Financial Engineering
Lecture 3

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General comments

- Questions or comments about the previous lecture and/or exercise set?
- There was typo in the exercise set.
- Exercise 3: It should be

$$\begin{aligned} i)\ & (\varphi, \theta) \cdot P_0 = 0, \\ ii)\ & \mathbb{P}\left[(\varphi, \theta) \cdot P_1 \geq 0\right] = 1, \\ iii)\ & \mathbb{P}\left[(\varphi, \theta) \cdot P_1 > 0\right] > 0. \end{aligned}$$

and $R_d < K_B(1) < R_u$.

Review of the previous lecture

What did we do in the previous lecture?

Review of the previous lecture

- Coupon Bonds.
- Risky Assets.
- Returns.
- One-Step Binomial model.
- Portfolios and strategies in discrete-time financial markets.

Outline for today

- Portfolio Return.
- Portfolio Allocation.
- Risk Measures:
    - Deviation Measures: The CAPM setting.
    - Coherent Risk Measures: (Conditional) Value at Risk.

Portfolio Allocation

Portfolios in a one-step financial market:

- We will concentrate in the one-step finite-horizon financial market:

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, \mathbb{P}), P = (B_t, S_t^{(1)} \ldots, S_t^{(d)})_{t=0,1} \right\},$$

consisting of

1. A probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
2. $S^{(j)}$ is a risky asset.
3. B is a bond, such that $B_0 = 1$ and $B_1 = (1 + r)$, for some $r > 0$.

Portfolios and Returns

- In this set-up a portfolio and strategy are the same: a $(d+1)$-dimensional vector

$$\Theta = (\varphi, \theta^{(1)}, \dots, \theta^{(d)}) \in \mathbb{R}^{d+1}.$$

- Moreover, the wealth process consists of our initial capital $V_0 = v_0$ and the random variable

$$V_1 = \varphi(1+r) + \sum_{j=1}^{d} \theta^{(j)} S^{(j)}.$$

- Therefore, the return process is also a single random variable

$$K_V = \frac{V_1}{V_0} - 1.$$

Portfolio Weights

Proposition (Proposition 1)

Let $\Theta = (\varphi, \theta^{(1)}, \ldots, \theta^{(d)}) \in \mathbb{R}^{d+1}$ be a portfolio in one-step finite-horizon financial market. Put

$$w_{j-1} := \frac{\Theta^j S_0^{(j)}}{V_0}, \quad j = 1, \ldots, d+1.$$

Then, $\sum_{j=1}^{d+1} w_{j-1} = 1$ and

$$K_V = \sum_{j=1}^{d+1} w_{j-1} K_{P^{(j)}} = \mathbf{w} \cdot \mathbf{K}_P,$$

where $\mathbf{w} = (w_0, \ldots, w_d)$ and $\mathbf{K}_P = (r, K_{S^{(1)}}, \ldots, K_{S^{(d)}})$.

Proof.

See the Exercise Set. ■

Portfolio Weights and Portfolio Allocation

- Suppose that we want to invest $V_0 > 0$ on the market

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, \mathbb{P}), S = (B_t, S_t^{(1)} \ldots, S_t^{(d)})_{t=0,1} \right\}.$$

- The previous result says that we only need to choose the percentage of V_0 that we want to invest in each asset.
- This procedure of choosing $(w_0, \ldots, w_d)$ is known as portfolio allocation.

Portfolio Weights and Portfolio Allocation

- The behavior of returns plays a crucial role in the allocation: They are unpredictable.
- An agent always will take a look at the utility that a particular investment generates:

$$U(\mathbf{w} \cdot \mathbf{K}_P),$$

  for some utility function U and decide whether it is convenient or not to take such a position.
- The natural question here is how risky is this investment?
- Even tho the word risk is intuitively clear for most of the people, its quantification is not a trivial task.

Allocation as an Optimization Problem

Optimization Problem:

$$\arg\min_{\mathbf{w}\in\mathbb{R}^{d+1}} \mathcal{R}(\mathbf{w}\cdot\mathbf{K}_P).$$

Subject to:

$$\sum_{j=0}^{d} w_j = 1,\ \ \mathbb{E}\left[U(\mathbf{w}\cdot\mathbf{K}_P)\right] = \mu,\ \ \mu\in\mathbb{R},$$

where

$\mathcal{R}$ is a measure of risk.

$U(\mathbf{w}\cdot\mathbf{K}_P)$ is the utility of such strategy.

How to Measure the Risk

- Unless the market allows arbitrage, it is clear that any strategy carries a risk: it's random so we cannot predict our profit or losses with certainty.
- How do we measure the risk?
- It depends on the agent!
- Some agents wants to maximizes profit while reducing the variation of such investment: Mean-Variance approach.
- Some agents would prefer to minimizes their chances of default: Value At Risk approach.

Portfolio Allocation: The CAPM set-up

- The classical way of measuring risk is via standard deviations:

$$\mathcal{R}(X) = \sigma(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{\mathbb{E}\left[(X - \mathbb{E}(X))^2\right]}.$$

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- In particular for a portfolio with weights $\mathbf{w}$, we have that (see the exercise set)

$$\mathcal{R}(\mathbf{w} \cdot \mathbf{K}_P) = \sqrt{\mathbf{w}' C \mathbf{w}},$$

where C is a $(d+1) \times (d+1)$ matrix given by

$$C^{i,j} = \mathrm{Cov}(\mathbf{K}_P^i, \mathbf{K}_P^j) = \sqrt{\mathbb{E}\left[\left(\mathbf{K}_P^i - \mathbb{E}(\mathbf{K}_P^i)\right)\left(\mathbf{K}_P^j - \mathbb{E}(\mathbf{K}_P^j)\right)\right]}.$$

Portfolio Allocation: The CAPM set-up

- In the CAPM set-up the utility function is the identity function, that is
$$U(x) = x.$$
- By letting
$$\mu_{\mathbf{K}} := \mathbb{E}\left[\mathbf{K}_P\right],$$
we see that
$$\mathbb{E}\left[U(\mathbf{w} \cdot \mathbf{K}_P)\right] = \mathbf{w} \cdot \mu_{\mathbf{K}}.$$

Portfolio Allocation: The CAPM set-up

In this situation, the way of allocating our portfolio reduces to solve

Problem (Optimization Problem 1)

$$\arg\min_{\mathbf{w}\in\mathbb{R}^{d+1}} \sqrt{\mathbf{w}' C \mathbf{w}}.$$

$$\textit{Subject to: } i) \sum_{j=0}^{d} w_j = 1.$$

$$ii)\, \mathbf{w}\cdot\mu_{\mathbf{KP}} = \mu,\ \ \mu\in\mathbb{R}.$$

Portfolio Allocation: The CAPM set-up

Theorem (Allocation in the CAPM set-up)

Let $C_S = (C^{i,j})_{i,j=1}^d$ *and put* $\mathbf{m} := (K_{S(1)}, \ldots, K_{S(d)})$ *as well as* $\mathbf{u}' = (1, \ldots, 1) \in \mathbb{R}^d$. *If* $\det C_S \neq 0$ *and* μ_S *and* $\mathbf{u}$ *are linearly independent, then the solutions to the Optimization Problem 1 are given by*

$$\mathbf{w}_x^* = (x, \tilde{\mathbf{w}}_x^*), \quad x \in \mathbb{R},$$

where

$$\tilde{\mathbf{w}}_x^* = C_S^{-1} \left(a_1(x)\mathbf{u} + a_2(x)\mathbf{m} \right),$$

in which

$$\begin{pmatrix} a_1(x) \\ a_2(x) \end{pmatrix} = \begin{bmatrix} \mathbf{u}' C_S^{-1} \mathbf{u} & \mathbf{m}' C_S^{-1} \mathbf{u} \\ \mathbf{u}' C_S^{-1} \mathbf{m} & \mathbf{m} C_S^{-1} \mathbf{m} \end{bmatrix}^{-1} \begin{pmatrix} 1 - x \\ \mu - r x \end{pmatrix}.$$

Proof.

On the blackboard. ∎

An illustrative example

- Consider the portfolios that generates the wealth

$$V_1^{(1)} = \begin{cases} 1 & \text{with probability } 1/2; \\ -9 & \text{with probability } 1/2, \end{cases}$$

and

$$V_1^{(2)} = \begin{cases} 5 & \text{with probability } 1/2; \\ -5 & \text{with probability } 1/2, \end{cases}$$

- Which one is riskier?

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- Which one is riskier?
- Their standard deviations are the same (see the exercise set), so they carry the same risk.

Risk Measures

Value at Risk

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$$-V_1 \leq x.$$

- Then to cover the risk of bankruptcy, we must have at least have the amount x in our bank account.

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- If V_1 represents the outcome of an investment, then $-V_1$ describe our losses.
- Therefore, if we know that for some $x \in \mathbb{R}$

$$-V_1 \leq x.$$

- Then to cover the risk of bankruptcy, we must have at least have the amount x in our bank account.
- In reality we can only quantify the chances of this happening with

$$\mathbb{P}(-V_1 \leq x).$$

- This is the key motivation for considering the Value at Risk as a risk measure.

Value at Risk

Definition

Let $0 < \alpha < 1$ and X be a random variable. The **Value at Risk** (VaR from now on) of X is defined and denoted by

$$\mathrm{VaR}_\alpha(X) := \inf\{x \in \mathbb{R} : \mathbb{P}\,(X + x \geq 0) \geq 1 - \alpha\}.$$

Remarks $\left(\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(-X \leq x) \geq 1 - \alpha\}\right)$

We observe the following:

1. If X represents the outcome of an investment, then $\mathrm{VaR}_\alpha(X)$ represents the extra amount of extra capital we need to reduce the probability of bankruptcy to α:

$$\mathbb{P}(-X \leq x) \geq 1 - \alpha \Longleftrightarrow \mathbb{P}(X + x < 0) \leq \alpha.$$

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3. In F_X is the cdf of X, then

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X < -x) \leq \alpha\} = -\inf\{x \in \mathbb{R} : F_X(x) > \alpha\}$$

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4. In particular, if F_X is strictly increasing, then

$$\mathrm{VaR}_\alpha(X) = -F_X^{-1}(\alpha).$$

Value at Risk: Basic Properties

Recall that

$$\mathrm{VaR}_\alpha(X) := \inf \underbrace{\{x \in \mathbb{R} : \mathbb{P}(X < -x) \leq \alpha\}}_{A_\alpha(X)}.$$

Proposition (Proposition 2)

Let X, Y be arbitrary random variables. Then, the following holds

1. If $X \geq 0$ almost surely, then $\mathrm{VaR}_\alpha(X) \leq 0$.
2. For all $y \in \mathbb{R}$ we have that $\mathrm{VaR}_\alpha(X + y) = \mathrm{VaR}_\alpha(X) - y$. In particular $\mathrm{VaR}_\alpha(X + \mathrm{VaR}_\alpha(X)) = 0$.
3. If $\lambda \geq 0$, then $\mathrm{VaR}_\alpha(\lambda X) = \lambda \mathrm{VaR}_\alpha(X)$.
4. If $X \geq Y$ almost surely, then $\mathrm{VaR}_\alpha(X) \leq \mathrm{VaR}_\alpha(Y)$.

Proof.

On the blackboard. ∎