Gauge Theories: Electroweak

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1 Introduction

Imagine a world without electroweak:

- Still have electromagnetism (EM), massive hadrons, atoms, gravity etc.
- Parity (P) and charge conjugation (C) are still good symmetries
- Flavour is always conserved: everything lasts forever (and always existed)
- No neutrinos (would be non-interacting)

Neutrinos were first hypothesised by Pauli (1930) to explain the missing energy in β-decay. They were first observed in 1956 (Cowan-Reines). Parity violation was first directly observed in 1956/7 (Lee-Yang-Wu).

1.1 Chirality

Define the projection operators $P_R \equiv \frac{1}{2}(1+\gamma^5)$ and $P_L \equiv \frac{1}{2}(1-\gamma^5)$. Recalling that $(\gamma^5)^2 = 1$ and $(\gamma^5)^\dagger = \gamma^5$, we can deduce the following properties:

- $P_R^2 = P_R$,
- $P_L^2 = P_L$,
- $P_L P_R = P_R P_L = 0$,
- $P_R + P_L = 1$,
- $P_R - P_L = \gamma^5$.

Any Dirac spinor can be split up into a right-handed and a left-handed component, $\psi = \psi_R + \psi_L$, using the projection operators to define $\psi_R = P_R\psi$ and $\psi_L = P_L\psi$. Since $\gamma^\mu P_L = P_R\gamma^\mu$, it follows that $\bar{\psi}P_R \equiv (\bar{\psi})_R = \bar{\psi}_L$ and $\bar{\psi}P_L \equiv (\bar{\psi})_L = \bar{\psi}_R$. So

$$\begin{gathered}\bar{\psi}\psi = \bar{\psi}(\psi_R + \psi_L) = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L \\ \text{and} \\ \bar{\psi}\gamma^\mu\psi = \bar{\psi}\gamma^\mu(\psi_R + \psi_L) = \bar{\psi}_R\gamma^\mu\psi_R\bar{\psi}_L\gamma^\mu\psi_L.\end{gathered} \tag{1}$$

The Dirac Lagrangian splits up like

$$\begin{aligned}\mathcal{L}_D &= \bar{\psi}(i\partial\!\!\!/ - m)\psi \\ &= \bar{\psi}_R i\partial\!\!\!/\psi_R + \bar{\psi}_L i\partial\!\!\!/\psi_L + m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L),\end{aligned} \tag{2}$$

so the mass term mixes ψ_R and ψ_L: if $m \to 0$, ψ_R and ψ_L are independent. In this scenario they are both 2-component spinors obeying the Weyl equation $i\partial\!\!\!/\psi_{R/L} = 0$.

1.2 Helicity

The spin operator can be expressed as $\Sigma^i = \frac{i}{2}\epsilon^{ijk}[\gamma^j, \gamma^k] = \gamma^5\gamma^0\gamma^i$. Then $[P_L, \Sigma^i] = [P_R, \Sigma^i]$: spin and chirality commute. Now consider *helicity*, defined

$$h \equiv \frac{2\underline{\Sigma}\cdot\underline{p}}{|\underline{p}|}. \tag{3}$$

h has eigenvalues ± 1, which follows from $(p\!\!\!/ - m)u^{\pm} = 0 \implies hu^{\pm} = \pm u^{\pm}$. But $p\!\!\!/ = E\gamma^0 - \underline{\gamma}\cdot\underline{p}$ and $h = \gamma^5\gamma^0(\underline{\gamma}\cdot\underline{p})/|\underline{p}| = \gamma^5\gamma^0(E\gamma^0 - p\!\!\!/)$. So

$$\begin{aligned}\gamma^5(E - \gamma^0 p\!\!\!/)u^{\pm} &= \pm u^{\pm} \\ \implies (P_R - P_L)(E - \gamma^0 m)u^{\pm} &= \pm p(P_R + P_L)u^{\pm} \\ \text{and } (E \mp p)u_R^{\pm} &= m\gamma^0 u_L^{\pm}, \\ (E \pm p)u_L^{\pm} &= m\gamma^0 u_R^{\pm}.\end{aligned} \tag{4}$$

Again, the mass term mixes R and L, but if $m \to 0$, $p = E + \mathcal{O}(\frac{m^2}{E})$ and $2Eu_R^- = 2Eu_L^+ = 0$ so $u_R^- = u_L^+ = 0$, i.e. u_R has helicity $+1$ and u_L has helicity -1.

Note that when $m = 0$, helicity is Lorentz invariant (no rest frame). For $m \neq 0$,

$$\begin{aligned}u_R^- = \frac{m\gamma^0}{E+p}u_L^- &\approx \frac{m}{2E}\gamma^0 u_L^- \\ \text{and } u_L^+ &\approx \frac{m}{2E}\gamma^0 u_R^+.\end{aligned} \tag{5}$$

You get similar expressions for negative energy solutions:

$$\begin{aligned}&\text{for } m = 0\text{: } v_R^- = v_L^+ = 0, \\ &\text{for } m \neq 0\text{: } v_R^- = -\frac{m\gamma^0}{2E}v_L^- \text{ and } v_L^+ = -\frac{m\gamma^0}{2E}v_R^-.\end{aligned} \tag{6}$$

1.3 The Chiral Representation

In the chiral representation the gamma matrices can be expressed as:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So

$$P_R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \implies \psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}.$$

Furthermore, the spin operator can be written

$$\Sigma^i = \gamma^5\gamma^0\gamma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix},$$

so the helicity eigenstates are $\left(\begin{smallmatrix} \psi_R^\pm \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ \psi_L^\pm \end{smallmatrix}\right)$. Now consider the positive and negative energy solutions. $(\not{p} - m)u = 0 \implies$

$$\begin{pmatrix} -m & E + \underline{\sigma} \cdot \underline{p} \\ E - \underline{\sigma} \cdot \underline{p} & -m \end{pmatrix} \begin{pmatrix} u_R \\ u_L \end{pmatrix} = 0.$$

But $\underline{\sigma} \cdot \underline{p}\ u_{L/R} = \pm p\ u^\pm_{L/R}$ so $(E \pm p)u_L^\pm = m\ u_R^\pm$. Using $E^2 = p^2 + m^2$:

$$\begin{aligned} (E \pm p)u_L^\pm &= \sqrt{(E+p)(E-p)}u_R^\pm \\ \implies \sqrt{E \pm p}\ u_L^\pm &= \sqrt{E \mp p}\ u_R^\pm \end{aligned} \tag{7}$$

and we can write

$$u^\pm = \begin{pmatrix} \sqrt{E \pm p}\ \xi^\pm \\ \sqrt{E \mp p}\ \xi^\pm \end{pmatrix},$$

where $\xi^+ = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\xi^- = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$.

Exercise: check that the normalisations $\bar{u}u = 2m$, $u^\dagger u = 2E$.

Similarly, $(\not{p} + m)v = 0$ and $(\underline{\sigma} \cdot \underline{p})v^\pm = \mp p\ v^\pm$, so

$$\begin{aligned} (E \mp p)v_L^\pm &= -m\ v_R^\pm \\ \implies \sqrt{E \mp p}\ v_L^\pm &= -\sqrt{E \pm p}\ v_R^\pm \end{aligned} \tag{8}$$

and

$$v^\pm = \begin{pmatrix} \sqrt{E \mp p}\ \xi^\pm \\ -\sqrt{E \pm p}\ \xi^\pm \end{pmatrix}.$$

You can relate $\xi^\pm$ to $\chi^\pm$ by charge conjugation: $v^\pm = i\gamma^2 u^{\pm *}$, where

$$i\gamma^2 = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}, \qquad -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

So $\chi^\pm = -i\sigma^2\xi^\pm$ and if $\xi^\pm = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ then $\chi^\pm = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)$ as before.

1.4 Parity

Under P, $\psi \to \psi_p = \gamma^0\psi$. We know that $P_L\gamma^0 = \gamma^0 P_R$, so $\psi_L \to \gamma^0\psi_L = P_R\gamma^0\psi_L = (\psi_p)_R$. In other words, $(\psi_L)_p = (\psi_p)_R$, so parity switches L and R.

Note that $[\gamma^0, \Sigma^i] = 0$ but under P $\underline{p} \to -\underline{p}$ so $h \to -h$, as expected. This means $u_R^+ tou_L^-$ etc.: parity flips helicity but not spin.

1.5 Charge conjugation

Under C, $\psi \to \psi_c = C\overline{\psi}^T$ where $C = i\gamma^2\gamma^0$. Then $P_L C = C P_L$ so $\psi_L \to C(\overline{\psi})_L^T$. This means that charge conjugation leaves the chirality unchanged. Helicity is also unchanged: C just takes particles $\leftrightarrow$ antiparticles.

1.6 Time reversal

Under T, $\psi \to \psi_T = B\psi$ where $B = i\gamma^1\gamma^3 = -i\gamma^5 C$ and $B^\dagger = B = B^{-1}$. Again, $P_L B = B P_L$ so $\psi_L \to B\psi_L$ and time reversal leaves chirality and helicity unchanged (it reverses both spin and momentum).

2 Charged Current Electroweak

2.1 Fermi Theory (1934)

Fermi theory is based on a point-like 4-fermion interaction

$$G_F(\bar{n}\gamma_\mu p)(\bar{\nu}\gamma^\mu e), \tag{9}$$

representing the process $n \to pe^-\bar{\nu}$, where n is a neutron, p is a proton, e^- is an electron and $\bar{\nu}$ is an antineutrino. More generally, the interaction can be written as $G_F J_\mu^\dagger J^\mu$ where J_μ is the "weak current" and is composed of leptonic and hadronic contributions: $J_\mu = \bar{\nu}\gamma_\mu e + \bar{p}\gamma_\mu n + ...$ Note that J_μ has $\Delta Q = +1$ so $J_\mu^\dagger$ has $\Delta Q = -1$ and electric charge is conserved.

Considering mass dimensions $[\cdot]$: $[m\psi\bar{\psi}] = +4$, $[\psi] = 3/2$ and $[J_\mu] = 3$ so $[G_F] = -2$. In other words, the coupling scales as $1/\text{mass}^2$. The mass scale is $\approx$ 300 GeV, so the interaction is weak. The original Fermi interaction conserves parity (and C and CP), just like QED. This turned out to be wrong, according to theory developed by Lee and Yang (1956) and the Wu experiment in 1957. In charged current weak interactions, P and C are violated but CP is conserved. More on this later.

2.2 V-A Theory

Developed by Masshart, Sudarshan, Feynman, Gell-Man ... in 1958. V-A theory is based on a vector current V_μ and an axial current A_μ:

$$\begin{aligned} V_\mu &= \bar{\nu}\gamma_\mu e + \bar{p}\gamma_\mu n + ..., \\ A_\mu &= \bar{\nu}\gamma_\mu\gamma^5 e + \bar{p}\gamma_\mu\gamma^5 n + ..., \end{aligned} \tag{10}$$

whose difference gives the overall current

$$\begin{aligned} \frac{1}{2}J_\mu = \frac{1}{2}(V_\mu - A_\mu) &= \bar{\nu}\gamma_\mu\frac{1}{2}(1-\gamma^5)e + \bar{p}\gamma_\mu\frac{1}{2}(1-\gamma^5)n + ... \\ &= \bar{\nu}_L\gamma_\mu e_L + \bar{p}_L\gamma_\mu n_L + ... \end{aligned} \tag{11}$$

So the weak interactions involve **only left-handed fields**, and maximally violate P and C. In fact, under P, C and CP the currents transform in the following ways:

$$\begin{aligned} &P: \quad V^\mu \to V_\mu, \quad A^\mu \to -A_\mu, \quad (V-A)^\mu \to (V+A)_\mu. \\ &C: \quad V^\mu \to -V^\mu, \quad A^\mu \to A^\mu, \quad (V-A)^\mu \to (-V-A)^\mu. \\ &CP: \quad V^\mu \to -V_\mu, \quad A^\mu \to -A_\mu, \quad (V-A)^\mu \to -(V-A)_\mu. \end{aligned} \tag{12}$$

So $(V-A)^{\mu\dagger}(V-A)_\mu$ is invariant under CP (and T).

Note that that neutrinos **only** interact weakly, so ν_R does not interact at all! If neutrinos are massless, there is no need for ν_R. From 1930-1998 neutrinos were always assumed to be massless: here we will assume $m_\nu = 0$ so ν are always left handed and $\bar{\nu}$ are always right handed. In reality $m_\nu \leq 0.3$ eV, which is very small.

So we have

$$\mathcal{L}_{4F} = -\frac{G_F}{\sqrt{2}}J_\mu^\dagger J^\mu, \tag{13}$$

where the $\sqrt{2}$ is historical. $J_\mu = J^l_\mu + J^h_\mu$ can be split up into a leptonic and a hadronic current, each of which comprises three generations. For example, the leptonic current,

$$\frac{1}{2}J^l_\mu = \bar{\nu}_e\gamma_\mu e_L + \bar{\nu}_{(\mu)}\gamma_\mu\mu_L + \bar{\nu}_\tau\gamma_\mu\tau_L. \tag{14}$$

Lepton number $L_e = N_{e^-} - N_{e^+} + N_{\nu_e} - N_{\bar{\nu}_e}$ and the corresponding L_μ and L_τ are conserved, according to Noether's Theorem.

The hadronic current,

$$\frac{1}{2}J^h_\mu = \bar{u}_L\gamma_\mu d'_L + \bar{c}_L\gamma_\mu s'_L + \bar{t}_L\gamma_\mu b'_L, \tag{15}$$

is simplest to express in terms of the quarks. The baryon number $B = \sum_{gen}(N_u - N_{\bar{u}} + N_d - N_{\bar{d}})$ is conserved. Things are complicated due to the effects of quark mixing - see later.

(V-A) Theory is quite complicated, but there is only one coupling, G_F: we call this universality. Explicitly,

$$\mathcal{L}_{4F} = -\frac{G_F}{\sqrt{2}}(J^{l\dagger}_\mu J^{l\mu} + (J^{h\dagger}_\mu J^{l\mu} + J^{l\dagger}_\mu J^{h\mu}) + J^{h\dagger}_\mu J^{h\mu}), \tag{16}$$

so there are three types of (charged current) weak interaction:

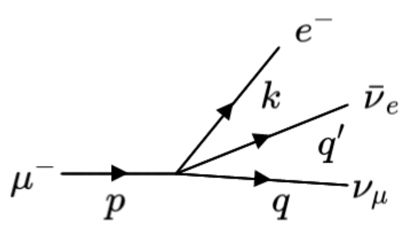
- **leptonic,** only leptons (and neutrinos) e.g. $\mu \to e\nu_\mu\bar{\nu}_e$;
- **semi-leptonic,** both leptons and quarks e.g. $\pi^- = d\bar{u} \to \mu^-\bar{\nu}_\mu$;
- **hadronic,** only quarks e.g. $\Lambda \to p\pi$.

But all of them violate P and C and conserve CP, only involve left-handed particles and only involve one coupling G_F.

Example: $\pi^+ \approx u\bar{d} \to \mu^+\nu_\mu$.
Fig. 1 demonstrates the sequential application of C and P to this process. Neither of the intermediate steps can happen, as they involve either a right-handed neutrino or a left-handed antineutrino.

2.3 Leptonic decays

Example: $\mu^- \to e^-\bar{\nu}_e\nu_\mu$.



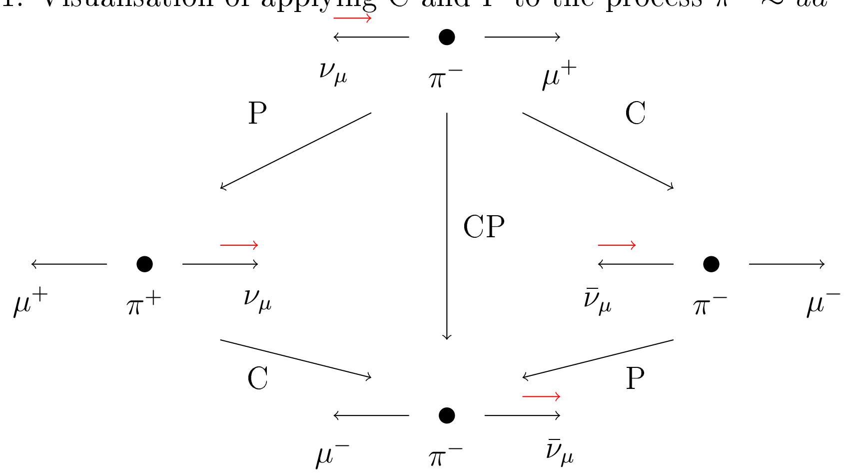
We will work with Dirac spinors everywhere and contract the spiniors as required. The matrix element can be written

$$\mathcal{M} = \langle e^-(k); \bar{\nu}_e(q')|\mathcal{L}_{4F}|\mu^-(p)\rangle, \tag{17}$$

and we can identify the Feynman rule for the four fermion vertex as

$$-i\frac{G_F}{\sqrt{2}}[\gamma_\mu(1-\gamma^5)]_{ab}[\gamma^\mu(1-\gamma^5)]_{cd} \tag{18}$$

Figure 1: Visualisation of applying C and P to the process $\pi^+ \approx u\bar{d} \to \mu^+\nu_\mu$.



so

$$\mathcal{M} = -i\frac{G_F}{\sqrt{2}}\Big(\bar{u}(k)\gamma^\mu(1-\gamma^5)v(q')\Big)\Big(\bar{u}(q)\gamma_\mu(1-\gamma^5)u(p)\Big), \tag{19}$$

where $\bar{u}(k)$, $v(q')$, $\bar{u}(q)$ and $u(p)$ refer to the electron, anti-electron neutrino, muon neutrino and muon respectively.

Average over initial spins and sum over final spins (assuming the neutrinos are massless):

$$\frac{1}{2}\sum_{spins}|\mathcal{M}|^2 = \frac{1}{4}G_F^2\, tr\Big((\not{k}+m_e)\gamma^\mu(1-\gamma^5)\not{q}'\gamma^\nu(1-\gamma^5)\Big)\, tr\Big(\not{q}\gamma_\mu(1-\gamma^5)(\not{p}+m_\mu)\gamma_\nu(1-\gamma^5)\Big), \tag{20}$$

where we call the first trace $\mathcal{M}^{\mu\nu}(k,q')$ and the second trace $\mathcal{M}_{\mu\nu}(q,p)$. Consider evaluating one of these traces, using $(1-\gamma^5)^2 = 2(1-\gamma^5)$ and $tr\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = 4i\epsilon^{\mu\nu\alpha\beta}$:

$$\begin{aligned}\mathcal{M}_{\mu\nu}(q,p) &= 2\ tr\Big(\not{q}\gamma_\mu(\not{p}+m_\mu)\gamma_\nu(1-\gamma^5)\Big)\\ &= 8(q_\mu p_\nu + p_\mu q_\nu - \eta_{\mu\nu}p\cdot q - i\epsilon_{\alpha\mu\beta\nu}q^\alpha p^\beta)\\ &= 8(q_\mu p_\nu + p_\mu q_\nu - \eta_{\mu\nu}p\cdot q + i\epsilon_{\mu\nu\alpha\beta}q^\alpha p^\beta),\end{aligned} \tag{21}$$

so

$$\begin{aligned}\frac{1}{2}\sum_{spins}|\mathcal{M}|^2 &= 16\ G_F^2(k^\mu q'^\nu + q'^\mu k^\nu - \eta^{\mu\nu}k\cdot q' - i\epsilon^{\mu\nu\alpha\beta}q'^\alpha k_\beta)(q_\mu p_\nu + p_\mu q_\nu - \eta_{\mu\nu}p\cdot q + i\epsilon_{\mu\nu\rho\sigma}q^\rho p^\sigma)\\ &= 16\ G_F^2(2p\cdot kq\cdot q' + 2p\cdot q'k\cdot q + 4p\cdot qk\cdot q' - 4k\cdot q'p\cdot q + \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\rho\sigma}q'_\alpha k_\beta q^\rho p^\sigma)\\ &= 64\ G_F^2 p\cdot q'k\cdot q.\end{aligned} \tag{22}$$

Now we want to evaluate the decay rate in the lab frame of the muon. Here $E(p) = m_\mu$.

$$
\begin{aligned}
d\Gamma &= \frac{1}{2m_\mu}\frac{1}{2}\sum_{spins}|\mathcal{M}|^2 (dPS)_3 \\
&= \frac{1}{2m_\mu}\int \frac{d^3k}{(2\pi)^3 2k^0}\frac{d^3q}{(2\pi)^3 2q^0}\frac{d^3q'}{(2\pi)^3 2q'^0}(2\pi)^4\delta^4(p-k-q-q')\frac{1}{2}\sum_{spins}|\mathcal{M}|^2 \\
&= \frac{G_F^2}{8m_\mu\pi^5}\int \frac{d^3k}{k^0}\frac{d^3q}{q^0}\frac{d^3q'}{q'^0}\delta^4(p-k-q-q')(p\cdot q')(k\cdot q).
\end{aligned}
\tag{23}
$$

Integrate over q and q', since these are unobserved. Write $Q = q + q' = p - k$, $Q^2 = 2q\cdot q' = 2(q^0 q'^0 - \underline{q}\cdot\underline{q}') \geq 0$. Then we need to evaluate the integral

$$
\mathcal{I}_{\mu\nu}(Q) \equiv \int \frac{d^3q}{q^0}\frac{d^3q'}{q'^0}\delta^4(Q-q-q')q_\mu q'_\nu. \tag{24}
$$

Appealing to Lorentz invariance, we can write $\mathcal{I}_{\mu\nu} = aQ_\mu Q_\nu + b\eta_{\mu\nu}Q^2$ for some a, b to be found. This is because these are the only two Lorentz invariant terms which can be constructed using the vectors and tensors available. In order to find the coefficients a and b, we can systematically contract the expression with $\eta^{\mu\nu}$ and $Q^\mu Q^\nu$. This gives

$$
\begin{aligned}
a + 4b &= \frac{1}{2}\mathcal{I}, \\
a + b &= \frac{1}{4}\mathcal{I}, \\
\text{where } \mathcal{I} &= \int \frac{d^3q}{|q|}\frac{d^3q'}{|q'|}\delta^4(Q-q-q')
\end{aligned}
\tag{25}
$$

and we have used $q^0 = |q|$, $Q\cdot qQ\cdot q' = (q\cdot q')^2 = 1/4\ Q^4$. Since $\mathcal{I}$ is a Lorentz scalar, we can evaluate it in any frame. Choose $Q^\mu = (Q^0, \underline{0})$, where $\underline{q} = -\underline{q}'$.

$$
\begin{aligned}
\mathcal{I} &= \int \frac{d^3\underline{q}}{|\underline{q}|^2}\ \delta(Q^0 - 2|\underline{q}|) \\
&= 4\pi\int_0^\infty dq\ \delta(Q^0 - 2q) \\
&= 2\pi,
\end{aligned}
\tag{26}
$$

so $a = \pi/3$ and $b = \pi/6$. Putting everything together,

$$
p^\mu k^\nu \mathcal{I}_{\mu\nu}(Q) = \frac{1}{6}\pi(2Q\cdot pQ\cdot k + p\cdot kQ^2), \tag{27}
$$

so

$$
\begin{aligned}
d\Gamma &= \frac{G_F^2}{3m_\mu(2\pi)^4)}\int\frac{d^3\underline{k}}{k^0}\left(2p\cdot(p-k)k\cdot(p-k) + p\cdot k(p-k)^2\right) \\
&= \frac{G_F^2}{3m_\mu(2\pi)^4)}\int\frac{d^3\underline{k}}{k^0}\left(-2p^2k^2 + 3(p^2+k^2)p\cdot k - 4(p\cdot k)^2\right).
\end{aligned}
\tag{28}
$$

In the muon rest frame, $p \cdot k = m_\mu E$, where $E \equiv k^0$, and $\frac{d^3 k}{k^0} \to 4\pi|\underline{k}|dE$. We can ignore $m_e \ll m_\mu$. Then

$$d\Gamma = \frac{2G_F^2}{3(2\pi)^3} m_\mu E^2(3m_\mu - 4E)dE, \tag{29}$$

but $Q^2 > 0$ so $m_\mu^2 - 2p \cdot k = m_\mu(m_\mu - 2e) > 0$, i.e. $E < m_\mu/2$. The total cross-section

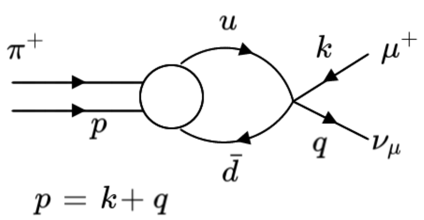
$$\begin{aligned} \Gamma &= \frac{G_F^2}{12\pi^3} m_\mu \int_0^{m_\mu/2} dE(3m_\mu E^2 - 4E^3) \\ &= \frac{G_F^2 m_\mu^5}{192\pi^3}. \end{aligned} \tag{30}$$

We can then use this to determine G_F: $\tau_\mu = 2.2 \times 10^{-6}$s and $\Gamma_\mu = 3.0 \times 10^{-19}$GeV, so $G_F \approx 1.2 \times 10^{-5}\text{GeV}^{-2}$.

You can use the same expression for $\tau^- \to \mu^- \bar{\nu}_\mu \nu_\tau$, $e^- \bar{\nu}_e \nu_\tau$ as a nice test of universality: $\Gamma(\tau^- \to \mu^- \bar{\nu}_\mu \nu_\tau) = \Gamma(\tau^- \to e^- \bar{\nu}_e \nu_\tau) = B\ \Gamma_\tau = (\frac{m_\tau}{m_\mu})^5\ \Gamma_\mu \approx 4 \times 10^{-13}$GeV.

2.4 Semi-leptonic decays

Example: $\pi^+ \to \mu^+ \nu_\mu$.



The matrix element can be written

$$\begin{aligned} \mathcal{M} &= \langle \mu^+(k)\nu_\mu(q)|\mathcal{L}_{4F}|\pi^+(p)\rangle \\ &= \frac{-iG_F}{\sqrt{2}} \bar{u}(q)\gamma^\mu(1-\gamma^5)v(k)\langle 0|\bar{d}\gamma_\mu(1-\gamma^5)u|\pi^+(p)\rangle, \end{aligned} \tag{31}$$

in other words, it factorises into a lepton piece and a quark piece, where the quark piece is of $V_\mu - A_\mu$ form. The pion is a pseudoscalar, so $\langle 0|V_\mu|\pi^+\rangle = 0$. The $\langle 0|A_\mu|\pi^+\rangle$ piece is a four-vector, so Lorentz invariance means it must take the form $\langle 0|A_\mu|\pi^+\rangle = if_\pi p_\mu$, with f_π some "pion decay constant" to be found. This means we can write

$$\mathcal{M} = \frac{G_F f_\pi}{\sqrt{2}} \bar{u}(q)\not{p}(1-\gamma^5)v(k). \tag{32}$$

But $p = k + q$, and using $\not{q}u(q) = 0$; $(\not{k} + m_\mu)v(k) = 0$ gives $\bar{u}(q)(\not{k} + \not{q})(1-\gamma^5)v(k) = -m_\mu \bar{u}(q)(1+\gamma^5)v(k)$. Now we would like to evaluate the spin-averaged matrix element squared, taking note that $(1+\gamma^5)m_\mu(1-\gamma^5) = 0$:

$$\begin{aligned} \sum_{spins} |\mathcal{M}|^2 &= \frac{G_F^2}{2} f_\pi^2 m_\mu^2 tr\ \not{q}(1+\gamma^5)\not{k}(1-\gamma^5) \\ &= 4G_F^2 f_\pi^2 m_\mu^2\ k \cdot q \\ &= 2G_F^2 f_\pi^2 m_\mu^2(m_\pi^2 - m_\mu^2). \end{aligned} \tag{33}$$

In the lab frame,

$$d\Gamma = \frac{1}{2m_\mu}\sum_{spins}|\mathcal{M}|^2(dPS)_2, \tag{34}$$

while

$$\begin{aligned}\int(dPS)_2 &= \frac{1}{(4\pi)^2}\frac{|\underline{k}|}{m_\pi}\int d\Omega \\ &= \frac{1}{4\pi}\frac{|\underline{k}|}{m_\pi},\end{aligned} \tag{35}$$

where $|\underline{k}| = (m_\pi^2 - m_\mu^2)/2m_\pi$. So

$$\Gamma = \frac{1}{8\pi}G_F^2 f_\pi^2 \frac{m_\mu^2}{m_\pi^3}(m_\pi^2 - m_\mu^2)^2. \tag{36}$$

Using the measured rate for $\tau = 2.6\times10^{-8}$s, we can calculate $f_\pi \approx 130$ MeV. Note that

$$\frac{\Gamma(\pi\to e\nu_e)}{\Gamma(\pi\to\mu\nu_\mu)} = \frac{m_e^2}{m_\mu^2}\frac{(m_\pi^2 - m_e^2)}{(m_\pi^2 - m_\mu^2)} \approx 10^{-4}. \tag{37}$$

This is a good test of lepton universality. The ratio vanishes as $m_e \to 0$: this is because of helicity conservation.

2.5 Cabibbo mixing (1963)

The Kaon decay $K^+(u\bar{s}) \to \mu^+\nu_\mu$ works similarly: $\langle 0|A_\mu|K^+(p)\rangle = if_k p_\mu$, and from experiment

$$\frac{\Gamma(K\to\mu\nu)}{\Gamma(\pi\to\mu\nu)} \approx 1.3. \tag{38}$$

But if $\mathcal{L}_{4F}$ contained $\bar{u}_L\gamma_\mu s_L$ (in an analogy with $\bar{u}_L\gamma_\mu d_L$) we expect

$$\frac{\Gamma(K\to\mu\nu)}{\Gamma(\pi\to\mu\nu)} = \frac{f_K^2}{f_\pi^2}\frac{m_\pi^3}{m_K^3}\frac{(m_K^2 - m_\mu^2)^2}{(m\pi^2 - m_\mu^2)^2} \approx 24, \tag{39}$$

since $f_K/f_\pi \approx 1.2$, i.e. SU(3) symmetry breaking is small.

The explanation is down to the form of the hadronic weak current. In actual fact this is

$$\frac{1}{2}J_\mu^h = \bar{u}_L\gamma_\mu(\cos\theta_c\ d_L + \sin\theta_c\ s_L) + ..., \tag{40}$$

where the term in brackets can be thought of as d_L', a *mixture* of d and s. Then

$$\Gamma(\pi\to\mu\nu) = \frac{G_F^2}{4\pi}f_\pi^2\frac{m_\mu^2}{m_\pi^3}(m_\pi^2 - m_\mu^2)^2\cos^2\theta_c \tag{41}$$

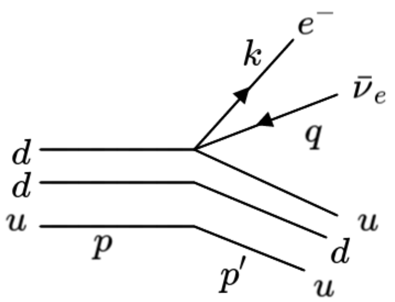
and

$$\Gamma(K\to\mu\nu) = \frac{G_F^2}{4\pi}f_K^2\frac{m_\mu^2}{m_K^3}(m_K^2 - m_\mu^2)^2\sin^2\theta_c. \tag{42}$$

The Cabibbo angle, $\theta_c \approx 13°$, so $\cos\theta_c = 0.97$ and $\sin\theta_c = 0.22$. This reduces the relative size of $\Gamma(K\to\mu\nu)$ by a factor of roughly 20. We will revisit quark mixing later - stay tuned!

2.6 Beta decay

$n \to p\ e^- \bar{\nu}_e$.



What's really going on is $d \to u\ e^- \bar{\nu}_e$: the other quarks are "spectators". Again, the matrix element factorises:

$$\begin{aligned}
\mathcal{M} &= \langle p(p') e^-(k) \bar{\nu}_e(q) | \mathcal{L}_{4F} | n(p) \rangle \\
&= \frac{-iG_F}{\sqrt{2}} \bar{u}(k) \gamma^\mu (1-\gamma^5) v(q) \\
&\quad \times \langle p(p') | \bar{d}' \gamma_\mu (1-\gamma^5) u | n(p) \rangle \cos\theta_c .
\end{aligned} \tag{43}$$

Considering the hadronic part,

$$\langle p(p') | (V_\mu - A_\mu) | n(p) \rangle = \cos\theta_c \bar{u}(p') \gamma_\mu (g_v - g_A \gamma^5) u(p), \tag{44}$$

where $g_V = 1$ and $g_A = 1.25$. Then

$$\begin{aligned}
\frac{1}{2}\sum_{spins} |\mathcal{M}|^2 &= \frac{1}{4} G_F^2 \cos^2\theta_c\ tr\Big((\not{k} + m_e) \gamma^\mu (1-\gamma^5) \not{q} \gamma^\mu (1-\gamma^5) \Big) \\
&\quad \times tr\Big((\not{p}' + m_p) \gamma_\mu (1 - g_A \gamma^5) (\not{p} + m_n) \gamma_\nu (1 - g_A \gamma^5) \Big) \\
&= \frac{1}{4} G_F^2 \cos^2\theta_c\ \mathcal{M}^{\mu\nu}(k,q) \tilde{\mathcal{M}}_{\mu\nu}(p',p).
\end{aligned} \tag{45}$$

Recalling the previous calculation for $\mu \to e\ \nu\bar{\nu}$,

$$\tilde{\mathcal{M}}_{\mu\nu}(p',p) = 4\Big((p'_\mu p_\nu + p_\mu p'_\nu - \eta_{\mu\nu} p \cdot p')(1+g_A^2) + i g_A \epsilon_{\mu\nu\alpha\beta} p'^\alpha p^\beta + m_n m_p (1-g_A^2) \eta_{\mu\nu} \Big) \tag{46}$$

so

$$\begin{aligned}
\frac{1}{2}\sum_{spins} |\mathcal{M}|^2 &= 8 G_F^2 \cos^2\theta_c \Big((2p\cdot k p'\cdot q + 2p\cdot q p'\cdot k)(1+g_A)^2 \\
&\quad - 4 g_A (k\cdot p q\cdot p' - k\cdot p' q\cdot p) - 2 m_n m_p (1-g_A^2) k\cdot q \Big) \\
&= 16 G_F^2 \cos^2\theta_c \Big(p\cdot k p'\cdot q (1-g_A)^2 + p\cdot q p'\cdot k (1+g_A)^2 - (1-g_A^2) m_n m_p k\cdot q \Big).
\end{aligned} \tag{47}$$

The decay rate

$$\begin{aligned}
d\Gamma &= \frac{1}{2m_n} \int \frac{d^3k}{(2\pi)^3 2k^0} \int \frac{d^3q}{(2\pi)^3 2q^0} \int \frac{d^3p'}{(2\pi)^3 2p'^0}\ (2\pi)^4 \delta^4(p - p' - k - q) \frac{1}{2}\sum_{spins} |\mathcal{M}|^2 \\
&= \frac{1}{16 m_n (2\pi)^5} \int \frac{d^3k}{k^0} \int \frac{d^3q}{q^0} \frac{1}{p'^0}\ \delta(p^0 - p'^0 - k^0 - q^0) \frac{1}{2}\sum_{spins} |\mathcal{M}|^2 .
\end{aligned} \tag{48}$$

Now write $p^0 = E_p$, $p'^0 = E_p$, $k^0 = E_e$, $q^0 = E_\nu$ and $d^3k = p_e E_e\ dE_e\ d\Omega_e$, $d^3q = E_\nu^2\ dE_\nu\ d\Omega_\nu$. Using $p_e^2 = E_e^2 - m_e^2$ we can rewrite $p_e\ dp_e = E_e\ dE_e$. So

$$\begin{aligned} d\Gamma &= \frac{1}{16 m_n (2\pi)^5} \int dE_e\ d\Omega_e\ d\Omega_\nu\ dE_\nu \left(\frac{p_e E_\nu}{E_p}\right) \delta(E_n - E_p - E_e - E_\nu) \frac{1}{2} \sum_{spins} |\mathcal{M}|^2 \\ \implies \frac{d\Gamma}{dE_e} &= \frac{1}{16 m_n (2\pi)^5} \int d\Omega_e\ d\Omega_\nu \left(\frac{p_e E_\nu}{E_p}\right) \frac{1}{2} \sum_{spins} |\mathcal{M}|^2. \end{aligned} \tag{49}$$

Now in the lab frame, $E_n = m_n$. It is useful to define $\Delta \equiv m_n - m_p = 1.29$ MeV $\ll m_n$. We can ignore nuclear recoil by using the Born-Oppenheimer approximation, i.e. $E_p \approx m_p$. Note that $m_e = 0.51$ MeV is of the same order as Δ. Then

$$\begin{aligned} & p \cdot k p' \cdot q \approx m_n E_e m_p E_\nu, \\ & p \cdot q p' \cdot k \approx m_n E_\nu m_p E_e, \\ & k \cdot q = E_e E_\nu - \underline{p}_e \cdot \underline{p}_\nu, \end{aligned} \tag{50}$$

so

$$\begin{aligned} \frac{d\Gamma}{dE_e} &= \frac{G_F^2 \cos^2\theta_c}{m_n (2\pi)^5} \int d\Omega_e\ d\Omega_\nu \frac{p_e E_\nu}{E_p} m_n\ m_p \Bigg(E_e E_\nu (1 - g_A)^2 \\ &\quad + E_\nu E_e (1 + g_A)^2 - E_e E_\nu (1 - g_A^2) \left(1 - \frac{\underline{p}_e \cdot \underline{p}_\nu}{E_e E_\nu}\right)\Bigg) \\ &= \frac{G_F^2 \cos^2\theta_c}{2\pi^3} (1 + 3g_A^2) p_e E_e (\Delta - E_e)^2, \quad m_e < E_e < \Delta, \end{aligned} \tag{51}$$

where the $\underline{p}_e \underline{p}_\nu$ term integrates to zero. This evaluates to

$$\Gamma = \frac{G_F^2 \cos^2\theta_c}{2\pi^3} (1 + 3g_A^2) \Delta^5 \times \varpi \tag{52}$$

where $\varpi = 0.47$ (left as an exercise).

Other semileptonic processes include:

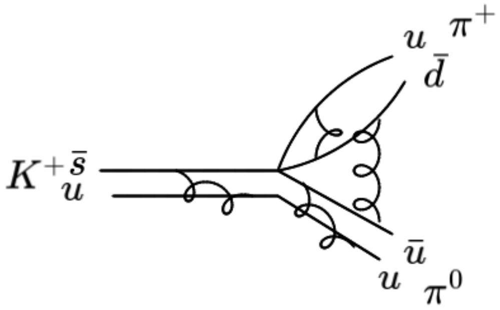
$$\begin{aligned} \Delta S = 0: &\quad \pi^+ \to \pi^0 e^+ \nu; \qquad \cos\theta_c \\ \Delta S = 1: &\quad K^+ \to \pi^0 \mu^+ \nu,\ K^0 \to \pi^- \mu^+ \nu \text{ etc.}; \qquad \sin\theta_c, \end{aligned} \tag{53}$$

and decays of the hyperons $\Sigma = uus$, $\Lambda = uds$, $\Xi = uss$, $\Omega = sss$ etc., which are all related by SU(3):

$$\begin{aligned} \Delta S = 0: &\quad \Sigma \to \Lambda e \bar\nu; \\ \Delta S = 1: &\quad \Lambda \to p e \bar\nu,\ \Sigma \to n e \bar\nu; \\ &\quad \Xi \to \Sigma e \bar\nu,\ \Omega \to \Xi e \bar\nu \text{ etc.} \end{aligned} \tag{54}$$

2.7 Nonleptonic decays

e.g. $K \to \pi\pi$, $K \to \pi\pi\pi$, $\Lambda \to p\pi$, $\Omega \to \Xi\pi$ etc.



Here factorisation no longer holds - there are strong interactions everywhere! We need to resort to non-perturbative methods such as lattice QCD (see later in the course).

3 Intermediate vector bosons

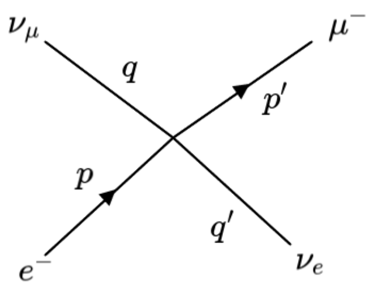
$\mathcal{L}_{4F}$ provides an excellent description of low-energy charged current weak interactions. But it runs into problems at higher energies.

3.1 Unitarity

Example: $\nu_\mu e^- \to \mu^- \nu_e$.
We can write the matrix element for this process

$$\begin{aligned}
\mathcal{M} &= -i\frac{G_F}{2}\bar{v}(q')\gamma_\mu(1-\gamma^5)u(p)\bar{u}(p')\gamma^\mu(1-\gamma^5)u(q) \\
\frac{1}{2}\sum_{spins}|\mathcal{M}|^2 &= \frac{1}{4}G_F\ tr\Big(\not{q}'\gamma_\mu(1-\gamma^5)(\not{p}+m_e)\gamma_\nu(1-\gamma^5)\Big) \\
&\quad\times tr\Big((\not{p}'+m_\mu)\gamma^\mu(1-\gamma^5)\not{q}\gamma^\nu(1-\gamma^5)\Big) \\
&= 64\ G_F^2(p\cdot q)(p'\cdot q') \\
&= 16\ G_F^2(s-m_e^2)(s-m_\mu^2)
\end{aligned} \tag{55}$$

where we used $s = (p+q)^2 = (p'+q')^2 = m_e^2 + 2p\cdot q = m_\mu^2 + 2p'\cdot q'$.



In the centre of mass frame, ignoring masses (i.e. at high energy)

$$\left(\frac{d\sigma}{d\Omega}\right)_{CoM} = \frac{1}{64\pi^2 s}\frac{1}{2}\sum_{spins}|\mathcal{M}|^2 = \frac{1}{4\pi}G_F^2 s. \tag{56}$$

Using $s = 4E^2$, $d\Omega = 2\pi\ d(\cos\theta)$,

$$\frac{d\sigma}{d\cos\theta} = \frac{2G_F^2}{\pi}E^2. \tag{57}$$

Compare this with, for example, the equivalent result for $e^+e^- \to \mu^+\mu^-$:

$$\frac{d\sigma}{d\cos\theta} = \frac{\alpha^2}{4\pi E^2}(\cos^4\theta/2 + \sin^4\theta/2) \tag{58}$$

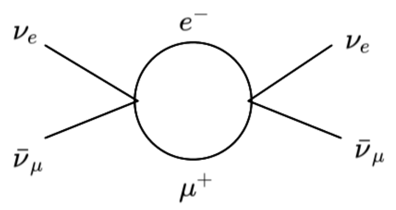
QED cross-sections scale as $1/E^2$ as $E \to \infty$. But Fermi theory cross-sections rise as E^2 as $E \to \infty$. This is a disaster! $|\mathcal{M}|^2$ is a probability, so it must be bounded. It can be shown quite generally that any cross section can be decomposed using the partial wave approximation:

$$\sigma = \frac{4\pi}{E^2}\sum_j(2j+1)|f_j|^2 \tag{59}$$

where $|f_j|^2 \leq 1$, which follows from $S^\dagger S = 1$. In other words, we define contributions $\sigma_j \leq \frac{4\pi}{E^2}(2j+1)$. We can see from dimensional arguments that the Lagrangian leads to unitarity violation. $[G_F] = -2$ and $[\sigma] = -2$ so because E is the only scale then $\sigma \sim G_F^2 E^2$. We can see that as $E \to \infty$ we will end up with infinite cross sections, violating unitarity.

3.2 Renormalisability

We need to draw all the possible Feynman diagrams. Consider, for example, $\nu_e\bar{\nu}_\mu \to \nu_e\bar{\nu}_\mu$.



The vertex receives contributions from loop diagrams like the one shown here. But the loop gives contributions to the amplitude of the form

$$\sim \int d^4k \frac{\not{k}}{k^2}\frac{\not{k}}{k^2} \sim \int \frac{d^4k}{k^2}. \tag{60}$$

In other words, the loop leads to quadratic divergences. But there is no $\bar{\nu}\nu\bar{\nu}\nu$ term in $\mathcal{L}$ to cancel this.

3.3 Power counting

Let's take QED as an example. Consider a graph with L loops, I_F fermion lines and I_B photon lines. Considering the structure of contributions from Feynman rules, we can define the "superficial degree of divergence", D, to be

$$D = 4L - I_F - 2I_B. \tag{61}$$

The logic here is that each loop contributes a $\int d^4k$ term, i.e. $+4$ powers of k. This gives the $4L$ term. Likewise, each fermion propagator contributes $\frac{\not{k}}{k^2}$, i.e. -1 powers of k, and each boson propagator contributes $\frac{1}{k^2}$, i.e. -2 powers of k. If $D \geq 0$ then we would naively expect the graphs to lead to divergences. We can also impose the general result from graph theory that for L loops, I internal lines and V vertices:

$$L = I - V + 1. \tag{62}$$

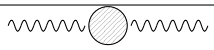
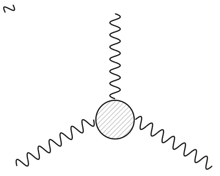
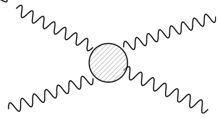
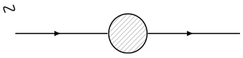
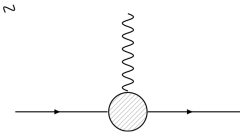
The proof is by induction and left as an exercise. This means that

$$D = 2I_B + 3I_F - 4V + 4. \tag{63}$$

Moreover, for QED we have the additional ammunition that there is only one type of vertex, joining two fermions and a photon. So, counting the ends of lines and labelling external lines E:

$$\begin{aligned} 2I_B + E_B &= V \\ 2I_F + E_F &= 2V \\ \text{so } D &= V - E_B + \frac{3}{2}(2V - E_F) - 4V + 4 \\ &= 4 - E_B - \frac{3}{2}E_F \end{aligned} \tag{64}$$

i.e. V cancels! D is independent of V and the only divergent graphs ($D \geq 0$) have corresponding counterterms which can regulate the divergences. The table enumerates the cases for QED where $D \geq 0$, explaining how each of these terms does in fact not lead to a divergence in this instance. In summary, QED is power-counting renormalisable.

E_B	E_F	Diagram	Comments
2	0		$F_{\mu\nu}F^{\mu\nu}$, i.e. Z_3
3	0		Photon is odd under C so this gives 0 contribution
4	0		Actually finite (gauge invariance)
0	2		$\bar{\psi}\partial\!\!\!/\psi + \bar{\psi}m\psi$, i.e. Z_2 and Z_m
2	1		$\bar{\psi}A\!\!\!/\psi$, i.e. Z_1

Note that if $E_F = 4$ and $E_B = 0$ then $D = -2$, i.e. the the four-fermion vertex is convergent.

Now let's move on from QED and consider $\mathcal{L}_{4F}$ instead. We can write the expression

$$2I_F + E_F = 4V \tag{65}$$

so, if we only have fermions

$$\begin{aligned} D &= \frac{3}{2}(4V - E_F) - 4V + 4 \\ &= 4 + 2V - \frac{3}{2}E_F. \end{aligned} \tag{66}$$

In other words,

1. every time we add a vertex, the graph becomes *more* divergent
2. graphs with *any* number of external fermions will diverge at a high enough order.

For example, if $E_F = 6$, there are divergences for $V = 3$. So $\mathcal{L}_{4F}$ is non-renormalisable. Adding bosons doesn't help the situation; we can trace the problem again to $[G_F] = -2$.

Theorem: A theory is only power counting renormalisable if all the couplings have mass dimension ≥ 0.
Proof: Consider a vertex V_{bf} with b boson lines, f fermion lines and p derivatives. The corresponding term in the Lagrangian is $g\phi^b\psi^f\partial^p$. The dimension of the coupling is

$$d^p_{bf} = 4 - b - \frac{3}{2}f - p \tag{67}$$

because $[\mathcal{L}] = 4$, $[\phi] = [A] = 1$, $[\psi] = 3/2$ and $[\partial] = 1$. However,

$$\begin{aligned}
2I_B + E_B &= \sum bV^p_{bf} \\
2I_F + E_F &= \sum fV^p_{bf} \\
\text{so } D &= \sum bV^p_{bf} - E_B + \frac{3}{2}\big(\sum fV^p_{bf} - E_F\big) - 4\sum V^p_{bf} + 4 + \sum pV^p_{bf} \\
&= \sum \big(b + \frac{3}{2}f + p - 4\big)V^p_{bf} - E_B - \frac{3}{2}E_F + 4 \\
&= -\sum d^p_{bf}V^p_{bf} - E_B - \frac{3}{2}E_F + 4.
\end{aligned} \tag{68}$$

This means that for renormalisability we need $d^p_{bf} \geq 0$ so that D increases as V increases.
Corollary: For a theory with scalar, fermion and vector fields, the only vertices allowed in a renormalisable Lorentz invariant Lagrangian are:

Pure scalar	ψ^3, ψ^4
Yang-Mills	$\partial_\mu A_\nu A^\mu A^\nu$, $A_\mu A_\nu A^\mu A^\nu$
Gauge-scalar interactions	$\phi A_\mu A^\mu$, $\phi^2 A_\mu A^\mu$, $\phi\partial_\mu\phi A^\mu$
Yukawa	$\phi\bar{\psi}\psi$
Gauge interactions	$\bar{\psi}\not{A}\psi$

Proof: $d^p_{bf} \geq 0$ only for the above. Note

- you need at least three fields
- $A_\mu A^\mu A^\nu$, $\phi^2\partial_\mu\phi$ and $\phi^2 A_\mu$ are not Lorentz invariant.

3.4 Intermediate vector bosons

Power counting renormalisability, chirality and Lorentz invariance suggest a vertex of the form $g\bar{\psi}_L\not{A}\psi_L$. Such a coupling is dimensionless so, following arguments from the previous section, unitarity is possible. This is the only possible coupling for left-handed fermions: the Yukawa coupling $\phi\bar{\psi}\psi = \phi(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$ so is not a left-handed-only coupling. The charged currents are

$$\begin{aligned}
\frac{1}{2}J_\mu &= \bar{\nu}_e\gamma_\mu e_L + ... + \bar{u}_L\gamma_\mu d'_L + ... \qquad \Delta Q = +1 \\
\frac{1}{2}J^\dagger_\mu &= \bar{e}_L\gamma_\mu\nu_e + ... + \bar{d}'_L\gamma_\mu u_L + ... \qquad \Delta Q = -1
\end{aligned} \tag{69}$$

so we need *two* charged vector fields

$$W^\mu_\pm = \frac{1}{\sqrt{2}}(W^\mu_1 \pm iW^m_2 u) \tag{70}$$

where W^μ_1 and W^μ_2 are real. Equivalently, we could define one complex field W^μ, such that $W^\mu_+ = W^\mu$ and $W^\mu_- = W^{\mu\dagger}$. The charged current Lagrangian can be written

$$\begin{aligned}
\mathcal{L}_{CC} &= \frac{g}{2\sqrt{2}}(J^\dagger_\mu W^\mu + W^\dagger_\mu J^\mu) \\
&= \frac{g}{\sqrt{2}}\bar{\nu}_e\not{W}e_L + \frac{g}{\sqrt{2}}\bar{e}_L\not{W}^\dagger\nu_e + ...
\end{aligned} \tag{71}$$

where the factor of $2\sqrt{2}$ is conventional. g is the (real) dimensionless coupling, and is the same for every term in J_μ (universality).

W_μ is different from the A_μ in electromagnetism:

- W_μ is charged, and thus complex
- W_μ must be massive, because CC weak interactions are short range

In 1936 the Romanian physicist Alexandru Proca wrote down a Lagrangian to describe W^μ, consisting of a kinetic term and a mass term:

$$\begin{aligned}\mathcal{L}_W &= -\frac{1}{2}(\partial_\mu W_\nu - \partial_\nu W_\mu)^\dagger(\partial^\mu W^\nu - \partial^\nu W^\mu) + m_W^2 W_\mu^\dagger W^\mu \\ &\equiv -\frac{1}{2}W_{\mu\nu}^\dagger W^{\mu\nu} + m_W^2 W_\mu^\dagger W^\mu.\end{aligned} \tag{72}$$

So this gives an equation of motion

$$\begin{aligned}&(\partial^2 + m_W^2)W^\mu - \partial^\mu\partial^\nu W_{\mu\nu} = 0 \\ &\implies \partial_\mu W^\mu = 0 \text{ if } m_W \neq 0.\end{aligned} \tag{73}$$

We can look for plane wave solutions of the form $W^\mu = \epsilon^\mu e^{-ik\cdot x}$:

$$\begin{aligned}(-k^2 + m_W^2)\epsilon^\mu + k^\mu k_\mu \epsilon^\nu &= 0 \\ \implies m_W^2 k_\mu \epsilon^\mu &= 0\end{aligned} \tag{74}$$

where we have taken the physical solution in the last line. Since W is massive it has three spin states: in the rest frame $k^\mu = (m_W, 0)$

$$\begin{aligned}&\epsilon_r^{\|} = (0, \underline{\epsilon}_r) \qquad r = 1,2,3 \\ \text{where} \qquad &\underline{\epsilon}_1 = (1,0,0) \\ &\underline{\epsilon}_2 = (0,1,0) \\ &\underline{\epsilon}_3 = (0,0,1).\end{aligned} \tag{75}$$

$\underline{\epsilon}_1$ and $\underline{\epsilon}_2$ are the transverse components and can be combined to form $\underline{\epsilon}_\pm = 1/\sqrt{2}(1, \pm i, 0)$, and $\underline{\epsilon}_3$ is the longitudinal component.

So if $k^\mu = (E, 0, 0, k)$, $\epsilon_3 = 1/m_W(k, 0, 0, E)$. N.B. for $E \gg m_W$, $\epsilon_3 \approx k^\mu/m_W + \mathcal{O}(m_W/E)$. Now let's evaluate the different polarisation sums: defining $\epsilon_0^\mu = k^\mu/m_W$ (so in the rest frame $\epsilon_0^\mu = (1, \underline{0})$ which is timelike, and therefore unphysical)

$$\begin{aligned}&\sum_{r=0,1,2,3} \eta^{rr}\epsilon_r^\mu \epsilon_r^{\nu *} = -\eta^{\mu\nu} \qquad \text{completeness: cf RQFT} \\ \text{so} \ &\sum_{r=1,2,3} \epsilon_r^\mu \epsilon_r^{\nu *} = -\eta^{\mu\nu} + \epsilon_0^\mu \epsilon_0^\nu = -\eta^{\mu\nu} + \frac{k^\mu k^\nu}{m_W^2}\end{aligned} \tag{76}$$

The propagator is

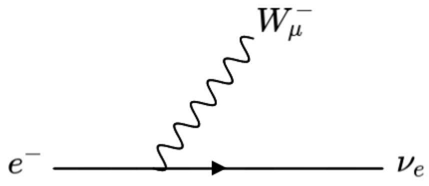
$$\Delta_W^{\mu\nu} = \frac{i(-\eta^{\mu\nu} + k^\mu k^\nu/m_W^2)}{k^2 - m_W^2 + i\epsilon} \tag{77}$$

because

$$((-k^2 + m_W^2)\eta_{\mu\nu} + k_\mu k_\nu)\Delta_W^{\nu\rho} = -i\delta_\mu^\rho \tag{78}$$

(check this as an exercise).

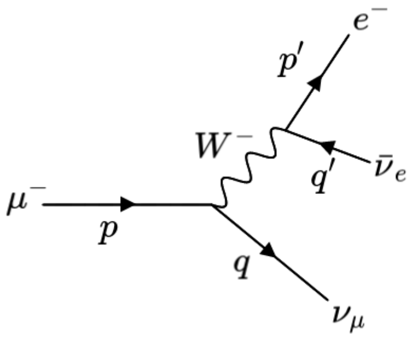
The Feynman rule for the interaction is



$$i\frac{g}{\sqrt{2}}\gamma_\mu\frac{1}{2}(1-\gamma^5) \tag{79}$$

How does this theory work in practice? The following are some examples.

3.4.1 $\mu^- \to e^-\bar{\nu}_e\nu_\mu$



The momentum of W^-, $k = p - q = p' + q'$.

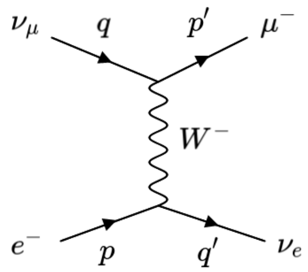
$$\begin{aligned} \mathcal{M} &= \frac{(ig)^2}{8}\bar{u}(p')\gamma_m u(1-\gamma^5)v(q')\frac{(-\eta^{\mu\nu}+k^\mu k^\nu/m_W^2)}{k^2-m_W^2}\bar{u}(q)\gamma_\nu(1-\gamma^5)u(p) \\ \text{But} \qquad & \bar{u}(q)\not{k}(1-\gamma^5)u(p) = \bar{u}(q)(\not{p}-\not{q})(1-\gamma^5)u(p) = m_\mu\bar{u}(q)(1+\gamma^5)u(p) \\ & \bar{u}(p')\not{k}(1-\gamma^5)v(q') = \bar{u}(p')(\not{p}'+\not{q}')(1-\gamma^5)v(q') = -m_e\bar{u}(p')(1+\gamma^5)v(q') \end{aligned} \tag{80}$$

so the $k^\mu k^\nu/m_W^2$ term in the propagator tends to $m_e m_\mu/m_W^2 \sim 10^{-8}$ ($m_W = 75$ GeV). Also $k^2 = m_\mu^2 - 2p\cdot q < m_\mu^2 \ll m_W^2$. If we ignore these terms,

$$\frac{1}{2}\sum_{spins}|\mathcal{M}|^2 = \frac{2g^4}{m_W^4}(p\cdot q')(p'\cdot q) \tag{81}$$

i.e. this is the same as in 4-Fermi theory, provided that $G_F/\sqrt{2} = g^2/8m_W^2$. So the weak force is weak because m_W is large ($\gg 1$ GeV). It is easy to see that the same arguments apply to *all* weak decays in Chapter 1.

3.4.2 $\nu_\mu e^- \to \mu^-\nu_e$



Here $k = p = q' = p' - q$ and

$$\mathcal{M} = \frac{(ig)^2}{8}\bar{u}(q')\gamma_\mu(1-\gamma^5)u(p)\frac{i(-\eta^{\mu\nu}+k^\mu k^\nu/m_W^2)}{k^2-m_W^2}\bar{u}(p')\gamma_\nu(1-\gamma^5)u(q). \tag{82}$$

As before, $k^\mu k^\nu/m_W^2 \to m_e m_\mu/m_W^2$ so we can ignore these terms. But $k^2 = (p-q')^2 = t$ so

$$\frac{d\sigma}{d\Omega} = \frac{g^4}{128\pi^2}\frac{s}{(t-m_W^2)^2} \tag{83}$$

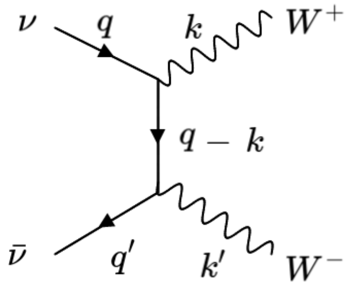
where $s = 4E^2$ and $t = -4E^2\sin^2\theta/2$. Integrating over ϕ,

$$\begin{aligned}\frac{d\sigma}{d\cos\theta} &= \frac{g^4 E^2}{64\pi(E^2\sin^2\theta/2 + m_W^2/4)^2} \\ &= \frac{g^4}{64\pi E^2}\csc^2\frac{\theta}{2}\left(1+\mathcal{O}(\frac{m_W^2}{E^2})\right).\end{aligned} \tag{84}$$

At very high energy, $E^2 \gg m_W^2$ so unitarity is restored.

3.4.3 $\nu\bar{\nu} \to W^+W^-$

Consider now the production of external $W^\pm$. In this case $q + q' = k + k'$.



$$\mathcal{M} = \frac{(ig)^2}{8}\bar{v}(q')\gamma_\mu(1-\gamma^5)\frac{i(\not{q}-\not{k}+m_e)}{(q-k)^2-m_e^2}\gamma_\nu(1-\gamma^5)u(q)\epsilon_r^\mu(k')^*\epsilon_r^\nu(k) \tag{85}$$

We want the behaviour of the cross-section at very high energy, so we can ignore m_e. Also consider the Ws to be longitudinally polarised, i.e. $\epsilon_3^\mu(k) \approx k^\mu/m_W + \mathcal{O}(m_W/E)$. So

$$\mathcal{M} = \frac{-ig^2}{4}\frac{1}{m_W^2(q-k)^2}\bar{v}(q')\not{k}'(\not{q}-\not{k})\not{k}(1-\gamma^5)u(q) \tag{86}$$

But $\not{q}u(q) = 0$ so

$$\begin{aligned} \not{k}u(q) &= -(\not{q}-\not{k})u(q) \\ \text{similarly} \qquad \bar{v}(q')\not{k}' &= \bar{v}(q')(\not{q}-\not{k})\end{aligned} \tag{87}$$

so if $p = q - k$, and recalling $\not{q}u(q) = 0$)

$$\begin{aligned}\mathcal{M} &= \frac{-ig^2}{4}\frac{1}{m_W^2 p^2}\bar{v}(q')\not{p}\not{p}\not{p}(1-\gamma^5)u(q) \\ &= +\frac{ig^2}{4}\bar{v}(q')\not{k}(1-\gamma^5)u(q)\end{aligned} \tag{88}$$