MAT3007: Optimization - Assignment 6

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Problem 1

1. $\nabla f(x, y, z) = (4x + y - 6, x + 2y + z - 7, y + 2z - 8)$

   Solve $\nabla f(x, y, z) = 0$, get $x = 1.2,\ y = 1.2,\ z = 3.4$, which is the candidate minimizer of $f(x, y, z)$.

2. $\nabla^2 f$ = [4 1 0; 1 2 1; 0 1 2]

   The eigenvalues of this matrix, 0.8299, 2.6889, and 4.4812 are all positive. Therefore, $\nabla^2 f$ is positive definite.

   Because $\nabla f(1.2, 1.2, 3.4) = 0$, and $\nabla^2 f$ is positive definite, then $(x, y, z) = (1.2, 1.2, 3.4)$ is the local minimizer of f for the unconstrained problem.

3. Because the domain of $f(x, y, z)$ is unbounded, it is an unconstrained problem. Then first-order necessary condition provides all the candidates for local minimizers, which is unique for this function. The second-order sufficient condtions shows that $(1.2, 1.2, 3.4)$ is indeed the only local minimizer. Since there is no boundary point that can be a minimizer, the only local minimizer should also be a global minimizer.

Problem 2

1. Let (x, y) denote the location of the fountain. Then the problem can be written as:

$$\min\ (\max\{\sqrt{x^2 + y^2},\ \sqrt{(x-1)^2 + (y-5)^2},$$
$$\sqrt{(x-2)^2 + (y-3)^2},\ \sqrt{(x-3)^2 + (y-1)^2}\})$$

   Now we can write this as:

$$\begin{aligned} \text{minimize} \quad & \sqrt{t} \\ \text{s.t.} \quad & x^2 + y^2 \le t \\ & (x-1)^2 + (y-5)^2 \le t \\ & (x-2)^2 + (y-3)^2 \le t \\ & (x-3)^2 + (y-1)^2 \le t \end{aligned}$$

2. First we associate the constraints with Lagrange multipliers μ_1, μ_2, μ_3 and μ_4 and construct the Lagrangian for this problem:

$$
\begin{aligned}
L(t,x,y,\boldsymbol{\mu}) = \sqrt{t} &+ \mu_1(x^2+y^2-t) + \mu_2[(x-1)^2+(y-5)^2-t] \\
&+ \mu_3[(x-2)^2+(y-3)^2-t] + \mu_4[(x-3)^2+(y-1)^2-t]
\end{aligned}
$$

Therefore the KKT conditions are:

1. Main Condition:

$$
\begin{aligned}
\frac{1}{2\sqrt{t}} - \mu_1 - \mu_2 - \mu_3 - \mu_4 = 0 \\
2\mu_1 x + 2\mu_2(x-1) + 2\mu_3(x-2) + 2\mu_4(x-3) = 0 \\
2\mu_1 y + 2\mu_2(y-5) + 2\mu_3(y-3) + 2\mu_4(y-1) = 0
\end{aligned}
$$

2. Primal Feasibility:

$$
\begin{aligned}
x^2+y^2 &\le t \\
(x-1)^2+(y-5)^2 &\le t \\
(x-2)^2+(y-3)^2 &\le t \\
(x-3)^2+(y-1)^2 &\le t
\end{aligned}
$$

3. Dual Feasibility:

$$
\mu_1 \ge 0,\ \mu_2 \ge 0,\ \mu_3 \ge 0,\ \mu_4 \ge 0
$$

4. Complementarity Condition:

$$
\begin{aligned}
\mu_1(x^2+y^2-t) = 0 \\
\mu_2[(x-1)^2+(y-5)^2-t] = 0 \\
\mu_3[(x-2)^2+(y-3)^2-t] = 0 \\
\mu_4[(x-3)^2+(y-1)^2-t] = 0
\end{aligned}
$$

Problem 3

First we associate the constraints with Lagrange multipliers μ , and construct the Lagrangian for this problem:

$$
L(\mathbf{x},\ \boldsymbol{\mu}) = \sum_{i=1}^{n} x_i log x_i + \mu(\sum_{i=1}^{n} a_i x_i - 1)
$$

Therefore the KKT conditions are:

1. Main Condition:

$$
log x_i + 1 + \mu a_i \ge 0,\ \forall i = 1,\dots,n
$$

2. Primal Feasibility:

$$\sum_{i=1}^{n} a_i x_i - 1 = 0$$

$$x_i \geq 0,\ \forall i = 1, \dots, n$$

3. Dual Feasibility:

   None, since μ is a free variable.

4. Complementarity Condition:

$$x_i(logx_i + 1 + \mu a_i) = 0,\ \forall i = 1, \dots n$$

Problem 4

Assume that the company don't have any stock at the first month. So let w_i be the amount of units stored for month i , $w_0 = 0$.

Let t_i denote the unmet demand of month i , $t_i = max\{d_i - (w_{i-1} + x_i),\ 0\}$

The optimization problem can be formulated as follows:

$$\begin{aligned} \text{minimize} \qquad & \sum_{i=1}^{12} \mathrm{x_i^2 + sw_i + kt_i} \\ \text{s.t.} \qquad \mathrm{x_i - r} &\leq 0 \qquad \forall\, \mathrm{i} = 1, \dots, 12 \\ \mathrm{w_{i-1} + x_i - d_i - w_i} &\leq 0 \qquad \forall\, \mathrm{i} = 1, \dots, 12 \\ \mathrm{w_{i-1} + x_i - w_i} &\geq 0 \qquad \forall\, \mathrm{i} = 1, \dots, 12 \\ \mathrm{t_i - d_i + w_{i-1} + x_i} &\geq 0 \qquad \forall\, \mathrm{i} = 1, \dots, 12 \\ \mathrm{t_i} &\geq 0 \qquad \forall\, \mathrm{i} = 1, \dots, 12 \\ \mathrm{w_i} &\geq 0 \qquad \forall\, \mathrm{i} = 1, \dots, 12 \\ \mathrm{x_i} &\geq 0 \qquad \forall\, \mathrm{i} = 1, \dots, 12 \end{aligned}$$

Associate the constraints with Lagrange multipliers $\lambda_i,\ \mu_i,\ \eta_i,\ \alpha_i$, and construct the Lagrangian for this problem:

$$L(\mathbf{x},\ \mathbf{w},\ \mathbf{t},\ \lambda_i,\ \boldsymbol{\mu}_i,\ \boldsymbol{\eta}_i,\ \alpha_i) = \sum_{i=1}^{12}(x_i^2 + sw_i + kt_i) + \sum_{i=1}^{12}\lambda_i(x_i - r) + \sum_{i=1}^{12}\mu_i(w_{i-1} + x_i - d_i - w_i) + \sum_{i=1}^{12}\eta_i(w_{i-1} + x_i - w_i) + \sum_{i=1}^{12}\alpha_i(t_i - d_i + w_{i-1} + x_i)$$

Therefore the KKT conditions are:

1. Main Condition:

$$\begin{aligned} 2x_i + \lambda_i + \mu_i + \eta_i + \alpha_i &\geq 0 \qquad \forall\, i = 1, \dots, 12 \\ s - \mu_i - \eta_i + \mu_{i+1} + \eta_{i+1} + \alpha_{i+1} &\geq 0 \qquad \forall\, i = 1, \dots, 11 \\ s - \mu_{12} - \eta_{12} &\geq 0 \\ k + a_i &\geq 0 \qquad \forall i = 1, \dots, 12 \end{aligned}$$

2. Primal Feasibility:

$$
\begin{aligned}
x_i - r \le 0 &\quad \forall\, i = 1, \dots, 12 \\
w_{i-1} + x_i - d_i - w_i \le 0 &\quad \forall\, i = 1, \dots, 12 \\
w_{i-1} + x_i - w_i \ge 0 &\quad \forall\, i = 1, \dots, 12 \\
t_i - d_i + w_{i-1} + x_i \ge 0 &\quad \forall\, i = 1, \dots, 12 \\
t_i \ge 0 &\quad \forall\, i = 1, \dots, 12 \\
w_i \ge 0 &\quad \forall\, i = 1, \dots, 12 \\
x_i \ge 0 &\quad \forall\, i = 1, \dots, 12
\end{aligned}
$$

3. Dual Feasibility:

$$\lambda_i \ge 0,\ \mu_i \ge 0,\ \eta_i \le 0,\ \alpha_i \le 0 \quad \forall i = 1, \dots, 12$$

4. Complementarity Condition:

$$
\begin{aligned}
\lambda_i(x_i - r) = 0 &\quad \forall i = 1, \dots, 12 \\
\mu_i(w_{i-1} + x_i - d_i - w_i) = 0 &\quad \forall i = 1, \dots, 12 \\
\eta_i(w_{i-1} + x_i - w_i) = 0 &\quad \forall i = 1, \dots, 12 \\
\alpha_i(t_i - d_i + w_{i-1} + x_i) = 0 &\quad \forall i = 1, \dots, 12 \\
x_i(2x_i + \lambda_i + \mu_i + \eta_i + \alpha_i) = 0 &\qquad \forall i = 1, \dots, 12 \\
w_i(s - \mu_i - \eta_i + \mu_{i+1} + \eta_{i+1} + \alpha_{i+1}) = 0 &\qquad \forall i = 1, \dots, 11 \\
w_{12}(s - \mu_{12} - \eta_{12}) = 0 & \\
t_i(k + a_i) = 0 &\qquad \forall i = 1, \dots, 12
\end{aligned}
$$

Problem 5

Associate the constraints with Lagrange multipliers $\mu_1,\ \mu_2$, and construct the Lagrangian for this problem:

$$L(\mathbf{x},\ \boldsymbol{\mu}) = 5x_1 + 2x_2 + 5x_3 + \mu_1(2x_1 + 3x_2 + x_3 - 4) + \mu_2(x_1 + 2x_2 + 3x_3 - 7)$$

Therefore the KKT conditions are:

1. Main condition:

$$
\begin{aligned}
5 + 2\mu_1 + \mu_2 \ge 0 \\
2 + 3\mu_1 + 2\mu_2 \ge 0 \\
5 + \mu_1 + 3\mu_2 \ge 0
\end{aligned}
$$

2. Primal Feasibility:

$$
\begin{aligned}
2x_1 + 3x_2 + x_3 \ge 4 \\
x_1 + 2x_2 + 3x_3 \ge 7 \\
x_1,\ x_2,\ x_3 \ge 0
\end{aligned}
$$

3. Dual Feasibiliy:

$$\mu_1 \leq 0,\ \mu_2 \leq 0$$

4. Complementary Condition:

$$\mu_1(2x_1 + 3x_2 + x_3 - 4) = 0$$
$$\mu_2(x_1 + 2x_2 + 3x_3 - 7) = 0$$
$$x_1(5 + 2\mu_1 + \mu_2) = 0$$
$$x_2(2 + 3\mu_1 + 2\mu_2) = 0$$
$$x_3(5 + \mu_1 + 3\mu_2) = 0$$