MAT3007: Optimization - Assignment 4

Ran Hu 116010078

24 October, 2018

Problem 1

1. Suppose that to achieve the ideal nutrition intake and the least cost, an adult should take x_1 bread, x_2 milk, x_3 fish, and x_4 potato. The optimization problem can be written as:

$$\begin{aligned} \text{minimize} \quad & 2x_1 + 3.5x_2 + 8x_3 + 1.5x_4 \\ \text{subject to} \quad & 4x_1 + 6x_2 + 20x_3 + x_4 \geq 25 \\ & 8x_1 + 12x_2 + 3x_4 \geq 40 \\ & 130x_1 + 120x_2 + 150x_3 + 70x_4 \geq 400 \\ & x_1,\ x_2,\ x_3,\ x_4 \geq 0 \end{aligned}$$

The first constraint means that the total intake of protein should be more than 25 grams; The second constraint means that the total intake of carbohydrates should be more than 40 grams; The third constraint means that the total intake of calories should be more than 400; and each of the decision variables, which is the intake amount of each one of the four types of food, should be greater than 0.

2. The MATLAB code is:

```
A = [4 6 20 1;
    8 12 0 30;
    130 120 150 70]
c = [25; 40; 400]
b = [2, 3.5, 8, 1.5]
n = 4

cvx_begin quiet
variable x(n);
    minimize b*x
    subject to
        A*x >= c;
        x >= 0
cvx_end

x
optval = cvx_optval
```

The optimal solution is: $x_1 = 1.55,\ x_2 = 0,\ x_3 = 0.894,\ x_4 = 0.92$. The optimal value is 11.632 .

3. The dual problem with three dual variables $y_1,\ y_2,\ y_3$ is:

$$\begin{aligned} \text{maximize} \quad & 25y_1 + 40y_2 + 400y_3 \\ \text{subject to} \quad & 4y_1 + 8y_2 + 130y_3 \le 2 \\ & 6y_1 + 12y_2 + 120y_3 \le 3.5 \\ & 20y_1 + 150y_3 \le 8 \\ & y_1 + 3y_2 + 70y_3 \le 1.5 \\ & y_1,\ y_2,\ y_3 \ge 0 \end{aligned}$$

A possible interpretation of the dual problem is as follows. A pharmaceutical company produces and cells nutrients in pill, where a gram of protein has a price y_1,a gram of carbohydrates has a price y_2, one calory has a price y_3. People will buy these pills instead of all the foods if they can get the nutrients with lower prices. At the same time, the pharmaceutical company wants to maximize their profit by maximizing the prices under the constraints, which is what the dual problem looks for.

4. The MATLAB code is:

```
A = A.'
c = c.'
b = b.'
m = 3

cvx_begin quiet
variable y(m);
    maximize c*y
    subject to
        A*y <= b;
        y >= 0
cvx_end

y
optval = cvx_optval
```

The optimal solution is: $y_1 = 0.3904,\ y_2 = 0.0340,\ y_3 = 0.0013$. The optimal value is 11.632 .

Problem 2

The dual problem with seven dual variables $y_1,\ y_2,\ y_3,\ y_4,\ y_5,\ y_6,\ y_7$ is:

$$\begin{aligned} \text{maximize} \quad & d_1y_1 + d_2y_2 + d_3y_3 + d_4y_4 + d_5y_5 + d_6y_6 + d_7y_7 \\ \text{subject to} \quad & y_1 + y_2 + y_3 + y_4 + y_5 \le 1 \\ & y_2 + y_3 + y_4 + y_5 + y_6 \le 1 \\ & y_3 + y_4 + y_5 + y_6 + y_7 \le 1 \\ & y_1 + y_4 + y_5 + y_6 + y_7 \le 1 \\ & y_1 + y_2 + y_5 + y_6 + y_7 \le 1 \\ & y_1 + y_2 + y_3 + y_6 + y_7 \le 1 \\ & y_1 + y_2 + y_3 + y_4 + y_7 \le 1 \\ & y_1,\ y_2,\ y_3,\ y_4,\ y_5,\ y_6,\ y_7 \ge 0 \end{aligned}$$

Here is the possible interpretation of the dual problem. Suppose there is a firm offering nursing service for hospitals. y_i is the price this firm charges for offering a service that one nurse do on day i . The firm wants to offer the highest price for one week's total services, and the dual problem finds that price. The price that the hospital pays the firm for a single service that one nurse do from day 1 to day 5 should be less than the original price the hospital pay for a nurse who work 5 days, so $y_1 + y_2 + y_3 + y_4 + y_5 \leq 1$. Similarly, it is the same for each of the day shifts.

Problem 3

1. Assume there is an imaginary node 5, with edges (5, 1) and (4, 5). There is no capacity constraint on those edges. One wants to maximize the flow from 5 to 1, which we denote by Δ . Let x_{ij} denote the amount of flow across edge (i, j), E denote the set of edges. The LP formulation is:

$$
\begin{array}{rl}
\text{maximize} & \Delta \\
\text{subject to} & \sum_{j:(j,i)\in E} x_{ji} - \sum_{j:(i,j)\in E} x_{ij} = 0, \ \forall\, i \neq s, t \\
& \sum_{j:(j,s)\in E} x_{js} - \sum_{j:(s,j)\in E} x_{sj} + \Delta = 0 \\
& \sum_{j:(j,t)\in E} x_{jt} - \sum_{j:(t,j)\in E} x_{tj} - \Delta = 0 \\
& x_{ij} \leq w_{ij}, \ \forall\, (i,j) \in E \\
& x_{ij} \geq 0, \ \forall\, (i,j) \in E
\end{array}
$$

The MATLAB code is:

```
W = [0 8 7 0;
     0 0 2 4;
     0 0 0 12;
     0 0 0 0];

cvx_begin quiet
variables x(4,4) D;
    maximize D
    subject to
        (x.' - x)*ones(4,1) == [-D; 0; 0; D];
        x <= W;
        x >= 0
cvx_end

x
D
```

The optimal solution is $x_{12} = 6, \ x_{13} = 7, \ x_{23} = 2, \ x_{24} = 4, \ x_{34} = 9$. The optimal value is 13.

2. The dual problem is:

$$\begin{array}{ll} \text{minimize} & \sum_{(i,j)\in E} w_{ij} z_{ij} \\ \text{subject to} & z_{ij} \geq y_i - y_j, \ \forall (i,j) \in E \\ & y_s - y_t = 1 \\ & z_{ij} \geq 0 \\ & 0 \leq y_i, y_j \leq 1, \ \forall (i,j) \in E \end{array}$$

where we want to find the minimum cut of edges.

```
cvx_begin quiet
variables z(4,4) y(4,1);
    minimize sum(sum(W.*z))
    subject to
        z >= y*ones(1,4) - (y*ones(1,4)).';
        y(1) - y(4) == 1;
        z >= 0;
        0 <= y <= 1
cvx_end

z
optval = cvx_optval
```

The optimal solution is $z_{13} = 1, \ z_{23} = 1, \ z_{24} = 1, \ and \ z_{ij} = 0, \ \forall (i,j) \in E \ and \ (i,j) \notin \{(1,3),(2,3),(2,4)\}$. The optimal value is 13.

3. The maximum flow and minimum cut of the solution are both 13.

Problem 4

The first one is an LP program:

$$\begin{array}{ll} \text{maximize} & 0 \\ \text{subject to} & Ax \leq b \end{array}$$

which has a dual problem:

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & y^T A = 0 \\ & y \geq 0 \end{array}$$

If the first one has a solution, then the dual problem should also has a solution.

But if there exists y such that $y^T A = 0, \ b^T y < 0, \ y \geq 0$, then by scaling this y, we can make $b^T y$ arbitrarily small, which means the dual problem must be unbounded. For an LP, if the dual problem is unbounded, then the primal problem must be infeasible, vise versa.

Therefore, these two cannot both have solutions.

If the second one is infeasible, which means $b^T y \geq 0$, then the LP problem

$$\begin{array}{rc} \text{minimize} & b^T y \\ \text{subject to} & y^T A = 0 \\ & y \geq 0 \end{array}$$

is feasible. This means that its dual problem, which is the primal problem $Ax \leq b$, is also feasible.

In conclusion, exactly one of the systems has a solution.

Problem 5

1. The dual problem is:

$$\begin{array}{rc} \text{maximize}_y & -c^T y \\ \text{subject to} & M^T y \leq c \\ & y \geq 0 \end{array}$$

Because $M = -M^T$, the dual problem is equivalent to

$$\begin{array}{rc} \text{minimize}_y & c^T y \\ \text{subject to} & My \geq -c \\ & y \geq 0 \end{array}$$

which is equivalent to the primal problem.

2. ”if" $\Rightarrow$

   Because the primal and the dual problem are equivalent, according to the weak duality theorem, it is impossible that one is unbounded (feasible) and the other is infeasible, or one is infeasible and the other is unbounded (feasible) . Therefore, both of them should have a feasible solution that make them optimal. According to the strong duality theorem, if x is primal feasible, y is dual feasible, $x = y$, and $c^T x = c^T y = -c^T y = 0$, then they are both optimal.

   “only if" $\Rightarrow$

   According to the LP fundamental theorem, if there is an optimal solution, there is an optimal solution that is a basic feasible solution, and a basic feasible solution is also a feasible solution.

Therefore, the problem has optimal solution if and only if there is a feasible solution.