MAT 3007
Assignment 7 Solution

Problem 1
1. False. Consider the following counterexample:

$$\begin{array}{ll} f(x) = -x, & \text{convex} \\ g(x) = x^2, & \text{convex} \\ f(g(x)) = -x^2, & \text{concave} \end{array}$$

2. True.
Proof Let $h(x) = f(g(x))$. Assuming $f(x)$ and $g(x)$ are differentiable, take the first order and second order derivative, we have

$$\begin{aligned} h'(x) &= f'(g(x))g'(x), \\ h''(x) &= f''(g(x))(g'(x))^2 + f'(g(x))g''(x). \end{aligned}$$

Since $f(x)$ is convex and nondecreasing, $g(x)$ is convex, we have

$$\begin{array}{l} f'(x) \geq 0, \\ f''(x) \geq 0, \\ g''(x) \geq 0. \end{array}$$

From this we can conclude that $h''(x) \geq 0$. Therefore, $f(g(x))$ is convex. $\square$

The result still holds if the functions are not differentiable. In this case, we can prove by definition.
Proof Take $\theta \in [0, 1]$. From convexity of $g(x)$, for all x, y, we have

$$g(\theta x + (1-\theta)y) \leq \theta g(x) + (1-\theta)g(y).$$

Since $f(x)$ is nondecreasing,

$$f\left(g(\theta x + (1-\theta)y)\right) \leq f\left(\theta g(x) + (1-\theta)g(y)\right).$$

By convexity of $f(x)$,

$$f\left(\theta g(x) + (1-\theta)g(y)\right) \leq \theta f(g(x)) + (1-\theta)f(g(y)).$$

Combine the last two inequalities,

$$f\left(g(\theta x + (1-\theta)y)\right) \leq \theta f(g(x)) + (1-\theta)f(g(y)).$$

By definition, $f(x)$ is convex. $\square$

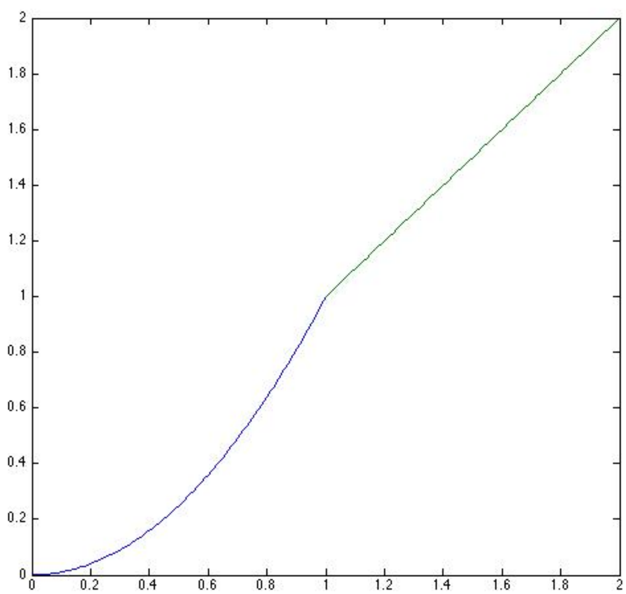
3. False. Consider the following counterexample:

$$\begin{array}{ll} f(x) = -x, & \text{concave and non-increasing;} \\ g(x) = x^2, & \text{convex;} \\ f(g(x)) = -x^2, & \text{concave.} \end{array}$$

4. False. Consider the following counter example:

$$f(x) = \begin{cases} x & x \in [0, 1], \\ 1 & x > 1. \end{cases}$$
$$xf(x) = \begin{cases} x^2 & x \in [0, 1], \\ x & x > 1. \end{cases}$$

We can see from the graph that $xf(x)$ is not convex.



Note here $xf(x)$ is not differentiable at 1; however, it is possible to come up with a function which is smooth at 1, but still keep the shape and characteristics.

Problem 2

1. Take the second order derivative of $f(x_1, x_2)$,

$$H = \begin{bmatrix} \frac{e^{x_1+x_2}}{(e^{x_1}+e^{x_2})^2} & -\frac{e^{x_1+x_2}}{(e^{x_1}+e^{x_2})^2} \\ -\frac{e^{x_1+x_2}}{(e^{x_1}+e^{x_2})^2} & \frac{e^{x_1+x_2}}{(e^{x_1}+e^{x_2})^2} \end{bmatrix}.$$

Here, $\det(H) \geq 0$, and the diagonal elements are positive. So, the Hessian matrix is positive semidefinite. Therefore, $f(x_1, x_2)$ is convex.
2. Let $u = \log x$, $v = \log y$, $w = \log z$, and take the logarithm of each of the constraints. The original problem is equivalent to,

$$\begin{array}{ll} \text{minimize} & u - v \\ \text{s.t.} & -10 \leq u \leq 3 \\ & \log(e^{2u} + e^{v-w}) \leq \frac{1}{2}v \\ & u - v = 2w \end{array}$$

Note that in this new problem, the first and third constraints are linear and thus are convex. For the second constraint, we use the property that any linear transformation of convex function is still convex.The second proposition on slide 23 in Lecture 16 states that, if f is convex, $g(x) = f(Ax + b)$ is convex. In this problem,

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

From part 1 we know that $\log(e^{x_1} + e^{x_2})$ is convex, thus, $\log(e^{2u} + e^{v-w})$ is convex in u, v, w.
3. The CVX syntax is as follows:

```
>> cvx_begin
variables u v w

minimize(u-v)

subject to

-10 <= u <= 3
log(exp(2*u)+exp(v-w))-0.5*v <= 0
u-v-2*w == 0

cvx_end
```

Problem 3

We want to show $\nabla^2 f(x)$ is positive definite. We can check that all the eigenvalues of the Hessian matrix are all positive.

$$H = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The eigenvalues of H are $4.481, 2.689, 0.830$. Therefore, the matrix H is positive definite. Thus the objective function is convex and the problem is a convex problem.
The CVX syntax is as follows:

```
>> cvx_begin
variables x y z
minimize(x^2 + 0.5*y^2 +(x+0.5*y)^2 + (z+0.5*y)^2 - 6*x - 7*y - 8*z - 9)
cvx_end
```

Problem 4

The objective function is convex. We can show $\nabla^2 f(x)$ is positive definite everywhere inside the domain.

$$H = \begin{bmatrix} \frac{1}{x_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{x_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{x_n} \end{bmatrix}$$

All the diagonal elements of the Hessian matrix are positive. Therefore the matrix H is positive definite and the objective function is convex. Since all the constraints are linear, the problem is also convex.

Problem 5

1. See Figure 1.

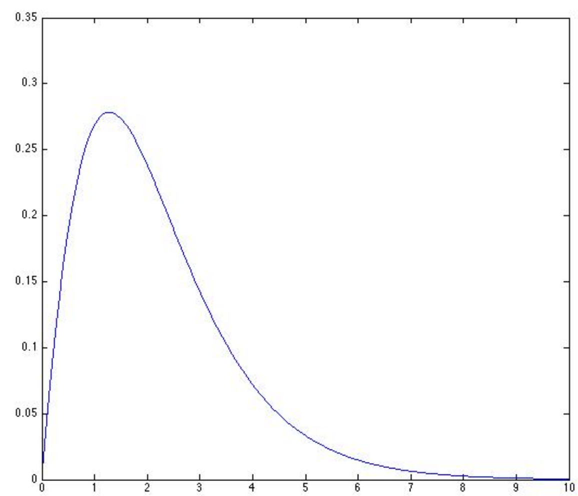


Figure 1: Problem 4.1

2. Rewrite p as a function of λ.

$$\begin{aligned} & \lambda(p) = \tfrac{e^{-p}}{1+e^{-p}}, \\ \implies \quad & p(\lambda) = \log(1-\lambda) - \log \lambda. \end{aligned}$$

Then the problem can be written as:

$$\text{maximize} \quad \lambda(\log(1-\lambda) - \log \lambda)$$

Next, we show that this objective function is concave in λ. To see this, let $\tilde{r}(\lambda) = \lambda(\log(1-\lambda) - \log\lambda)$, take the derivatives, we have

$$\begin{aligned} \tilde{r}'(\lambda) &= \log(1-\lambda) - \tfrac{\lambda}{1-\lambda} - \log\lambda - 1, \\ \tilde{r}''(\lambda) &= -\tfrac{1}{1-\lambda} - \tfrac{1}{(1-\lambda)^2} - \tfrac{1}{\lambda} < 0. \end{aligned}$$

The last inequality is because by definition, $0 < \lambda < 1$. Therefore, $\tilde{r}(\lambda)$ is concave. Maximizing $\tilde{r}(\lambda)$ is a convex problem.

3. Optimal condition for λ,

$$\tilde{r}'(\lambda) = \log(1-\lambda^*) - \tfrac{\lambda^*}{1-\lambda^*} - \log\lambda^* - 1 = 0$$

Transform it back to optimal condition of p, we get

$$p^* - e^{-p^*} - 1 = 0$$