Probability Model For An Ordered Logistic Regression Model

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1 Ordered Logistic Regression Model

The ordered logistic regression is a regression model for ordinal dependent variables. Assume the observable dependent variable is y, which is a function of another unobserved, continuous latent variable y^* in the ordered logit model described as below.

$$y = \begin{cases} 1, & \text{if } y^* \leq c_1, \\ 2, & \text{if } c_1 < y^* \leq c_2, \\ \vdots & \\ K, & \text{if } c_{K-1} < y^* \end{cases}$$

And

$$\begin{aligned} y^* &= X\beta + \epsilon \\ \epsilon &\sim \text{Logistic}\,(0, 1) \end{aligned}$$

where $y_n \in \{1, \ldots, K\}\,, n = 1, \ldots, N$; $y^* \in \mathbb{R}^N$; $X \in \mathbb{R}^{N \times P}$ is the predictor matrix, where N is the number of observations, and D is the number of predictors; $\beta \in \mathbb{R}^P$ is the coefficient vector; and $c \in \mathbb{R}^{K-1}$ is a sequence of cutpoints sorted so that $c_k < c_{k+1}$.

Therefore,

$$y^* | \, \beta \sim \text{Logistic}\,(X\beta, 1)$$

$$\Pr\left(y_n^* \leq c_k\right) = \frac{1}{1 + \exp\left(-\left(c_k - \sum_{p=1}^{P} X_{n,p}\beta_p\right)\right)}, \; n = 1, \ldots, N$$

where

$$\frac{1}{1 + \exp\left(-\left(c_k - \sum_{p=1}^{P} X_{n,p}\beta_p\right)\right)} = \text{logit}^{-1}\left(c_k - \sum_{p=1}^{P} X_{n,p}\beta_p\right) \tag{1}$$

Assuming $c_0 = -\infty$, $c_K = \infty$, $\text{logit}^{-1}(-\infty) = 0$, and $\text{logit}^{-1}(\infty) = 1$, the probability of y_n is therefore computed as

$$\begin{aligned}
\Pr(y_n = k) &= \Pr(c_{k-1} < y_n^* \le c_k) \\
&= \Pr(y_n^* \le c_k) - \Pr(y_n^* \le c_{k-1}) \\
&= \text{logit}^{-1}\left(c_k - \sum_{p=1}^{P} X_{n,p}\beta_p\right) - \text{logit}^{-1}\left(c_{k-1} - \sum_{p=1}^{P} X_{n,p}\beta_p\right)
\end{aligned}$$

where $n = 1, \ldots, N$, and $k = 1, \ldots, K$.

2 Probability Model

c and β are parameters in the ordered logit model. Stan allows us to use improper priors if we don't have any prior knowledge about the parameters. We can therefore start with a simple model by assuming improper priors for c and β.

$$\begin{aligned}
&c_k \sim \text{Uniform}(-\infty, \infty), \ c_k \le c_{k+1} \\
&\beta_p \sim \text{Uniform}(-\infty, \infty) \\
&k = 1, \ldots, K-1, \ p = 1, \ldots, P
\end{aligned}$$

Putting it all together, the probability model for the ordered logistic regression model is:

$$\begin{aligned}
y_n &\sim \text{Categorical}(\mathbf{p}_n) \\
\mathbf{p}_n &= (p_{n1}, \ldots, p_{nK}) \\
p_{nk} &= \text{logit}^{-1}\left(c_k - \sum_{p=1}^{P} X_{n,p}\beta_p\right) - \text{logit}^{-1}\left(c_{k-1} - \sum_{p=1}^{P} X_{n,p}\beta_p\right) \\
\beta_p &\sim \text{Uniform}(-\infty, \infty) \\
c_k &\sim \text{Uniform}(-\infty, \infty), \ k = 1, \ldots, K-1 \\
c_0 &= -\infty, \ c_K = \infty \\
c_k &< c_{k+1}, \ k = 1, \ldots, K-1
\end{aligned}$$