Probability Model For An ARCH(m) Model

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1 ARCH(m) Model

Asset volatility is perhaps the most commonly used risk measure in finance [1]. It has some characteristics that are commonly seen in asset returns [1] such as: 1) Volatility evolves over time in a continuous manner; 2) Volatility is often an stationary process; etc. which are important in volatility modeling.

Let r_t be the log return of an asset at time t. $\{r_t\}$ is either serially uncorrelated or with minor lower order serial correlations, but it is a dependent series. Let F_{t-1} denote the information available at time $t-1$, the conditional mean of asset return $\mu_t = E\left(r_t \,|F_{t-1}\right)$ could be modeled as following a stationary ARMA(p, q) model. Here, we assume $\mu_t = \mu$, a constant, for simplicity but without losing generality when discussing the ARCH model. The conditional standard deviation σ_t of r_t is the volatility to be modeled that $\sigma_t^2 = \mathrm{Var}\left(r_t \,|F_{t-1}\right)$. Therefore we have

$$r_t = \mu + a_t \tag{1}$$

$$\sigma_t^2 = \mathrm{Var}\left(r_t \,|F_{t-1}\right) = \mathrm{Var}\left(a_t \,|F_{t-1}\right) \tag{2}$$

where a_t is referred to as the shock of an asset return at time t.

There are different conditional heteroscedastic models that use different dynamic equations to govern the time evolution of the conditional variance σ_t^2 of the asset return.

The ARCH model of Engle [2] is the first model that provides a systematic framework for volatility modeling.

An ARCH(K) model is defined as

$$a_t = \sigma_t \epsilon_t \tag{3}$$

$$\sigma_t^2 = \alpha_0 + \sum_{k=1}^{K} \alpha_k a_{t-k}^2 \tag{4}$$

where $\{\epsilon_t\}$ is a sequence of iid standard normal random variables, $\alpha_0 > 0$, and $\alpha_i \geq 0$ for all $i > 0$. To ensure stationarity of the time series, $\sum\limits_{k=1}^{K} \alpha_k < 1$.

The unconditional mean and variance of a_t can be obtained as

$$E\left(a_t\right) = E\left[E\left(a_t \left| F_{t-1}\right.\right)\right] = E\left[\sigma_t E\left(\epsilon_t\right)\right] = 0 \tag{5}$$

$$\operatorname{Var}\left(a_t\right) = \frac{\alpha_0}{1 - \sum\limits_{k=1}^{K} \alpha_k} \tag{6}$$

Given $\epsilon_t \sim N(0,1)$, we have

$$a_t \left| \{\mu, \alpha_0, \alpha_1, \ldots, \alpha_K, r_{t-1}, \ldots, r_{t-K}\} \right. \sim N\left(0, \sigma_t\right) \tag{7}$$

$$r_t \left| \{\mu, \alpha_0, \alpha_1, \ldots, \alpha_K, r_{t-1}, \ldots, r_{t-K}\} \right. \sim N\left(\mu, \sigma_t\right) \tag{8}$$

2 Probability Model

Here we use an improper prior for μ and $\alpha_k, k = 0, \ldots, K$ to illustrate the idea. A weakly informative prior or informative prior could be added if more knowledge of the parameters is available.

The probability model for this ARCH(K) model is described as (we dropped the condition for likelihood for simplicity):

$$\begin{aligned}
r_t &\sim \text{Normal}\left(\mu, \sigma_t\right) \\
\sigma_t^2 &= \alpha_0 + \sum_{k=1}^{K} \alpha_k a_{t-k}^2 \\
a_{t-k} &= r_{t-k} - \mu \\
\mu &\sim \text{Uniform}\left(-\infty, \infty\right) \\
\alpha_0 &\sim \text{Uniform}\left(0, \infty\right) \\
\alpha_k &\sim \text{Uniform}\left(0, 1 - \sum_{j=1, j \neq k}^{K} \alpha_j\right) \\
k &= 1, \ldots, K
\end{aligned}$$

References

[1] R. S. Tsay, *An introduction to analysis of financial data with R.* John Wiley & Sons, 2014.

[2] R. F. Engle, "Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation," *Econometrica: Journal of the Econometric Society*, pp. 987–1007, 1982.