Financial Engineering

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1 Bonds and the Value of Money Through Time

Here you can explain what type of interests we studied and how the fair-price of bonds depends on these types of interest and viceversa.

1.1 Bonds

I will begin by giving the definition of a bond.

Definition 1.1 (Bond)**.** A bond is a financial security that pays the owner a chain of predetermined payments.

Using definition 1.1 we can interpret a bond as a financial asset which have no risk. In order words we know for a fact how much money we will make on this investment, disregarding the inherent risk of default.

1.2 Value of Money Through Time

The value of money changes through time. Assume you were offered an amount of money with value M, furthermore you would have the choice of recieving the money today or 1 year from now. Choosing to recieve the money now you would be able to spend them straightaway which could be preferable compared to waiting a year. Prices could also have risen during the waiting time, further diminishing the value. Last but not least, recieving the money today would enable you to put the money in the bank and thereby recieve a premium. All these arguments serve the purpose of showing that the value of money changes through time, and that the value M of the money would decrease by waiting a year before recieving the money.

1.3 Interest Rate

The premium recieved by depositing money at bank mentioned earlier is described by an interest rate. This interest rate will in the following be described by a positive number

$$r > 0.$$

Different types of interest exists and i will describe the following three types

1. simple interest,
2. compouned interest,
3. continuously compounded interes.

Common for all three types is that they represent a fraction of an investment that is paid either ones or through several periods.

However to further dive into the different types of interest we will first define the notion of a wealth process.

Definition 1.2 (Wealth Process)**.** The evolution of an investment over time is called the wealth process of that investment and is denoted by

$$V = (V_t)_{0 \leq t \leq T}. \tag{1.1}$$

The initial capital is denoted by v_0, and we assume that V is a real-valued stochastic process on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

With this definition in place we are now ready to define the three different types of interest starting with simple interest.

Definition 1.3 (Simple Interest)**.** Let $v_0 \in \mathbb{R}$ be our initial capital. An interest on v_0 is said to be simple if it follows the wealth process

$$V_t = (1 + rt)v_0, \quad 0 \leq t \leq T. \tag{1.2}$$

I will now show that the wealth process in (1.2) is indeed a stochastic process in any probability space. Any stochastic process X on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfies

$$\{\omega \in \Omega \,:\, X(\omega) \leq x\} \in \mathcal{F}, \quad \forall x \in \mathbb{R}. \tag{1.3}$$

Suppose $v_0 > 0$ and $x \geq (1 + rt)v_0$ then

$$\{\omega \in \Omega \,:\, (1 + rt)v_0 \leq x\} = \{\Omega\} \in \mathcal{F}, \tag{1.4}$$

on the other hand if $x < (1 + rt)v_0$

$$\{\omega \in \Omega \,:\, (1 + rt)v_0 \leq x\} = \{\emptyset\} \in \mathcal{F}. \tag{1.5}$$

As both Ω and $\emptyset$ is contained in any σ-algebra we have shown that the wealth process in (1.2) is a stochastic process in any probability space. Next i will define the notion of compund interest.

Definition 1.4 (Compounded Interest)**.** Let $v_0 \in \mathbb{R}$ be our initial capital. An interest on v_0 is said to be compunded over $m \in \mathbb{N}$ periods if it follows the wealth process

$$V_t = \left(1 + \frac{r}{m}\right)^{mt} v_0, \quad 0 \leq t \leq T. \tag{1.6}$$

As stated in the definition m can be any natural number, however if $m = 1, 12, 365$ the interest is refered to as annually, monthly and daily compounded, respectively. Note that we have the following properties $\forall\, 0 \leq t \leq T$

i) $V_{t+1} = \left(1 + \frac{r}{m}\right)^{m} V_t,$

ii) If $m_1 > m_2, v_0 > 0 \Rightarrow \left(1 + \frac{r}{m_1}\right)^{m_1 t} v_0 > \left(1 + \frac{r}{m_2}\right)^{m_2 t} v_0,$

iii) If $m_1 > m_2, v_0 < 0 \Rightarrow \left(1 + \frac{r}{m_1}\right)^{m_1 t} v_0 < \left(1 + \frac{r}{m_2}\right)^{m_2 t} v_0$.

From this is follows that for an *investor* compund interest is more attractive as it pays more, however as a *debtor* it is less attractive as he or she will have to pay more on his or hers debt.

At last i can turn to continuously compounded interest which i will present as the limit of (1.6) as $m \to \infty$. Note that by the following definition of e

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e, \tag{1.7}$$

by letting $x = r/m$ in the above the limit of the wealth process of compounded interest can be seen as

$$\left[\left(1 + \frac{r}{m}\right)^{\frac{m}{r}}\right]^{rt} v_0 \to (e)^{rt} v_0, \quad \text{as } m \to \infty. \tag{1.8}$$

This leads to the definition of continuously compounded interest

Definition 1.5 (Continuously Compounded Interest)**.** Let v_0 be our initial capital. An interest on v_0 is said to be continuously compounded at rate $r > 0$ if the wealth process

$$V_t = e^{rt} v_0, \quad 0 \leq t \leq T. \tag{1.9}$$

There exists the following relation between the different types of interest

$$(1 + r) \leq \left(1 + \frac{r}{m}\right)^m < e^r. \tag{1.10}$$

To show that the relation indeed holds i will show that the sequence

$$a_m = \left(1 + \frac{r}{m}\right)^m, \tag{1.11}$$

is increasing.

Write whiteboard notes

1.4 Types of Bonds

A bond is a financial security that pays to the owner, a chain of predetermined payments. However there are different types of bonds. I will discuss the following two types here

1. zero-coupon bonds,
2. coupon bonds.

A **zero-coupon bond** is a bond with a single payment $F > 0$ at time $T > 0$. The pay-off F is called the face value and T the maturity time. The next question i will answer is how much i will be willing to pay for such a financial assest. This depends on the way the time value of money is measured. Consider for example the following setup; let $B_0 \geq 0$ be the value of the zero-coupon bond with face value $F > 0$ and maturity time $T > 0$. Suppose that only annual compound interest at rate $r > 0$ is available. From a buyers perspective what if

$$B_0 > \frac{F}{(1+r)^T}, \tag{1.12}$$

would i buy the bond? No, of course not because the right side of (1.12) denote the amount of money i would have to put in the bank today to recieve exactly F at time T. When the aforementioned amount of monwy is less than the price of the bond, i would surely choose to put money in the bank instead of buying the bond, as the bond will pays exactly F at time T. Suppose now that we flip the inequality and look from a sellers perspective, that is if

$$B_0 < \frac{F}{(1+r)^T}, \tag{1.13}$$

would i sell the bond? No of course not because the right side of (1.13) denotes the amount at i can borrow at time 0 if i agree to pay exactly F at time T. For the bond i would also have to pay F at time zero however i would only recieve B_0 at time 0. Therefore i would not agree to sell the bond in this situation.

This implies that the only price a buyer and seller can agree to is in the situation where

$$B_0 = \frac{F}{(1+r)^T}. \tag{1.14}$$

Now i will consider the situatuion where at time $1 \leq t \leq T$ i want to get rid of a bond, but i what to determine what price i should sell it to. At this time the bond can be considered a new zero-coupon bond with face value $F > 0$ and maturity time $T - t$. Thus we have from the previous argumentation that

$$B_t = \frac{F}{(1+r)^{T-t}}, \quad 0 \leq t \leq T. \tag{1.15}$$

The chain of argumenation holds also when the time value of money is different, if a compounded interest over m periods where considered then the fair price of a zero-coupon bond at time t would be

$$B_t = \frac{F}{\left(1+\frac{r}{m}\right)^{m(T-t)}}. \tag{1.16}$$

If we consider the continuously compounded case the fair price would be

$$B_t = \frac{F}{e^{r(T-t)}}. \tag{1.17}$$

A **coupon bond** guarantees the owner a payment $F > 0$ at time $T > 0$, as well as a fixed amount $C > 0$ at $t = 1, 2, \dots, T$. To find a fair price for such a bond i will follow the same argumentation as above, namely i will compare the price of the bond with the cost of recieving the same benefits with a bank account. I other words how much money will i have to deposit in my bank account today if i want to

$$\begin{aligned}
1.\quad & \text{withdraw } C > 0 \text{ after 1 year} && (1.18)\\
2.\quad & \text{withdraw } C > 0 \text{ after 2 years} && (1.19)\\
& \vdots && (1.20)\\
T-1.\quad & \text{withdraw } C > 0 \text{ after } T-1 \text{ years} && (1.21)\\
T.\quad & \text{withdraw } F + C \text{ after } T \text{ years} && (1.22)\\
& && (1.23)
\end{aligned}$$

and have nothing left in the bank account afterwards.

In order to be able to get $C > 0$ after one year i have to put

$$\frac{C}{1+r} \tag{1.24}$$

in the bank. In order to be able to get $C > 0$ after two years i have to put

$$\frac{C}{(1+r)^2} \tag{1.25}$$

in the bank. Generalizing this argumentat tells me that in order to $C > 0$ after t years i have to put

$$\frac{C}{(1+r)^t} \tag{1.26}$$

in the bank. Lastly in order to get $F + C$ after T years i have to put

$$\frac{F+C}{(1+r)^T} = \frac{F}{(1+r)^T} + \frac{C}{(1+r)^T} \tag{1.27}$$

in the bank. Adding up all these amounts it is concluded that i have to make a deposit of

$$\sum_{i=1}^{T} \frac{C}{(1+r)^i} + \frac{F}{(1+r)^T}. \tag{1.28}$$

The agreeable price of a coupon bond is thus given by

$$B_0 = \sum_{i=1}^{T} \frac{C}{(1+r)^i} + \frac{F}{(1+r)^T} = \frac{\xi_T}{(1+r)^T}. \tag{1.29}$$

Where the pay-off ξ_T at time T is given by

$$\xi_T := \sum_{i=1}^{T} C(1+r)^{T-i} + F, \tag{1.30}$$

in other words the fair price of the coupon bond (as well as the zero-coupon bond) can be written as the discounted price of the total pay-off.

2 Portfolio Allocation and Risk Measures

Try to explain what a portfolio is and how we can create (allocate) portfolio based on risk measures. In your discussion about the latter, remember to include the differences/benefits/drawbacks of considering the standard deviation, VaR and CVaR as risk measures.

2.1 Portfolio

First and foremost i will give the definition of a portfolio.

Definition 2.1 (Portfolio and Strategies)**.** Let

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, \mathbb{P}), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)} \right)_{0 \leq t \leq T} \right\}, \tag{2.1}$$

be a finite-horizon financial market. A portfolio in $\mathfrak{M}$ is a $(d+1)$-dimensional vector

$$\Theta_t = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)} \right), \tag{2.2}$$

in which

$$\Theta_t^j = \text{Number of shares of the } j\text{'th asset held between time } t-1 \text{ and } t. \tag{2.3}$$

for $j = 1, \ldots, d+1$. The collection $\Theta = (\Theta_t)_{0 \leq t \leq T}$, with the convention that $\Theta_0 = \Theta_1$ is termed a strategy.

For every strategy on market there is an associated wealth process. The wealth process for Θ is defined and denoted by

$$V_t^{\Theta} = \varphi_t B_t + \sum_{j=1}^{d} \theta_t^{(j)} S_t^{(j)} = \Theta_t \cdot P_t, \quad 0 \leq t \leq T. \tag{2.4}$$

2.2 Risk Measures

Any strategy on a given market inherently carries a risk because the return is random, there is in other words no way to predict our profit or losses. There is no one way to measure the risk associated with a strategy, however in the next to sections i will explore two approaches. Both of these is based on portfolio allocation as an optmization problem. The problem is given in this way; solve

$$\underset{\mathbf{w} \in \mathbb{R}^{d+1}}{\operatorname{argmin}} \mathcal{R}\left(\mathbf{w} \cdot \mathbf{K}_P \right), \tag{2.5}$$

subject to:

$$\sum_{j=0}^{d} w_j = 1, \ \mathbb{E}\left[U(\mathbf{w} \cdot \mathbf{K}_P) \right] = \mu, \ \mu \in \mathbb{R}, \tag{2.6}$$

where $\mathcal{R}$ is a measure of risk and $U(\mathbf{w} \cdot \mathbf{K}_P)$ is the utility of the strategy.

2.2.1 Mean-Variance

If an agent would like to maximize profit while minimizing the variance, he or she could take the mean-variance approach. In this approach risk is measured using standard deviations, i.e.

$$\mathcal{R}(X) = \sigma(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{\mathbb{E}\left[(X - \mathbb{E}(X))^2\right]}. \tag{2.7}$$

For a portfolio with weights $\mathbf{w}$, we have that

$$\mathcal{R}(\mathbf{w} \cdot \mathbf{K}_P) = \sqrt{\mathbf{w}^\top C \mathbf{w}}, \tag{2.8}$$

where C is a $(d+1) \times (d+1)$ matrix given by

$$C^{i,j} = \mathrm{Cov}\left(\mathbf{K}_P^i, \mathbf{K}_P^j\right) = \sqrt{\mathbb{E}\left[\left(\mathbf{K}_P^i - \mathbb{E}(\mathbf{K}_P^i)\right)\left(\mathbf{K}_P^j - \mathbb{E}(\mathbf{K}_P^j)\right)\right]}. \tag{2.9}$$

Write exercise from lecture 2

The mean variance approach assumes the uitility function as the identity, that is

$$U(x) = x. \tag{2.10}$$

By letting

$$\mu_{\mathbf{K}} := \mathbb{E}\left[\mathbf{K}_P\right], \tag{2.11}$$

it follows that

$$\mathbb{E}\left[U(\mathbf{w} \cdot \mathbf{K}_P)\right] = \mathbf{w} \cdot \mu_{\mathbf{K}}. \tag{2.12}$$

Thus the optimization problem, in the mean-variance approach becomes

Problem 2.2 (Optimization Problem Mean-Variance)**.**

$$\underset{\mathbf{w} \in \mathbb{R}^{d+1}}{\operatorname{argmin}} \sqrt{\mathbf{w}^\top C \mathbf{w}} \tag{2.13}$$

$$\text{Subject to:} \tag{2.14}$$

$$1.) \quad \sum_{j=0}^{d} w_j = 1, \tag{2.15}$$

$$2.) \quad \mathbf{w} \cdot \mu_{\mathbf{K}} = \mu, \quad \mu \in \mathbb{R}. \tag{2.16}$$

An illustrative example to demonstrate why one might consider a different risk measure than the standard deviation is the following. Consider two portfolios that generate the following wealth

$$V_1^{(1)} = \begin{cases} 1 & \text{with probability } 1/2 \\ -9 & \text{with probability } 1/2 \end{cases} \tag{2.17}$$

and

$$V_1^{(2)} = \begin{cases} 5 & \text{with probability } 1/2 \\ -5 & \text{with probability } 1/2 \end{cases} \tag{2.18}$$

According to their standard deviation these two portfolios carries the same risk, but one could argue that the first is riskier that the latter. The next risk measure i will consider i concerned with controlling losses rather than the variation of the return.

2.2.2 Value at Risk

If i denote the outcome of an investment with V_1 then our potential losses will be given by $-V_1$. Suppose that for some $x \in \mathbb{R}$

$$-V_1 \leq x. \tag{2.19}$$

Then to cover my risk of bankruptcy i must keep at least the amount x in my bank account. In reality the only thing i can quantify is the chance of the happening, which is denoted by

$$\mathbb{P}\left(-V_1 \leq x\right). \tag{2.20}$$

This is the motivation behind the risk measure value at risk.

Definition 2.3 (Value at Risk)**.** Let $0 < \alpha < 1$ and X be a random variable. The Value at Risk (VaR) of X is defined and denoted by

$$\mathrm{VaR}_\alpha(X) := \inf\left\{x \in \mathbb{R} : \mathbb{P}(X + x \geq 0) \geq 1 - \alpha\right\}. \tag{2.21}$$

In other words the $\mathrm{VaR}_\alpha(X)$ represents the amount of extra capital i need to hold in order to reduce my risk of bankruptcy to $1 - \alpha$. An alternative representation of VaR can be formulated using the fact that

$$\mathbb{P}(-X \leq x) \geq 1 - \alpha \iff \mathbb{P}(X + x < 0) \leq \alpha, \tag{2.22}$$

this lets us formulate an equivalent representation given by

$$\mathrm{VaR}_\alpha(X) := \inf\left\{x \in \mathbb{R} : \mathbb{P}(X < -x) \leq \alpha\right\}. \tag{2.23}$$

The following proposition describes some properties for the VaR.

Proposition 2.4 (Properties of VaR)**.** *Let X, Y be arbitrary random variables. Then, the following holds*

1. *If $X \geq 0$ almost surely, then $\mathrm{VaR}_\alpha(X) \leq 0$.*
2. *For all $y \in \mathbb{R}$ we have that $\mathrm{VaR}_\alpha(X + y) = \mathrm{VaR}_\alpha(X) - y$. In particular $\mathrm{VaR}_\alpha(X + \mathrm{VaR}_\alpha(X)) = 0$.*
3. *If $\lambda \geq 0$, then $\mathrm{VaR}_\alpha(\lambda X) = \lambda\,\mathrm{VaR}_\alpha(X)$.*
4. *If $X \geq Y$ almost surely, then $\mathrm{VaR}_\alpha(X) \leq \mathrm{VaR}_\alpha(Y)$.*

Proof. **Get back to this proof**

□

2.3 Coherent Risk Measures

There are one important drawback to measuring risk using VaR and that is its lack of diversification.

Note that when using the standard deviation as a risk measure we get the following

$$\sigma(X+Y)^2 = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\rho_{X,Y}\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}, \tag{2.24}$$

where $\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} \leq 1$. Then

$$\begin{aligned}\sigma(X+Y)^2 &= \sigma(X)^2 + \sigma(Y)^2 + 2\rho_{X,Y}\sigma(X)\sigma(Y) &(2.25)\\ &\leq \sigma(X)^2 + \sigma(Y)^2 + 2\sigma(Y)\sigma(X) = [\sigma(X)+\sigma(Y)]^2, &(2.26)\end{aligned}$$

which would imply that

$$\sigma(X+Y) \leq \sigma(X) + \sigma(Y). \tag{2.27}$$

This means that, under the risk measure σ, investing in $X + Y$ carries less risk than investing in X or Y seperately. The phenomenon is better known as diversification, and it is common belief that diversification of a portfolio lowers the risk. However VaR as a risk measure is not able to reproduce this, that is in general we do not have that

$$\mathrm{VaR}_\alpha(X+Y) \leq \mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y). \tag{2.28}$$

As diversification is a desired i will know introduce the concept of Coherent Risk Measures.

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, i will use the following notation

$$L^p = \{X : \Omega \to \mathbb{R} : X \text{ r.v. such that } \mathbb{E}(|X|^p) < \infty\}. \tag{2.29}$$

I will now give the definition

Definition 2.5 (Coherent Risk Measure)**.** A function $\rho : L^1 \to \mathbb{R}$ is said to be a Coherent Risk Measure if

1. If $X \geq 0$ almost surely, then $\rho(X) \leq 0$.
2. For all $y \in \mathbb{R}$ we have that $\rho(X+y) = \rho(X) - y$.
3. If $\lambda \geq 0$, then $\rho(\lambda X) = \lambda\rho(X)$.
4. We have that $\rho(X+Y) \leq \rho(X) + \rho(Y)$.

2.3.1 Conditional Value at Risk

The Conditional Value at Risk (CVAR) is a common example of a coherent risk measure. This section will explore this particular risk measure further.

Given a random variable X, we will write

$$q_\alpha(X) := \inf\{x : \mathbb{P}(X \le x) \ge \alpha\}. \tag{2.30}$$

With this notation now introduced i can present the definition of CVaR

Definition 2.6 (Condition Value at Risk)**.** Let $0 < \alpha < 1$ and $X \in L^1$. The Conditional Value at Risk or Expected Shortfall of X is defined and denoted buy

$$\mathrm{CVaR}_\alpha := -\frac{1}{\alpha}\int_0^\alpha q_r(X)\,dr. \tag{2.31}$$

The name Expected Shortfall comes from the fact that if X has a continuous distribution, then

$$\mathrm{CVaR}_\alpha(X) = -\mathbb{E}[X \mid X + \mathrm{VaR}_\alpha(X) \le 0]. \tag{2.32}$$

Thus, CVaR_α measures the expected losses given that $\mathrm{VaR}_\alpha(X)$ was not enough to cover our position on X.

Theorem 2.7 (CVaR as a Coherent Risk Measure)**.** *The CVaR, i.e.*

$$\mathrm{CVaR}_\alpha(X) = -\frac{1}{\alpha}\int_0^\alpha q_r(X)\,dr, \tag{2.33}$$

where

$$q_\beta(X) := \inf\{x : \mathbb{P}(X \le x) \ge \beta\} \tag{2.34}$$

is a coherent risk measure.

Proof. **Get back to this proof**

□

3 The Multi-Step Binomial Model

Describe the dynamics of the model. Try to discuss, among other things, the way that the market information is described through time, its absence of arbitrage, and how to find a risk-neutral measure in such model.

3.1 Model Setup

Consider a model where only two assets are traded: A bond and a risky asset. The bond satisfies that

$$B_t = (1 + r), \quad t = 0, 1, \dots, T. \tag{3.1}$$

The price process $(S_t)_{t=0,1,\dots,T}$ satisfies that $S_0 > 0$ (non-random) and

$$S_t = S_{t-1}(1 + K_s(t)), \quad t = 1, \dots, T. \tag{3.2}$$

The returns are independent and identically distributed with

$$K_s(t) = \begin{cases} R_u & \text{with probability } p; \\ R_d & \text{with probability } 1 - p, \end{cases} \tag{3.3}$$

with the relation

$$R_u < R_d. \tag{3.4}$$

3.2 Market Information

I am considering the multi-step financial market

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, \mathbb{P}), P = \left(B_t, S_t^{(1)}, S_t^{(2)}, \dots, S_t^{(d)} \right)_{t=0,1,\dots,T} \right\}. \tag{3.5}$$

In order to further explain how information flows in the market i need to be more specific about the definition of the probability space. I will start by letting

$$\Omega = \left\{ \left(\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(T)} \right) \, : \, \omega^{(t)} \in \{u, d\}, \, t = 1, 2, \dots, T \right\}. \tag{3.6}$$

Take for example the two-step binomial model, that is $T = 2$. The vector (d, u) represent the event in which the price went down in the first period and then up in the second period.

I will consider all possible events, which means that the σ-algebra in this case is

$$\mathscr{F} = 2^{\Omega}. \quad \text{(The power set)} \tag{3.7}$$

Using this set-up the return process can be written as

$$K_s(t)(\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(T)}) = \begin{cases} R_u & \text{if } \omega^{(t)} = u; \\ R_d & \text{if } \omega^{(t)} = d. \end{cases} \tag{3.8}$$

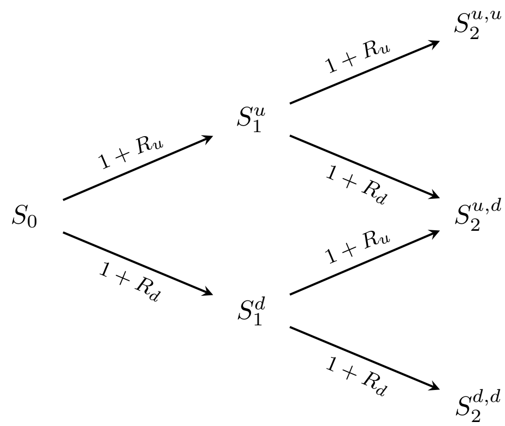

Figure 1: A graphical representation of the 2-step binomial model.

To ensure that the reurns are independent and can go up and down with probability p and $(1-p)$ respectively, i let

$$\mathbb{P}\left(\left\{\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(d)}\right\}\right) = p^k (1-p)^{T-k}, \tag{3.9}$$

where

$$k = \text{number of } u\text{'s in } \left(\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(d)}\right). \tag{3.10}$$

As an example lets consider the two-step binomial model, that is $T = 2$ and we have that

$$\Omega = \{uu, ud, du, dd\}, \tag{3.11}$$

using the notational convention that

$$\left(\omega^{(1)}, \omega^{(2)}\right) = \omega^{(1)}\omega^{(2)}. \tag{3.12}$$

At time $t = 0$ no information is available but as time goes by, the price movements reveal information about future outcomes. Say for example that at time $t = 1$ the price went down, the the future outcomes are restricted to the set

$$B_d := \{du, dd\}, \tag{3.13}$$

as the the events (uu) and (ud) are no longer possible.

The aforementioned set is related to the information generated by the return process. Consider the σ-field generated by a random variable, which is given by

$$\sigma(X) = \left\{X^{-1}(A) \,:\, A \in \mathscr{B}(\mathbb{R}).\right\} \tag{3.14}$$

Recall that for $t = 1, 2$

$$K_S(t)\left(\omega^{(1)}, \omega^{(2)}\right) = \begin{cases} R_u & \text{if } \omega^{(t)} = u; \\ R_d & \text{if } \omega^{(t)} = d. \end{cases} \tag{3.15}$$

This tells me that, in the event that $K_S(1) = R_d$ it follows that

$$\left\{\left(\omega^{(1)}, \omega^{(2)}\right) \,:\, K_S(1) = R_d\right\} = \{du, dd\} = B_d. \tag{3.16}$$

On the other hand if $K_S(1) = R_u$ it follows that

$$\left\{\left(\omega^{(1)}, \omega^{(2)}\right) \,:\, K_S(1) = R_u\right\} = \{uu, ud\} = B_u. \tag{3.17}$$

Therefore

$$B_u, B_d \subseteq \sigma\left(K_S(1)\right). \tag{3.18}$$

It is in fact so, that because $K_S(1)$ only takes two values

$$\sigma(K_S(1)) = \{B_u, B_d, \Omega, \emptyset\}. \tag{3.19}$$

At the end of the trading period, in this case $t = 2$, the outcome of the price movements are known and stored in

$$\mathscr{F} = \sigma\left(K_S(1), K_S(2)\right). \tag{3.20}$$

Furthermore there has been constructed a family of sub-σ-algebras of $\mathscr{F}$;

$$\mathscr{F}_1 = \sigma(K_S(1)) = \{B_u, B_d, \Omega, \emptyset\}. \tag{3.21}$$

$$\mathscr{F}_2 = \sigma(K_S(1), K_S(2)) = \mathscr{F}. \tag{3.22}$$

Note that by construction, $\mathscr{F}$ contains the whole information available on the price movements up until time t. Mathematically this means that the price process S_t is adapted to $\mathscr{F}$;

$$\sigma(S_t) \subseteq \mathscr{F}_t, \quad t = 0, 1, 2 \text{ where } \mathscr{F}_0 = \{\Omega, \emptyset\}, \tag{3.23}$$

and moreover

$$\mathscr{F}_{t-1} \subseteq \mathscr{F}_t, \quad t = 1, 2. \tag{3.24}$$

A collection of sub-σ-algebras satisfying the previous relation is called a filtration. A formal defintion is now given.

Definition 3.1 (Filtrations and Adapted Process)**.** Fix a probability space $(\Omega, \mathscr{F}, \mathbb{R})$. A collection of σ-algebras $\mathbb{F} = (\mathscr{F}_t)_{t=0,1,\ldots,T}$ is called a filtration if for all $1 \leq t \leq T$,

$$\mathscr{F}_{t-1} \subseteq \mathscr{F}_t \subseteq \mathscr{F}. \tag{3.25}$$

The quadruplet

$$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P}), \tag{3.26}$$

is termed a filteres probability space.

Furthermore, a stochastic process $(X_t)_{0\le t\le T}$ is said to be adapted to a filtration $\mathbb{F}$ if

$$\sigma(X_t) \subseteq \mathscr{F}_t, \quad \forall 0 \le t \le T. \tag{3.27}$$

I am now able to give a formal definition of a financial market with information.

Definition 3.2 (Financial Markets with Information)**.** Fix $T, d \in \mathbb{N}$. A finite-horizon financial market with information is the pair

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F})_{0\le t\le T}, \mathbb{P}\right), P = \left(B_t, S_t^{(1)}, \dots, S_t^{(d)}\right)_{0\le t\le T} \right\}, \tag{3.28}$$

consisting of

1. A filtered probability space $\left(\Omega, \mathscr{F}, (\mathscr{F})_{0\le t\le T}, \mathbb{P}\right)$
2. P_t is adapted.
3. $S^{(j)}$ is the price process of the j'th asset traded in the market
4. B is a numéraire (e.g. a bond), i.e.

$$\mathbb{P}(B_t > 0) = 1, 0 \le t \le T. \tag{3.29}$$

3.3 Absence of Arbitrage

An arbitrage is understood as a way to generate positive wealth with zero investment and zero risk. In the one-step binomial model it is only allowed to allocate wealth ones, namely at time $t = 0$. Returns are then measured at time $t = T = 1$.

In the multi-step binomial model agents will redesign their strategies based on the available information. In order to create an arbitrage the portfolio need to be updated every time the price changes. Therefore, instead of considering a single portfolio, an arbitrage now consists of a collection of portfolios $(\Theta_t)_{0\le t\le T}$. I will now give a formal definition of a portfolio and a strategy.

Definition 3.3 (Portfolio and Strategy)**.** A portfolio in $\mathfrak{M}$ is a $(d+1)$-dimensional vector $\Theta_t = \left(\varphi_t, \theta_t^{(1)}, \dots, \theta_t^{(d)}\right)$, in which

$$\Theta_t^{(j)} = \text{ Number of shares of the } j\text{'th asset held between time } t-1 \text{ and } t. \tag{3.30}$$

The collection $\Theta = (\Theta_t)_{0 \leq t \leq T}$, with the convention that $\Theta_0 = \Theta_1$ is termed a strategy.

The wealth process associated to $\Theta = (\Theta_t)_{0 \leq t \leq T}$ is defined and denoted by

$$V_t^{\Theta} := \varphi_t B_t + \sum_{j=1}^{d} \theta_t^{(j)} S_t^{(j)} = \Theta_t \cdot P_t, \quad 0 \leq t \leq T \tag{3.31}$$

$$\Rightarrow V_0^{\Theta} = \varphi_0 B_0 + \sum_{j=1}^{d} \theta_0^{(j)} S_0^{(j)} = \varphi_1 B_0 + \sum_{j=1}^{d} \theta_1^{(j)} S_0^{(j)} \tag{3.32}$$

The arbitrage approach from the one-step binomial model can not be used when considering the multi-step binomial model as many strategies that from an intuitive point of view should not be thought of as an arbitrage, will be believed to be an arbitrage. Consider for example the following situation

1. Injection of capital at $t \geq 1$.
2. Priviliged information.
3. Unilimited credit.

I therefore introduce the notion of an admissible strategy.

Definition 3.4 (Admissible Strategy)**.** Let $\Theta = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)}\right)_{0 \leq t \leq T}$ be strategy on the market

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F})_{0 \leq t \leq T}, \mathbb{P}\right), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)}\right)_{0 \leq t \leq T} \right\}. \tag{3.33}$$

We will say that Θ is admissible if:

1. **It is self-financed**: It only requires initial capital.
2. **Non-anticipative**: It is build up only on current information.
3. **It has a limited credit line**: There is a non-random constant $C > 0$, such that

$$V_t^{\Theta} \geq -C, \ \forall\, 0 \leq t \leq T. \tag{3.34}$$

With this definition now in place i am able to define the concept of arbitrage in the multi-step binomial model

Definition 3.5 (Arbitrage)**.** Consider the market

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F})_{0 \leq t \leq T}, \mathbb{P}\right), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)}\right)_{0 \leq t \leq T} \right\}. \tag{3.35}$$

An admissible strategy $\Theta = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)}\right)_{0 \leq t \leq T}$ is said to be an arbitrage if:

1. It has zero initial capital:

$$V_0^{\Theta} = \varphi_0 B_0 + \sum_{j=1}^{d} \theta_0^{(j)} \cdot S_0^{(j)} = 0. \tag{3.36}$$

2. At time $t = T$, i am out of debts with 100 % certainty: Almost surely

$$V_T^{\Theta} = \varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} \geq 0. \tag{3.37}$$

3. I have a chance to make a profit:

$$\mathbb{P}\left(\varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} > 0\right) > 0. \tag{3.38}$$

3.4 Risk-Neutral Measure

To begin this section i will define the notion of a martingale. Fix a filtered probability space

$$\left(\Omega, \mathscr{F}, (\mathscr{F})_{0 \leq t \leq T}, \mathbb{P}\right). \tag{3.39}$$

Recall that this means that $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ is a probability space and $(\mathscr{F}_t)_{0 \leq t \leq T}$ is a filtration, i.e. a nested collection of sub-σ-algebras of $\mathscr{F}$: In symbols

$$\mathscr{F}_{t-1} \subseteq \mathscr{F}_t \subseteq \mathscr{F}, \quad \forall\, 1 \leq t \leq T. \tag{3.40}$$

We are given a discrete time stochastic process

$$X = (X_t)_{0 \leq t \leq T}. \tag{3.41}$$

Under this framework i give the definition of a martingale.

Definition 3.6 (Martingale)**.** The collection $\{(X_t, \mathscr{F}_t) : t = 0, 1, \ldots, T\}$ is said to be a martingale if

1. X is adapted, i.e. $\sigma(X_t) \subseteq \mathscr{F}_t, \quad \forall\, 0 \leq t \leq T$.
2. X_t has finite first moment, that is $\mathbb{E}(|X_t|) < \infty$.
3. For all $0 \leq t \leq T-1$ we have that almost surely $\mathbb{E}(X_{t+1} | \mathscr{F}_t) = X_t$.

3.4.1 Binomial Model under Risk-Neutrality

Consider the risk neutral Multi-step binomial model: $B_0 = 1$, $S_0 > 0$ is a non-random constant and

$$B_t = (1+r)^t; \quad S_t = S_{t-1}(1 + K_S(t)), \quad t = 1, \dots, T. \tag{3.42}$$

$(K_S(t))_{t=1,\dots,T}$ are i.i.d. with

$$K_S(t) = \begin{cases} R_u & \text{with probability } q^* \\ R_d & \text{with probability } 1 - q* \end{cases} \tag{3.43}$$

with the relation $R_d < r < R_u$ and

$$q^* = \frac{r - R_d}{R_u - R_d}; \quad R_d < r < R_u. \tag{3.44}$$

I will now show that the discounted price

$$X_t := \frac{S_t}{B_t}, \quad 0 \le t \le T, \tag{3.45}$$

is a martingale under the filtration

$$\mathscr{F}_0 = \{\emptyset, \Omega\} \tag{3.46}$$
$$\mathscr{F}_t = \sigma(K_S(1), \dots, K_S(t)), \quad 1 \le t \le T. \tag{3.47}$$

In order to show this, i need to verify that the discounted price satisfies all three conditions in definiton 3.6 (Martingale)

Adapted

Get back to this lecture 6 part 5

Finite First Moment

Get back to this lecture 6 part 5

Martingale Property By definition

$$\mathbb{E}\left(X_{t+1} | \mathscr{F}_t\right) = \mathbb{E}\left(\frac{S_{t+1}}{B_{t+1}} \,\middle|\, \mathscr{F}_t\right) \tag{3.48}$$
$$= \mathbb{E}\left(\frac{S_t(1 + K_S(t+1))}{B_{t+1}} \,\middle|\, \mathscr{F}_t\right) \tag{3.49}$$
$$= \mathbb{E}\left(\frac{S_t(1 + K_S(t+1))}{(1+r)^{t+1}} \,\middle|\, \mathscr{F}_t\right) \tag{3.50}$$
$$= \mathbb{E}\left(\frac{S_t}{(1+r)^{t+1}}(1 + K_S(t+1)) \,\middle|\, \mathscr{F}_t\right) \tag{3.51}$$

$$
\begin{aligned}
\text{(By the Product Rule)} &= \frac{S_t}{(1+r)^{t+1}}\mathbb{E}\left((1+K_S(t+1)) \,\middle|\, \mathscr{F}_t\right) && (3.52)\\
&= \frac{S_t}{(1+r)^t}\frac{1}{(1+r)}\mathbb{E}\left((1+K_S(t+1)) \,\middle|\, \mathscr{F}_t\right) && (3.53)\\
&= X_t\frac{1}{(1+r)}\mathbb{E}\left((1+K_S(t+1)) \,\middle|\, \mathscr{F}_t\right). && (3.54)
\end{aligned}
$$

I am now left to show that

$$
\mathbb{E}\left((1+K_S(t+1) \,\middle|\, \mathscr{F}_t\right) = 1, \tag{3.55}
$$

to do this i will use the independence of the returns but let me start by considering $t = 0$. In this case

$$
\begin{aligned}
\mathbb{E}\left((1+K_S(1) \,|\, \mathscr{F}_0\right) &= \mathbb{E}\left((1+K_S(1) \,|\, \{\emptyset, \Omega\}\right) && (3.56)\\
\text{(Conditioning on the trivial } \sigma\text{-algebra)} &= \mathbb{E}\left((1+K_S(1)\right) && (3.57)\\
\text{(From Risk-Neutrality)} &= (1+r). && (3.58)
\end{aligned}
$$

Let me now consider the case when $t \geq 1$

$$
\begin{aligned}
\mathbb{E}\left((1+K_S(t+1) \,|\, \mathscr{F}_t\right) &= \mathbb{E}\left((1+K_S(t+1) \,|\, K_S(1), \ldots, K_S(t)\right) && (3.59)\\
\text{(Indepence prop. of the cond. expectation)} &= \mathbb{E}\left((1+K_S(1)\right) && (3.60)\\
&= (1+r). && (3.61)
\end{aligned}
$$

4 The First Fundamental Theorem of Asset Pricing

Focus on explaining the hypothesis in the theorem (martingale measure, admissible strategies, arbitrage, etc.) and its applications to the pricing of financial derivatives.

The Fundamental theorem of asset pricing gives necessary and sufficient conditions for the absence of arbitrage in the market with information, given by

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F})_{0 \le t \le T}, \mathbb{P} \right), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)} \right)_{0 \le t \le T} \right\}. \tag{4.1}$$

Theorem 4.1 (FFTAP)**.** *The market given by* (4.1) *is arbitrage free if and only if there is a martingale measure.*

5 Pricing in the Binomial Model

Explain what is the typical methodology for pricing derivatives and how such a methodology works in the Binomial model. Remember to include an example (e.g. the price of call-option).

6 The Second Fundamental Theorem of Asset Pricing

Focus on explaining the hypothesis in the theorem (martingale measure, admissible strategies, arbitrage, completeness, etc.) and its consequences on the pricing of financial derivatives. You can, in particular, show that the Binomial Model is complete while the Trinomial Model is not.