Probing the interior of the Schwarzschild black hole

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Abstract

The probing of the black hole interior is studied beginning with the black hole metric [1] where the shear parameter and the vorticity parameter can be computed. From the following set of parameters, the rate of change also known as the Raychaudhuri equation are able to be determined. It is found that the vorticity computed is zero which is consistent with a black hole of zero spins. Classical equations of motion are also plotted as a function of τ.

1 Spacetime Metric as a 4x4 matrix and its inverse

The spacetime metric in this region is written as

$$g_{ab}dx^a dx^b = -N^2dt^2 + \Lambda^2(dr + N^r dt)^2 + R^2 d\Omega^2 \tag{1}$$

where, N is the lapse function that determines the foliation and $d\Omega^2 = d\theta^2 + sin^2\theta d\varphi^2$

The spacetime metric is then define as

$$g_{uv} = \begin{bmatrix} -N^2(t) + (N^r)^2(r,t)\Lambda^2(r,t) & N^r(r,t)\Lambda^2(r,t) & 0 & 0 \\ N^r(r,t)\Lambda^2(r,t) & \Lambda^2(r,t) & 0 & 0 \\ 0 & 0 & R^2(r,t) & 0 \\ 0 & 0 & 0 & R^2(r,t)sin^2(\theta) \end{bmatrix} \tag{2}$$

The inverse becomes

$$g^{uv} = \begin{bmatrix} -\frac{1}{N^2(t)} & \frac{N^r(r,t)}{N^{2(t)}} & 0 & 0 \\ \frac{N^r(r,t)}{N^{2(t)}} & \frac{1}{\Lambda^2(r,t)} - \frac{N^r(r,t)}{N^{2(t)}} & 0 & 0 \\ 0 & 0 & \frac{1}{R^2(r,t)} & 0 \\ 0 & 0 & 0 & \frac{1}{R^2(r,t)sin^2(\theta)} \end{bmatrix} \tag{3}$$

Multiplying them together gives us

$$g_{uv}g^{uv} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

2 Christoffel Symbols

Christoffel symbols of the second kind as a function of the metric tensor is defined as

$$\Gamma^{i}{}_{kl} = \frac{1}{2} g^{im}\left(\frac{\partial g_{mk}}{\partial x^{l}} + \frac{\partial g_{ml}}{\partial x^{k}} - \frac{\partial g_{kl}}{\partial x^{m}}\right) = \frac{1}{2} g^{im}\left(g_{mk,l} + g_{ml,k} - g_{kl,m}\right) \qquad (5)$$

where g^{jk} is the inverse of the matrix g_{jk}

The non -zero terms of the computed Christoffel symbols are

$$\Gamma^{0}{}_{00} = \frac{N(t)\frac{\partial}{\partial t}N(t) - (N^{r})^{3}(r,t)\Lambda(r,t)\frac{\partial}{\partial r}\Lambda(r,t) - (N^{r})^{2}(r,t)\Lambda^{2}(r,t)\frac{\partial}{\partial r}N^{r}(r,t) + (N^{r})^{2}(r,t)\Lambda(r,t)\frac{\partial}{\partial t}\Lambda(r,t)}{N^{2}(t)}$$

$$\Gamma^{0}{}_{01} = \frac{\left(-N^{r}(r,t)\frac{\partial}{\partial r}\Lambda(r,t) - \Lambda(r,t)\frac{\partial}{\partial r}N^{r}(r,t) + \frac{\partial}{\partial t}\Lambda(r,t)\right)N^{r}(r,t)\Lambda(r,t)}{N^{2}(t)}$$

$$\Gamma^{0}{}_{10} = \frac{\left(-N^{r}(r,t)\frac{\partial}{\partial r}\Lambda(r,t) - \Lambda(r,t)\frac{\partial}{\partial r}N^{r}(r,t) + \frac{\partial}{\partial t}\Lambda(r,t)\right)N^{r}(r,t)\Lambda(r,t)}{N^{2}(t)}$$

$$\Gamma^{0}{}_{11} = \frac{\left(-N^{r}(r,t)\frac{\partial}{\partial r}\Lambda(r,t) - \Lambda(r,t)\frac{\partial}{\partial r}N^{r}(r,t) + \frac{\partial}{\partial t}\Lambda(r,t)\right)\Lambda(r,t)}{N^{2}(t)}$$

$$\Gamma^{0}{}_{22} = \frac{\left(-N^{r}(r,t)\frac{\partial}{\partial r}R(r,t) + \frac{\partial}{\partial r}R(r,t)\right)R(r,t)}{N^{2}(t)}$$

$$\Gamma^{0}{}_{33} = \frac{\left(-N^{r}(r,t)\frac{\partial}{\partial r}R(r,t) + \frac{\partial}{\partial r}R(r,t)\right)R(r,t)\,sin^{2}(\theta)}{N^{2}(t)}$$

$$\Gamma^{1}{}_{00} = \frac{\begin{gathered}-\left(N^{2}(r,t) - (N^{r})^{2}(r,t)\Lambda^{2}(r,t)\right)\left((N^{r})^{2}(r,t)\frac{\partial}{\partial r}\Lambda(r,t) + N^{r}(r,t)\Lambda(r,t)\frac{\partial}{\partial r}N^{r}(r,t) - 2N^{r}(r,t)\frac{\partial}{\partial t}\Lambda(r,t) - \Lambda(r,t)\frac{\partial}{\partial r}N^{r}(r,t)\right)\\ + \left(-N(t)\frac{\partial}{\partial t}N(t) + (N^{r})^{2}(r,t)\Lambda(r,t)\frac{\partial}{\partial t}\Lambda(r,t) + N^{r}(r,t)\Lambda^{2}(r,t)\frac{\partial}{\partial t}N^{r}(r,t)\right)N^{r}(r,t)\Lambda(r,t)\end{gathered}}{N^{2}(t)\Lambda(r,t)}$$

$$\Gamma^{1}{}_{01} = \frac{\left(N^{r}(r,t)\frac{\partial}{\partial r}\Lambda(r,t) + \Lambda(r,t)\frac{\partial}{\partial r}N^{r}(r,t)\right)(N^{r})^{2}(r,t)\Lambda^{2}(r,t) + \left(N^{2}(t) - (N^{r})^{2}(r,t)\Lambda^{2}(r,t)\right)\frac{\partial}{\partial t}\Lambda(r,t)}{N^{2}(t)\Lambda(r,t)}$$

$$\Gamma^{1}{}_{10} = \frac{\left(N^{r}(r,t)\frac{\partial}{\partial r}\Lambda(r,t) + \Lambda(r,t)\frac{\partial}{\partial r}N^{r}(r,t)\right)(N^{r})^{2}(r,t)\Lambda^{2}(r,t) + \left(N^{2}(t) - (N^{r})^{2}(r,t)\Lambda^{2}(r,t)\right)\frac{\partial}{\partial t}\Lambda(r,t)}{N^{2}(t)\Lambda(r,t)}$$

$$\Gamma^{1}{}_{11} = \frac{N^{2}(t)\frac{\partial}{\partial r}\Lambda(r,t) + (N^{r})^{2}(r,t)\Lambda^{2}(r,t)\frac{\partial}{\partial r}\Lambda(r,t) + N^{r}(r,t)\Lambda^{3}(r,t) - \frac{\partial}{\partial r}N^{r}(r,t) - N^{r}(r,t)\Lambda^{2}(r,t)\frac{\partial}{\partial t}\Lambda(r,t)}{N^{2}(t)\Lambda(r,t)}$$

$$\Gamma^{1}{}_{22} = -\frac{\left(\left(N^{2}(t) - (N^{r})^{2}(r,t)\Lambda^{2}(r,t)\right)\frac{\partial}{\partial r}R(r,t) + N^{r}(r,t)\Lambda^{2}(r,t)\frac{\partial}{\partial t}R(r,t)\right)R(r,t)}{N^{2}(t)\Lambda^{2}(r,t)}$$

$$\Gamma^{1}{}_{33} = -\frac{\left(\left(N^{2}(t) - (N^{r})^{2}(r,t)\Lambda^{2}(r,t)\right)\frac{\partial}{\partial r}R(r,t) + N^{r}(r,t)\Lambda^{2}(r,t)\frac{\partial}{\partial t}R(r,t)\right)R(r,t)\sin^{2}(\theta)}{N^{2}(t)\Lambda^{2}(r,t)}$$

$$\Gamma^2{}_{02} = \Gamma^2{}_{12} = \Gamma^2{}_{20} = \Gamma^2{}_{21} = \Gamma^3{}_{03} = \Gamma^3{}_{13} = \Gamma^3{}_{30} = \Gamma^3{}_{31} = -\frac{\frac{\partial}{\partial t}\mathrm{R}(r,t)}{R(r,t)}$$

$$\Gamma^2{}_{33} = -\frac{1}{2}\sin^2(2\theta)$$

$$\Gamma^3{}_{23} = \Gamma^3{}_{32} = \frac{1}{\tan(\theta)}$$

3 Components of the tangent vectors to radial timelike or null geodesics

Rearranging our metric from Eq.1 gives

$$g_{uv} = -N^2dt^2 + \Lambda^2dr^2 + 2\Lambda^2N^r drdt + \Lambda^2N^{2r}dt^2 + R^2d\Omega \tag{6}$$

Consider timelike, radial geodesics:

$$g_{uv}\frac{dx^\mu}{d\tau}\left(\frac{dx^\nu}{d\tau}\right) = -1 \tag{7}$$

This gives us

$$\left(N^2\frac{dt^2}{d\tau^2} - 1\right) = \Lambda^2\frac{dr^2}{d\tau^2} + 2\Lambda^2N^2\frac{dr}{d\tau}\frac{dt}{d\tau} + \Lambda^2N^{2r}\frac{dt^2}{d\tau^2} \tag{8}$$

Factorizing the right- hand side of the equation

$$N^2\frac{dt^2}{d\tau^2} - 1 = \left(\Lambda N^r\frac{dt}{d\tau} + \Lambda\frac{dr}{d\tau}\right)^2 \tag{9}$$

Given that $U^\mu = \frac{dx^\mu}{d\tau} = (\, U^0, U^1, U^2, U^3\,)$, we know from the above equation that

$$N^2(U^0)^2 - 1 = (\Lambda N^r U^0 + \Lambda U^1)^2 \tag{10}$$

Solving for U^1

$$U^1 = \frac{1}{\Lambda}\left(\pm\sqrt{N^2(U^0)^2 - 1} - \Lambda N^r U^0\right) \tag{11}$$

Then the tangent vectors to radial timelike geodesic is

$$U^\mu = \left(\, U^0, \frac{1}{\Lambda}\left(\pm\sqrt{N^2(U^0)^2 - 1} - \Lambda N^r U^0\right), 0, 0\right) \tag{12}$$

3.1 Choosing suitable U^0 to simplify U^μ components

Simplifying our equation, we choose a suitable choice for U^0 as

$$U^0 = \frac{1}{N} \tag{13}$$

to obtain

$$U^{\mu} = \left(\frac{1}{N(t)}, -\frac{N^{r}(r,t)}{N(t)}, 0,0 \right) \tag{14}$$

4 Transverse metric

The transverse metric is defined by

$$h_{\mu\nu} = g_{\mu\nu} - U_{\mu}\, U_{\nu} \tag{15}$$

where $U_{\mu} = g_{\mu\nu} U^{\nu}$

Using this form of vector from Eq.14, we can easily obtain the transverse metric as

$$h_{\mu\nu} = \begin{bmatrix} \left(\frac{-N^2(t) + (N^r)^2\Lambda^2(r,t)}{N(t)} - \frac{(N^r)^2(r,t)\Lambda^2(r,t)}{N(t)} \right)^2 - N^2(t) + (N^r)(r,t)\Lambda^2(r,t) & N^r(r,t)\Lambda^2(r,t) & 0 & 0 \\ N^r(r,t)\Lambda^2(r,t) & \Lambda^2(r,t) & 0 & 0 \\ 0 & 0 & R^2(r,t) & 0 \\ 0 & 0 & 0 & R^2(r,t) sin^2(\theta) \end{bmatrix} \tag{16}$$

5 Covariant Derivative

The covariant derivate $B_{\mu\nu}$ is defined by

$$B_{\mu\nu} = \Delta_{\nu} U_{\mu} = \partial_{\nu} U_{\mu} - \Gamma^{\alpha}_{\nu\mu} U_{\alpha} \tag{17}$$

which becomes

$$B_{\mu\nu} = \begin{bmatrix} \frac{(N^r)^2(t)\Lambda(t)\frac{\partial}{\partial t}\Lambda(t)}{N(t)} & \frac{N^r(t)\Lambda(t)\frac{\partial}{\partial t}\Lambda(t)}{N(t)} & 0 & 0 \\ \frac{N^r(t)\Lambda(t)\frac{\partial}{\partial t}\Lambda(t)}{N(t)} & \frac{\Lambda(t)\frac{\partial}{\partial t}\Lambda(t)}{N(t)} & 0 & 0 \\ 0 & 0 & \frac{R(t)\frac{\partial}{\partial t}R(t)}{N(t)} & 0 \\ 0 & 0 & 0 & \frac{R(t)\sin^2(\theta)\frac{\partial}{\partial t}R(t)}{N(t)} \end{bmatrix} \tag{18}$$

6 Expansion scalar

Now, the expansion scalar is defined as

$$\theta = B^{\mu}\mu = g^{\mu\nu} B_{\mu\nu} \tag{19}$$

Using Eq.18 and metric from Eq.3, the expansion scalar becomes

$$\theta = \frac{2.0\left(-N^r(r,t)\frac{\partial}{\partial r}R(r,t) + \frac{\partial}{\partial t}R(r,t) \right)}{N(t)R(r,t)} + \frac{\left(-N^r(r,t)\frac{\partial}{\partial r}\Lambda(r,t) - \Lambda(r,t)\frac{\partial}{\partial r}N^r(r,t) + \frac{\partial}{\partial t}\Lambda(r,t) \right)}{N(t)\Lambda(r,t)} \tag{20}$$

7 Shear Tensor

For shear tensor, it is given by

$$\sigma_{\mu\nu} = B_{(\mu\nu)} - \frac{1}{3} h_{\mu\nu}\theta \tag{21}$$

where $B_{(\mu\nu)} = \frac{1}{2}(B_{\mu\nu} + B_{\nu\mu})$.

Shear σ^2 is defined by

$$\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} \sigma_{\mu\nu}\sigma_{\alpha\beta} \tag{22}$$

From Eq.22, $\sigma^{\mu\nu}$ is identifies as

$$\sigma^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} \sigma_{\alpha\beta} \tag{23}$$

By solving the above equation, the shear obtain is

$$\begin{aligned}
\sigma^2 = {} & 0.444444444444445(1.0 \Big(N^r(r,t) \frac{\partial}{\partial r} R(r,t) - \frac{\partial}{\partial t} R(r,t) \Big) \Lambda(r,t) - \Big(N^r(r,t) \frac{\partial}{\partial r} \Lambda(r,t) + \\
& \Lambda(r,t) \frac{\partial}{\partial r} N^r(r.t) - \frac{\partial}{\partial t} \Lambda(r,t) \Big)^2 ((N^r)^2(r,t) + 1) \ \Big(\big(N^2(t) - (N^r)^2(r,t)\Lambda^2(r,t) \big)^2 + \big(N^2(t) - \\
& (N^r)^2(r,t)\Lambda^2(r,t) - \Lambda^2(r,t) \big)(N^r)^2(r,t)\Lambda^2(r,t) + (N^r)^2(r,t)\Lambda^4(r,t) \Big) + \\
& 0.444444444444445 \Bigg(1.0 \Big(N^r\,(r,t) \frac{\partial}{\partial r} R(r,t) - \frac{\partial}{\partial t} R(r,t) \Big) \Lambda(r,t) - \Big(N^r(r,t) \frac{\partial}{\partial r} \Lambda(r,t) + \\
& \Lambda(r,t) \frac{\partial}{\partial r} N^r(r,t) - \frac{\partial}{\partial t} \Lambda(r,t) \Big) R(r,t) \Bigg)^2 ((N^r)^2(r,t) + 1) \ \big(((N^r)^2(r,t) + 1)\Lambda^2(r,t) + N^2(t) - \\
& (N^r)^2(r,t)\Lambda^2(r,t) - \Lambda^2(r,t) \big)(N^r)^2(r,t)\Lambda^2(r,t) + \\
& \frac{0.222222222222222 \Big(\Big(N^r(r,t) \frac{\partial}{\partial r} R(r,t) - \frac{\partial}{\partial t} R(r,t) \Big) \Lambda(r,t) - 1.0 \Big(N^r(r,t) \frac{\partial}{\partial r} \Lambda(r,t) + \Lambda(r,t) \frac{\partial}{\partial r} N^r(r,t) - \frac{\partial}{\partial t} \Lambda(r,t) \Big) R(r,t) \Big)^2 N^4(t)}{N^6(t) R^2(r,t) \Lambda^2(r,t)}
\end{aligned} \tag{24}$$

8 Vorticity or Rotation Tensor

Vorticity is given by

$$\omega_{\mu\nu} = B_{[\mu\nu]} = \frac{1}{2}\big(B_{\mu\nu} - B_{\nu\mu}\big) \tag{25}$$

whereby

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{26}$$

Vorticity ω^2 is defined by

$$\omega^2 = \omega_{\mu\nu}\omega^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} \omega_{\mu\nu}\omega_{\alpha\beta} \tag{27}$$

Similarly, from Eq.25, $\omega^{\mu\nu}$ is identified as

$$\omega^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} \omega_{\alpha\beta} \tag{28}$$

Since $\omega_{\mu\nu}$ is a null matrix, ω^2 is therefore

$$\omega^2 = 0 \tag{29}$$

9 Raychaudhuri Equation

The behavior of geodesics on how a congruence of timelike or null geodesic evolves over time is one way to probe the classical and effective structure of spacetime. This analysis is closely linked to the geodesic deviation and the expansion scalar and its rate of change that is the Raychaudhuri equation where it is found to be

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{ab}U^{\mu}U^{\nu} \tag{30}$$

where $\sigma^2 = \ \sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}$ is the shear parameter, $\omega^2 = \omega_{\mu\nu}\omega^{\mu\nu}$ is the vorticity parameter and τ is the proper time along the geodesic.

The Raychaudhuri equation is then computed which yields

$$\begin{aligned}\frac{d\theta}{d\tau} = {}& 0.444444444444445(1.0 \Big(N^r(r,t)\frac{\partial}{\partial r}R(r,t) - \frac{\partial}{\partial t}R(r,t)\Big)\Lambda(r,t) - \Big(N^r(r,t)\frac{\partial}{\partial r}\Lambda(r,t) - \Big(N^r(r,t)\frac{\partial}{\partial r}\Lambda(r,t) + \Lambda(r,t)\frac{\partial}{\partial t}\Lambda(r,t)\Big)R(r,t)\Big)^2 ((N^r)^2(r,t) \\ & +1)\Big(\big(N^2(t) - (N^r)^2(r,t)\Lambda^2(r,t)\big)^2 - \big(-N^2(t) + (N^r)^2(r,t)\Lambda^2(r,t) + \Lambda^2(r,t)\big)(N^r)^2(r,t)\Lambda^2(r,t) + (N^r)^2(r,t)\Lambda^4(r,t)\Big) \\ & +0.444444444444445\Big(1{,}0\Big(N^r(r,t)\frac{\partial}{\partial r}R(r,t) - \frac{\partial}{\partial t}R(r,t)\Big)\Lambda(r,t) - \Big(N^r(r,t)\frac{\partial}{\partial t}\Lambda(r,t) + \Lambda(r,t)\frac{\partial}{\partial r}N^r(r,t) - \frac{\partial}{\partial t}\Lambda(r,t)\Big)R(r,t)\Big)^2 ((N^r)^2(r,t)+1) \\ & ((N^r)^2(r,t)+1)\Lambda^2(r,t) + N^2(t) - (N^r)^2(r,t)\Lambda^2(r,t) - \Lambda^2(r,t))(N^r)^2(r,t)\Lambda^2(r,t) + 0.222222222222222(\Big(N^r(r,t)\frac{\partial}{\partial r}R(r,t) - \frac{\partial}{\partial t}R(r,t)\Big)\Lambda(r,t) \\ & -1.0\Big(N^r(r,t)\frac{\partial}{\partial r}\Lambda(r,t) + \Lambda(r,t)\frac{\partial}{\partial r}N^r(r,t) - \frac{\partial}{\partial t}\Lambda(r,t)\Big)^2 N^4(t) + 1.33333333333333(\Big(N^r(r,t)\frac{\partial}{\partial r}R(r,t) - \frac{\partial}{\partial t}R(r,t)\Big)\Lambda(r,t) \\ & + \frac{0.5\Big(N^r(r,t)\frac{\partial}{\partial r}\Lambda(r,t) + \Lambda(r,t)\frac{\partial}{\partial r}N^r(r,t) - \frac{\partial}{\partial t}\Lambda(r,t)\Big)R(r,t))^2 N^4(t)}{N^6(t)R^2(r,t)\Lambda^2(r,t)}\end{aligned} \tag{31}$$

10 Plotting the Classical Equation

The following phase space trajectories obtained from the classical Hamiltonian is

$$R_c(\tau) = 2Gme^{\tau/2Gm} \tag{32a}$$

$$\Lambda_{c(\tau)} = \pm\sqrt{e^{-\tau/2Gm} - 1} \tag{32b}$$

$$P_{R_c}(\tau) = \frac{1}{2G}\left[2 - e^{-\tau/2Gm}\right] \tag{32c}$$

$$P_{\Lambda_c}(\tau) = \ \mp 2me^{\frac{\tau}{4Gm}}\sqrt{1 - e^{\frac{\tau}{2Gm}}} \tag{32d}$$

whereby the range is $-\infty < \tau < 0$ where τ corresponds to the black hole's event horizon and $\tau = -\infty$ corresponds to the classical singularity.

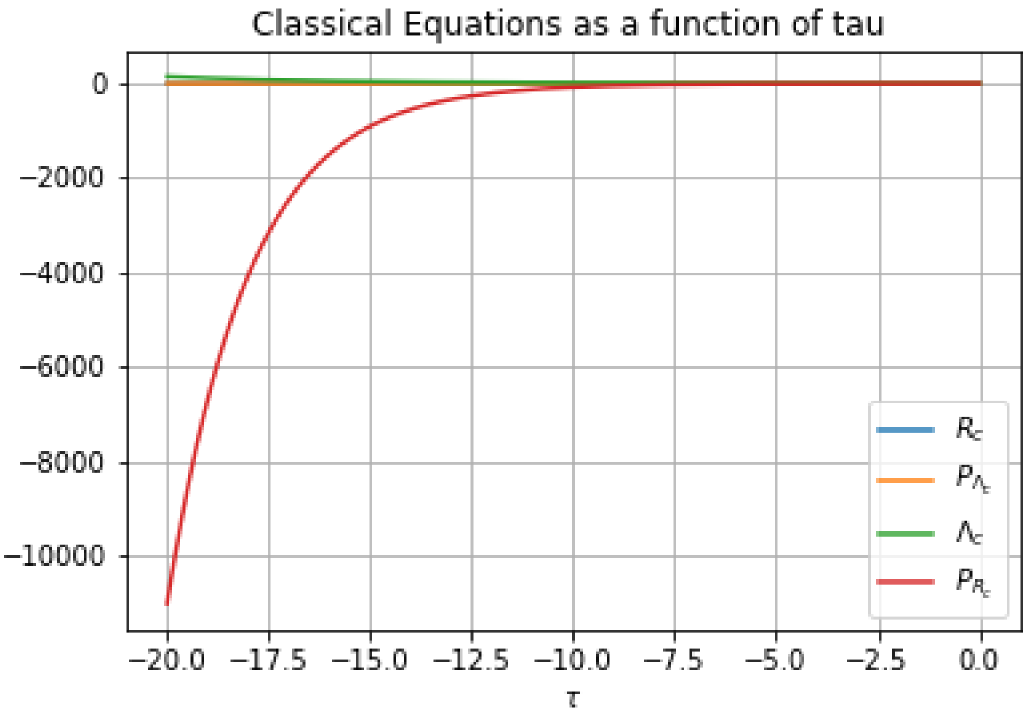

Figure 1: Plotted graph of classical equations as a function of tau

To solve them numerically, more research is required but nevertheless the quantum equation should converge with the classical equations at the event horizon [1].

References

[1] E.Alesci, S.Bahrami, D.Pranzetti (2020), *Assymptotically de Sitter universe inside a Schwarzschild black hole*, Phys.Rev.D 102, 066010

[2] S.Rastgoo (unpublished paper)

[3] S.M. Caroll, (June,2003), *An Introduction to General Relativity Spacetime and Geometry*,