MAT 3007
Assignment 6 Solution

Problem 1
1. By first order necessary condition $\nabla f(x^*) = 0$, we have

$$\begin{aligned}
&4x + y - 6 = 0 \\
&x + 2y - 7 + z = 0 \\
&2z - 8 + y = 0
\end{aligned}$$

Solving the above equations, we get a candidate point of the minimizer: $(\frac{6}{5}, \frac{6}{5}, \frac{17}{5})$.

2. We want to show $\nabla^2 f(x^*)$ is positive definite. We can check that all the eigenvalues of the Hessian matrix are all positive.

$$H = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The eigenvalues of H are $4.481, 2.689, 0.830$. Therefore, the matrix H is positive definite. By second order sufficient condition, the candidate point is indeed a local minimum.

3. Since this function is convex, the local minimizer is also the global minimizer.

Problem 2
1. Let r be the radius. Let (x, y) be the center of the circle. Let $R = r^2$. Problem formulation:

$$\begin{array}{ll}
\text{minimize}_{R,x,y} & R \\
\text{s.t.} & R \geq x^2 + y^2 \\
& R \geq (x-1)^2 + (y-5)^2 \\
& R \geq (x-2)^2 + (y-3)^2 \\
& R \geq (x-3)^2 + (y-1)^2
\end{array}$$

2. Let λ_i be the Lagrangian multiplier of the ith inequality. Lagrangian Function:

$$\begin{aligned}
\mathcal{L}(x, y, R, \lambda) = \; & R + \lambda_1(R - x^2 - y^2) + \lambda_2(R - (x-1)^2 - (y-5)^2) \\
& + \lambda_3(R - (x-2)^2 - (y-3)^2) + \lambda_4(R - (x-3)^2 - (y-1)^2)
\end{aligned}$$

KKT condition:

$$
\begin{aligned}
&\text{Main Condition:}\\
&1+\lambda_1+\lambda_2+\lambda_3+\lambda_4=0\\
&-2\lambda_1 x-2\lambda_2(x-1)-2\lambda_3(x-2)-2\lambda_4(x-3)=0\\
&-2\lambda_1 y-2\lambda_2(y-5)-2\lambda_3(y-3)-2\lambda_4(y-1)=0\\
&\text{Dual Feasibility:}\\
&\lambda_i\leq 0\\
&\text{Complementary Condition:}\\
&\lambda_1(R-x^2-y^2)=0\\
&\lambda_2(R-(x-1)^2-(y-5)^2)=0\\
&\lambda_3(R-(x-2)^2-(y-3)^2)=0\\
&\lambda_4(R-(x-3)^2-(y-1)^2)=0\\
&\text{Primal Feasibility:}\\
&R\geq x^2+y^2\\
&R\geq (x-1)^2+(y-5)^2\\
&R\geq (x-2)^2+(y-3)^2\\
&R\geq (x-3)^2+(y-1)^2
\end{aligned}
$$

Problem 3

Let ν be the Lagrangian multiplier for the equality constraint.
Lagrangian Function:

$$\mathcal{L}(x,\nu)=\textstyle\sum_i x_i\log x_i+\nu(\sum_i a_i x_i-1)$$

KKT conditions:

$$
\begin{aligned}
&\text{Main Condition:}\\
&\log x_i+1+\nu a_i\geq 0\quad\forall i\\
&\text{Complementary Condition:}\\
&x_i(\log x_i+1+\nu a_i)=0\\
&\text{Primal Feasibility:}\\
&\textstyle\sum_{i=1}^n a_i x_i-1=0\\
&x_i\geq 0
\end{aligned}
$$

Problem 4

Let y_i be the units of products after producing and meeting demands in the n-th month, also let z_i be the unmet demand. We have

$$
\begin{aligned}
&y_0=0\\
&y_{i+1}=\max(y_i+x_{i+1}-d_{i+1},0),\forall i=0,1,..11\\
&z_{i+1}=\max(d_{i+1}-y_i-x_{i+1},0),\forall i=0,1,..11
\end{aligned}
$$

The optimization problem can be formulated as:

$$
\begin{array}{ll}
\text{minimize}_{x_i,y_i,z_i} & \sum_{i=1}^{12}(x_i^2 + s\cdot y_i + k\cdot z_i) \\
\text{s.t.} & x_i \geq 0, \forall i = 1,2,..12 \\
 & x_i \leq r, \forall i = 1,2,..12 \\
 & y_0 = 0 \\
 & y_i \geq 0, \forall i = 1,2,..12 \\
 & y_i - (y_{i-1} + x_i - d_i) \geq 0, \forall i = 1,2,..12 \\
 & z_i \geq 0, \forall i = 1,2,..12 \\
 & z_i - (d_i - y_{i-1} - x_i) \geq 0, \forall i = 1,2,..12
\end{array}
$$

Lagrangian Function:

$$\mathcal{L}(x,\lambda,\eta) = \sum_{i=1}^{12}(x_i^2 + s\cdot y_i + k\cdot z_i + \lambda_i(x_i - r) + \eta_{1,i}(y_i - (y_{i-1} + x_i - d_i)) + \eta_{2,i}(z_i - (d_i - y_{i-1} - x_i)))$$

KKT condition:

Main Condition:

$$
\begin{array}{l}
2x_i + \lambda_i - \eta_{1,i} + \eta_{2,i} \geq 0, \forall i \\
s + \eta_{1,i} - \eta_{1,i+1} + \eta_{2,i+1} \geq 0, \forall i = 1,..11 \\
s + \eta_{1,12} \geq 0 \\
k + \eta_{2,i} \geq 0, \forall i
\end{array}
$$

Dual Feasibility:

$$
\begin{array}{l}
\lambda_i \geq 0 \\
\eta_{1,i}, \eta_{2,i} \leq 0
\end{array}
$$

Complementary Condition:

$$
\begin{array}{l}
\lambda_i(x_i - r) = 0 \\
\eta_{1,i}(y_i - (y_{i-1} + x_i - d_i)) = 0 \\
\eta_{2,i}(z_i - (d_i - y_{i-1} - x_i)) = 0 \\
x_i(2x_i + \lambda_i - \eta_{1,i} + \eta_{2,i}) = 0 \\
y_i(s + \eta_{1,i} - \eta_{1,i+1} + \eta_{2,i+1}) = 0, \forall i = 1,...11 \\
y_{12}(s + \eta_{1,12}) = 0 \\
z_i(k + \eta_{2,i}) = 0
\end{array}
$$

Primal Feasibility:

$$
\begin{array}{l}
x_i \geq 0, \forall i = 1,2,..12 \\
x_i \leq r, \forall i = 1,2,..12 \\
y_0 = 0 \\
y_i \geq 0, \forall i = 1,2,..12 \\
y_i - (y_{i-1} + x_i - d_i) \geq 0, \forall i = 1,2,..12 \\
z_i \geq 0, \forall i = 1,2,..12 \\
z_i - (d_i - y_{i-1} - x_i) \geq 0, \forall i = 1,2,..12
\end{array}
$$

Problem 5

Let λ_i be the Lagrangian multiplier of the ith constraint.

Lagrangian Function:

$$\mathcal{L}(x,\lambda) = 5x_1 + 2x_2 + 5x_3 + \lambda_1(2x_1 + 3x_2 + x_3 - 4) + \lambda_2(x_1 + 2x_2 + 3x_3 - 7)$$

KKT condition:

Main Condition:
$5 + 2\lambda_1 + \lambda_2 \geq 0$
$2 + 3\lambda_1 + 2\lambda_2 \geq 0$
$5 + \lambda_1 + 3\lambda_2 \geq 0$
Dual Feasibility:
$\lambda_i \leq 0$
Complementary Condition:
$\lambda_1(2x_1 + 3x_2 + x_3 - 4) = 0$
$\lambda_2(x_1 + 2x_2 + 3x_3 - 7) = 0$
$x_1(5 + 2\lambda_1 + \lambda_2) = 0$
$x_2(2 + 3\lambda_1 + 2\lambda_2) = 0$
$x_3(5 + \lambda_1 + 3\lambda_2) = 0$
Primal Feasibility:
$2x_1 + 3x_2 + x_3 \geq 4$
$x_1 + 2x_2 + 3x_3 \geq 7$
$x_i \geq 0$