Probing the interior of the Schwrazschild black hole with expansion: a review of recent works

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Abstract

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1 Introduction

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One of the ways to probe the classical and effective structure of spacetime is by investigating the behavior of geodesics. More precisely, how a congruence of timelike or null geodesic evolves over time. This analysis is intimately related to the geodesic deviation and to the so-called expansion scalar and its rate of change, the Raychaudhuri equation, as we will see below.

Consider a family of curves in a region of spacetime such that through each point in that region, one and only one geodesic passes. This is called a congruence. If every curve in the congruence is timelike/null, then conguence is called timelike/null. In what follows we briefly review both type of these congruences and how they evolve over time.

2.1 Timelike congruence

Suppose we have a congruence of timelike geodesics with unit timelike tangent vectors[1] $\{U^a\}$, where

$$g_{ab}U^aU^b =-1, \qquad\qquad U^a\nabla_a U^b =0. \tag{1}$$

Using these curves, we can decompose the spacetime metric g_{ab} as

$$g_{ab} = h_{ab} - U_aU_b, \tag{2}$$

where h_{ab} is called the transverse metric. The metric h_{ab} is spatial in the sense that it is orthogonal to to timelike U^a

$$h_{ab}U^b = 0 = h_{ab}U^a \tag{3}$$

[1]We use lower case Latin letter for abstract indices and Greek indices for components.

which is simple to check from (2). It is also essentially a three dimensional metric on the hypersurfaces transverse to U^a, and this can be seen by taking the trace of (2) which leads to $h^a{}_a = 3$. The $(1,1)$ tensor $h^a{}_b$ is a projection operator onto the transverse hypersurfaces to U^a since $B^a{}_c B^c{}_b = B^a{}_b$, which can also be seen from (2).

In order to study the evolution of the congruence, we consider the tensor

$$B_{ab} = \nabla_b U_a, \tag{4}$$

which is sometimes called the expansion tensor, and its $B^a{}_b$ version measures the amount of failure of the deviation vector between the geodesics in the congruence from being parallel transported. This tensor is also spatial since it is orthogonal to U^a

$$B_{ab}U^b = 0 = B_{ab}U^a. \tag{5}$$

This is the result of the curves being geodesics, i.e., (1).

We algebraically decompose B_{ab} into its trace part, symmetric traceless part and antisymmetric part as

$$B_{ab} = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab}. \tag{6}$$

Here

$$\theta = B^a{}_a \tag{7}$$

is the trace of B_{ab} and is called the expansion scalar, and describes the fractional change of the area of the cross-section area of the congruence per unit time. The symmetric traceless part

$$\sigma_{ab} = B_{(ab)} - \frac{1}{3}\theta h_{ab} \tag{8}$$

is called the shear tensor. It measures how the cross-section is deformed from a circle. The antisymmetric part

$$\omega_{ab} = B_{[ab]} \tag{9}$$

is called the vorticity (or rotation) tensor, and encodes the overall rotation of the cross-section while the area remains unchanged. The factor $\frac{1}{3}$ is the result of the transverse hypersurfaces being three dimensional. Both σ_{ab} and ω_{ab} are spatial tensors, i.e., and their contraction with U^a vanishes.

These quantities and their rates of change along the geodesic in proper time incorporate important information about the structure of spacetime particularly, the geodesics incompleteness and singularities. The most important of these rates of change is the rate of change of the expansion scalar along the geodesics, $\frac{d\theta}{d\tau}$, where τ is the proper time along the geodesic. It can be computed to yield

$$U^a\nabla_a\theta \coloneqq \frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}U^aU^b. \tag{10}$$

Here, $\sigma^2 = \sigma_{ab}\sigma^{ab}$ is called the shear parameter and $\omega^2 = \omega_{ab}\omega^{ab}$ is the vorticity parameter. In the presence of matter that obeys strong energy condition the last term $R_{ab}U^aU^b$ is always nonnegative, and it vanishes for the vacuum case.

The above equation is called the Raychaudhuri equation, and is a purely geometrical identity. Since σ_{ab} and ω_{ab} are spatial tensors, $\sigma^2 > 0$, $\omega^2 > 0$. So, in cases where the strong energy condition holds, the first second and the fourth terms on the right hand side of (10) (including the signs behind them) are all negative and contribute to the convergence of geodesics as we move along them. The only term with a positive sign is the third term ω^2, which contributes to divergence of geodesics. Hence, if it was not for the vorticity parameter, the rate of change of the expansion scalar would always have been negative, which leads to the geodesics increasingly converge as we move along them. In fact this is the case where U^a are hypersurface-orthogonal. In that case we have

$$\frac{d\theta}{d\tau} \leq -\frac{1}{3}\theta^2 \tag{11}$$

which can be solved to yield

$$\frac{1}{\theta(\tau)} \geq \frac{1}{\theta(\tau_0)} + \frac{1}{3}\tau. \tag{12}$$

Then if we have an initially-converging congruence, i.e., $\theta(\tau_0) < 0$, we arrive at a caustic point where $\theta \to -\infty$ in a finite proper time

$$\tau \leq -3\frac{1}{\theta(\tau_0)}. \tag{13}$$

This caustic point could be the result of bad coordinates or a real physical singularity in spacetime. The Raychaudhuri equation is the backbone of the theorems of Hawking and Penrose about geodesic incompleteness and singularities in general spacetimes.

2.2 Null congruence

For the null congruences with geodesics parametrized by λ, we consider the subspace normal to the null vector field k^a tangent to the geodesics. To do that, introduce a auxiliary null vector field l^a such that

$$k_a l^a = -1. \tag{14}$$

Using these two null vectors, we can decompose the metric as

$$g_{ab} = h_{ab} - 2k_{(a}l_{b)}, \tag{15}$$

where once again h_{ab} is the transverse metric on the hypersurface transverse to k^a and l^a. This hypersurface and its metric h_{ab} is two dimensional which can be confirmed by noticing $h^a{}_a = 2$. The tensor $h^a{}_b$ is again a projection operator onto these transverse hypersurfaces.

We can once again define

$$B_{ab} = \nabla_b k_a, \tag{16}$$

but while this tensor is orthogonal to k^a, it is not orthogonal to l^a. It turns out the purely transverse part B_{ab},

$$\tilde{B}_{ab} = h^c{}_a h^d{}_b B_{cd} \tag{17}$$

is the tensor that is orthogonal to both k^a and l^a. This tensor can explicitly written as

$$\tilde{B}_{ab} = B_{ab} + k_a l^c B_{cb} + k_b B_{ac} l^c + k_a k_b B_{cd} l^c l^d. \tag{18}$$

Once again we can decompose $\tilde{B}_{ab}$ as

$$\tilde{B}_{ab} = \frac{1}{2}\tilde{\theta} h_{ab} + \tilde{\sigma}_{ab} + \tilde{\omega}_{ab}, \tag{19}$$

where

$$\tilde{\theta} = \tilde{B}^a{}_a \tag{20}$$

$$\tilde{\sigma}_{ab} = \tilde{B}_{(ab)} - \frac{1}{2}\theta h_{ab} \tag{21}$$

$$\tilde{\omega}_{ab} = \tilde{B}_{[ab]} \tag{22}$$

are the expansion scalar, and shear and vorticity tenors respectively. The factor $\frac{1}{2}$ is the result of the transverse hypersurfaces being two dimensional. Again, both $\tilde{\sigma}_{ab}$ and $\tilde{\omega}_{ab}$ are spatial tensors.

The Raychaudhuri equation is now derived by considering the evolution of $\tilde{\theta}$ along the geodesics

$$k^a \nabla_a \tilde{\theta} \coloneqq \frac{d\tilde{\theta}}{d\lambda} = -\frac{1}{2}\tilde{\theta}^2 - \tilde{\sigma}_{ab}\tilde{\sigma}^{ab} + \tilde{\omega}_{ab}\tilde{\omega}^{ab} - R_{ab} k^a k^b. \tag{23}$$

Notice that $\tilde{\theta}$ is unique and independent of l^a since $\tilde{\theta} = \tilde{B}^a{}_a = B^a{}_a = \nabla_a k^a$, and so is the Raychaudhuri equation above. The interpretation of (23) is similar to the to the timelike case and leads to the existence of caustic points in many situations.

In what follows we will use both the expansion and the Raychaudhuri equation for both the null and timelike cases to study the effective behavior of the interior of the Schwarzschild black hole.

3 General Schwarzschild interior

Given that the radial spacelike and timelike coordinates swtch their causal nature one we cross the horizon in the Schwarzschild black hole, we can simply switch $t \leftrightarrow r$ in the Schwarzschild metric to obtain the metric of the interior as

$$ds^2 = -\left(\frac{2GM}{t} - 1\right)^{-1} dt^2 + \left(\frac{2GM}{t} - 1\right) dr^2 + t^2 d\Omega^2, \tag{24}$$

where t, r, θ, ϕ are the standard Schwarzschild coordinates and $d\Omega^2 = d\theta^2 + \sin^2(\theta) d\phi^2$. As it is seen, t^2 now plays the role of the radius of the infalling

2-spheres. Notice that this model is not a field theory anymore since the metric components (and hence the degrees of freedom) are independent of r. So we are dealing with a system with finite degrees of freedom, i.e., a minisuperspace model. The above metric is a special case of the Kantowski-Sachs cosmological model

$$ds_{KS}^2 = -N(t)^2dt^2 + g_{xx}(t)dx^2 + g_{\theta\theta}(t)d\theta^2 + g_{\phi\phi}(t)d\phi^2, \tag{25}$$

which describes a homogeneous but anisotropic spacetime.

In order to obtain a general result for such models, we consider a metric of the form

$$ds^2 = -N(t)^2dt^2 + X^2(t)dr^2 + Y^2(t)d\Omega^2. \tag{26}$$

3.1 Timelike case

Let us consider a radial timelike congruence of geodesics where their velocity vector in the coordinates given in (26) is

$$U^\mu = \left(U^0, U^1, 0, 0\right). \tag{27}$$

Given that U^a is a unit timlike vector field, the above vector can be written as

$$U^\mu = \left(U^0, \frac{\sqrt{-1 + N^2\left(U^0\right)^2}}{X}, 0, 0\right). \tag{28}$$

Hence, to simplify our analysis we choose the free component U^0 as

$$U^0 = \frac{1}{N}, \tag{29}$$

to obtain

$$U^\mu = \left(\frac{1}{N}, 0, 0, 0\right). \tag{30}$$

Using this form of the velocity vectors, we can easily obtain the transverse metric from (2) as

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & X^2 & 0 & 0 \\ 0 & 0 & Y^2 & 0 \\ 0 & 0 & 0 & Y^2\sin^2(\theta) \end{pmatrix}. \tag{31}$$

The expansion tensor corresponding to (30) also becomes

$$B_{\mu\nu} = \nabla_\nu U_\mu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{X\dot{X}}{N} & 0 & 0 \\ 0 & 0 & \frac{Y\dot{Y}}{N} & 0 \\ 0 & 0 & 0 & \frac{Y\dot{Y}}{N}\sin^2(\theta) \end{pmatrix}. \tag{32}$$

Using this tensor, the metric and the transverse metric (31), it is straightforward to obtain

$$\theta = \frac{\dot{X}}{NX} + 2\frac{\dot{Y}}{NY}, \tag{33}$$

$$\sigma^2 = \frac{2}{3N^2}\left(\frac{\dot{X}}{X} - \frac{\dot{Y}}{Y}\right)^2, \tag{34}$$

$$\omega_{ab} = 0. \tag{35}$$

It is clear from here that in order to be able to find these quantities, we need to obtain the equations of motions, i.e., the Einstein's equations. Here is where the difference between the classical and the effective cases show up. As we will see later, either the Hamiltonian or the canonical algebra of the interior is changed and this leads to modified equations of motion, which consequently results in modified expansion scalar and its rate of change.

We can now compute the Raychaudhuri equation (10) either by finding the Ricci tensor components and replacing them in the last term of (10), or simply by using the chain role $\frac{d\theta}{d\tau} = \frac{d\theta}{dt}\frac{dt}{d\tau} = \frac{1}{N}\frac{d\theta}{dt}$. The result is

$$\frac{d\theta}{d\tau} = -\frac{\dot{N}}{N^3}\frac{\dot{X}}{X} + \frac{1}{N^2}\frac{\ddot{X}}{X} - \frac{1}{N^2}\left(\frac{\dot{X}}{X}\right)^2 - 2\frac{\dot{N}}{N^3}\frac{\dot{Y}}{Y} + \frac{2}{N^2}\frac{\ddot{Y}}{Y} - \frac{2}{N^2}\left(\frac{\dot{Y}}{Y}\right)^2 \tag{36}$$

3.2 Null case

In this case we choose a congruence of radial null geodesics and due to the null property of their tangent vector k^a, we obtain

$$k^\mu = \left(k^0, -\frac{Nk^0}{X}, 0, 0\right). \tag{37}$$

A simplifying choice for k^0 is thus $k^0 = \frac{1}{N}$ which results in

$$k^\mu = \left(\frac{1}{N}, -\frac{1}{X}, 0, 0\right). \tag{38}$$

The auxiliary radial null vector field l^a has two nonvanishing components that can be fixed by using the null property of l^a and the condition (14). This way we obtain

$$l^\mu = \left(\frac{1}{2N}, \frac{1}{2X}, 0, 0\right) \tag{39}$$

Using these vectors and the spacetime metric, we can find the transverse metric (15) as

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Y^2 & 0 \\ 0 & 0 & 0 & Y^2\sin^2(\theta) \end{pmatrix} \tag{40}$$

which is a two dimensional metric as it should be. Next, we can compute B_{ab} as in (16) and then find $\tilde{B}_{ab}$ using B_{ab} and k^a, l^a above as

$$\tilde{B}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{Y\dot{Y}}{N} & 0 \\ 0 & 0 & 0 & \frac{Y\dot{Y}}{N}\sin^2(\theta) \end{pmatrix}. \tag{41}$$

As mentioned before, we can use these data to compute the expansion scalar and shear and vorticity parameters as

$$\tilde{\theta} = 2\frac{\dot{Y}}{NY}, \tag{42}$$

$$\tilde{\sigma}^2 = 0, \tag{43}$$

$$\tilde{\omega}_{ab} = 0. \tag{44}$$

While quantities are simpler compared to the timelike case, we still need the equations of motion in order to be able to compute the expansion. The Raychaudhuri equation can be computed as before by using the Ricci tensor as

$$\frac{d\theta}{d\lambda} = -\frac{2\dot{N}}{N^3}\frac{\dot{Y}}{Y} - \frac{2}{N^2}\frac{\dot{X}}{X}\frac{\dot{Y}}{Y} + \frac{2}{N^2}\frac{\ddot{Y}}{Y} - \frac{2}{N^2}\left(\frac{\dot{Y}}{Y}\right)^2. \tag{45}$$

4 Effective Schwarzschild interior

The main idea in this section is to find the modified equations of motion of the interior and using them, compute the effective expansion and Raychaudhuri equation for null and timelike cases. We consider two models, one coming from loop quantum gravity (LQG) and the other one from generalized uncertainty principle (GUP).