Financial Engineering

Exam

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F20

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Bonds

Definition (Bond)

A bond is a financial security that pays the owner a chain of predetermined payments.

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A bond is a financial security that pays the owner a chain of predetermined payments.

- Financial asset with no risk

Interest Rate

- Predetermined payments are also known as interest
- Fraction of an investment paid either ones for several periods
- Different types of interest
  1. Simple
  2. Compounded
  3. Continuously compounded

Definition (Wealth Process)

The evolution of an investment over time is called the wealth process of that investment and is denoted by

$$V = (V_t)_{0 \leq t \leq T}. \tag{1}$$

The initial capital is denoted by v_0, and we assume that V is a real-valued stochastic process on a given probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Simple Interest)

Let $v_0 \in \mathbb{R}$ be our initial capital. An interest on v_0 is said to be simple if it follows the wealth process

$$V_t = (1 + rt)v_0, \quad 0 \leq t \leq T. \tag{2}$$

Interest Rate

I will now show that the wealth process in (2) is indeed a stochastic process in any probability space. Any stochastic process X on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ satisfies

$$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathscr{F}, \quad \forall x \in \mathbb{R}. \tag{3}$$

Suppose $v_0 > 0$ and $x \geq (1 + rt)v_0$ then

$$\{\omega \in \Omega : (1 + rt)v_0 \leq x\} = \{\Omega\} \in \mathscr{F}, \tag{4}$$

on the other hand if $x < (1 + rt)v_0$

$$\{\omega \in \Omega : (1 + rt)v_0 \leq x\} = \{\emptyset\} \in \mathscr{F}. \tag{5}$$

As both Ω and $\emptyset$ is contained in any σ-algebra we have shown that the wealth process in (2) is a stochastic process in any probability space.

Definition (Compounded Interest)

Let $v_0 \in \mathbb{R}$ be our initial capital. An interest on v_0 is said to be compunded over $m \in \mathbb{N}$ periods if it follows the wealth process

$$V_t = \left(1 + \frac{r}{m}\right)^{mt} v_0, \quad 0 \leq t \leq T. \tag{6}$$

Note that we have the following properties $\forall\, 0 \leq t \leq T$

1. $V_{t+1} = \left(1 + \frac{r}{m}\right)^m V_t$,
2. If $m_1 > m_2, v_0 > 0 \Rightarrow \left(1 + \frac{r}{m_1}\right)^{m_1 t} v_0 > \left(1 + \frac{r}{m_2}\right)^{m_2 t} v_0$,
3. If $m_1 > m_2, v_0 < 0 \Rightarrow \left(1 + \frac{r}{m_1}\right)^{m_1 t} v_0 < \left(1 + \frac{r}{m_2}\right)^{m_2 t} v_0$.

From this is follows that for an *investor* compund interest is more attractive as it pays more, however as a *debtor* it is less attractive as he or she will have to pay more on his or hers debt.

At last i can turn to continuously compounded interest which i will present as the limit of (6) as $m \to \infty$. Note that by the following definition of e

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e, \tag{7}$$

by letting $x = r/m$ in the above the limit of the wealth process of compounded interest can be seen as

$$\left[\left(1 + \frac{r}{m}\right)^{\frac{m}{r}}\right]^{rt} v_0 \to (e)^{rt} v_0, \quad \text{as } m \to \infty. \tag{8}$$

This leads to the definition of continuously compounded interest.

Definition (Continuously Compounded Interest)

Let v_0 be our initial capital. An interest on v_0 is said to be continuously compounded at rate $r > 0$ if the wealth process

$$V_t = e^{rt} v_0, \quad 0 \leq t \leq T. \tag{9}$$

There exists the following relation between the different types of interest

$$(1 + r) \leq \left(1 + \frac{r}{m}\right)^m < e^r. \tag{10}$$

To show that the relation indeed holds i will show that the sequence

$$a_m = \left(1 + \frac{r}{m}\right)^m, \tag{11}$$

is increasing.

Using the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

we have

$$\begin{aligned} \left(1+\frac{r}{m}\right)^m &= \sum_{k=0}^{m} \binom{m}{k} 1^{m-k} \left(\tfrac{r}{m}\right)^k \\ &= \sum_{k=0}^{m} \binom{m}{k} \left(\tfrac{r}{m}\right)^k := \clubsuit \end{aligned}$$

Each term is of the form

$$\binom{m}{k}\left(\tfrac{r}{m}\right)^k = \prod_{l=0}^{k-1} \frac{m-l}{k-l}\left(\tfrac{r}{m}\right)$$

Each term is of the form

$$\frac{m-l}{k-l}\frac{r}{m} = \frac{rm-lr}{m(k-l)} = \frac{m(r-lr/m)}{m(k-l)} = \frac{r-lr/m}{k-l} := \bigstar$$

The term $\bigstar$ increases with m and thus the product increases with m and thus the sum $\clubsuit$ increases with m and therefore it is an increasing sequence.

Types of Bonds

I will discuss the following two types here

1. zero-coupon bonds,
2. coupon bonds.

A **zero-coupon bond** is a bond with a single payment $F > 0$ at time $T > 0$. The pay-off F is called the face value and T the maturity time. The next question i will answer is how much i will be willing to pay for such a financial assest. This depends on the way the time value of money is measured. Consider for example the following setup; let $B_0 \geq 0$ be the value of the zero-coupon bond with face value $F > 0$ and maturity time $T > 0$. Suppose that only annual compound interest at rate $r > 0$ is available.

From a buyers perspective what if

$$B_0 > \frac{F}{(1+r)^T}, \tag{12}$$

would i buy the bond?

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Suppose now that we flip the inequality and look from a sellers perspective, that is if

$$B_0 < \frac{F}{(1+r)^T}, \tag{13}$$

would i sell the bond?

Now i will consider the situatuion where at time $1 \leq t \leq T$ i want to get rid of a bond, but i what to determine what price i should sell it to. At this time the bond can be considered a new zero-coupon bond with face value $F > 0$ and maturity time $T - t$. Thus we have from the previous argumentation that

$$B_t = \frac{F}{(1+r)^{T-t}}, \quad 0 \leq t \leq T. \tag{14}$$

The chain of arguments holds also when the time value of money is different, if a compounded interest over m periods where considered then the fair price of a zero-coupon bond at time t would be

$$B_t = \frac{F}{\left(1 + \frac{r}{m}\right)^{m(T-t)}}. \tag{15}$$

If we consider the continuously compounded case the fair price would be

$$B_t = \frac{F}{e^{r(T-t)}}. \tag{16}$$

how much money will i have to deposit in my bank account today if i want to

$$
\begin{aligned}
1.\quad & \text{withdraw } C > 0 \text{ after 1 year} && (17)\\
2.\quad & \text{withdraw } C > 0 \text{ after 2 years} && (18)\\
& \vdots \\
T-1.\quad & \text{withdraw } C > 0 \text{ after } T-1 \text{ years} && (19)\\
T.\quad & \text{withdraw } F + C \text{ after } T \text{ years} && (20)
\end{aligned}
$$

and have nothing left in the bank account afterwards.

Types of Bonds

In order to be able to get $C > 0$ after one year i have to put

$$\frac{C}{1+r} \tag{21}$$

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In order to be able to get $C > 0$ after two years i have to put

$$\frac{C}{(1+r)^2} \tag{22}$$

in the bank.

Generalizing this argument tells me that in order to recieve $C > 0$ after t years i have to put

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Generalizing this argument tells me that in order to recieve $C > 0$ after t years i have to put

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in the bank.
Lastly in order to get $F + C$ after T years i have to put

$$\frac{F+C}{(1+r)^T} = \frac{F}{(1+r)^T} + \frac{C}{(1+r)^T} \tag{24}$$

in the bank.

Types of Bonds

Adding up all these amounts it is concluded that i have to make a deposit of

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The agreeable price of a coupon bond is thus given by

$$B_0 = \sum_{i=1}^{T} \frac{C}{(1+r)^i} + \frac{F}{(1+r)^T} = \frac{\xi_T}{(1+r)^T}. \tag{26}$$

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in other words the fair price of the coupon bond (as well as the zero-coupon bond) can be written as the discounted price of the total pay-off.

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