Probability Model For A Multilevel 2PL Item-response Model

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1 2PL Item-response Model

In the 1PL (Rasch) model, there is only parameter for questions. In the 2PL item-response model, we generalize the model by adding a discrimination parameter γ to model how noisy a question is and by adding multilevel priors for student ability parameters, the question difficulty and discrimination parameters.

Suppose J students are given a test with K questions, with $y_{jk} = 1$ if the response of student j to question k is correct, and $y_{jk} = 0$ otherwise. Assume that there are N individual responses (observations) with each response n associated with a person j_n and a question k_n, where $j_n \in 1, \ldots, J$, and $k_n \in 1, \ldots, K$, and α_j is the ability of student j, β_k is the difficulty of question k, γ_k is the discrimination of question k, then the 2PL model can be written as

$$\Pr\left(y_n = 1\right) = \operatorname{logit}^{-1}\left(\gamma_{k_n}\left(\alpha_{j_n} - \beta_{k_n}\right)\right) \tag{1}$$

2 Probability Model

We assume the following multilevel priors for α_j, β_k, and γ_k

$$\begin{aligned}
\alpha_j &\sim \operatorname{Normal}(0, 1) \\
\beta_k &\sim \operatorname{Normal}(\mu_\beta, \sigma_\beta) \\
\mu_\beta &\sim \operatorname{Cauchy}(0, 5) \\
\sigma_\beta &\sim \operatorname{Cauchy}_+(0, 5) \\
\gamma_k &\sim \operatorname{Lognormal}(0, \sigma_\gamma) \\
\sigma_\gamma &\sim \operatorname{Cauchy}_+(0.5)
\end{aligned}$$

where $j = 1, \ldots, J$, and $k = 1, \ldots, K$.

Putting it all together, we have the probability model for this multilevel 2PL model as:

$$
\begin{aligned}
y_n &\sim \text{Bernoulli}\left(p_n\right) \\
p_n &= \text{logit}^{-1}\left(\gamma_{k_n}\left(\alpha_{j_n} - \beta_{k_n}\right)\right) \\
\alpha_j &\sim \text{Normal}\left(0, 1\right) \\
\beta_k &\sim \text{Normal}\left(\mu_\beta, \sigma_\beta\right) \\
\mu_\beta &\sim \text{Cauchy}\left(0, 5\right) \\
\sigma_\beta &\sim \text{Cauchy}_+\left(0, 5\right) \\
\gamma_k &\sim \text{Lognormal}\left(0, \sigma_\gamma\right) \\
\sigma_\gamma &\sim \text{Cauchy}_+\left(0.5\right)
\end{aligned}
$$

where $j_n \in (1, \ldots, J)$, $k_n \in (1, \ldots, K)$, $j = 1, \ldots, J$, and $k = 1, \ldots, K$.