MAT3007: Optimization - Assignment 2

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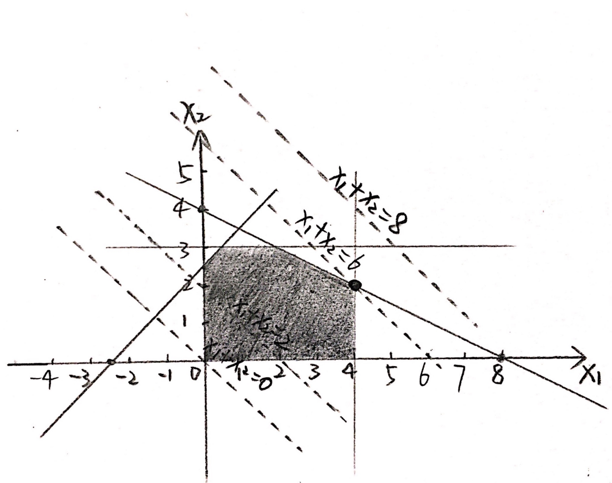
Problem 1

1. False. If the objective function is not related to solution , then any $x \geq 0$ are optimal solutions. For example, minimize 0, subject to $x \geq 0$, the optimal solution set is $[0, \infty)$, which is unbounded.

2. False. The objective function of LP in its standard form is minimize $c^T x$, if c $= 0$, which means c is a zero vector, then there can be $[0, n]$ $m \leq n$ variables that are positive.

3. True. If there are two optimal solutions, say x and $x^{'}$, then any convex combination of them are optimal solutions. Which means that if there is more than one optimal solution, then there are uncountably many optimal solutions.

Problem 2



Let x_1 be the horizontal axis, x_2 be the vertical axis.
The optimal solution is $x_1 = 4, x_2 = 2$, which gives an optimal result: 6.
Constrains $x_1 \leq 4$ and $x_1 + 2x_2 \leq 8$ are active at optimal solution.
The vertices of the feasible region are $(0,0), (0,2.5), (0.5,3), (2,3), (4,2), (4,0)$.

Problem 3

1. The optimization problem can be written as the following standard form:

$$\begin{aligned} \text{minimize} \quad & -\mathrm{x}_1 - 4\mathrm{x}_2 - \mathrm{x}_3 \\ \text{s.t.} \quad & 2\mathrm{x}_1 + 2\mathrm{x}_2 + \mathrm{x}_3 + \mathrm{s}_1 = 4 \\ & x_1 - x_3 - s_2 = 1 \\ & x_1, x_2, x_3, s_1, s_2 \geq 0 \end{aligned}$$

2. The constrain can also be written as $Ax = b$, where A $= \begin{bmatrix} 2 & 2 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & -1 \end{bmatrix}$, x $= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{bmatrix}$, b $= \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Because of the theorem that if there is an optimal solution, there is an optimal solution that is a basic feasible solution, I only need to prove that there exist a basic feasible solution with no more than 2 positive variables.
   Notice that there are two constrains. So we can choose any two independent columns of A to form A_B. Then $x_B = A_B^{-1}b$. The entries in x_B are the only non-zero entries in the solution x, where the other three entries in x are all zero. Therefore, x can have no more than 2 positive variables. So there must exist an optimal solution with no more than 2 positive variables.

3. Basic solutions are $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{5}{3} \\ 0 \\ \frac{2}{3} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2.5 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ -1 \end{bmatrix}$. Basic feasible solutions are $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{5}{3} \\ 0 \\ \frac{2}{3} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

4. The optimal solution is $x_1 = 1, x_2 = 1, x_3 = 0$, which gives an optimal value -5.

Problem 4

1. I use $\boldsymbol{x}$ to denote the number of shares of each security to purchase. A_i is the i th column vector of A. The optimization problem can be formed as follows:

$$\begin{aligned} \mathrm{maximize_x} \quad & \mathrm{min_i} - \pi^{\mathrm{T}}\mathrm{x} + \mathrm{A_i^T x}, \ \mathrm{i} = 1,2,3,4,5 \\ s.t. \quad & 0 \leq x \leq q \end{aligned}$$

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%% Problem 4

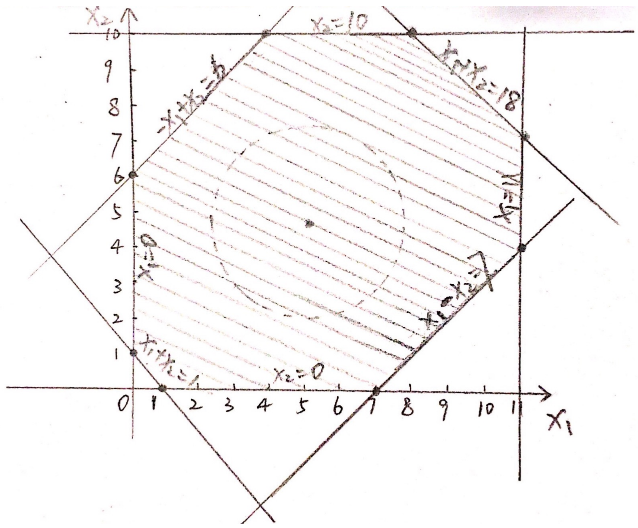
A = [1 1 1 0 0;
    0 0 0 1 1;
    1 0 1 0 1;
    1 1 1 1 0;
    0 1 0 1 1];
p = [0.75; 0.35; 0.40; 0.75; 0.65];
q = [10; 5; 10; 10; 5];
n = 5;

cvx_begin quiet
variables x(n) t;
    maximize (t - p'*x)
    subject to
        0 <= x <= q;
        t * ones(n,1) <= A'*x
cvx_end

x
t
optval = cvx_optval
```

2. The optimal solution is $\mathrm{x} = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \\ 5 \end{bmatrix}$. The optimal value is \$1. This means that buying 5 Security 3, 5 Security 4, 5 Security 5 will get an optimal payoff of \$1.

Problem 5



Let $\boldsymbol{y} = (y_1, y_2)$ be the center of the fountain, r be the radius. The optimization problem can be formed as:

$$\text{maximize}_{\mathrm{y_1,y_2}} \quad \mathrm{r}$$

$$\begin{aligned} \text{subject to} \quad \mathrm{y_1 + y_2 - 1} &\geq \sqrt{2}\mathrm{r} \\ 7 - y_1 + y_2 &\geq \sqrt{2}r \\ -y_1 + 11 &\geq r \\ 18 - y_1 - y_2 &\geq \sqrt{2}r \\ 6 + y_1 - y_2 &\geq \sqrt{2}r \\ 10 - y_2 &\geq r \\ y_2 &\geq r \\ y_1 &\geq r \\ r &\geq 0 \end{aligned}$$

The optimal solution is $\boldsymbol{y}$ = (5.3753, 4.8753). The optimal value is r = 4.5962. Attached is the MatLab code:

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%% Problem 5

a = [1 1 -1;
    -1 1 7;
    -1 0 11;
    -1 -1 18;
    1 -1 6;
    0 -1 10;
    0 1 0;
    1 0 0];
b = [sqrt(2); sqrt(2); 1; sqrt(2); sqrt(2); 1; 1; 1]

cvx_begin quiet
variables y(2) r;
    maximize r
    subject to
        a * [y(1);y(2);1] >= b * r
        r >= 0
cvx_end

y
optval = cvx_optval
```