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IE 5531 – Midterm #1 Solutions

Prof. John Gunnar Carlsson

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Before you begin: This exam has 8 pages and a total of 5 problems. Make sure that all pages are present. To obtain credit for a problem, you must show all your work. if you use a formula to answer a problem, write the formula down. Do not open this exam until instructed to do so.

Problem 1: The simplex method (20 points) Use the simplex method to solve the following linear program:

$$\begin{aligned} \underset{x_1,x_2,x_3}{\text{maximize}}\; 3x_1 - 2x_2 + x_3 & \qquad s.t. \\ x_1 + 5x_2 - x_3 &\leq 4 \\ 2x_1 - 2x_2 + 4x_3 &\leq 6 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The starting simplex tableau is

B	-3	2	-1	0	0	0
4	1	5	-1	1	0	4
5	2	-2	4	0	1	6

followed by

B	0	-1	5	0	3/2	9
4	0	6	-3	1	-1/2	1
1	1	-1	2	0	1/2	3

and finally

B	0	0	9/2	1/6	17/12	55/6
2	0	1	-1/2	1/6	-1/12	1/6
1	1	0	3/2	1/6	5/12	19/6

Problem 2: Resource management (30 points) An insurance company is introducing three products: special risk insurance, mortgage insurance, and long-term care insurance. The expected profit is \$500 per unit on special risk insurance, \$250 per unit on mortgage insurance and \$600 per unit on long term care insurance. The work requirements are as follows:

Department	Working hours per unit			Working hours available
	Special risk	Mortgage	Long-term care	
Underwriting	2	1	1	240
Administration	3	1	2	150
Claims	1	2	4	180

Management wants to establish sales quotas for each product to maximize their total expected profit.

1. Formulate this problem as a linear program. Specify the decision variables, objective function, and constraints.

   The linear program is

$$\begin{aligned} \text{maximize } \mathbf{c}^T\mathbf{x} & \quad s.t. \\ A\mathbf{x} &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

   with $\mathbf{c} = (500; 250; 600)$, $A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 240 \\ 150 \\ 180 \end{pmatrix}$.

2. Complete one iteration of the simplex method and show the new tableau.

   The initial tableau is

B	-500	-250	-600	0	0	0	0
4	2	1	1	1	0	0	240
5	3	1	2	0	1	0	150
6	1	2	4	0	0	1	180

   Let x_3 (for example) be the incoming variable and x_6 the outgoing. Pivoting, we find that the next tableau is

B	-350	50	0	0	0	150	27000
4	1.75	0.5	0	1	0	-0.25	195
5	2.5	0	0	0	1	-0.5	60
3	0.25	0.5	1	0	0	0.25	45

3. After solving the problem, the final tableau is

B	0	50	0	0	140	80	35400
4	0	0.5	0	1	-0.7	0.1	153
1	1	0	0	0	0.4	-0.2	24
3	0	0.5	1	0	-0.1	0.3	39

   Show the dual variables corresponding to the services of the three departments. Explain why mortgage insurance is not sold.

   From the final tableau, we see tha the shadow prices are \$0, \$140, and \$80 respectively. If the company sells a unit of Mortgage, it requires 1 hour for underwriting, 1 hour for administration, and 2 hours for claims. So, the cost to sell 1 unit of Mortgage is $\$140 + 2 \cdot \$80 = \$300$. However, the profit for one unit of Mortgage is only \$250 per unit. Therefore, the company cannot make any net profit by selling Mortgage.

4. Find the range of working hours available for claims to keep the current basis optimal.

   We have $B = \{4, 1, 3\}$, and we need to determine the range of λ such that

$$\bar{\mathbf{b}} + \lambda A_B^{-1}\mathbf{e}_3 \geq \mathbf{0}$$

   The constraint matrix for our problem in standard form is $A = \left(A', I\right)$, where

$$A' = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix}$$

and the left-hand side of the final tableau is precisely $A_B^{-1}A = A_B^{-1}\left(A', I\right) = \left(A_B^{-1}A', A_B^{-1}\right)$. Therefore, we find that A_B^{-1} is just the last three columns of the tableau, so

$$A_B^{-1} = \begin{pmatrix} 1 & -0.7 & 0.1 \\ 0 & 0.4 & -0.2 \\ 0 & -0.1 & 3 \end{pmatrix}$$

It is then straightforward to verify that we must have $120 \geq \lambda \geq -130$ and therefore that $b_3 \in [50, 300]$.

5. Find the range of the expected profit on special risk insurance such that the current basis remains optimal.

   The optimal basis remains the same if and only if $\mathbf{r}_N - \lambda A_N^T \left(A_B^{-1}\right)^T \mathbf{e}_1 \geq \mathbf{0}$. We have computed all relevant quantities and it is therefore straightforward to verify that $400 \geq \lambda \geq -350$, and therefore that $c_1 \in [150, 900]$.

Problem 3: Parametric linear programming (20 points) Consider the problem

$$\underset{x_1,x_2}{\text{minimize}}\, 3x_1 - 2x_2 \qquad s.t.$$

$$\begin{aligned} x_1 - x_2 &\geq 7+t \\ -x_1 &\geq -5+t \\ x_1, x_2 &\geq 0\,. \end{aligned}$$

Find a value of t for which this problem is feasible. Starting from this value, find the solutions of this problem for all values of t (the graphical method will probably be useful here, and a grid is provided on the following pages).

We first consider the values of t for which this problem is feasible. The two constraints imply that

$$\begin{aligned} x_1 &\geq 7+t \\ x_1 &\leq 5-t \end{aligned}$$

and we observe that if $t = -1$ then both constraints are tight at $x_1 = 6$ and $x_2 = 0$. The solution $x_1 = 6$, $x_2 = 0$ remains feasible for all $t < -1$, and if $t > -1$ then $7 + t > 5 - t$ and the system is infeasible. Using the graphical method, we see that the optimal solution is to set $x_1 = 7 + t$, $x_2 = 0$ for $t \in [-7, -1]$ and $x_1 = 0$, $x_2 = -7 - t$ for $t < -7$.

Problem 4: Duality (10 points) Consider the linear program

$$\begin{array}{rcl} \text{minimize } \mathbf{c}^T\mathbf{x} & & s.t. \\ A\mathbf{x} & \leq & \mathbf{b} \end{array}$$

where A is ***square*** and nonsingular (i.e. invertible). Show that the optimal value is $\mathbf{c}^T A^{-1}\mathbf{b}$ if $A^{-T}\mathbf{c} \leq \mathbf{0}$ and $-\infty$ otherwise (Hint: take the dual).

This linear program is always feasible, since setting $\mathbf{x} = A^{-1}\mathbf{b}$ is always feasible. The dual is

$$\begin{array}{rcl} \text{maximize } \mathbf{b}^T\mathbf{y} & & s.t. \\ A^T\mathbf{y} & = & \mathbf{c} \\ \mathbf{y} & \leq & \mathbf{0} \end{array}$$

Since A is square and nonsingular, we must have $\mathbf{y} = A^{-T}\mathbf{c}$, which is feasible if and only if $A^{-T}\mathbf{c} \leq \mathbf{0}$. The objective function is $\mathbf{b}^T\left(A^{-T}\mathbf{c}\right) = \mathbf{c}^T A^{-1}\mathbf{b}$ as desired.

Problem 5: Special LPs (20 points) Recall that the ℓ_1 and ℓ_∞ norms of a vector $\mathbf{v} \in \mathbb{R}^n$ are defined by

$$\begin{aligned} \|\mathbf{v}\|_1 &= |v_1| + \cdots + |v_n| \\ \|\mathbf{v}\|_\infty &= \max\{|v_1|, \cdots, |v_n|\}. \end{aligned}$$

1. Compute the optimal solutions of the two problems below in terms of $\mathbf{c}$:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}}\, \mathbf{c}^T\mathbf{x} & \quad s.t. \\ \|\mathbf{x}\|_1 &\le 1 \end{aligned}$$

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}}\, \mathbf{c}^T\mathbf{x} & \quad s.t. \\ \|\mathbf{x}\|_\infty &\le 1 \end{aligned}$$

   The optimal solution to the first problem is to set $x_i = -\text{sign}(c_i)$ and all other entries 0, where c_i is the entry of $\mathbf{c}$ with the largest absolute value. The optimal solution to the second problem is to set $x_i = -\text{sign}(c_i)$ for all i.

2. Reformulate the first problem as a linear program, construct its dual, and write down the optimal solution to the dual.

   The first problem, written as a linear program, is

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}}\, \mathbf{c}^T\mathbf{x} & \quad s.t. \\ \mathbf{e}^T\mathbf{z} &\le 1 \\ z_i &\ge x_i \,\forall i \\ z_i &\ge -x_i \,\forall i \end{aligned}$$

   its dual is

$$\begin{aligned} \text{maximize}\ y & \quad s.t. \\ s_i - t_i &= c_i \,\forall i \\ y - s_i - t_i &= 0 \,\forall i \\ y, \mathbf{s}, \mathbf{t} &\le \mathbf{0} \end{aligned}$$

   which we can re-write as

$$\begin{aligned} -\text{minimize}\ y & \quad s.t. \\ t_i &\ge c_i \,\forall i \\ t_i &= \frac{y + c_i}{2} \,\forall i \\ y, \mathbf{t} &\ge \mathbf{0}. \end{aligned}$$

   The optimal solution to this problem is to set $y = |c_k|$ where k is the index of the element of $\mathbf{c}$ with the largest absolute value. We then find that setting $t_i = \frac{c_i + |c_k|}{2} \ge c_i$ gives a feasible solution with the same objective value as the primal.

3. Consider the approximation problem

$$\begin{aligned} \underset{x_1, x_2}{\text{minimize}} \left\| \begin{pmatrix} x_1 + 2x_2 - 1 \\ 3x_1 + x_2 + 3 \end{pmatrix} \right\| & \quad s.t. \\ x_1 &\ge 0 \\ x_2 &\ge 0. \end{aligned}$$

   Express this problem as a linear program for the cases in which the norm in the objective is an ℓ_1 norm or an ℓ_∞ norm (do not try to solve these problems).

The first formulation is

$$
\begin{aligned}
\underset{x_1,x_2,z_1,z_2}{\text{minimize}}\, z_1 + z_2 & \qquad s.t. \\
z_1 &\geq x_1 + 2x_2 - 1 \\
z_1 &\geq -(x_1 + 2x_2 - 1) \\
z_2 &\geq 3x_1 + x_2 + 3 \\
z_2 &\geq -(3x_1 + x_2 + 3) \\
x_1 &\geq 0 \\
x_2 &\geq 0\,.
\end{aligned}
$$

and the second formulation is

$$
\begin{aligned}
\underset{x_1,x_2,z}{\text{minimize}}\, z & \qquad s.t. \\
z &\geq x_1 + 2x_2 - 1 \\
z &\geq -(x_1 + 2x_2 - 1) \\
z &\geq 3x_1 + x_2 + 3 \\
z &\geq -(3x_1 + x_2 + 3) \\
x_1 &\geq 0 \\
x_2 &\geq 0\,.
\end{aligned}
$$