Finding the 4-vector U^μ that is tangent to the congruence of radial, timelike geodesics

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The metric is:

$$g_{\mu\nu} = -N^2 dt^2 + \Lambda^2 (dr + N^r dt)^2 + R^2 d\Omega^2$$

Rearranging:

$$g_{\mu\nu} = -N^2 dt^2 + \Lambda^2 dr^2 + 2\Lambda^2 N^r dr dt + \Lambda^2 N^{2r} dt^2 + R^2 d\Omega^2$$

We consider timelike, radial geodesics:

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -1$$

$$-1 = -N^2 \frac{dt^2}{d\tau^2} + \Lambda^2 \frac{dr^2}{d\tau^2} + 2\Lambda^2 N^r \frac{dr\,dt}{d\tau\, d\tau} + \Lambda^2 N^{2r} \frac{dt^2}{d\tau^2}$$

$$\left(N^2 \frac{dt^2}{d\tau^2} - 1\right) = \Lambda^2 \frac{dr^2}{d\tau^2} + 2\Lambda^2 N^r \frac{dr\,dt}{d\tau\, d\tau} + \Lambda^2 N^{2r} \frac{dt^2}{d\tau^2}$$

Factorizing the RHS:

$$\left(N^2 \frac{dt^2}{d\tau^2} - 1\right) = \left(\Lambda N^r \frac{dt}{d\tau} + \Lambda \frac{dr}{d\tau}\right)^2$$

Note that $\frac{dx^\mu}{d\tau} = U^\mu = \{U^0, U^1, U^2, U^3\}$:

$$N^2 (U^0)^2 - 1 = (\Lambda N^r U^0 + \Lambda U^1)^2$$

$$U^1 = \frac{1}{\Lambda}\left(\pm\sqrt{N^2 (U^0)^2 - 1} - \Lambda N^r U^0\right)$$

Then the tangent 4-vector to the timelike, radial geodesics is:

$$U^{\mu} = \left\{ U^0 ,\ \frac{1}{\Lambda}\left(\pm\sqrt{N^2 (U^0)^2 - 1} - \Lambda N^r U^0 \right),\ 0 ,\ 0 \right\}$$