MAT 3007
Assignment 2 Solution

Problem 1

1. False. Consider the following counterexample:

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{s.t.} & x \geq 0 \end{array}$$

then the optimal solution set is unbounded.

2. False. Consider the following counterexample: *minimize* 0. Then any feasible x is optimal no matter how many positive components it has.

3. True. If x_1 and x_2 are optimal solution, then, any convex combination of x_1, x_2 is also optimal.

Problem 2

From Figure 1, we can see the optimal solution is $(4, 2)$, the optimal value can be calculated using this point, which is 6.

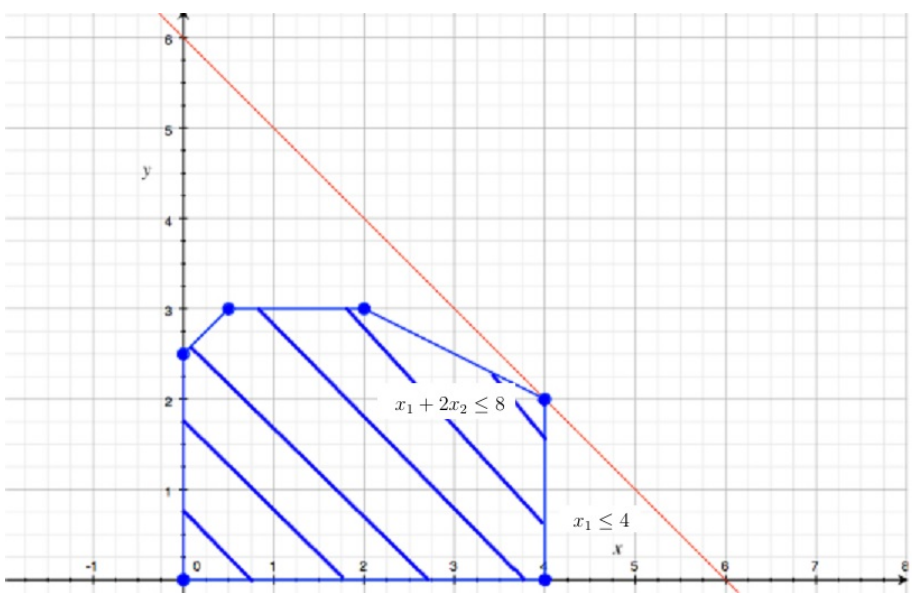

Figure 1: Problem 2

Since the optimal solution is at $(4, 2)$, the two lines that intersect at that point are the active constraints.

$$x_1 + 2x_2 \leq 8$$
$$x_1 \leq 4$$

From Figure 1, we can find all the vertices of the feasible region: $(0,0), (0,2.5), (0.5,3), (2,3), (4,2), (4,0)$.

Problem 3

- Standard form is as follows.

$$\begin{array}{ll} \text{minimize} & -x_1 - 4x_2 - x_3 \\ \text{s.t.} & 2x_1 + 2x_2 + x_3 + x_4 = 4 \\ & -x_1 + x_3 + x_5 = -1 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

- First, we will argue that there exist an optimal solution for this problem. From the constraints we know the feasible set is bounded. The objective function is to maximize the sum of bounded variables, so an optimal solution must exist. Moreover, since all the rows of the constraint matrix are linearly independent with $m = 2$ and $n = 5$, there must exists an optimal solution with no more than 2 positive variables.

- See Table1 for the basic solutions and basic feasible solutions.

B	{1, 2}	{1, 3}	{1, 4}	{1, 5}	{2, 3}	{2, 5}	{3, 4}	{3, 5}	{4, 5}
x_B	(1, 1)	($\frac{5}{3}$, $\frac{2}{3}$)	(1, 2)	(2, 1)	($\frac{5}{2}$, -1)	(2, -1)	(-1, 5)	(4, -5)	(4, -1)
Obj. Val.	-5	$-\frac{7}{3}$	-1	-2	-	-	-	-	-
BFS	Y	Y	Y	Y	N	N	N	N	N

Table 1: Basic solutions and basic feasible solutions

- The optimal solution is (1, 1, 0, 0, 0)

Problem 4

- let $\mathbf{x}$ be the vector of how many shares of each security to purchase, a is the variable to denote the income of worst case.

- we formulate the problem as follows.

$$
\begin{array}{ll}
\text{maximize} & a - \pi^T \mathbf{x} \\
\text{s.t.} & A^T \mathbf{x} - a\mathbf{1} \geq \mathbf{0} \\
& \mathbf{x} - \mathbf{q} \leq \mathbf{0} \\
& \mathbf{x}, a \geq 0
\end{array}
$$

- the code is:

```
A = [1 1 1 0 0;
    0 0 0 1 1;
    1 0 1 0 1;
    1 1 1 1 0;
    0 1 0 1 1];
q = [10 5 10 10 5]';
p = [0.75 0.35 0.4 0.75 0.65]';

cvx_begin
variables x(5) a t


minimize -t
subject to
        t == a - p'*x;
        A'* x - a >= 0;
        x - q <= 0;
        x>=0;
        a>=0;
cvx_end
```

- The optimal solution is (0, 0, 5, 5, 5), the optimal objective function value is 1

Problem 5

The polygon can be described using the following inequalities:

$$
\begin{aligned}
&-x_1 - x_2 \leq -1 \\
&-x_1 + x_2 \leq 6 \\
&x_1 + x_2 \leq 18 \\
&x_1 - x_2 \leq 7 \\
&x_1 \leq 11 \\
&x_2 \leq 10 \\
&-x_1 \leq 0 \\
&-x_2 \leq 0
\end{aligned}
$$

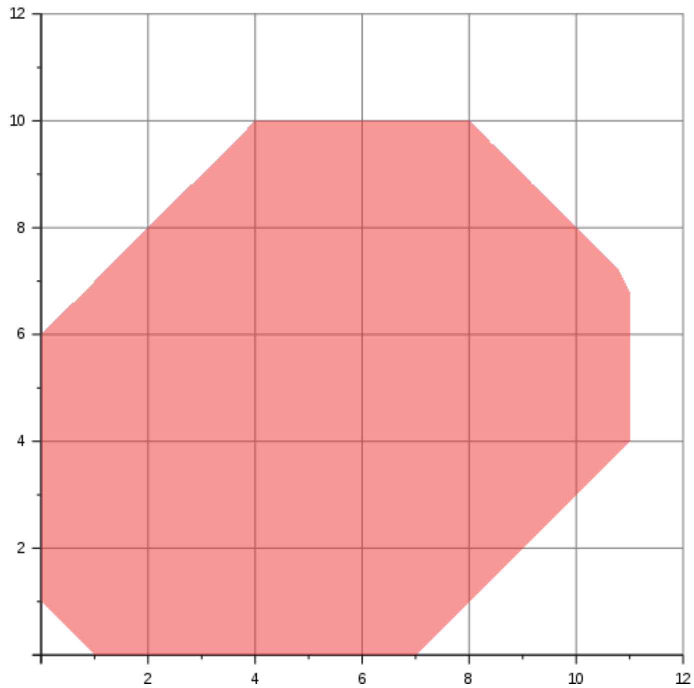

Figure 2: Problem 4

Let (y_1, y_2) be the center of the circle, and let r be the radius. This problem can be solved via a linear program.

$$
\begin{array}{ll}
\text{maximize} & r \\
\text{s.t.} & -y_1 - y_2 + r\sqrt{(-1)^2 + (-1)^2} \leq -1 \\
& -y_1 + y_2 + r\sqrt{(-1)^2 + (1)^2} \leq 6 \\
& y_1 + y_2 + r\sqrt{(1)^2 + (1)^2} \leq 18 \\
& y_1 - y_2 + r\sqrt{(1)^2 + (-1)^2} \leq 7 \\
& y_1 + r\sqrt{(1)^2} \leq 11 \\
& y_2 + r\sqrt{(1)^2} \leq 10 \\
& -y_1 + r\sqrt{(-1)^2} \leq 0 \\
& -y_2 + r\sqrt{(-1)^2} \leq 0 \\
& r \geq 0
\end{array}
$$

The objective is to maximize the radius, which is equivalent to maximizing the area. The constraints give a feasible region for the center of the circle. In order for the circle to be within the shaded area, the distance from the center of the circle to the line defined by the extreme points must be greater than the radius r. For example, in order for the circle to be above the line "$-y_1 - y_2 + 1 = 0$", we need to have the distance from the center to this line, i.e., $\frac{y_1+y_2-1}{\sqrt{(1^2+1^2)}}$, to be greater than r. That gives us the first constraint. All other constrains can be obtained using the similar argument.

Using MATLAB, we get the optimal solution $r = 4.596$, and one possible solution of the center (y_1, y_2) is $(5.8, 5.3)$. Note that the center is not unique. It lies on the line segment $y_1 - y_2 = 0.5$ with interval $y_1 \in [5.096, 5.904]$.