MAT3007
Assignment 4 Solution

Problem 1
1. Let x_i denote the amount of food $i = 1, \ldots, 4$. The LP formulation is as follows:

$$\begin{array}{ll} \text{minimize} & 2x_1 + 3.5x_2 + 8x_3 + 1.5x_4 \\ \text{s.t.} & 4x_1 + 6x_2 + 20x_3 + x_4 \geq 25 \\ & 8x_1 + 12x_2 + 30x_4 \geq 40 \\ & 130x_1 + 120x_2 + 150x_3 + 70x_4 \geq 400 \\ & x_i \geq 0, \quad \forall i = 1, \ldots, 4 \end{array}$$

The first three constraints are the minimum daily value requirements for each nutrient.

2. The optimal solution is $(1.55, 0, 0.894, 0.92)$, and the objective value is 11.632 at optimal.

3. Dual problem:

$$\begin{array}{ll} \text{maximize} & 25y_1 + 40y_2 + 400y_3 \\ \text{s.t.} & 4y_1 + 8y_2 + 130y_3 \leq 2 \\ & 6y_1 + 12y_2 + 120y_3 \leq 3.5 \\ & 20y_1 + 150y_3 \leq 8 \\ & y_1 + 30y_2 + 70y_3 \leq 1.5 \\ & y_i \geq 0, \quad \forall i = 1, \ldots, 3 \end{array}$$

Dual interpretation: A pharmaceutical company makes 3 kinds of pills. Each kind of pill contains 1 gram of a type of nutrient. The dual variable y_i represent the price for each pill of nutrient i. The company wants to sell the pills and maximize its profit. The objective function is the revenue of the company when consumers are buying the pills to satisfy their daily nutrient requirements. The constraints are restrictions on the prices of the pills to make it attractive to the customers. The total cost that can purchase nutrients equivalent to the food should not exceed the price of buying the food directly.

4. The optimal solution is (0.3904, 0.034, 0.0013), and the objective value is 11.632 at optimal.

Problem 2

Let y_i denote the dual variable for the ith constraint.

$$
\begin{array}{ll}
\text{maximize} & d_1y_1 + d_2y_2 + d_3y_3 + d_4y_4 + d_5y_5 + d_6y_6 + d_7y_7 \\
\text{s.t.} & y_1 + y_2 + y_3 + y_4 + y_5 \le 1 \\
& y_2 + y_3 + y_4 + y_5 + y_6 \le 1 \\
& y_3 + y_4 + y_5 + y_6 + y_7 \le 1 \\
& y_1 + y_4 + y_5 + y_6 + y_7 \le 1 \\
& y_1 + y_2 + y_5 + y_6 + y_7 \le 1 \\
& y_1 + y_2 + y_3 + y_6 + y_7 \le 1 \\
& y_1 + y_2 + y_3 + y_4 + y_7 \le 1 \\
& y_i \ge 0 \quad \forall i = 1, \ldots, 7
\end{array}
$$

Dual interpretation: Suppose a nurse service company is providing nursing service to the hospital. The company charges the hospital on daily basis. The dual variable y_i is the fraction of full time salary for a nurse to work on day i. So, if the full time salary is 1, the company charges the hospital with price y_i per nurse on day i. The company's goal is to maximize the total income. The dual problem finds the optimal prices the company should charge the hospital in order to maximize its revenue while being competitive. The objective function maximizes the total income from all the nurses providing nursing service. The constraints require the total charges for a nurse working from day i to day $i+5$ should not exceed the hospital's cost of hiring a nurse directly.

Problem 3

1. Let x_{ij} denote the flow on edge (i, j). Let Δ denote the flow from imaginary node 0 to 1. The LP formulation of this problem is as follows:

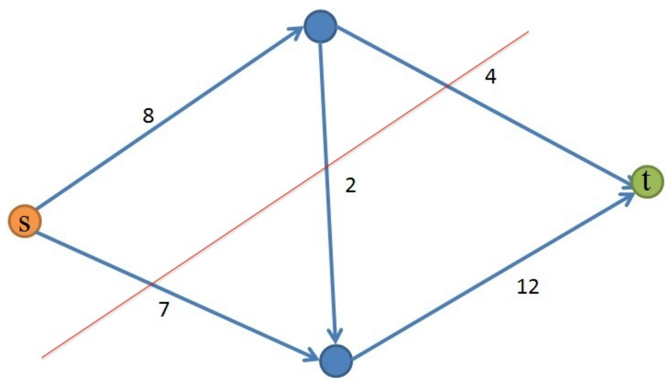
$$
\begin{array}{ll}
\text{maximize} & \Delta \\
\text{s.t.} & x_{12} - x_{23} - x_{24} = 0 \\
& x_{13} + x_{23} - x_{34} = 0 \\
& \Delta - x_{12} - x_{13} = 0 \\
& x_{24} + x_{34} - \Delta = 0 \\
& 0 \le x_{12} \le 8 \\
& 0 \le x_{13} \le 7 \\
& 0 \le x_{23} \le 2 \\
& 0 \le x_{24} \le 4 \\
& 0 \le x_{34} \le 12
\end{array}
$$

The optimal solutions are $x_{12} = 6, x_{13} = 7, x_{23} = 2, x_{24} = 4, x_{34} = 9$, and the objective value is 13 at optimal

2. Dual problem:

$$\begin{array}{ll} \text{minimize} & 8z_{12} + 7z_{13} + 2z_{23} + 4z_{24} + 12z_{34} \\ \text{s.t.} & z_{12} \geq y_1 - y_2 \\ & z_{13} \geq y_1 - y_3 \\ & z_{23} \geq y_2 - y_3 \\ & z_{24} \geq y_2 - y_4 \\ & z_{34} \geq y_3 - y_4 \\ & y_1 - y_4 = 1 \\ & z_{ij} \geq 0 \end{array}$$

Solving the problem using MATLAB, we get $y = (1, 1, 0, 0)$, $z = (0, 1, 1, 1, 0)$ and the optimal value is 13. The z_{ij} variables represent whether or not each edge is in the minimum cut. The cut is shown in the picture below.



Problem 4

First, we show that the two systems can't both have solutions. If so, we have

$$0 = \boldsymbol{y}^T A\boldsymbol{x} \leq \boldsymbol{y}^T \boldsymbol{b} < 0,$$

which is a contradiction.

Second, we show that if the second system is infeasible, then the first system must be feasible. We consider the following pair of linear optimization problems:

$$\begin{array}{ll} \min & \boldsymbol{b}^T \boldsymbol{y} \\ s.t. & A^T \boldsymbol{y} = \boldsymbol{0} \\ & \boldsymbol{y} \geq \boldsymbol{0}. \end{array}$$

The dual of this problem is

$$\begin{array}{ll} \max & 0 \\ s.t. & A\boldsymbol{x} \leq \boldsymbol{b} \end{array}$$

If the second system does not have a solution, then the primal problem can't attain negative objective value. In the meantime, $\boldsymbol{y} = 0$ is always a feasible solution for the primal problem with objective value 0. Therefore, $\boldsymbol{y} = 0$ must be an optimal solution to the primal problem. Then by the strong duality theorem, the dual problem must also be feasible. Thus, the result is proved.

Problem 5

1. The dual is as follows:

$$\begin{aligned} \max \quad & -\boldsymbol{c}^T\boldsymbol{y} \\ s.t. \quad & M^T\boldsymbol{y} \le \boldsymbol{c} \\ & \boldsymbol{y} \ge \boldsymbol{0}. \end{aligned}$$

Since $M = -M^T$, $M^T\boldsymbol{y} \le \boldsymbol{c}$ is equivalent to $M\boldsymbol{y} \ge -\boldsymbol{c}$. Therefore, we can reformulate the dual problem as a minimization problem as follows:

$$\begin{aligned} \min \quad & \boldsymbol{c}^T\boldsymbol{y} \\ s.t. \quad & M\boldsymbol{y} \ge -\boldsymbol{c} \\ & \boldsymbol{y} \ge \boldsymbol{0} \end{aligned}$$

Therefore, the dual problem is equivalent to the primal problem.

2. First, it is obvious that if the problem has optimal solution, then it must have a feasible solution. Now we prove the other direction. If the problem has a feasible solution $\boldsymbol{x}$, then $\boldsymbol{y} = \boldsymbol{x}$ is also feasible to the dual problem. Therefore, both the primal and dual problems are feasible. According to the weak duality theorem, they both have finite optimal solution.