Suppose we have

$$
\begin{aligned}
O(W') &= \sqrt{\frac{|\langle \Psi_{W'}|\Phi\rangle|^2}{\langle \Psi_{W'}|\Psi_{W'}\rangle\langle\Phi|\Phi\rangle}} \\
&= \sqrt{\left\langle \frac{\Phi(B)}{\Psi_{W'}(B)} \right\rangle_\Psi \left\langle \frac{\Psi_{W'}(B)}{\Phi(B)} \right\rangle_\Phi^*}
\end{aligned}
$$

where $\langle f(B)\rangle_A \equiv \sum_B f(B)\frac{|A(B)|^2}{\sum_B |A(B)|^2}$ is the expectation value over variational state A.

Define Loss function as $L(W') = -\log O(W')$, we want to calculate the derivative.

$$
\partial_{p_k} L(W') = -\frac{1}{2}\partial_{p_k}[\log(\left\langle \frac{\Phi(B)}{\Psi_{W'}(B)} \right\rangle_\Psi) + \log(\left\langle \frac{\Psi_{W'}(B)}{\Phi(B)} \right\rangle_\Phi^*)]
$$

Lemma 1

Define $O_k(B) \equiv \partial_{p_k}\log(\Psi(B))$ and $O_k^*(B) \equiv \partial_{p_k}\log(\Psi^*(B))$, and

$$
\partial_{p_k}\langle f\rangle_\Psi = \langle \partial_{p_k} f\rangle_\Psi + \langle f(B)(O_k(B) + O_k^*(B))\rangle_\Psi - \langle f\rangle_\Psi\langle O_k(B) + O_k^*(B)\rangle_\Psi
$$

Proof:

$$
\langle f\rangle_\Psi = \frac{\sum_B f(B)|\Psi(B)|^2}{\sum_B |\Psi(B)|^2}
$$

Then the derivative can be decomposed into 3 parts, where

$$
\begin{aligned}
\partial_{p_k}\langle f\rangle_\Psi = &\frac{\sum_B[\partial_{p_k} f(B)]|\Psi(B)|^2}{\sum_B |\Psi(B)|^2} \\
&+ \frac{\sum_B f(B)[\partial_{p_k}|\Psi(B)|^2]}{\sum_B |\Psi(B)|^2} \\
&+ \frac{\sum_B f(B)|\Psi(B)|^2}{\partial_{p_k}\sum_B |\Psi(B)|^2}
\end{aligned}
$$

The first part is $\langle \partial_{p_k} f\rangle_\Psi$
The second part:

$$
\begin{aligned}
\frac{\sum_B f(B)[\partial_{p_k}|\Psi(B)|^2]}{\sum_B |\Psi(B)|^2} &= \frac{\sum_B f(B)[\Psi^*(B)\partial_{p_k}\Psi(B) + \Psi(B)\partial_{p_k}\Psi^*(B)]}{\sum_B |\Psi(B)|^2} \\
&= \frac{\sum_B f(B)|\Psi(B)|^2\partial_{p_k}[\log(\Psi(B)) + \log(\Psi^*(B))]}{\sum_B |\Psi(B)|^2} \\
&= \langle f(B)(O_k(B) + O_k^*(B))\rangle_\Psi
\end{aligned}
$$

The third part:

$$
\begin{aligned}
\frac{\sum_B f(B)|\Psi(B)|^2}{\partial_{p_k}\sum_B |\Psi(B)|^2} &= -\frac{\sum_B f(B)|\Psi(B)|^2}{(\sum_B |\Psi(B)|^2)^2}(\partial_{p_k}\sum_B |\Psi(B)|^2) \\
&= -\frac{\sum_B f(B)|\Psi(B)|^2}{\sum_B |\Psi(B)|^2}\frac{\partial_{p_k}\sum_B |\Psi(B)|^2}{\sum_B |\Psi(B)|^2} \\
&= -\langle f\rangle_\Psi \frac{\sum_B |\Psi(B)|^2\partial_{p_k}(\log(\Psi(B)) + \log(\Psi^*(B)))}{\sum_B |\Psi(B)|^2} \\
&= -\langle f\rangle_\Psi \langle O_k(B) + O_k^*(B)\rangle_\Psi
\end{aligned}
$$

Continue calculate

$$
\partial_{p_k}L(W') = -\frac{1}{2}\partial_{p_k}[\log(\left\langle \frac{\Phi(B)}{\Psi_{W'}(B)} \right\rangle_\Psi) + \log(\left\langle \frac{\Psi_{W'}(B)}{\Phi(B)} \right\rangle_\Phi^*)]
$$

Calculate the first part

Using Lemma 1 to the first part $\partial_{p_k}\log(\left\langle \frac{\Phi(B)}{\Psi_{W'}(B)} \right\rangle_\Psi)$, then

$$
\begin{aligned}
\partial_{p_k} \log(\left\langle \frac{\Phi(B)}{\Psi_{W'}(B)} \right\rangle_\Psi) &= \frac{1}{\left\langle \frac{\Phi}{\Psi} \right\rangle_\Psi} \partial_{p_k} \left\langle \frac{\Phi(B)}{\Psi_{W'}(B)} \right\rangle_\Psi \\
&= \frac{1}{\left\langle \frac{\Phi}{\Psi} \right\rangle_\Psi} \left\langle \partial_{p_k} \frac{\Phi(B)}{\Psi_{W'}(B)} \right\rangle_\Psi \\
&+ \frac{1}{\left\langle \frac{\Phi}{\Psi} \right\rangle_\Psi} \left\langle \frac{\Phi(B)}{\Psi_{W'}(B)} \partial_{p_k} (\log(\Psi(B)) + \log(\Psi^*(B))) \right\rangle_\Psi \\
&- \frac{1}{\left\langle \frac{\Phi}{\Psi} \right\rangle_\Psi} \left\langle \frac{\Phi(B)}{\Psi_{W'}(B)} \right\rangle_\Psi \langle \partial_{p_k} (\log(\Psi(B)) + \log(\Psi^*(B))) \rangle_\Psi \\
&= -\frac{1}{\left\langle \frac{\Phi}{\Psi} \right\rangle_\Psi} \left\langle \frac{\Phi(B)}{\Psi_{W'}(B)} \partial_{p_k} \log(\Psi(B)) \right\rangle_\Psi \\
&+ \frac{1}{\left\langle \frac{\Phi}{\Psi} \right\rangle_\Psi} \left\langle \frac{\Phi(B)}{\Psi_{W'}(B)} \partial_{p_k} (\log(\Psi(B)) + \log(\Psi^*(B))) \right\rangle_\Psi \\
&- \langle \partial_{p_k} (\log(\Psi(B)) + \log(\Psi^*(B))) \rangle_\Psi \\
&= \frac{\left\langle \frac{\Phi(B)}{\Psi_{W'}(B)} \partial_{p_k} \log(\Psi^*(B)) \right\rangle_\Psi}{\left\langle \frac{\Phi}{\Psi} \right\rangle_\Psi} - \langle \partial_{p_k} (\log(\Psi(B)) + \log(\Psi^*(B))) \rangle_\Psi
\end{aligned}
$$

Write it in the observable form, we get

$$
\partial_{p_k} \log(\left\langle \frac{\Phi(B)}{\Psi_{W'}(B)} \right\rangle_\Psi) = \frac{\left\langle \frac{\Phi(B)}{\Psi_{W'}(B)} O_k^*(B) \right\rangle_\Psi}{\left\langle \frac{\Phi}{\Psi} \right\rangle_\Psi} - \langle O_k(B) + O_k^*(B) \rangle_\Psi
$$

Calculate the second part

The second part $\partial_{p_k} \log(\left\langle \frac{\Psi_{W'}(B)}{\Phi(B)} \right\rangle_\Phi^*)$ can be easily calculated by

$$
\partial_{p_k} \log(\left\langle \frac{\Psi_{W'}(B)}{\Phi(B)} \right\rangle_\Phi^*) = \frac{\left\langle \frac{\partial_{p_k} \Psi_{W'}(B)}{\Phi(B)} \right\rangle_\Phi^*}{\left\langle \frac{\Psi_{W'}(B)}{\Phi(B)} \right\rangle_\Phi^*}
$$

Combining together

Combine the first and second parts together, we get

$$\partial_{p_k} L(W') = \frac{1}{2}\left\langle O_k(B) + O_k^*(B)\right\rangle_\Psi - \frac{1}{2}\frac{\left\langle \frac{\Phi(B)}{\Psi(B)} O_k^*(B)\right\rangle_\Psi}{\left\langle \frac{\Phi}{\Psi}\right\rangle_\Psi} - \frac{1}{2}\frac{\left\langle \frac{\partial_{p_k}\Psi_{W'}(B)}{\Phi(B)}\right\rangle_\Phi^*}{\left\langle \frac{\Psi_{W'}(B)}{\Phi(B)}\right\rangle_\Phi^*}$$

Which is not consistent with the result in the original paper, that is

$$\partial_{p_k} L(\Psi_{\mathscr{W}'}) = \langle \mathscr{O}_k^\star(\mathscr{B})\rangle_\Psi - \frac{\left\langle \frac{\Phi(\mathscr{B})}{\Psi(\mathscr{B})} \mathscr{O}_k^\star(\mathscr{B})\right\rangle_\Psi}{\left\langle \frac{\Phi(\mathscr{B})}{\Psi(\mathscr{B})}\right\rangle_\Psi}, \tag{B3}$$

Possible solution

Maybe we should turn into loss function as

$$L(W') = \frac{\langle \Psi_{W'}|\Phi\rangle}{\langle \Psi_{W'}|\Psi_{W'}\rangle}\sqrt{\langle \Psi_{W'}|\Psi_{W'}\rangle}$$

and this time the derivative should give us expectation only depends on distribution $\Psi_{W'}$

Appendix: Result in original paper

$$\begin{aligned}
\mathrm{O}(\mathscr{W}') &= \sqrt{\frac{|\langle \Psi_{\mathscr{W}'}|\Phi\rangle|^2}{\langle \Psi_{\mathscr{W}'}|\Psi_{\mathscr{W}'}\rangle\langle \Phi|\Phi\rangle}} \\
&= \sqrt{\left\langle \frac{\Phi(\mathscr{B})}{\Psi_{\mathscr{W}'}(\mathscr{B})}\right\rangle_\Psi \left\langle \frac{\Psi_{\mathscr{W}'}(\mathscr{B})}{\Phi(\mathscr{B})}\right\rangle_\Phi^\star},
\end{aligned} \tag{B1}$$

$$L(\mathscr{W}') = -\log \mathrm{O}(\mathscr{W}'), \tag{B2}$$

$$\partial_{p_k} L(\Psi_{\mathscr{W}'}) = \langle \mathscr{O}_k^\star(\mathscr{B}) \rangle_\Psi - \frac{\left\langle \frac{\Phi(\mathscr{B})}{\Psi(\mathscr{B})} \mathscr{O}_k^\star(\mathscr{B}) \right\rangle_\Psi}{\left\langle \frac{\Phi(\mathscr{B})}{\Psi(\mathscr{B})} \right\rangle_\Psi}, \tag{B3}$$

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