MAT3007 Assignment 5 胡皓达 116010078.

Problem 1

1. Let x_1, x_2, x_3 be the number of production process 1, 2, 3, respectively.
The linear optimization problem is:

$$\begin{aligned} \text{maximize} \quad & 38(4x_1+x_2+3x_3)+33(3x_1+x_2+4x_3)-51x_1-11x_2-40x_3 \\ \text{subject to} \quad & 3x_1+x_2+5x_3 \le 8000 \\ & 5x_1+x_2+3x_3 \le 5000 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

The standard form is:

$$\begin{aligned} \text{minimize} \quad & -200x_1-60x_2-206x_3 \\ \text{Subject to} \quad & 3x_1+x_2+5x_3+s_1=8000 \\ & 5x_1+x_2+3x_3+s_2=5000 \\ & x_1, x_2, x_3, s_1, s_2 \ge 0 \end{aligned}$$

The simplex tableau is:

B	-200	-60	-206	0	0	0
4	3	1	5	1	0	8000
5	5	1	3	0	1	5000
	0	-20	-86	0	40	200000
4	0	$\frac{2}{5}$	$\frac{16}{5}$	1	$-\frac{3}{5}$	5000
1	1	$\frac{1}{5}$	$\frac{3}{5}$	0	$\frac{1}{5}$	1000
	100	0	-26	0	60	300000
4	-2	0	2	1	-1	3000
2	5	1	3	0	1	5000
	74	0	0	13	47	339000
3	-1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	1500
2	8	1	0	$-\frac{3}{2}$	$\frac{5}{2}$	500

The optimal solution is $x_1=0, x_2=500, x_3=1500, s_1=0, s_2=0$. So the company should have 500 process 2 and 1500 process 3.

2. Let the rise of price of gasoline be λ, r_N^T is the original reduced cost.
$\Delta c = \lambda e_j$ $\quad A_B^{-1} = \begin{bmatrix} -\frac{3}{2} & \frac{5}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$, $A_N = \begin{bmatrix} 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$

the reduced cost becomes.
$$\begin{aligned} & c_N^T - 4\lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T - \left(c_B^T - \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T - 3\lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T\right) A_B^{-1} A_N \\ = {} & c_N^T - c_B^T A_B^{-1} A_N - 4\lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T + \left(\lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T + 3\lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T\right) A_B^{-1} A_N \\ = {} & r_N^T - 4\lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} 5\lambda \\ 0 \\ \lambda \end{bmatrix}^T \\ = {} & \begin{bmatrix} 74 \\ 13 \\ 47 \end{bmatrix}^T - \begin{bmatrix} 4\lambda \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} 5\lambda \\ 0 \\ \lambda \end{bmatrix}^T \ge 0 \end{aligned}$$

$$\begin{cases} 74+\lambda \ge 0 \\ 47+\lambda \ge 0 \end{cases} \Rightarrow \lambda \ge -47$$

Therefore, the highest selling price of gasoline that will not cause the optimal solution to change is unbounded, which means no matter how high it goes, the optimal solution will not change.

3. The optimization problem becomes:

$$\begin{aligned}
\text{minimize} \quad & -200x_1 - 60x_2 - 206x_3 \\
\text{subject to} \quad & 3x_1 + x_2 + 5x_3 + S_1 = 8000 \\
& 5x_1 + x_2 + 3x_3 + S_2 = 5000 \\
& 4x_1 + 3x_2 + 5x_3 + S_3 = 10000 \\
& x_1, x_2, x_3, S_1, S_2, S_3 \geq 0
\end{aligned}$$

The optimal solution is $x_1 = 0, x_2 = 500, x_3 = 1500, S_1 = 0, S_2 = 0, S_3 = 100$.
So the company should have 500 process 2 and 1500 process 3 to maximize net revenue and also ensure waste disposal.

Problem 2

1. Let x_1, x_2, x_3 be the quotas of special risk, mortagage, Long-term care insurance, respectively. The linear optimization problem is:

$$\begin{aligned}
\text{maximize}_{x_1,x_2,x_3} \quad & 500x_1 + 250x_2 + 600x_3 \\
\text{subject to} \quad & 2x_1 + x_2 + x_3 \leq 240 \\
& 3x_1 + x_2 + 2x_3 \leq 150 \\
& x_1 + 2x_2 + 4x_3 \leq 180 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

The decision variables are x_1, x_2, x_3.
The objective function is $f = 500x_1 + 250x_2 + 600x_3$
The constraints are $2x_1 + x_2 + x_3 \leq 240$, $3x_1 + x_2 + 2x_3 \leq 150$, $x_1 + 2x_2 + 4x_3 \leq 180$.

2. The standard form is:

$$\begin{aligned}
\text{minimize} \quad & -500x_1 - 250x_2 - 600x_3 \\
\text{subject to} \quad & 2x_1 + x_2 + x_3 + S_1 = 240 \\
& 3x_1 + x_2 + x_3 + S_2 = 150 \\
& x_1 + 2x_2 + 4x_3 + S_3 = 180 \\
& x_1, x_2, x_3, S_1, S_2, S_3 \geq 0
\end{aligned}$$

The dual problem is:

$$\begin{aligned}
\text{maximize} \quad & 240y_1 + 150y_2 + 180y_3 \\
\text{subject to} \quad & 2y_1 + 3y_2 + y_3 \leq -500 \\
& y_1 + y_2 + 2y_3 \leq -250 \\
& y_1 + y_2 + 4y_3 \leq -600 \\
& y_1, y_2, y_3 \leq 0
\end{aligned}$$

$y=(A_B^{-1})^T C_B$; reduced cost is: $c^T - C_B^T A_B^{-1} A = c^T - y^T A$.

$\therefore$ The dual variables are $y_1=0, y_2=-140, y_3=-80$.

The complementarity condition is $x_i(A_i^T y - c_i)=0, \forall i$

when $i=2$, $A_2^T y - c_2 = -50 \neq 0$ $\therefore x_2 = 0$

Therefore, in order to satisfy the complementariy condition of mortgage insurance, whose reduced cost is not 0, the value of x_2 should be 0, which means it is not sold.

3. let $\Delta b = \lambda e_i$, x^* be the optimal solution,

$$\tilde{x_B} = x^* + A_B^{-1}\Delta b = \begin{bmatrix} 24 \\ 39 \\ 153 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0.4 & -0.2 \\ 0 & -0.1 & 0.3 \\ 1 & -0.7 & 0.1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 24 - 0.2\lambda \\ 39 + 0.3\lambda \\ 153 + 0.1\lambda \end{bmatrix} \geq 0$$

$\therefore -130 \leq \lambda \leq 120$.

$b_3 \in [180-130, 120+180] = [50, 300]$.

The range of working hours for claims to keep the current basis optimal is $b_3 \in [50, 300]$.

Let the increase of profit on special risk insurance be λ.

4. The reduced cost is: $r_N^T - \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0.4 & -0.2 \\ 0 & -0.1 & 0.3 \\ 1 & -0.7 & 0.1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 50 \\ 140 \\ 80 \end{bmatrix}^T + \begin{bmatrix} 0 \\ 0.4\lambda \\ -0.2\lambda \end{bmatrix}^T \geq 0.$$

$\therefore -350 \leq \lambda \leq 400$

$c_1 \in [500-350, 500+400] = [150, 900]$

$\therefore$ The range of the expected profit on special risk insurance is $c_1 \in [150, 900]$.