Gauge Theories: Electroweak

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January 22, 2019

1 Introduction

Imagine a world without electroweak:

- Still have electromagnetism (EM), massive hadrons, atoms, gravity etc.
- Parity (P) and charge conjugation (C) are still good symmetries
- Flavour is always conserved: everything lasts forever (and always existed)
- No neutrinos (would be non-interacting)

Neutrinos were first hypothesised by Pauli (1930) to explain the missing energy in β-decay. They were first observed in 1956 (Cowan-Reines). Parity violation was first directly observed in 1956/7 (Lee-Yang-Wu).

1.1 Chirality

Define the projection operators $P_R \equiv \frac{1}{2}(1+\gamma^5)$ and $P_L \equiv \frac{1}{2}(1-\gamma^5)$. Recalling that $(\gamma^5)^2 = 1$ and $(\gamma^5)^\dagger = \gamma^5$, we can deduce the following properties:

- $P_R^2 = P_R$
- $P_L^2 = P_L$
- $P_L P_R = P_R P_L = 0$
- $P_R + P_L = 1$
- $P_R - P_L = \gamma^5$.

Any Dirac spinor can be split up into a right-handed and a left-handed component, $\psi = \psi_R + \psi_L$, using the projection operators to define $\psi_R = P_R\psi$ and $\psi_L = P_L\psi$. Since $\gamma^\mu P_L = P_R\gamma^\mu$, it follows that $\bar{\psi}P_R \equiv (\bar{\psi})_R = \bar{\psi}_L$ and $\bar{\psi}P_L \equiv (\bar{\psi})_L = \bar{\psi}_R$. So

$$\begin{gathered} \bar{\psi}\psi = \bar{\psi}(\psi_R + \psi_L) = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L \\ \text{and} \\ \bar{\psi}\gamma^\mu\psi = \bar{\psi}\gamma^\mu(\psi_R + \psi_L) = \bar{\psi}_R\gamma^\mu\psi_R\bar{\psi}_L\gamma^\mu\psi_L. \end{gathered} \tag{1}$$

The Dirac Lagrangian splits up like

$$\begin{aligned}\mathcal{L}_D &= \bar{\psi}(i\partial\!\!\!/ - m)\psi \\ &= \bar{\psi}_R i\partial\!\!\!/\psi_R + \bar{\psi}_L i\partial\!\!\!/\psi_L + m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)\end{aligned} \tag{2}$$

so the mass term mixes ψ_R and ψ_L: if $m \to 0$, ψ_R and ψ_L are independent. In this scenario they are both 2-component spinors obeying the Weyl equation $i\partial\!\!\!/\psi_{R/L} = 0$.

1.2 Helicity

The spin operator can be expressed as $\Sigma^i = \frac{i}{2}\epsilon^{ijk}[\gamma^j, \gamma^k] = \gamma^5\gamma^0\gamma^i$. Then $[P_L, \Sigma^i] = [P_R, \Sigma^i]$: spin and chirality commute. Now consider *helicity*, defined

$$h \equiv \frac{2\underline{\Sigma} \cdot \underline{p}}{|\underline{p}|}. \tag{3}$$

h has eigenvalues ± 1, which follows from $(p\!\!\!/ - m)u^\pm = 0 \implies hu^\pm = \pm u^\pm$. But $p\!\!\!/ = E\gamma^0 - \underline{\gamma} \cdot \underline{p}$ and $h = \gamma^5\gamma^0(\underline{\gamma} \cdot \underline{p})/|\underline{p}| = \gamma^5\gamma^0(E\gamma^0 - p\!\!\!/)$. So

$$\begin{aligned} \gamma^5(E - \gamma^0 p\!\!\!/)u^\pm &= \pm u^\pm \\ \implies (P_R - P_L)(E - \gamma^0 m)u^\pm &= \pm p(P_R + P_L)u^\pm \\ \text{and } (E \mp p)u_R^\pm &= m\gamma^0 u_L^\pm \\ (E \pm p)u_L^\pm &= m\gamma^0 u_R^\pm. \end{aligned} \tag{4}$$

Again, the mass term mixes R and L, but if $m \to 0$, $p = E + \mathcal{O}(\frac{m^2}{E})$ and $2Eu_R^- = 2Eu_L^+ = 0$ so $u_R^- = u_L^+ = 0$, i.e. u_R has helicity $+1$ and u_L has helicity -1.

Note that when $m = 0$, helicity is Lorentz invariant (no rest frame). For $m \neq 0$,

$$\begin{aligned} u_R^- = \frac{m\gamma^0}{E+p}u_L^- &\approx \frac{m}{2E}\gamma^0 u_L^- \\ \text{and } u_L^+ &\approx \frac{m}{2E}\gamma^0 u_R^+. \end{aligned} \tag{5}$$

You get similar expressions for negative energy solutions:

$$\begin{aligned} &\text{for } m = 0\text{: } v_R^- = v_L^+ = 0 \\ &\text{for } m \neq 0\text{: } v_R^- = -\frac{m\gamma^0}{2E}v_L^- \text{ and } v_L^+ = -\frac{m\gamma^0}{2E}v_R^-. \end{aligned} \tag{6}$$

1.3 The Chiral Representation

In the chiral representation the gamma matrices can be expressed as:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so

$$P_R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad P_L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \implies \psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}.$$

Furthermore, the spin operator can be written

$$\Sigma^i = \gamma^5\gamma^0\gamma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

so the helicity eigenstates are $\left(\begin{smallmatrix} \psi_R^\pm \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ \psi_L^\pm \end{smallmatrix}\right)$. Now consider the positive and negative energy solutions. $(\not{p} - m)u = 0 \implies$

$$\begin{pmatrix} -m & E + \underline{\sigma}\cdot\underline{p} \\ E - \underline{\sigma}\cdot\underline{p} & -m \end{pmatrix} \begin{pmatrix} u_R \\ u_L \end{pmatrix} = 0.$$

But $\underline{\sigma}\cdot\underline{p}\ u_{L/R} = \pm p\ u^\pm_{L/R}$ so $(E \pm p)u^\pm_L = m\ u^\pm_R$. Using $E^2 = p^2 + m^2$:

$$\begin{aligned} (E \pm p)u^\pm_L &= \sqrt{(E+p)(E-p)}u^\pm_R \\ \implies \sqrt{E \pm p}\ u^\pm_L &= \sqrt{E \mp p}\ u^\pm_R \end{aligned} \tag{7}$$

and we can write

$$u^\pm = \begin{pmatrix} \sqrt{E \pm p}\ \xi^\pm \\ \sqrt{E \mp p}\ \xi^\pm \end{pmatrix}$$

where $\xi^+ = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\xi^- = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$.

Exercise: check that the normalisations $\bar{u}u = 2m$, $u^\dagger u = 2E$.

Similarly, $(\not{p} + m)v = 0$ and $(\underline{\sigma}\cdot\underline{p})v^\pm = \mp p\ v^\pm$, so

$$\begin{aligned} (E \mp p)v^\pm_L &= -m\ v^\pm_R \\ \implies \sqrt{E \mp p}\ v^\pm_L &= -\sqrt{E \pm p}\ v^\pm_R \end{aligned} \tag{8}$$

and

$$v^\pm = \begin{pmatrix} \sqrt{E \mp p}\ \xi^\pm \\ -\sqrt{E \pm p}\ \xi^\pm \end{pmatrix}.$$

You can relate $\xi^\pm$ to $\chi^\pm$ by charge conjugation: $v^\pm = i\gamma^2 u^{\pm *}$, where

$$i\gamma^2 = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}; \qquad -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so $\chi^\pm = -i\sigma^2\xi^\pm$ and if $\xi^\pm = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ then $\chi^\pm = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)$ as before.

1.4 Parity

Under P, $\psi \to \psi_p = \gamma^0\psi$. We know that $P_L\gamma^0 = \gamma^0 P_R$, so $\psi_L \to \gamma^0\psi_L = P_R\gamma^0\psi_L = (\psi_p)_R$. In other words, $(\psi_L)_p = (\psi_p)_R$, so parity switches L and R.

Note that $[\gamma^0, \Sigma^i] = 0$ but under P $\underline{p} \to -\underline{p}$ so $h \to -h$, as expected. This means $u^+_R to u^-_L$ etc.: parity flips helicity but not spin.

1.5 Charge conjugation

Under C, $\psi \to \psi_c = C\overline{\psi}^T$ where $C = i\gamma^2\gamma^0$. Then $P_L C = C P_L$ so $\psi_L \to C(\overline{\psi})_L^T$. This means that charge conjugation leaves the chirality unchanged. Helicity is also unchanged: C just takes particles $\leftrightarrow$ antiparticles.

1.6 Time reversal

Under T, $\psi \to \psi_T = B\psi$ where $B = i\gamma^1\gamma^3 = -i\gamma^5 C$ and $B^\dagger = B = B^{-1}$. Again, $P_L B = B P_L$ so $\psi_L \to B\psi_L$ and time reversal leaves chirality and helicity unchanged (it reverses both spin and momentum).

2 Charged Current Electroweak

2.1 Fermi Theory (1934)

Fermi theory is based on a point-like 4-fermion interaction

$$G_F(\bar{n}\gamma_\mu p)(\bar{\nu}\gamma^\mu e) \tag{9}$$

representing the process $n \to pe^-\bar{\nu}$, where n is a neutron, p is a proton, e^- is an electron and $\bar{\nu}$ is an antineutrino. More generally, the interaction can be written as $G_F J_\mu^\dagger J^\mu$ where J_μ is the "weak current" and is composed of leptonic and hadronic contributions: $J_\mu = \bar{\nu}\gamma_\mu e + \bar{p}\gamma_\mu n + ...$ Note that J_μ has $\Delta Q = +1$ so $J_\mu^\dagger$ has $\Delta Q = -1$ and electric charge is conserved.

Considering mass dimensions $[\cdot]$: $[m\psi\bar{\psi}] = +\psi$, $[\psi] = 3/2$ and $[J_\mu] = 3$ so $[G_F] = -2$. In other words, the coupling scales as $1/\text{mass}^2$. The mass scale is $\approx$ 300 GeV, so the interaction is weak. The original Fermi interaction conserves parity (and C and CP), just like QED. This turned out to be wrong, according to theory developed by Lee and Yang (1956) and the Wu experiment in 1957. In weak interactions, P and C are violated by CP is conserved. More on this later.

2.2 V-A Theory

Developed by Masshart, Sudarshan, Feynman, Gell-Man ... in 1958. V-A theory is based on a vector current V_μ and an axial current A_μ,

$$\begin{aligned} V_\mu &= \bar{\nu}\gamma_\mu e + \bar{p}\gamma_\mu n + ... \\ A_\mu &= \bar{\nu}\gamma_\mu\gamma^5 e + \bar{p}\gamma_\mu\gamma^5 n + ... \end{aligned} \tag{10}$$

whose difference gives the overall current

$$\begin{aligned} \frac{1}{2}J_\mu = \frac{1}{2}(V_\mu - A_\mu) &= \bar{\nu}\gamma_\mu\frac{1}{2}(1-\gamma^5)e + \bar{p}\gamma_\mu\frac{1}{2}(1-\gamma^5)n + ... \\ &= \bar{\nu}_L\gamma_\mu e_L + \bar{p}_L\gamma_\mu n_L + ... \end{aligned} \tag{11}$$

So the weak interactions involve **only left-handed fields**, and maximally violate P and C. In fact, under P, C and CP the currents transform in the following ways:

$$\begin{array}{llll} P & V^\mu \to V_\mu, & A^\mu \to -A_\mu, & (V-A)^\mu \to (V+A)_\mu \\ C & V^\mu \to -V^\mu, & A^\mu \to A^\mu, & (V-A)^\mu \to (-V-A)^\mu \\ CP & V^\mu \to -V_\mu, & A^\mu \to -A_\mu, & (V-A)^\mu \to -(V-A)_\mu \end{array} \tag{12}$$

so $(V-A)^{\mu\dagger}(V-A)_\mu$ is invariant under CP (and T).

Note that that neutrinos **only** interact weakly, so ν_R does not interact at all! If neutrinos are massless, there is no need for ν_R. From 1930-1998 neutrinos were always assumed to be massless: here we will assume $m_\nu = 0$ so ν are always left handed and $\bar{\nu}$ are always right handed. In reality $m_\nu \leq 0.3$ eV, which is very small.

So we have

$$\mathcal{L}_{4F} = -\frac{G_F}{\sqrt{2}} J_\mu^\dagger J^\mu \tag{13}$$

where the $\sqrt{2}$ is historical. $J_\mu = J_\mu^l + J_\mu^h$ can be split up into a leptonic and a hadronic current, each of which comprises three generations. For example, the leptonic current

$$\frac{1}{2} J_\mu^l = \bar{\nu}_e \gamma_\mu e_L + \bar{nu}_{(\mu)} \gamma_\mu \mu_L + \bar{\nu}_\tau \gamma_\mu \tau_L. \tag{14}$$

Lepton number $L_e = N_{e^-} - N_{e^+} - N_{\nu_e} + N_{\bar{\nu}_e}$ and the corresponding L_μ and L_τ are conserved, according to Noether's Theorem.

The hadronic current

$$\frac{1}{2} J_\mu^h = \bar{u}_L \gamma_\mu d'_L + \bar{c}_L \gamma_\mu s'_L + \bar{t}_L \gamma_\mu b'_L \tag{15}$$

is simplest to express in terms of the quarks. The baryon number $B = \sum_{gen}(N_u - N_{\bar{u}} - N_d + N_{\bar{d}})$ is conserved. Things are complicated due to the effects of quark mixing - see later.

(V-A) Theory is quite complicated, but there is only one coupling, G_F: we call this universality. Explicitly,

$$\mathcal{L}_{4F} = -\frac{G_F}{\sqrt{2}} (J_\mu^{l\dagger} J^{l\mu} + (J_\mu^{h\dagger} J^{l\mu} + J_\mu^{l\dagger} J^{h\mu}) + J_\mu^{h\dagger} J^{h\mu}) \tag{16}$$

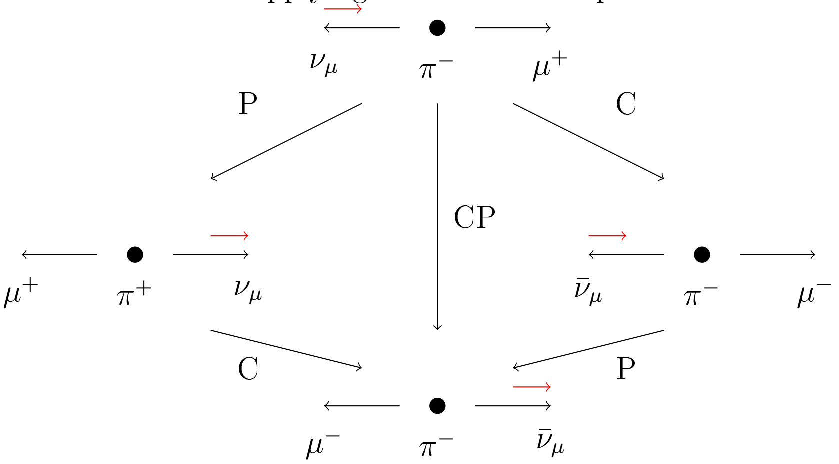
so there are three types of (charged current) weak interaction

- **leptonic:** only leptons (and neutrinos) e.g. $\mu \to e \nu_\mu \bar{\nu}_e$
- **semi-leptonic:** both leptons and quarks e.g. $\pi^- = d\bar{u} \to \mu^- \bar{\nu}_\mu$
- **hadronic:** only quarks e.g. $\Lambda \to p\pi$

But all of them violate P and C and conserve CP, only involve left-handed particles and only involve one coupling G_F.

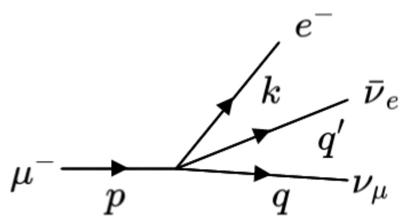
Example: $\pi^+ \approx u\bar{d} \to \mu^+ \nu_\mu$.
Fig. 1 demonstrates the sequential application of C and P to this process. Neither of the intermediate steps can happen, as they involve either a right-handed neutrino or a left-handed antineutrino.

Figure 1: Visualisation of applying C and P to the process $\pi^+ \approx u\bar{d} \to \mu^+\nu_\mu$.



2.3 Leptonic decays

Example: $\mu^- \to e^-\bar{\nu}_e\nu_\mu$.



We will work with Dirac spinors everywhere and contract the spiniors as required. The matrix element can be written

$$\mathcal{M} = \langle e^-(k); \bar{\nu}_e(q')|\mathcal{L}_{4F}|\mu^-(p)\rangle \tag{17}$$

and we can identify the Feynman rule for the four fermion vertex as

$$-i\frac{G_F}{\sqrt{2}}[\gamma_\mu(1-\gamma^5)]_{ab}[\gamma^\mu(1-\gamma^5)]_{cd} \tag{18}$$

so

$$\mathcal{M} = -i\frac{G_F}{\sqrt{2}}\Big(\bar{u}(k)\gamma^\mu(1-\gamma^5)v(q')\Big)\Big(\bar{u}(q)\gamma_\mu(1-\gamma^5)u(p)\Big) \tag{19}$$

where $\bar{u}(k)$, $v(q')$, $\bar{u}(q)$ and $u(p)$ refer to the electron, anti-electron neutrino, muon neutrino and muon respectively.

Average over initial spins and sum over final spins (assuming the neutrinos are massless):

$$\frac{1}{2}\sum_{spins}|\mathcal{M}|^2 = \frac{1}{4}G_F^2\, tr\Big((\not{k}+m_e)\gamma^\mu(1-\gamma^5)\not{q}'\gamma^\nu(1-\gamma^5)\Big)\; tr\Big(\not{q}\gamma_\mu(1-\gamma^5)(\not{p}+m_\mu)\gamma_\nu(1-\gamma^5)\Big) \tag{20}$$

where we call the first trace $\mathcal{M}^{\mu\nu}(k,q')$ and the second trace $\mathcal{M}_{\mu\nu}(q,p)$. Consider evaluating one of these traces