at $E_b/N_0 = -1.6$ dB. It is not possible in practice to reach the Shannon limit, because as k increases without bound, the bandwidth requirement and the implementation complexity increases without bound. Shannon's work provided a theoretical proof for the existence of codes that could improve the P_B performance, or reduce the E_b/N_0 required, from the levels of the uncoded binary modulation schemes to levels approaching the limiting curve. For a bit error probability of 10^{-5}, binary phase-shift-keying (BPSK) modulation requires an E_b/N_0 of 9.6 dB (the optimum uncoded binary modulation). Therefore, for this case, Shannon's work promised the existence of a theoretical performance improvement of 11.2 dB over the performance of optimum uncoded binary modulation, through the use of coding techniques. Today, most of that promised improvement (as much as 10 dB) is realizable with turbo codes (see Section 8.4). Optimum system design can best be described as a search for rational compromises or trade-offs among the various constraints and conflicting goals. The modulation and coding trade-off, that is, the selection of modulation and coding techniques to make the best use of transmitter power and channel bandwidth, is important, since there are strong incentives to reduce the cost of generating power and to conserve the radio spectrum.

9.4.2 Entropy

To design a communications system with a specified message handling capability, we need a metric for measuring the information content to be transmitted. Shannon [3] developed such a metric, H, called the entropy of the message source (having n possible outputs). *Entropy* is defined as the average amount of information per source output and is expressed by

$$H = -\sum_{i=1}^{n} p_i \log_2 p_i \quad \text{bits/source output} \tag{9.8}$$

where p_i is the probability of the ith output and $\Sigma p_i = 1$. In the case of a binary message or a source having only two possible outputs, with probabilities p and $q = (1 - p)$, the entropy is written

$$H = -(p \log_2 p + q \log_2 q) \tag{9.9}$$

and is plotted versus p in Figure 9.5.

The quantity H has a number of interesting properties, including the following:

1. When the logarithm in Equation (9.8) is taken to the base 2, as shown, the unit for H is average bits per event. The unit *bit,* here, is a measure of *information content* and is not to be confused with the term "bit," meaning "binary digit."
2. Ther term "entropy" has the same uncertainty connotation as it does in certain formulations of statistical mechanics. For the information source with two equally likely possibilities (e.g., the flipping of a fair coin), it can be seen from Figure 9.5 that the uncertainty in the event, and hence the average infor-

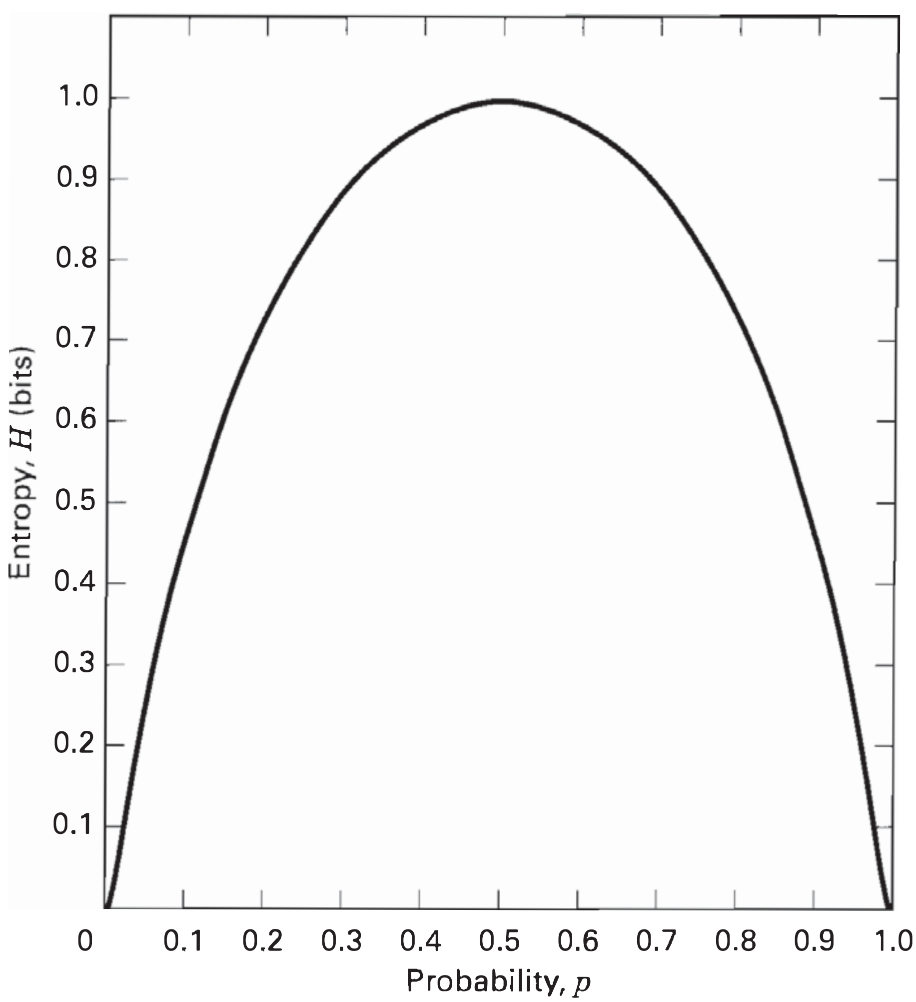

Figure 9.5 Entropy versus probability (two events).