Financial Engineering
Lecture 4

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General comments

- Questions or comments about the previous lecture and/or exercise set?
- We will have a self-study session in 2 weeks (12 of March).
- The topic will be "Conditional Expectation".
- I am planing to write a plan on moodle by the next week: references, topics and exercises.
- I will, of course, be around if you need help.
- The Fundamental Theorems of Asset Pricing relies on the concept of martingale.
- It is really important that you understand the general concept of conditional expectations.

Review of the previous lecture

What did we do in the previous lecture?

Review of the previous lecture

- Portfolio Allocation:

$$\arg\min_{\mathbf{w}\in\mathbb{R}^{d+1}} \mathcal{R}(\mathbf{w}\cdot\mathbf{K}_P), \quad \mathbf{K}_p = \text{Returns of } P.$$

$$\text{Subject to: } i)\ \sum_{j=0}^{d} w_j = 1, \ \ ii)\ \mathbb{E}\left[U(\mathbf{w}\cdot\mathbf{K}_P)\right] = \mu.$$

- Explicit solutions when $\mathcal{R}(X) = \sigma(X) = \sqrt{\mathrm{Var}(X)}$.
- Value at Risk: The minimum extra capital we need to reduce the probability of bankruptcy to $0 < \alpha < 1$:

$$\begin{aligned}\mathrm{VaR}_\alpha(X) &:= \inf\{x\in\mathbb{R} : \mathbb{P}(X + x \geq 0) \geq 1-\alpha\} \\ &= \inf\{x\in\mathbb{R} : \mathbb{P}(X < -x) \leq \alpha\} \\ &= -\inf\{x\in\mathbb{R} : F_X(x) > \alpha\}\end{aligned}$$

Outline for today

- Coherent Risk Measures.
- Conditional Value at Risk.
- Financial Derivatives.
- Pricing Financial Derivatives in the One-Step Binomial model.

Coherent Risk Measures

Value at Risk: Basic Properties

In the last lecture we prove that:

Proposition (Proposition 2)

Let X, Y be arbitrary random variables. Then, the following holds

1. If $X \geq 0$ almost surely, then $\mathrm{VaR}_\alpha(X) \leq 0$.
2. For all $y \in \mathbb{R}$ we have that $\mathrm{VaR}_\alpha(X + y) = \mathrm{VaR}_\alpha(X) - y$.
3. If $\lambda \geq 0$, then $\mathrm{VaR}_\alpha(\lambda X) = \lambda \mathrm{VaR}_\alpha(X)$.
4. If $X \geq Y$ almost surely, then $\mathrm{VaR}_\alpha(X) \leq \mathrm{VaR}_\alpha(Y)$.

Lack of diversification

- Recall that $\sigma(X) = \sqrt{\mathrm{Var}(X)}$. In view that

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(X) + 2\rho_{X,Y}\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)},$$

where $\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} \leq 1$

- Then,

$$\begin{aligned}\sigma(X+Y)^2 =& \sigma(X)^2 + \sigma(Y)^2 + 2\rho_{X,Y}\underbrace{\sigma(Y)\sigma(X)}_{\geq 0}\\ \leq& \sigma(X)^2 + \sigma(Y)^2 + 2\sigma(Y)\sigma(X) = [\sigma(X)+\sigma(Y)]^2\end{aligned}$$

- Hence

$$\sigma(X+Y) \leq \sigma(X) + \sigma(Y).$$

Lack of diversification

- If we measure risk via σ, this means that investing in $X + Y$ carries less risk than investing separately on X and Y.
- This strategy is referred to as diversification, which is a common believe while dealing with risky investments.
- However, VaR_α is not able to reproduce this, i.e. in general we do **NOT** have that

$$\mathrm{VaR}_\alpha(X+Y) \leq \mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y).$$

Lack of diversification

- Suppose that a bank loans (with no interest) 100,000 DKK to a company that either will default on the loan with probability 0.008 or manages to pay its debt.
- The outcome of this strategy is described as

$$X = \begin{cases} -100000 & \text{with probability } 0.008; \\ 0 & \text{with probability } 0.992. \end{cases}$$

- Let $\alpha = 0.01$, then

$$\mathrm{VaR}_\alpha(X) = -\inf\{x \in \mathbb{R} : F_X(x) > 0.01\} = 0.$$

Lack of diversification

- Now, suppose that the bank makes two loans, each of 50,000 DKK to two different (independent) companies which both have a default probability of 0.008.

Lack of diversification

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- If X, Y represent the debt that each company has with the bank at the end of the loan period, then we have that

$$X+Y = \begin{cases} -100,000 & \text{with probability } 0.0064; \\ -50,000 & \text{with probability } 2(0.008)(0.992) = 0.015872; \\ 0 & \text{with probability } (0.992)^2. \end{cases}$$

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- Thus,

$$\mathrm{VaR}_{0.01}(X+Y) = -\inf\{x \in \mathbb{R} : F_X(x) > 0.01\} = 50,000.$$

- Consequently,

$$50,000 = \mathrm{VaR}_{0.01}(X+Y) > \mathrm{VaR}_{0.01}(X) + \mathrm{VaR}_{0.01}(Y) = 0.$$

Some notation

- Diversification is a desired property when we measure risk.
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- Diversification is a desired property when we measure risk.
- To do this, we introduce the concept of Coherent Risk Measures.
- Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
- We will use the following notation

$$L^p = \{X : \Omega \to \mathbb{R} : X \text{ r.v. such that } \mathbb{E}\left(|X|^p\right) < \infty\}.$$

Coherent Risk Measures

Definition (Coherent Risk Measures)

A function $\rho : L^1 \to \mathbb{R}$ is said to be a **Coherent Risk Measure** if

1. If $X \geq 0$ almost surely, then $\rho(X) \leq 0$.
2. For all $y \in \mathbb{R}$ we have that $\rho(X + y) = \rho(X) - y$.
3. If $\lambda \geq 0$, then $\rho(\lambda X) = \lambda \rho(X)$.
4. We have that $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

with:

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Coherent Risk Measures: Basic Properties

Proposition (Proposition 3)

Let $\rho : L^1 \to \mathbb{R}$ be a coherent risk measure. Then, the following holds

1. If $X \geq Y$ almost surely, then $\rho(X) \leq \rho(Y)$. In particular, if $a \leq X \leq b$ almost surely, then $-b \leq \rho(X) \leq -a$.
2. $\rho(X + \rho(X)) = 0$.

Proof.

On the blackboard. ∎

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Conditional Value at Risk

- Our main example and one of the most common risk measures is the so-called Conditional Value at Risk (CVaR from now on).
- Given a random variable X, we will write

$$q_\alpha(X) := \inf\{x : \mathbb{P}(X \leq x) \geq \alpha\}$$

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Definition (CVaR)

Let $0 < \alpha < 1$ and $X \in L^1$. The **Conditional Value at Risk or Expected Shortfall** of X is defined and denoted by

$$\mathrm{CVaR}_\alpha(X) := -\frac{1}{\alpha}\int_0^\alpha q_r(X)dr.$$

Conditional Value at Risk

- The name Expected Shortfall comes from the fact that if X has a continuous distribution, then

$$\mathrm{CVaR}_\alpha(X) = -\mathbb{E}\left[X \middle| X + \mathrm{VaR}_\alpha(X) \leq 0\right].$$

- Thus, CVaR_α measures the expected losses given that $\mathrm{VaR}_\alpha(X)$ was not enough to cover our position on X.

CVaR as a Coherent Risk Measure

Theorem (CVaR as a Coherent Risk Measure)

The CVaR, i.e. $\mathrm{CVaR}_\alpha(\mathrm{X}) = -\frac{1}{\alpha}\int_0^\alpha q_r(X)dr$, *where*

$$q_\beta(X) := \inf \underbrace{\{x : \mathbb{P}(X \le x) \ge \beta\}}_{A_\beta(X)},$$

is a coherent risk measure.

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Financial Derivatives

Financial Derivatives

- A financial derivative is a contract between two or more parties.
- Typical one of the parts is guaranteed a pay-off based on the value of one or several financial assets, e.g. stock prices.

Financial derivatives

European Options These type of contracts give to the owner the right, but not the obligation, for buying or selling a security at a fixed time $T > 0$ (known as the maturity date) at some fixed price K (called the strike price).

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Asian Options Similar rights as the European, but it can only be bought or sold at time $T > 0$ if the average of the price in $[0, T]$ is over/under the strike price K, respectively.

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Forwards It is a contract between two financial agents where, at time 0, no money is exchanged and at time $T > 0$ one part is obligated to sell/buy an asset with price S_T for an agreed F_T units of money.

Financial Derivatives

- There are three components on the contract:
  1. The underlying assets.
  2. The pay-off.
  3. The delivery time.
- How do we mathematized these concepts?

Financial Derivatives

Definition (DefFD)

Consider the discrete-time market

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, \mathbb{P}), P = (B_t, S_t^{(1)} \dots, S_t^{(d)})_{t=0,1,\dots,T} \right\}.$$

We will say that a random variable ξ is **a European financial derivative (contingent claim)** with date of maturity (exercise date) T if there exists a (measurable) function $\Phi : \prod_{t=0}^{T} \mathbb{R}^{d+1} \to \mathbb{R}$ such that

$$\xi = \Phi\left(P_0, P_1, \dots, P_T\right).$$

, ξ is called **simple** if it only depends on P_T, i.e.

$$\xi = \phi(P_T),$$

for some (measurable) function $\phi : \mathbb{R}^{d+1} \to \mathbb{R}$. will refer to the function Φ (or ϕ) as the pay-off function.

Forwards

- Recall that a forward is a contract in which at time $T > 0$ the parts involved are obligated to sell/buy an asset with price S_T for an agreed price $F_T > 0$.
- T as usual is the delivery time.
- $F_T > 0$ is known as the forward price and it is settled at the time the contract is written.
- This is a derivative, so what is its pay-off function?

Forwards

- Long Position: If you are obligated to buy the asset, your earning is

$$\underbrace{S_T}_{\text{Value received}} \underbrace{-F_T}_{\text{Payment}} .$$

- Short Position: If you are obligated to sell the asset, your earning is

$$\underbrace{F_T}_{\text{Payment Received}} - \underbrace{S_T}_{\text{Actual Value}} .$$

- We conclude that this contract pay-off

$$\text{Pay-off or earning} = \begin{cases} S_T - F_T & \text{for a long position;} \\ F_T - S_T & \text{for a short position.} \end{cases}$$

Forwards

- Forwards are simple derivatives with pay-off function given by

$$\phi(x) := \begin{cases} x - F_T & \text{for a long position;} \\ F_T - x & \text{for a short position.} \end{cases}$$

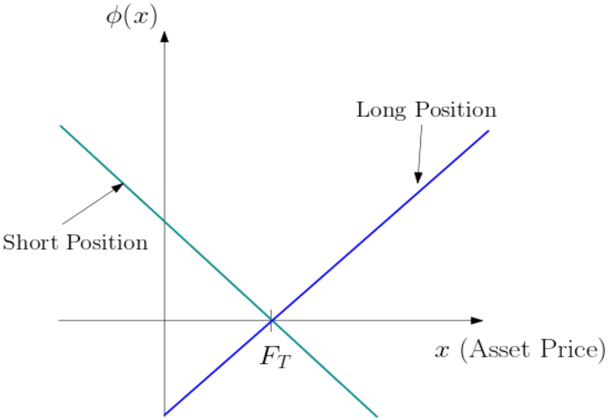

Figure: Pay-off function of forward contracts.

European Options

- Recall that European options give to the owner the right, but not the obligation, for buying or selling a security at a fixed time $T > 0$ (known as the maturity date) at some fixed price K (called the strike price).
- There are two type of options: Call and Put.
- In the call option the owner of the option has the right but not the obligation to buy the asset.
- In contrast, put options consider only the situations in which the owner sells the underlying asset.

Call Options

- Since you do not have the obligation to buy the asset you will compare whether it is convenient or not to buy it at price K, so
- If $S_T > K$ then we use our right to buy it and our earning is

$$S_T - K.$$

- If $S_T = K$, then it is irrelevant whether we buy the asset by means of the option or in the market.
- Either way your profit in this situation is 0.
- If $S_T < K$ you definitely won't buy because it is expensive compared to the market price, so once again your earnings are o.

Call Options

- The pay-off in this situation can be written as

$$\begin{aligned}\text{Pay-off or earnings} &= \begin{cases} S_T - K & \text{if } S_T > K; \\ 0 & \text{otherwise.} \end{cases} \\ &= \max\{S_T - K, 0\} \\ &=: (S_T - K)^+ .\end{aligned}$$

- It is clear from here that a call option is a derivative in the sense of Definition DefFD with pay-off function given by

$$\phi(x) = (x - K)^+.$$

Call Options

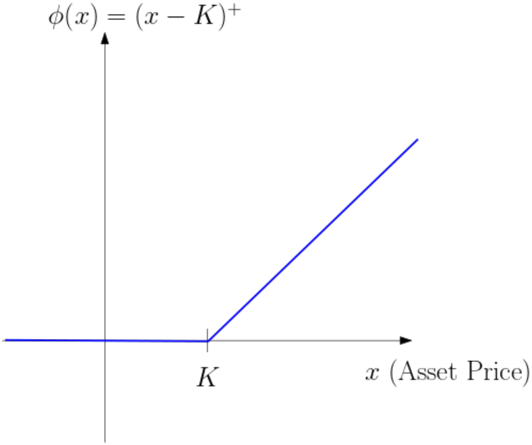

Figure: Pay-off function of a call option with strike price K.

Put Options

- By interchanging the roles in the reasoning developed for the call option we deduce that the pay-off for a put option satisfies that

$$\begin{aligned}\text{Pay-off or earnings} &= \begin{cases} K - S_T & \text{if } K > S_T; \\ 0 & \text{otherwise.} \end{cases} \\ &= \max\{K - S_T, 0\} \\ &= (K - S_T)^+ .\end{aligned}$$

- In this situation the pay-off function is given by

$$\phi(x) = (x - K)^+.$$

Put Options

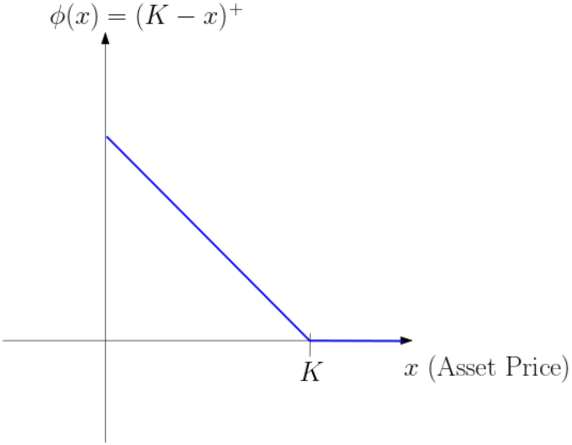

Figure: Pay-off function of a put option with strike price K.

Asian Options

- Unlike European options, Asian options are cash settled, only money is interchanged at expiration time $T > 0$.

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- The pay-off for Asian options is based on the average of the price and the strike price K:

$$\text{Pay-off} = \begin{cases} \left(\frac{1}{T}\sum_{t=1}^{T} S_t - K\right)^+ & \text{Call} \\ \left(K - \frac{1}{T}\sum_{t=1}^{T} S_t\right)^+ & \text{Put} \end{cases}$$

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- Asian options are **not** simple derivatives: The pay-off depends on the whole path of the price.
- The pay-off function is given by

$$\Phi(x_1, \ldots, x_T) = \begin{cases} \left(\frac{1}{T}\sum_{t=1}^{T} x_t - K\right)^+ & \text{Call} \\ \left(K - \frac{1}{T}\sum_{t=1}^{T} x_t\right)^+ & \text{Put} \end{cases}$$

Put Options