# MAT3007: Optimization - Assignment 7

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## Problem 1

1. False.

   Let $f(x) = x^2$ ; $g(x) = -lnx,\ x > 0$ , so both $f(x)$ and $g(x)$ are convex.

   Get the second derivative:

   $$f(g(x)) = (lnx)^2$$
   $$f'(g(x)) = \frac{2}{x} lnx$$
   $$f''(g(x)) = \frac{2}{x^2}(1 - lnx)$$

   (i) when $1 - lnx \geq 0$ , i.e. $0 < x \leq e$ , $f''(g(x)) \geq 0$ , so $f(g(x))$ is convex;

   (ii) when $1 - lnx \leq 0$, i.e. $x \geq e$ , $f''(g(x)) \leq 0$ , so $f(g(x))$ is concave.

   Therefore, if $f(x)$ is convex, $g(x)$ is convex, $f(g(x))$ is not necessarily convex.

2. True.

   Because $g(x)$ is convex, for every $x_1,\ x_2 \in \Omega$ ($g(x)$ is on $\Omega$) and any $0 \leq \alpha \leq 1$ ,

   $$g(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha g(x_1) + (1 - \alpha)g(x_2)$$

   Because $f(x)$ is nondecreasing and convex, for every $\alpha g(x_1),\ (1 - \alpha)g(x_2) \in \Omega'$ ($f(x)$ is on $\Omega'$) we have

   $$f(g(\alpha x_1 + (1 - \alpha)x_2)) \leq f(\alpha g(x_1) + (1 - \alpha)g(x_2)) \leq \alpha f(g(x_1)) + (1 - \alpha)f(g(x_2))$$

   Therefore, if $f(x)$ is convex and nondecreasing, $g(x)$ is convex, as long as $g(x) \in \Omega'$ , then $f(g(x))$ is convex.

3. False.

   Because $g(x)$ is convex, for every $x_1,\ x_2 \in \Omega$ ($g(x)$ is on $\Omega$) and any $0 \leq \alpha \leq 1$ ,

   $$g(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha g(x_1) + (1 - \alpha)g(x_2)$$

   Because $f(x)$ is nonincreasing and concave, for every $\alpha g(x_1),\ (1 - \alpha)g(x_2) \in \Omega'$ ($f(x)$ is on $\Omega'$) we have

   $$f(g(\alpha x_1 + (1 - \alpha)x_2)) \geq f(\alpha g(x_1) + (1 - \alpha)g(x_2)) \geq \alpha f(g(x_1)) + (1 - \alpha)f(g(x_2))$$

   Therefore, if $f(x)$ is convex and nondecreasing, $g(x)$ is convex, as long as $g(x) \in \Omega'$ , then $f(g(x))$ is concave.

4. False.

Since $f(x)$ is increasing and non-negative, $f'(x) > 0, f(x) \geq 0, x \geq 0$ .

(a) If $f(x)$ is continuously differentiable on $x \geq 0$ ,

(i) when $x > 0$ ,

$$(xf(x))' = f(x) + xf'(x) > 0 \Rightarrow f'(x) > -\frac{f(x)}{x}$$
$$(xf(x))'' = 2f'(x) + xf''(x) > -\frac{2f(x)}{x} + xf''(x)$$

if $(xf(x))'' \geq 0$ , $f''(x) \geq \frac{-2f'(x)}{x}$ ; if $f''(x) \geq \frac{2f(x)}{x^2}$ , $(xf(x))'' > 0$ .

(ii) when $x = 0,\ (xf(x))'' = 2f'(x) > 0$

Therefore, $xf(x)$ is convex on $x \geq 0$ only if $f''(x) \geq \frac{-2f'(x)}{x}$ . If $f''(x) \leq \frac{-2f'(x)}{x}$, then $xf(x)$ is concave.

(b) If $f(x)$ are segmented functions on $x \geq 0$ , a counter example can be $f(x) = x^2, 0 \leq x < 1;\ f(x) = x,\ x \geq 1$ .

In this case, $xf(x)$ is not convex, since it does not satisfy $f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha)f(x_2)$ when $x_1 \in [0,\ 1],\ x_2 \in [1, \infty)$ .

# Problem 2

1. For every $x_1',\ x_1'',\ x_2',\ x_2'' \in R$ and any $0 \leq \alpha \leq 1$ ,

$$\begin{aligned} f(\alpha x_1' + (1-\alpha)x_1'',\ \alpha x_2' + (1-\alpha)x_2'') &= log(e^{\alpha x_1' + (1-\alpha)x_1''} + e^{\alpha x_2' + (1-\alpha)x_2''}) \\ &= log(e^{\alpha x_1'} e^{(1-\alpha)x_1''} + e^{\alpha x_2'} e^{(1-\alpha)x_2''}) \\ &\leq log((e^{x_1'} + e^{x_2'})^{\alpha} (e^{x_1''} + e^{x_2''})^{(1-\alpha)}) \\ &= \alpha log(e^{x_1'} + e^{x_2'}) + (1-\alpha) log(e^{x_1''} + e^{x_2''}) \\ &= \alpha f(x_1',\ x_2') + (1-\alpha) f(x_1'',\ x_2'') \end{aligned}$$

Because $f(\alpha x_1' + (1-\alpha)x_1'',\ \alpha x_2' + (1-\alpha)x_2'') \leq \alpha f(x_1',\ x_2') + (1-\alpha)f(x_1'',\ x_2'')$ , $f$ is a convex function in $(x1,\ x2)$ .

2. let $x = e^a,\ y = e^b,\ z = e^c$ . The convex optimization problem is:

$$\begin{aligned} \text{minimize} \quad & \mathrm{e^{2c}} \\ \text{s. t.} \quad & \mathrm{a} \geq -10 \\ & \mathrm{a} \leq 3 \\ & \log(\mathrm{e}^{2\mathrm{a} - \frac{\mathrm{b}}{2}} + \mathrm{e}^{\mathrm{b} - \frac{\mathrm{a}}{2}}) \leq 0 \\ & \mathrm{a - b - 2c} = 0 \end{aligned}$$

3.

```
cvx_begin quiet
    variables a b c;
    minimize exp(2*c)
    subject to
        3 >= a >= -10
        a-b-2*c == 0
        log(exp(2*a)+exp(b-c)) <= b/2
cvx_end

[a b c]
cvx_opt = cvx_optval
```

The results are: $a=-10,\ b=-5,\ c=-2.5$ , which means $x=e^{-10},\ y=e^{-5},\ z=e^{-2.5}$ . The optimal solution is 0.00673795 .

## Problem 3

Its Hessian matrix is:

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

The eigenvalues are 0.8299, 2.6889, and 4.4812, which shows that the Hessian matrix is positive semi-definite (PSD) throughout the defined region. So $f(x,y,z)=2x^2+xy+y^2+yz+z^2-6x-7y-8z-9$ is convex.

```
cvx_begin quiet
    variables x y z
    minimize ((x+1/2*y)^2 + (y/2+z)^2 + (x-3)^2 + 1/2*(y-7)^2 - 8*z - 85/2)
cvx_end

[x y z]
cvx_opt = cvx_optval
```

The optimal solution is $x=1.2,\ y=1.2,\ z=3.4$ . The optimal value is $-30.4$ .

## Problem 4

The optimization problem is:

$$\begin{aligned} \text{minimize} \qquad & \sum_{i=1}^{n} x_i \log x_i \\ \text{s.t.} \qquad & \sum_{i=1}^{n} a_i x_i = 1 \\ & x_i \geq 0 \end{aligned}$$

Let $f(x) = xlogx$ , then $f'(x) = logx + 1, f''(x) = \frac{1}{x}$ . When $x \geq 0$ , $f''(x) > 0$ , so $f(x)$ is convex.

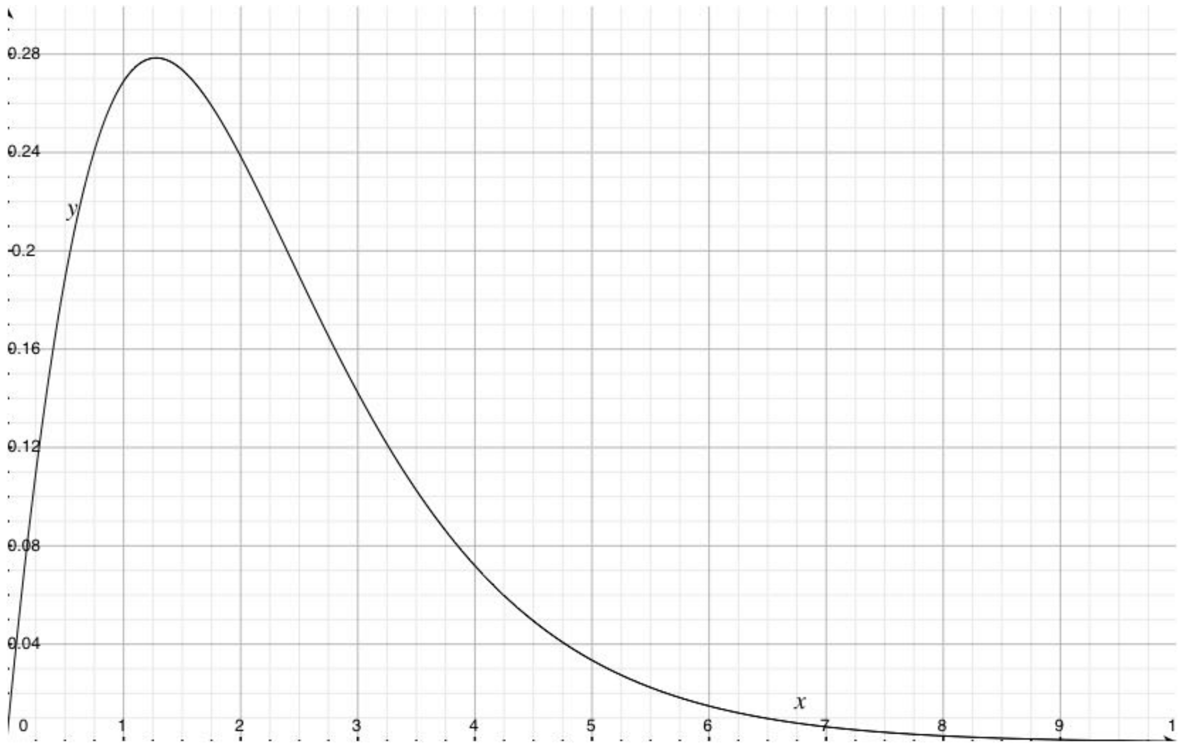
Since $f(x_i) = x_i logx_i, \ \forall i \in [1, \ n]$ are convex functions, $\sum_{i=1}^{n} f(x_i)$ is a convex function.

Additionally, $\sum_{i=1}^{n} a_i x_i = 1$ is a linear constraint, so it is a convex constraint. $g(x) = x$ is concave, so $x_i \geq 0$ are all convex constraints.

Therefore, the constraints is a convex feasible region.

In conclusion, it is a convex optimization problem, because it minimizes a convex function over a convex feasible region.

## Problem 5

1. 


The picture shows that $r(p) = p\lambda(p)$ is not concave.

2.

$$p = log\frac{1-\lambda}{\lambda}$$

$$\tilde{r}(\lambda) = \lambda log\frac{1-\lambda}{\lambda}$$

$$\tilde{r}'(\lambda) = log\frac{1-\lambda}{\lambda} - \frac{1}{1-\lambda}$$

$$\tilde{r}''(\lambda) = -\frac{1}{(1-\lambda)^2\lambda}$$

Since $p \geq 0,\ \lambda(p) = \frac{e^{-p}}{1+e^{-p}} \in (0,\ \frac{1}{2})$ , $\tilde{r}''(\lambda) < 0$ . Therefore, $\tilde{r}''(\lambda)$ is concave in λ.

4. The optimization problem is:

$$\begin{aligned} \text{maximize}_\lambda \quad & \lambda \text{log}\frac{1-\lambda}{\lambda} \\ \text{s. t.} \quad & \lambda - \frac{1}{2} \leq 0 \\ & \lambda \geq 0 \end{aligned}$$

   Associate the constraints with Lagrange multipliers $\mu$ , and construct the Lagrangian for this problem:

$$L(\lambda,\ \mu) = \lambda log\frac{1-\lambda}{\lambda} + \mu(\lambda - \frac{1}{2})$$

Therefore the KKT conditions are:

1. Main Condition:

$$log\frac{1-\lambda}{\lambda} - \frac{1}{1-\lambda} + \mu \geq 0$$

2. Dual Feasibility:

$$\mu \geq 0$$

3. Complementary Condition:

$$\mu(\lambda - \frac{1}{2}) = 0$$

$$\lambda(log\frac{1-\lambda}{\lambda} - \frac{1}{1-\lambda} + \mu) = 0$$

4. Primal Feasibility:

$$\lambda - \frac{1}{2} \leq 0$$

$$\lambda \geq 0$$

Transform it back to an optimal condition in p:

1. Main Condition:

$$p - e^{-p} - 1 + \mu \geq 0$$

2. Dual Feasibility:

$$\mu \geq 0$$

3. Complementary Condition:

$$\mu(-\frac{1}{1+e^{-p}}+\frac{1}{2})=0$$
$$\frac{e^{-p}(p-e^{-p}-1+\mu)}{1+e^{-p}}=0$$

4. Primal Feasibility:

$$p \geq 0$$