PARTIAL PORTFOLIO

ROSE RAHIMI

1. Direct Proof

Homework 3 #2. Prove that for all positive real numbers x, the sum of x and it's reciprocal is greater than or equal 2.

Proof. Suppose $x \in \mathbb{R}$ such that $x > 0$. Since $x \in \mathbb{R}$, we know that $x - 1 \in \mathbb{R}$ and so $(x-1)^2 \geq 0$. Therefore,

$$\begin{aligned} x^2 - 2x + 1 &\geq 0 \\ x^2 + 1 &\geq 2x. \end{aligned}$$

Since $x > 0$, we can divide the both sides of inequality by x. Thus,

$$\begin{aligned} \frac{x^2+1}{x} &\geq \frac{2x}{x} \\ x + \frac{1}{x} &\geq 2. \end{aligned}$$

Therefore, the sum of x and its reciprocal is greater than or equal to 2 for all positive real numbers x. □

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2. Contradiction proof

Homework 3 #7. Let $a_1, a_2, \ldots, a_{2000} \in \mathbb{N}$ such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2000}} = 1.$$

Prove that at least one of the numbers $a_1, a_2, \ldots, a_{2000}$ is even.

Lemma 1. *For any positive integer n, the product of n odd integers is odd.*

Proof. We proceed by mathematical induction on n.
(Base Case). Let $n = 1$. The product of one odd integer is itself odd by definition of odd integers.
(Inductive Step). Assume that for some integer $N \geq 1$, the product of any N odd integers is odd. Consider $N + 1$ odd integers $a_1 \cdot a_2 \cdot a_3 \cdots a_N \cdot a_{N+1}$. Let

$$P = (a_1 \cdot a_2 \cdot a_3 \cdots a_N) \cdot a_{N+1},$$

By inductive hypothesis, We know that $a_1 \cdot a_2 \cdot a_3 \cdots a_N$ is odd, and so $a_1 \cdot a_2 \cdot a_3 \cdots a_N = 2b + 1$ for some $b \in \mathbb{Z}$. Also, $a_{N+1} = 2c + 1$ for some $c \in \mathbb{Z}$. Hence,

$$\begin{aligned} P &= (2b + 1)(2c + 1) \\ &= 4bc + 2b + 2c + 1 \\ &= 2(2bc + b + c) + 1. \end{aligned}$$

Let $d = 2bc + b + c$. Since b and c are integers, we know that d is an integer. Thus,

$$P = 2d + 1.$$

Therefore, by principles of mathematical induction, the product of $N+$ 1 odd integers is odd. □

Proof. Suppose, for the sake of contradiction, that all of $a_1, a_2, \ldots, a_{2000} \in \mathbb{N}$ are odd and that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2000}} = 1.$$

Let $P = a_1 a_2 \cdots a_{2000}$. Then P is odd by Lemma 1, since it is a product of 2000 odd integers. By multiplying the given equation by P, we get

$$P(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2000}}) = P$$

$$\frac{P}{a_1} + \frac{P}{a_2} + \cdots + \frac{P}{a_{2000}} = P. \tag{1}$$

For each $k = 1, 2, 3, \cdots, 2000$, note that

$$a_1 a_2 \cdots \widehat{a_k} \cdots a_{2000} = \frac{a_1 a_2 \cdots a_{2000}}{a_k} = \frac{P}{\widehat{a_k}}$$

is product of 1999 odd integers. By lemma 1, each $\frac{P}{\widehat{a_k}}$ is odd. Thus, the left hand side of equation (1) is sum of 2000 odd integers. Since 2000 is even, the sum of an even number of odd integers is even. Therefore, the left hand side of equation (1) is even. That is, P is even. But earlier we establish that P is odd. That is a contradiction. Therefore, at least one of the numbers $a_1, a_2, \ldots, a_{2000}$ must be even. $\square$

3. Contrapositive proof

Homework 4 #5. If A, B, C are arbitrary sets and $A - C \nsubseteq A - B$, then $B \nsubseteq C$.

Proof. We will prove the contrapositive statement: if A, B, C are arbitrary sets and $B \subseteq C$, then $A - C \subseteq A - B$. Let $x \in A - C$. By the definition of set difference,

$$x \in A \quad \text{and} \quad x \notin C.$$

Since $B \subseteq C$, we know that every element of B is in C. Equivalently by contrapositive, any element not in C is not in B. Thus, from $x \notin C$, we obtain $x \notin B$. Therefore,

$$x \in A \quad \text{and} \quad x \notin B$$

and so, by definition of set difference, $x \in A - B$. Since x was arbitrary element of $A - C$, we know $A - C \subseteq A - B$. Therefore, by contrapositive, if $A - C \nsubseteq A - B$, then $B \nsubseteq C$. $\square$

4. If and only if (equivalence)

Exam 1 #5. Let $n \in \mathbb{Z}$. Use definitions of even or odd numbers to prove that $4n^2 - n + 5$ is odd if and only if n is even. Prove any results, theorems, propositions, etc. about even or odd numbers you use.

Proof. **($\Longrightarrow$).** We will prove the contrapositive statement: If n is odd, then $4n^2 - n + 5$ is even. Suppose n is odd. By the definition of odd numbers, $n = 2k + 1$ for some integer k. Thus,

$$\begin{aligned} 4n^2 - n + 5 &= 4(2k+1)^2 - (2k+1) + 5 \\ &= 4(4k^2 + 4k + 1) - 2k - 1 + 5 \\ &= 16k^2 + 16k + 4 - 2k + 4 \\ &= 16k^2 + 14k + 8 \\ &= 2(8k^2 + 7k + 4). \end{aligned}$$

Let $l = 8k^2 + 7k + 4$. Since $k \in \mathbb{Z}$, we know that $l \in \mathbb{Z}$. Hence,

$$4n^2 - n + 5 = 2l$$

Thus, by the definition of even numbers, $4n^2 - n + 5$ is even. Therefore, by contrapositive, we have proven that if $4n^2 - n + 5$ is odd, then n is even.

($\Longleftarrow$). Suppose $n \in \mathbb{Z}$ and n is even. By the definition of even numbers $n = 2k$ for some integer k. Thus,

$$\begin{aligned} 4n^2 - n + 5 &= 4(2k)^2 - (2k) + 5 \\ &= 16k^2 - 2k + 4 + 1 \\ &= 2(8k^2 - k + 2) + 1. \end{aligned}$$

Let $m = 8k^2 - k + 2$. Since $k \in \mathbb{Z}$, we know that $m \in \mathbb{Z}$. Hence,

$$4n^2 - n + 5 = 2m + 1$$

By the definition of odd numbers, we know that $4n^2 - n + 5$ is odd. Therefore, if n is even, then $4n^2 - n + 5$ is odd.

Since we have proven that if $4n^2 - n + 5$ is odd, then n is even and if n is even, then $4n^2 - n + 5$ is odd, it must be true that $4n^2 - n + 5$ is odd if and only if n is even. $\square$

5. Existence and Uniqueness Proof

Homework 4 #7. Prove that for all integers a, b, c, d with $a \neq c$ and $ad - bc \neq 0$, there exists a unique rational number r such that

$$\frac{ar + b}{cr + d} = 1.$$

Proof. **Existence.** Suppose $r = \frac{d-b}{a-c}$. Since a, b, c, d are integers and $a \neq c$, we know this is a ratio of integers with non-zero denominator. Hence, $r \in \mathbb{Q}$. By substituting r back into the expression, we have

$$\begin{aligned}
\frac{ar + b}{cr + d} &= \frac{a\left(\frac{d-b}{a-c}\right) + b}{c\left(\frac{d-b}{a-c}\right) + d} \\
&= \frac{\dfrac{a(d-b) + b(a-c)}{a-c}}{\dfrac{c(d-b) + d(a-c)}{a-c}} \\
&= \frac{a(d-b) + b(a-c)}{c(d-b) + d(a-c)} \\
&= \frac{ad - ab + ab - bc}{cd - bc + ad - cd} \\
&= \frac{ad - bc}{ad - bc} \\
&= 1.
\end{aligned}$$

Thus, such an r exists and satisfies $\dfrac{ar + b}{cr + d} = \dfrac{ar + b}{cr + d} = 1.$

Uniqueness. Suppose $r_1, r_2 \in \mathbb{Q}$ such that

$$\frac{ar_1 + b}{cr_1 + d} = 1 = \frac{ar_2 + b}{cr_2 + d}.$$

Thus,

$$\begin{aligned}
\frac{ar_1+b}{cr_1+d} &= \frac{ar_2+b}{cr_2+d} \\
(ar_1+b)(cr_2+d) &= (ar_2+b)(cr_1+d) \\
\cancel{acr_1r_2} + adr_1 + bcr_2 + \cancel{bd} &= \cancel{acr_1r_2} + adr_2 + bcr_1 + \cancel{bd} \\
(ad)r_1 + (bc)r_2 &= (ad)r_2 + (bc)r_1 \\
(ad)r_1 - (bc)r_1 &= (ad)r_2 - (bc)r_2 \\
r_1(\cancel{ad-bc}) &= r_2(\cancel{ad-bc}) \\
r_1 &= r_2.
\end{aligned}$$

Thus, r is unique.

Therefore, for all integers a, b, c, d with $a \neq c$ and $ad - bc \neq 0$, there exists one unique rational number r such that $\frac{ar+b}{cr+d} = 1$.

$\square$

6. Proof involving sets

Homework 5 #3. Fix a set U. Prove that there is a unique set $A \subseteq U$ such that

$$A \cup B = A \quad \text{for all } B \subseteq U.$$

Proof. **Existence.** Suppose $A = U$ and let $B \subseteq U$. If $x \in U \cup B$, then, by definition of set union, $x \in U$. Conversely, if $x \in U$, then $x \in U \cup B$ and so $U \cup B \subset U$ and $U \subset U \cup B$. Therefore, $U \cup B = U = A$. Note that $A = U$ and so we can replace the the A with U in $A \cup B = A$. Since $A = U$, we know that $A \subseteq U$. Hence, $A \cup B = A$ for all $B \subseteq U$.

Uniqueness. Suppose A', A are subsets of U such that for all $B \subseteq U$,

$$A \cup B = A \quad \text{and} \quad A' \cup B = A'.$$

Note that considering $B = A'$ (since $A' \subseteq U$) in the first equation gives $A \cup A' = A$. Similarly, taking $B = A$ (since $A \subseteq U$) in the second equation gives $A' \cup A = A'$. Since $A = A \cup A' = A' \cup A = A'$, it must be true that $A = A'$. Thus, the set A is unique.

Therefore, there is only one unique set $A \subseteq U$ such that $A \cup B = A$ for all $B \subseteq U$.

$\square$

7. Induction proof

Homework 5 #7. Prove that $2^n + 1 \leq 3^n$ for every positive integer n.

Proof. We proceed by mathematical induction on n.
(Base Case). If $n = 1$, we have

$$2^1 + 1 = 3 = 3^1.$$

Thus, the base case holds.
(Inductive Step). Suppose for all $N \geq 1$, we have $2^N + 1 \leq 3^N$. We must show that $2^{N+1} + 1 \leq 3^{N+1}$. By multiplying the both sides of inductive hypothesis, we have

$$2 \cdot 2^N + 2 \leq 2 \cdot 3^N$$
$$2^{N+1} + 2 \leq 2 \cdot 3^N.$$

Since $2 < 3$, we have $2 \cdot 3^N \leq 3 \cdot 3^N = 3^{N+1}$ and so,

$$2^{N+1} + 2 \leq 3^{N+1}.$$

Since $2^{N+1} + 1 < 2^{N+1} + 2$, it follows that

$$2^{N+1} + 1 < 3^{N+1},$$

which implies $2^{N+1}+1 \leq 3^{N+1}$ Therefore, by principles of mathematical induction, $2^n + 1 < 3^n$ for all positive integers n. $\square$

8. Proof of an Equivalence Relation

Homework 7 #4. Suppose P is a partition of a set A. Define a relation R on A by declaring if $a, b \in A$, then aRb if and only if there exists $X \in P$ such that $a, b \in X$. prove that R is an equivalence relation on A.

Proof. **(Reflexive).** Let a be an arbitrary element in set A. Since P is a partition, the union of all sets in P is A. Thus, there exists some $X \in P$ such that $a, a \in X$ and aRa. Therefore, R is reflexive.

(Symmetry). Suppose $a, b \in A$ and aRb. Thus, there exists $X \in P$ such that $a, b \in X$. Note that $b, a \in X$ is same as $a, b \in X$. Hence, definition of R, bRa. Therefore, R is symmetric.

(Transitive). Suppose $a, b, c \in A$ such that aRb and bRc. By definition of Partition, there exists some $X \in P$ such that $a, b \in X$ and some $Y \in P$ such that $b, c \in Y$. Since P is a partition, we know that if $X \neq Y$, then $X \cap Y = \emptyset$. Consider the contrapositive statement: if $X \cap Y \neq \emptyset$, then $X = Y$. Since $b \in X \cap Y$, it follows that $X \cap Y \neq \emptyset$ and so $X = Y$. Thus a, b, c all belong to same subset $X \in P$ and so aRc. Therefore, R is transitive.

Since R is reflexive, symmetric, and transitive, we know R is an equivalence relation on A. $\square$

9. Injectivity and Subjectivity of a Function

Homework 8 #6. Prove that the function $\Theta : \{0,1\} \times \mathbb{N} \to \mathbb{Z}$ defined by $\Theta(a,b) = a - 2ab + b$ for all $(a,b) \in \{0,1\} \times \mathbb{N}$ is bijective.

Proof. **(Injective).** suppose $\Theta(a,b) = \Theta(c,d)$ for some $(a,b),(c,d) \in \{0,1\} \times \mathbb{N}$. Thus, $a - 2ab + b = c - 2cd + d$. Since the first coordinate of any ordered pair in the domain comes from the set $\{0,1\}$, the only possibilities are that

$$a = c = 1 \quad \text{or} \quad a = c = 0.$$

Suppose, for sake of a contradiction, $a \neq c$. Without loss of generality, let $a = 1$ and $c = 0$. By putting in the values of a and c in our function definition, we have

$$\begin{aligned} a - 2ab + b &= c - 2cd + d \\ 1 - 2 \cdot 1 \cdot b + b &= 0 - 2 \cdot 0 \cdot d + d \\ 1 - 2b + b &= d \\ 1 - b &= d. \end{aligned}$$

Since $d \in \mathbb{N}$, the expression $1 - b$ must also be a natural number. However, $1 - b \notin \mathbb{N}$ for all $b \geq 2$ and so d is a negative number. This contradicts our initial assumption that d is a natural number. Therefore, our assumption $a \neq c$ must be false, and we conclude that $a = c$. Given $a = c$, there remain two possibilities of $a = c = 1$ or $a = c = 0$.

Case 1. Suppose $a = c = 1$. By putting it into our function definition, we have

$$\begin{aligned} a - 2ab + b &= c - 2cd + d \\ 1 - 2 \cdot 1 \cdot b + b &= 1 - 2 \cdot 1 \cdot d + d \\ 1 - 2b + b &= 1 - 2d + d \\ 1 - b &= 1 - d \\ b &= d. \end{aligned}$$

Case 2. Suppose $a = c = 0$. By putting it into our function definition, we have

$$\begin{aligned} a - 2ab + b &= c - 2cd + d \\ 0 - 2 \cdot 0 \cdot b + b &= 0 - 2 \cdot 0 \cdot d + d \\ b &= d. \end{aligned}$$

In both cases, assuming $\Theta(a, b) = \Theta(c, d)$ forces $a = c$ and $b = d$. Thus Θ is injective.

(Surjective). To show that Θ is surjective, we must prove that for every $x \in \mathbb{Z}$ there exists a pair $(a, b) \in \{0, 1\} \times \mathbb{N}$ such that $\Theta(a, b) = x$. Recall that $\Theta(a, b) = a - 2ab + b$. Since $a \in \{0, 1\}$, we consider both possible values.

Case 1. Suppose $x \geq 1$. Let $a = 0$ and $b = x$. Since $x \geq 1$ and $b \in \mathbb{N}$, we have

$$\Theta(0, x) = 0 - 2 \cdot 0 \cdot x + x = x$$

.

Case 2. Suppose $x \leq 0$ and let $a = 1$ and $b = 1 - x$. Since $x \leq 0$, we have $b = 1 - x \geq 1$ and so $b \in \mathbb{N}$. Thus,

$$\begin{aligned}\Theta(1, 1 - x) &= 1 - 2 \cdot 1 \cdot (1 - x) + (1 - x) \\ &= 1 - 2(1 - x) + 1 - x \\ &= x.\end{aligned}$$

In both cases, whenever the chosen b lies in $\mathbb{N}$, we obtain a pair (a, b) satisfying $\Theta(a, b) = x$. Thus, for every $x \in \mathbb{Z}$ there exists $(a, b) \in \{0, 1\} \times \mathbb{N}$ with $\Theta(a, b) = x$, so Θ is surjective.

Since Θ is both injective and surjetive, we can conclude that Θ is bijective. $\square$

10. Proof involving the image or pre-image of a function

Exam 2 #4. Let A and B be sets and let $f : A \to B$ be a function. If $B_1, B_2 \subseteq B$, prove that

$$f^{-1}(B_1 - B_2) = f^{-1}(B_1) - f^{-1}(B_2)$$

only using definitions.

Proof. **($\subseteq$).** Let $b \in f^{-1}(B_1 - B_2)$. By definition of preimage,

$$f^{-1}(B_1 - B_2) = \{x \in A : f(x) \in B_1 - B_2\}.$$

Since $f(b) \in B_1 - B_2$, it follows $f(b) \in B_1 \quad \text{and} \quad f(b) \notin B_2$. Since $b \in A$ and $f(b) \in B_1$, we know $b \in f^{-1}(B_1)$, and since $f(b) \notin B_2$, we know $b \notin f^{-1}(B_2)$. Thus, $b \in f^{-1}(B_1) - f^{-1}(B_2)$ and so,

$$f^{-1}(B_1 - B_2) \subseteq f^{-1}(B_1) - f^{-1}(B_2).$$

($\supseteq$). Let $b \in f^{-1}(B_1) - f^{-1}(B_2)$. By the set difference definition, we know $b \in f^{-1}(B_1)$ or $b \notin f^{-1}(B_2)$. By definition of pre image,

$$f^{-1}(B_1) = \{x \in A : f(x) \in B_1\}.$$

Thus, $b \in A$ and $f(b) \in B_1 - B_2$. Thus, $b \in f^{-1}(B_1 - B_2)$.Therefore, $f^{-1}(B_1) - f^{-1}(B_2) \subseteq f^{-1}(B_1 - B_2)$.

Since we have both inclusions, it must be true that $f^{-1}(B_1 - B_2) = f^{-1}(B_1) - f^{-1}(B_2)$. $\square$

11. Proof that two infinite sets have the same cardinality

Home Work 10 #7. Prove that $|\mathcal{F}(\mathbb{R}, \{0,1\})| = |\mathcal{P}(\mathbb{R})|$.

Proof. Define the function $\Phi : \mathcal{P}(\mathbb{R}) \to F(\mathbb{R}, \{0,1\})$ by $\Phi(A) = \chi_A$for each $A \subseteq \mathbb{R}$, where the characteristic function $\chi_A : \mathbb{R} \to \{0,1\}$ is given by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

We must show that Φ is a bijection.
(Injective). Suppose $A, B \subseteq \mathbb{R}$ and $\Phi(A) = \Phi(B)$. Then, $\chi_A = \chi_B$ as functions from $\mathbb{R}$ to $\{0,1\}$. Hence, for all $x \in \mathbb{R}$, we have

$$\chi_A(x) = \chi_B(x).$$

If $x \in A$, then $\chi_A(x) = 1$. Since $\chi_A(x) = \chi_B(x)$, we know that $\chi_B(x) = 1$, which implies $x \in B$. Thus, $A \subseteq B$. Conversely, if $x \in B$, then $\chi_B(x) = 1$. Since $\chi_A(x) = \chi_B(x)$, we know that $\chi_A(X) = 1$ which implies $x \in A$. Thus, $B \subseteq A$. Therefore, $A = B$ and so ϕ is injective.
(Surjective). Suppose $f \in F(\mathbb{R}, \{0,1\})$ be an arbitrary function from $\mathbb{R}$ to $\{0,1\}$. Define the set

$$A = f^{-1}(\{1\}) = x \in \mathbb{R} : f(x) = 1.$$

Thus, $A \subseteq \mathbb{R}$ and so $A \in \mathcal{P}(\mathbb{R})$. Let $\phi(A) = f$. For any $x \in \mathbb{R}$, if $x \in A$, then by definition $f(x) = 1$, and so $\chi_A(x) = 1$. Moreover, if $x \notin A$, then $f(x) = 0$ (since f maps in to $\{0,1\}$) and so $\chi_A(x) = 0$. Thus, $\chi_A(x) = f(x)$ for all $x \in \mathbb{R}$ which implies $\chi_A = f$. Therefore, $\phi(A) = f$ and so ϕ is surjective.

Since Φ is both injective and surjective, there must be a bijection between $\mathcal{F}(\mathbb{R}, \{0,1\})$ and $\mathcal{P}(\mathbb{R})$. Therefore,

$$|\mathcal{F}(\mathbb{R}, \{0,1\})| = |\mathcal{P}(\mathbb{R})|.$$

$\square$

12. Proof that something is a group

Home Work 10 #7. Prove that $\mathbb{R}-\{1\}$, together with the operation $\circ$ defined by

$$x \circ y = \frac{1}{2}(x+y-xy+1)$$

for all $x, y \in \mathbb{R}-\{1\}$, forms an abelian group. (Recall that a group is abelian if its operation is commutative.) Be sure to verify that $\circ$ is a well-defined operation on $\mathbb{R}-\{1\}$.

Proof. Let $\circ$ be the operation on $\mathbb{R}-\{1\}$ defined by

$$x \circ y = \frac{1}{2}(x+y-xy+1).$$

Closure. Let $x, y \in \mathbb{R}-\{1\}$. Suppose, for sake of contradiction, that

$$x \circ y = 1.$$

Then, $\frac{1}{2}(x+y-xy+1) = 1$, which implies $x+y-xy+1 = 2$. Hence, $xy - x - y = -1$ which can be rewritten as $(x-1)(y-1) = 0$. Note that either $x = 1$ or $y = 1$ which contradicts the assumption that $x, y \in \mathbb{R}-\{1\}$. Therefore, $x \circ y \neq 1$ and so $x \circ y \in \mathbb{R}-\{1\}$. Hence $\circ$ is closed.

Associativity. Let $x, y, z \in \mathbb{R}-\{1\}$. Thus,

$$\begin{aligned}(x \circ y) \circ z &= \frac{1}{2}\left(\frac{1}{2}(x+y-xy+1) + z - \frac{1}{2}(x+y-xy+1)z + 1\right) \\ &= \frac{1}{4}\big(x+y+z-xy-xz-yz+xyz+1\big).\end{aligned}$$

Similarly,

$$\begin{aligned}x \circ (y \circ z) &= \frac{1}{2}\left(x + \frac{1}{2}(y+z-yz+1) - x\frac{1}{2}(y+z-yz+1) + 1\right) \\ &= \frac{1}{4}\big(x+y+z-xy-xz-yz+xyz+1\big).\end{aligned}$$

Note that $(x \circ y) \circ z = x \circ (y \circ z)$ and so $\circ$ is associative.

Identity. We seek $e \in \mathbb{R}-\{1\}$ such that $x \circ e = x$ for all $x \in \mathbb{R}-\{1\}$. Then,

$$\frac{1}{2}(x+e-xe+1) = x.$$

Multiplying both sides by 2 and rearranging gives

$$\begin{aligned}x+e-xe+1 &= 2x \\ e(1-x) &= x-1.\end{aligned}$$

Note that for $x \neq 1$, we get $e = -1$. Since $-1 \in \mathbb{R} - \{1\}$ and

$$x \circ (-1) = (-1) \circ x = x.$$

Thus, the identity element is -1.

Inverse. Let $x \in \mathbb{R} - \{1\}$. We seek $y \in \mathbb{R} - \{1\}$ such that

$$x \circ y = -1.$$

Then

$$\frac{1}{2}(x + y - xy + 1) = -1,$$

which gives

$$x + y - xy + 1 = -2.$$

Rewriting,

$$y(1 - x) = -3 - x,$$

so

$$y = \frac{x + 3}{x - 1}.$$

Since $x \neq 1$, this value is well-defined and satisfies $y \neq 1$. Hence every element has an inverse.

Commutativity. Let $x, y \in \mathbb{R} - \{1\}$. Then,

$$x \circ y = \frac{1}{2}(x + y - xy + 1) = \frac{1}{2}(y + x - yx + 1) = y \circ x.$$

Thus $\circ$ is commutative.

Since $\mathbb{R} - \{1\}$ is closed under $\circ$, $\circ$ is associative, has an identity element, every element has an inverse, and $\circ$ is commutative, it follows that $(\mathbb{R} - \{1\}, \circ)$ is an abelian group.

$\square$