Financial Engineering
Lecture 8

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Outline

- Pricing in the Multi-Step Binomial model: Examples and the price function.
- Second Fundamental Theorem of Asset Pricing.

Pricing in the Multi-Step Binomial model

The First Fundamental Theorem of Asset Pricing

Definition (Martingale Measures)

A **martingale measure** $\mathbb{Q}$ on the market

$$\mathfrak{M} = \left\{(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leq t\leq T}, \mathbb{P}), P = (B_t, S_t^{(1)}, \ldots, S_t^{(d)})_{0\leq t\leq T}\right\}$$

is a probability measure on $(\Omega, \mathscr{F})$, under which **the discounted price** $(\tilde{P}_t^{(j)})_{0\leq t\leq T}$ **is a martingale** w.r.t. $(\mathscr{F}_t)_{0\leq t\leq T}$, for every $j = 1, 2, \ldots, d+1$.

The First Fundamental Theorem of Asset Pricing

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Theorem (FFTAP)

The market

$$\mathfrak{M} = \left\{(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P}), P = (B_t, S_t^{(1)}, \dots, S_t^{(d)})_{0\le t\le T}\right\}$$

is arbitrage free if and only if there is a martingale measure.

The First Fundamental Theorem of Asset Pricing

Corollary (Pricing Financial Derivatives)

Consider

$$\mathfrak{M} = \left\{(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P}), P = (B_t, S_t^{(1)}, \ldots, S_t^{(d)})_{0\le t\le T}\right\}$$

to be the market that is arbitrage free (exists a martingale measure $\mathbb{Q}$). Let $\xi \geq 0$ be the pay-off of a financial derivative with delivery time $T > 0$, i.e.

$$\xi = \Phi\left(P_0, P_1, \ldots, P_T\right),$$

for some (measurable) function $\Phi : \prod_{t=0}^{T} \mathbb{R}^{d+1} \to \mathbb{R}$. If we let

$$\xi_t := \mathbb{E}_*\left(\frac{B_t}{B_T}\xi \,\middle|\, \mathscr{F}_t\right), \quad 0 \le t \le T,$$

then the extended market

$$\tilde{\mathfrak{M}} = \left\{(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P}), \tilde{P} = (B_t, S_t^{(1)}, \ldots, S_t^{(d)}, \xi_t)_{0\le t\le T}\right\},$$

is arbitrage free.

Methodology for Pricing

1. Define a model for the prices: Specify the dynamics for

$$P = (B_t, S_t^{(1)} \dots, S_t^{(d)})_{0 \leq t \leq T},$$

i.e. describe how the distribution, evolution and behavior of the prices should be... according to some empirical facts observed in the data.

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4. Dynamics in the risk-neutral world: Investigate what is the distribution of the price process under the martingale measure.
5. Price the derivative: Compute the price of the derivative by

$$\xi_t := \mathbb{E}_* \left(\frac{B_t}{B_T} \xi \middle| \mathscr{F}_t \right).$$

Define a model for the prices

- In this part we will consider Multi-Step Binomial model, i.e. only a bond and a risky asset are traded.
- The bond price is given by

$$B_t = (1+r)^t, \quad t = 0, 1, \ldots, T.$$

- The price of the risky asset is given by $S_0 > 0$ (non-random) and

$$S_t = S_{t-1}(1 + K_S(t)), \; t = 1, \ldots, T.$$

where

$$K_S(t) = \begin{cases} R_u & \text{with probability } p; \\ R_d & \text{with probability } 1-p, \end{cases}$$

with the relation

$$R_d < R_u.$$

Indicate a set of information

- We assume that the market movements are exclusively determined by the behavior of the returns, thus

$$\mathscr{F}_0 = \{\emptyset, \Omega\}, \;\; \mathscr{F}_t = \sigma\left(K_S(1), \ldots, K_S(t)\right), \;\; 1 \leq t \leq T.$$

Check for arbitrage opotunities

- We have checked several times that the market does not admit an arbitrage if and only if
$$R_d < r < R_u.$$
- In this situation a martingale measure exists: Under $\mathbb{Q}$ the returns are i.i.d. and satisfies that
$$\mathbb{Q}(K_S(t) = R_u) = q^*; \ \mathbb{Q}(K_S(t) = R_d) = 1 - q^*.$$
- Where
$$q^* = \frac{r - R_d}{R_u - R_d}.$$

Dynamics in the risk-neutral world

Proposition

In the Multi-Step Binomial model it holds that for all $t = 1, \ldots, T$

$$S_t = S_0(1+R_u)^{N_t}(1+R_d)^{t-N_t},$$

where

$$\mathbb{Q}(N_t = x) = \binom{t}{x} (q^*)^x (1-q^*)^{t-x}, \; x = 0, \ldots, t.$$

In particular

$$\mathbb{Q}\left[S_t = S_0(1+R_u)^x(1+R_d)^{t-x}\right] = \binom{t}{x} (q^*)^x (1-q^*)^{t-x}, \; x = 0, \ldots, t.$$

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Remark: N_t represents the number of times that the price went up during the periods $1, 2, \ldots, t$. In particular, under the risk-neutral measure

$$N_t \sim \mathrm{Bin}(t, q^*).$$

Proof

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$$\mathbb{Q}(K_S(v) = R_u) = q^*, \quad v = 1, \ldots, t.$$

- It follows that N_t is the the sum of i.i.d. independent Bernoulli r.v.'s with parameter q^*, from which it is clear that

$$N_t \sim \mathrm{Bin}(t, q^*),$$

so

$$\mathbb{Q}(N_t = x) = \binom{t}{x} (q^*)^x (1 - q^*)^{t-x}, \; x = 0, \ldots, t.$$

Proof $\left(S_t = S_0(1+R_u)^{N_t}(1+R_d)^{t-N_t}\right)$

- On the other hand, by definition

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\prod_{v=1}^{t}(1+K_S(v)) = (1+R_u)^{N_t}(1+R_d)^{t-N_t}, \ \forall\, t = 1, \ldots, T.
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- Let us proceed by induction.

Proof $\left(\prod_{v=1}^{t}(1+K_S(v)) = (1+R_u)^{N_t}(1+R_d)^{t-N_t}\right)$

- The case $t=1$ follows from the fact that $\forall\ t = 1, \ldots, T$

$$(1+K_S(t)) = (1+R_u)^{\mathbf{1}_{K_S(t)=R_u}} \times (1+R_d)^{\mathbf{1}_{K_S(t)=R_d}}$$

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Proof $\left(\prod_{v=1}^{t}(1+K_S(v)) = (1+R_u)^{N_t}(1+R_d)^{t-N_t};\;\; N_t = \sum_{v=1}^{t} \mathbf{1}_{K_S(v)=R_u}\right)$

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$$\prod_{v=1}^{t+1}(1+K_S(v)) = \left[\prod_{v=1}^{t}(1+K_S(v))\right] \times (1+K_S(t+1))$$

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&\times \underline{(1+R_u)^{\mathbf{1}_{K_S(t+1)=R_u}}}(1+R_d)^{1-\mathbf{1}_{K_S(t+1)=R_u}} \\
&= (1+R_u)^{N_t+\mathbf{1}_{K_S(t+1)=R_u}} \times (1+R_d)^{t+1-(N_t+\mathbf{1}_{K_S(t+1)=R_u})}
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&= (1+R_u)^{N_{t+1}} \times (1+R_d)^{t+1-N_{t+1}},
\end{aligned}$$

Initial Price of a Call Option in the Bin Model

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$$\xi = (S_T - C)^+ = \max\{S_T - C, 0\}.$$

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- Independently of the model, we know by the FFTAP that the arbitrage-free initial price of such option is

$$\xi_0 = \mathbb{E}_* \left(\frac{B_0}{B_T} (S_T - C)^+ \right).$$

Initial Price of a Call Option in the Bin Model

- In the Multi-Step Binomial model we have

$$\xi_0 = \mathbb{E}_* \left(\frac{B_0}{B_T} (S_T - C)^+ \right) = \frac{1}{(1+r)^T} \mathbb{E}_* \left((S_T - C)^+ \right).$$

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- Using that under $\mathbb{Q}$ for all $x = 0, \ldots, T$

$$\begin{aligned} p_T(x; q^*) &= \mathbb{Q}\left[S_T := S_0 (1+R_u)^x (1+R_d)^{T-x} \right] \\ &= \binom{T}{x} (q^*)^x (1-q^*)^{T-x}. \end{aligned}$$

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- It follows that

$$\mathbb{E}_* \left((S_T - C)^+ \right) = \sum_{x=0}^{T} \underbrace{\left(S_0 (1+R_u)^x (1+R_d)^{T-x} - C \right)^+}_{\text{This can vanish}} p_T(x; q^*).$$

Initial Price of a Call Option in the Bin Model

- Observe that

$$\left(S_0(1+R_u)^x(1+R_d)^{T-x} - C\right)^+ \neq 0,$$

if and only if

$$S_0(1+R_u)^x(1+R_d)^{T-x} > C.$$

Initial Price of a Call Option in the Bin Model

- Let (with the conventions that $\min\{\emptyset\} = +\infty$)

$$x_0(S_0, R_u, R_d, T) = \min\{0 \leq x \leq T : S_0(1+R_u)^x(1+R_d)^{T-x} > C\}.$$

- Then

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\end{aligned}$$

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&= S_0 \sum_{x=x_0}^{T} (1+R_u)^x(1+R_d)^{T-x} p_T(x; q^*) \\
&\quad - C \sum_{x=x_0}^{T} p_T(x; q^*).
\end{aligned}$$

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- Since

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$$\begin{aligned}(1+R_u)^x(1+R_d)^{T-x}p_T(x;q^*) &= (1+R_u)^x(1+R_d)^{T-x} \\ &\quad \times \binom{T}{x}(q^*)^x(1-q^*)^{T-x} \\ &= \binom{T}{x}\left[(1+R_u)q^*\right]^x \\ &\quad \times \left[(1+R_d)(1-q^*)\right]^{T-x}.\end{aligned}$$

Initial Price of a Call Option in the Bin Model

- All in all implies that the arbitrage-free initial price of a Call Option in the Multi-Step Binomial model is given by the formula

$$\mathbb{E}_* \left(\frac{(S_T - C)^+}{(1+r)^T} \right) = S_0 \Psi_1(T, S_0, C, q^*) - \frac{C}{(1+r)^T} \Psi_2(T, S_0, C, q^*),$$

where

$$\Psi_1(T, S_0, C, q^*) := \sum_{x=x_0}^{T} \binom{T}{x} \left[\frac{1+R_u}{1+r} q^* \right]^x \left[\frac{1+R_d}{1+r} (1-q^*) \right]^{T-x};$$

$$\Psi_2(T, S_0, C, q^*) := \sum_{x=x_0}^{T} p_T(x; q^*),$$

in which

$$x_0 = x_0(S_0, R_u, R_d, T) = \min\{0 \le x \le T : S_0(1+R_u)^x(1+R_d)^{T-x} > C\}$$

- This expression is known as the Cox-Ross-Rubinstein formula.

Price Function for simple derivatives

Theorem (Price function)

Within the framework of the Multi-Step Binomial model, let ξ be a simple derivative, that is $\xi = \varphi(S_T)$. Put

$$F(t, y) := \frac{1}{(1+r)^{T-t}} \mathbb{E}_* \left\{ \varphi \left(S^y_{T-t} \right) \right\}, \;\; y \geq 0, 0 \leq t \leq T,$$

where $(S^y_t)_{0 \leq t \leq T}$ is a process satisfying that $S^y_0 = y$ and

$$S^y_t = S^y_{t-1}(1 + K_S(t)), \;\; t = 1, \ldots, T.$$

Then, almost surely

$$F(t, S_t) = \frac{1}{(1+r)^{T-t}} \mathbb{E}_* \left(\varphi(S_T) \middle| \mathscr{F}_t \right), \;\; 0 \leq t \leq T.$$

In other words, the arbitrage-free price of the simple derivative $\xi = \varphi(S_T)$ at time $0 \leq t \leq T$ is given by $F(t, S_t)$.

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Remark: The function F is known as the price function associated to the pay-off function φ.

Proof $\left(F(t,S_t) = \frac{1}{(1+r)^{T-t}}\mathbb{E}_*\left(\varphi(S_T)\middle|\mathscr{F}_t\right)\right)$

- The case $t = 0$ is trivial, so suppose that $1 \le t \le T$.

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- The case $t = 0$ is trivial, so suppose that $1 \leq t \leq T$.
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- Therefore, there must exists a deterministic function $\psi : \mathbb{R}^t \to \mathbb{R}$ such that

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- Such a function also satisfies that for all $k_1, \ldots, k_t \in \{R_u, R_d\}$

$$\psi(k_1, \ldots, k_t) = \mathbb{E}_*\left(\varphi(S_T)\middle|\,K_S(1) = k_1, \ldots, K_S(t) = k_t\right).$$

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- It is enough to show that

$$\psi(k_1, \ldots, k_t) = \mathbb{E}_*\left\{\varphi\left(S^y_{T-t}\right)\right\}, \quad y = S_0 \prod_{v=1}^{t}(1 + k_v).$$

Proof $\quad \left(A_t = \{K_S(1) = k_1, \ldots, K_S(t) = k_t\}\right)$

- Recall that for all $1 \leq t \leq T$

$$S_T = S_0 \prod_{v=1}^{t} (1 + K_S(v)) \times \overbrace{\prod_{v=t+1}^{T} (1 + K_S(v))}^{Z_{t+1}^{T}}.$$

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$$\psi(k_1, \dots, k_t) = \mathbb{E}_* \left(\varphi(S_T) \middle| A_t \right) = \mathbb{E}_* \left\{ \varphi\left(S_{T-t}^{y}\right) \right\}.$$

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- Therefore

$$\frac{\mathbb{E}_* \left(\varphi(S_T) | \, K_S(1), \ldots, K_S(t) \right)}{(1+r)^{T-t}} = \frac{\psi(K_S(1), \ldots, K_S(t))}{(1+r)^{T-t}}$$
$$= F(t, \underbrace{S_0 \prod_{v=1}^{t} (1 + K_S(v))}_{S_t})$$

Interpretation of the Price Function

Theorem (Price function)

In the Multi-Step Binomial model, it holds that

$$F(t, S_t) = \frac{1}{(1+r)^{T-t}} \mathbb{E}_* \left(\varphi(S_T) \middle| \mathscr{F}_t \right), \quad 0 \le t \le T.$$

where

$$F(t, y) := \frac{1}{(1+r)^{T-t}} \mathbb{E}_* \left\{ \varphi\left(S^y_{T-t} \right) \right\}, \quad y \ge 0, 0 \le t \le T,$$

and $(S^y_t)_{0 \le t \le T}$ *is a process satisfying that* $S^y_0 = y$ *and*

$$S^y_t = S^y_{t-1}(1 + K_S(t)), \quad t = 1, \dots, T.$$

Interpretation: The function $F(t, y)$ can be thought as **the initial price of a simple derivative** with maturity time $T - t$ and pay-off function φ under the circumstance that the initial price of the risky asset is $y \ge 0$.

General Price of a Call Option in the Bin Model

- According to the previous theorem the price of a Call Option in the Multi-Step Binomial model at time $t = 0, 1, \ldots, T$ is given by

$$\xi_t = F(t, S_t) = \frac{1}{(1+r)^{T-t}} \mathbb{E}_* \left((S_T - C)^+ \middle| \mathscr{F}_t \right),$$

where

$$F(t, y) = \mathbb{E}_* \left[\frac{\left(S^y_{T-t} - C\right)^+}{(1+r)^{T-t}} \right], \quad y \geq 0, 0 \leq t \leq T.$$

in which $(S^y_t)_{0 \leq t \leq T}$ is a process satisfying that $S^y_0 = y$ and

$$S^y_t = S^y_{t-1}(1 + K_S(t)), \quad t = 1, \ldots, T.$$

General Price of a Call Option in the Bin Model

- The Cox-Ross-Rubinstein formula dictates that

$$
\begin{aligned}
F(t, S_0) =& \mathbb{E}_* \left(\frac{(S_{T-t} - C)^+}{(1+r)^{T-t}} \right) \\
=& S_0 \Psi_1(T - t, S_0, C, q^*) - \frac{C}{(1+r)^{T_0}} \Psi_2(T - t, S_0, C, q^*).
\end{aligned}
$$

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- Replacing S_0 by y the Cox-Ross-Rubinstein formula we get that

$$F(t, y) = y \Psi_1(T - t, y, C, q^*) - \frac{C}{(1+r)^{T-t}} \Psi_2(T - t, y, C, q^*).$$

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- Hence the price Call Option in the Multi-Step Binomial model is

$$\xi_t = F(t, S_t) = S_t \Psi_1(T - t, S_t, C, q^*) - \frac{C}{(1+r)^{T-t}} \Psi_2(T - t, S_t, C, q^*).$$