Financial Engineering
Exam

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F20

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Definition (Bond)

A bond is a financial security that pays the owner a chain of predetermined payments.

Definition (Wealth Process)

The evolution of an investment over time is called the wealth process of that investment and is denoted by

$$V = (V_t)_{0 \leq t \leq T}. \tag{1.1}$$

The initial capital is denoted by v_0, and we assume that V is a real-valued stochastic process on a given probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Bonds and the Value of Money Through Time

Interest Rate

▶ Predetermined payments are also known as interest
▶ Fraction of an investment paid either ones or in several periods
▶ Different types of interest
  1. Simple
  2. Compounded
  3. Continuously compounded

Definition (Simple Interest)

Let $v_0 \in \mathbb{R}$ be our initial capital. An interest on v_0 is said to be simple if it follows the wealth process

$$(1.2)$$

I will now show that the wealth process in (1.2) is indeed a
stochastic process in any probability space. Any stochastic process
X on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ satisfies

(1.3)

Suppose $v_0 > 0$ and $x \geq (1 + rt)v_0$ then

(1.4)

on the other hand if $x < (1 + rt)v_0$

(1.5)

Definition (Compounded Interest)

Let $v_0 \in \mathbb{R}$ be our initial capital. An interest on v_0 is said to be compunded over $m \in \mathbb{N}$ periods if it follows the wealth process

$$(1.6)$$

Note that we have the following properties $\forall\, 0 \leq t \leq T$

1. $V_{t+1} = \left(1 + \dfrac{r}{m}\right)^m V_t,$

2. If $m_1 > m_2, v_0 > 0 \Rightarrow \left(1 + \dfrac{r}{m_1}\right)^{m_1 t} v_0 > \left(1 + \dfrac{r}{m_2}\right)^{m_2 t} v_0,$

3. If $m_1 > m_2, v_0 < 0 \Rightarrow \left(1 + \dfrac{r}{m_1}\right)^{m_1 t} v_0 < \left(1 + \dfrac{r}{m_2}\right)^{m_2 t} v_0.$

At last i can turn to continuously compounded interest which i will present as the limit of (1.6) as $m \to \infty$. Note that by the following definition of e

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e, \qquad (1.7)$$

by letting $x = r/m$ in the above the limit of the wealth process of compounded interest can be seen as

$$\left[\left(1 + \frac{r}{m}\right)^{\frac{m}{r}}\right]^{rt} v_0 \to (e)^{rt} v_0, \quad \text{as } m \to \infty. \qquad (1.8)$$

This leads to the definition of continuously compounded interest.

There exists the following relation between the different types of interest

$$(1 + r) \leq \left(1 + \frac{r}{m}\right)^m < e^r. \qquad (1.9)$$

To show that the relation indeed holds i will show that the sequence

$$a_m = \left(1 + \frac{r}{m}\right)^m, \qquad (1.10)$$

is increasing.

Using the binomial theorem

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

we have

$$\left(1 + \frac{r}{m}\right)^m = \sum_{k=0}^{m} \binom{m}{k} 1^{m-k} \left(\frac{r}{m}\right)^k$$

$$= \sum_{k=0}^{m} \binom{m}{k} \left(\frac{r}{m}\right)^k$$

Each term is of the form

$$\binom{m}{k} \left(\tfrac{r}{m}\right)^k = \prod_{l=0}^{k-1} \frac{m-l}{k-l} \left(\tfrac{r}{m}\right)$$

Each term is of the form

$$\frac{m-l}{k-l}\frac{r}{m} = \frac{rm-lr}{m(k-l)}$$
$$= \frac{m(r-lr/m)}{m(k-l)}$$
$$= \frac{r-lr/m}{k-l}$$

I will discuss the following two types here

1. zero-coupon bonds,
2. coupon bonds.

Bonds and the Value of Money Through Time

Types of Bonds

A **zero-coupon bond** is a bond with a single payment $F > 0$ at time $T > 0$. The pay-off F is called the face value and T the maturity time.

The next question i will answer is how much i will be willing to pay for such a financial assest. This depends on the way the time value of money is measured.

Consider for example the following setup; let $B_0 \geq 0$ be the value of the zero-coupon bond with face value $F > 0$ and maturity time $T > 0$. Suppose that only annual compound interest at rate $r > 0$ is available.

From a buyers perspective what if

$$B_0 > \frac{F}{(1+r)^T}, \qquad (1.11)$$

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Suppose now that we flip the inequality and look from a sellers perspective, that is if

$$B_0 < \frac{F}{(1+r)^T}, \qquad (1.12)$$

would i sell the bond?

Now i will consider the situation where at time $1 \leq t \leq T$ i want to get rid of a bond, but i what to determine what price i should sell it to. At this time the bond can be considered a new zero-coupon bond with face value $F > 0$ and maturity time $T - t$. Thus we have from the previous argumentation that

$$B_t = \frac{F}{(1+r)^{T-t}}, \quad 0 \leq t \leq T. \tag{1.13}$$

The chain of arguments holds also when the time value of money is different, if a compounded interest over m periods where considered then the fair price of a zero-coupon bond at time t would be

$$B_t = \frac{F}{\left(1 + \frac{r}{m}\right)^{m(T-t)}}. \tag{1.14}$$

If we consider the continuously compounded case the fair price would be

$$B_t = \frac{F}{e^{r(T-t)}}. \tag{1.15}$$

Bonds and the Value of Money Through Time

Types of Bonds

A **coupon bond** guarantees the owner a payment $F > 0$ at time $T > 0$, as well as a fixed amount $C > 0$ at $t = 1, 2, \ldots, T$. To find a fair price for such a bond i will follow the same argumentation as above, namely i will compare the price of the bond with the cost of recieving the same benefits with a bank account. In other words how much money will i have to deposit in my bank account today if i want to

$$
\begin{align}
1. \quad &\text{withdraw } C > 0 \text{ after 1 year} \tag{1.16} \\
2. \quad &\text{withdraw } C > 0 \text{ after 2 years} \tag{1.17} \\
&\qquad\qquad \vdots \\
T-1. \quad &\text{withdraw } C > 0 \text{ after } T-1 \text{ years} \tag{1.18} \\
T. \quad &\text{withdraw } F + C \text{ after } T \text{ years} \tag{1.19}
\end{align}
$$

and have nothing left in the bank account afterwards.

In order to be able to get $C > 0$ after one year i have to put

$$\frac{C}{1+r} \tag{1.20}$$

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In order to be able to get $C > 0$ after two years i have to put

$$\frac{C}{(1+r)^2} \qquad (1.21)$$

in the bank.

Generalizing this argument tells me that in order to recieve $C > 0$
after t years i have to put

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in the bank.

Lastly in order to get $F + C$ after T years i have to put

$$\frac{F+C}{(1+r)^T} = \frac{F}{(1+r)^T} + \frac{C}{(1+r)^T} \tag{1.23}$$

in the bank.

Types of Bonds

Adding up all these amounts it is concluded that i have to make a deposit of

$$\sum_{i=1}^{T} \frac{C}{(1+r)^i} + \frac{F}{(1+r)^T}. \qquad (1.24)$$

Types of Bonds

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The agreeable price of a coupon bond is thus given by

$$B_0 = \sum_{i=1}^{T} \frac{C}{(1+r)^i} + \frac{F}{(1+r)^T} = \frac{\xi_T}{(1+r)^T}. \tag{1.25}$$

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Where the pay-off ξ_T at time T is given by

$$\xi_T := \sum_{i=1}^{T} C(1+r)^{T-i} + F, \tag{1.26}$$

in other words the fair price of the coupon bond (as well as the zero-coupon bond) can be written as the discounted price of the total pay-off.

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Portfolio Allocation and Risk Measures
Portfolio

Let

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, \mathbb{P}), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)} \right)_{0 \leq t \leq T} \right\}, \qquad (2.1)$$

be a finite-horizon financial market.

Definition (Portfolio and Strategies)

A portfolio in $\mathfrak{M}$ is a $(d+1)$-dimensional vector

$$\Theta_t = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)} \right), \qquad (2.2)$$

in which

$\Theta_t^j =$ Number of shares of the j'th asset held between time $t-1$ and t. $\qquad (2.3)$

for $j = 1, \ldots, d+1$. The collection $\Theta = (\Theta_t)_{0 \leq t \leq T}$, with the convention that $\Theta_0 = \Theta_1$, is termed a strategy.

For every strategy on market there is an associated wealth process. The wealth process for Θ is defined and denoted by

$$V_t^\Theta = \varphi_t B_t + \sum_{j=1}^{d} \theta_t^{(j)} S_t^{(j)} = \Theta_t \cdot P_t, \quad 0 \leq t \leq T. \qquad (2.4)$$

Any strategy on a given market inherently carries a risk because the return is random, there is in other words no way to predict our profit or losses with certainty. There is no one way to measure the risk associated with a strategy, however in the next two sections i will explore two approaches. Both of these is based on portfolio allocation as an optimization problem.

Portfolio Allocation and Risk Measures

Risk Measures

The problem is given in this way; solve

$$\underset{\mathbf{w}\in\mathbb{R}^{d+1}}{\operatorname{argmin}} \mathcal{R}\left(\mathbf{w} \cdot \mathbf{K}_P\right), \qquad (2.5)$$

subject to:

$$\sum_{j=0}^{d} w_j = 1, \ \mathbb{E}\left[U(\mathbf{w} \cdot \mathbf{K}_P)\right] = \mu, \ \mu \in \mathbb{R}, \qquad (2.6)$$

where $\mathcal{R}$ is a measure of risk and $U(\mathbf{w} \cdot \mathbf{K}_P)$ is the utility of the strategy.

The mean variance approach assumes the uitility function as the identity, that is

$$U(x) = x. \tag{2.7}$$

By letting

$$\mu_{\mathsf{K}} := \mathbb{E}\left[\mathsf{K}_P\right], \tag{2.8}$$

it follows that

$$\mathbb{E}\left[U(\mathsf{w} \cdot \mathsf{K}_P)\right] = \mathsf{w} \cdot \mu_{\mathsf{K}}. \tag{2.9}$$

Thus the optimization problem, in the mean-variance approach becomes

Problem (Optimization Problem Mean-Variance)

$$\underset{\mathsf{w}\in\mathbb{R}^{d+1}}{\operatorname{argmin}} \sqrt{\mathsf{w}^\top C \mathsf{w}} \qquad (2.10)$$

Subject to:

$$1.) \ \sum_{j=0}^{d} w_j = 1, \qquad (2.11)$$

$$2.) \ \mathsf{w} \cdot \mu_{\mathsf{K}} = \mu, \quad \mu \in \mathbb{R}. \qquad (2.12)$$

Risk Measures

An illustrative example to demonstrate why one might consider a different risk measure than the standard deviation is the following. Consider two portfolios that generate the following wealth

$$V_1^{(1)} = \begin{cases} 1 & \text{with probability } 1/2 \\ -9 & \text{with probability } 1/2 \end{cases} \tag{2.13}$$

and

$$V_1^{(2)} = \begin{cases} 5 & \text{with probability } 1/2 \\ -5 & \text{with probability } 1/2 \end{cases} \tag{2.14}$$

According to their standard deviation these two portfolios carries the same risk, but one could argue that the first is riskier that the latter. The next risk measure i will consider i concerned with controlling losses rather than the variation of the return.

If i denote the outcome of an investment with V_1 then my potential losses will be given by $-V_1$. Suppose that for some $x \in \mathbb{R}$

$$-V_1 \leq x. \qquad (2.15)$$

Then to cover my risk of bankruptcy i must keep at least the amount x in my bank account. In reality the only thing i can quantify is the chance of that happening, which is denoted by

$$\mathbb{P}\left(-V_1 \leq x\right). \qquad (2.16)$$

This is the motivation behind the risk measure Value at Risk.

Definition (Value at Risk)

Let $0 < \alpha < 1$ and X be a random variable. The Value at Risk (VaR) of X is defined and denoted by

$$\text{VaR}_\alpha(X) := \inf \left\{ x \in \mathbb{R} \, : \, \mathbb{P}(X + x \geq 0) \geq 1 - \alpha \right\}. \qquad (2.17)$$

In other words the $\text{VaR}_\alpha(X)$ represents the amount of extra capital i need to hold in order to reduce my risk of bankruptcy to $1 - \alpha$.

An alternative representation of VaR can be formulated using the fact that

$$\mathbb{P}(-X \leq x) \geq 1 - \alpha \iff \mathbb{P}(X + x < 0) \leq \alpha, \qquad (2.18)$$

this lets us formulate an equivalent representation given by

$$\mathsf{VaR}_{\alpha}(X) := \inf \{x \in \mathbb{R} \, : \, \mathbb{P}(X < -x) \leq \alpha\}. \qquad (2.19)$$

Proposition (Properties of VaR)

Let X, Y be arbitrary random variables. Then, the following holds

1. *If $X \geq 0$ almost surely, then $\mathrm{VaR}_\alpha(X) \leq 0$.*
2. *For all $y \in \mathbb{R}$ we have that $\mathrm{VaR}_\alpha(X + y) = \mathrm{VaR}_\alpha(X) - y$. In particular $\mathrm{VaR}_\alpha(X + \mathrm{VaR}_\alpha(X)) = 0$.*
3. *If $\lambda \geq 0$, then $\mathrm{VaR}_\alpha(\lambda X) = \lambda \, \mathrm{VaR}_\alpha(X)$.*
4. *If $X \geq Y$ almost surely, then $\mathrm{VaR}_\alpha(X) \leq \mathrm{VaR}_\alpha(Y)$.*

Proof:

Coherent Risk Measures

Note that when using standard deviation as a risk measure we get the following

$$\sigma(X + Y)^2 = \text{Var}(X) + \text{Var}(Y) + 2\rho_{X,Y}\sqrt{\text{Var}(X)\,\text{Var}(Y)}, \tag{2.20}$$

where $\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}} \leq 1$. Then

$$\sigma(X + Y)^2 = \sigma(X)^2 + \sigma(Y)^2 + 2\rho_{X,Y}\sigma(X)\sigma(Y) \tag{2.21}$$
$$\leq \sigma(X)^2 + \sigma(Y)^2 + 2\sigma(Y)\sigma(X) = [\sigma(X) + \sigma(Y)]^2, \tag{2.22}$$

which would imply that

$$\sigma(X + Y) \leq \sigma(X) + \sigma(Y). \tag{2.23}$$

However VaR as a risk measure is not able to reproduce this, that is in general we do not have that

$$\mathsf{VaR}_\alpha(X + Y) \leq \mathsf{VaR}_\alpha(X) + \mathsf{VaR}_\alpha(Y). \qquad (2.24)$$

As diversification is a desired i will know introduce the concept of Coherent Risk Measures.

Definition (Coherent Risk Measure)

A function $\rho : L^1 \to \mathbb{R}$ is said to be a Coherent Risk Measure if

1. If $X \geq 0$ almost surely, then $\rho(X) \leq 0$.
2. For all $y \in \mathbb{R}$ we have that $\rho(X + y) = \rho(X) - y$.
3. If $\lambda \geq 0$, then $\rho(\lambda X) = \lambda \rho(X)$.
4. We have that $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

Coherent Risk Measures

The Conditional Value at Risk (CVAR) is a common example of a coherent risk measure.

Given a random variable X, we will write

$$q_\alpha(X) := \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) \ge \alpha\}. \qquad (2.25)$$

With this notation now introduced i can present the definition of CVaR.

Portfolio Allocation and Risk Measures

Coherent Risk Measures

The Conditional Value at Risk (CVAR) is a common example of a coherent risk measure.
Given a random variable X, we will write

$$q_\alpha(X) := \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) \ge \alpha\}. \qquad (2.25)$$

With this notation now introduced i can present the definition of CVaR.

Definition (Condition Value at Risk)

Let $0 < \alpha < 1$ and $X \in L^1$. The Conditional Value at Risk or Expected Shortfall of X is defined and denoted buy

$$\text{CVaR}_\alpha := -\frac{1}{\alpha} \int_0^\alpha q_r(X) \, dr. \qquad (2.26)$$

The name Expected Shortfall comes from the fact that if X has a continuous distribution, then

$$\mathsf{CVaR}_\alpha(X) = -\mathbb{E}[X \mid X + \mathsf{VaR}_\alpha(X) \leq 0]. \qquad (2.27)$$

Thus, CVaR_α measures the expected losses given that $\mathsf{VaR}_\alpha(X)$ was not enough to cover our position on X.

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The Multi-Step Binomial Model

Model Setup

Consider a model where only two assets are traded: A bond and a risky asset. The bond satisfies that

$$(3.1)$$

The price process $(S_t)_{t=0,1,\ldots,T}$ satisfies that $S_0 > 0$ (non-random) and

$$(3.2)$$

The returns are independent and identically distributed with

$$(3.3)$$

with the relation

$$(3.4)$$

I am considering the multi-step financial market

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, \mathbb{P}), P = \left(B_t, S_t^{(1)}, S_t^{(2)}, \ldots, S_t^{(d)} \right)_{t=0,1,\ldots,T} \right\}.$$
(3.5)

In order to further explain how information flows in the market i need to be more specific about the definition of the probability space. I will start by letting

$$\Omega = \left\{ \left(\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(T)} \right) \, : \, \omega^{(t)} \in \{u, d\}, \, t = 1, 2, \ldots, T \right\}.$$
(3.6)

Take for example the two-step binomial model, that is $T = 2$.

The Multi-Step Binomial Model
Market Information

I will consider all possible events, which means that the σ-algebra in this case is

$$\mathscr{F} = 2^{\Omega}. \quad \text{(The power set)} \tag{3.7}$$

Using this set-up the return process can be written as

$$K_s(t)(\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(T)}) = \begin{cases} R_u & \text{if } \omega^{(t)} = u; \\ R_d & \text{if } \omega^{(t)} = d. \end{cases} \tag{3.8}$$

To ensure that the returns are independent and can go up and down with probability p and $(1 - p)$ respectively, i let

$$\mathbb{P}\left(\left\{\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(T)}\right\}\right) = p^k(1 - p)^{T-k}, \qquad (3.9)$$

where

$$k = \text{number of } u\text{'s in } \left(\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(T)}\right). \qquad (3.10)$$

As an example lets consider the two-step binomial model, that is $T = 2$ and we have that

$$\Omega = \{uu, ud, du, dd\}, \tag{3.11}$$

using the notational convention that

$$\left(\omega^{(1)}, \omega^{(2)}\right) = \omega^{(1)}\omega^{(2)}. \tag{3.12}$$

The Multi-Step Binomial Model
Market Information

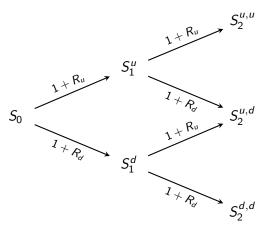

Figure: A graphical representation of the 2-step binomial model.

At time $t = 0$ no information is available but as time goes by, the price movements reveal information about future outcomes. Say for example that at time $t = 1$ the price went down, the future outcomes are restricted to the set

$$B_d := \{du, dd\}, \qquad (3.13)$$

as the the events (uu) and (ud) are no longer possible.

The aforementioned set is related to the information generated by the return process. Consider the σ-field generated by a random variable, which is given by

$$\sigma(X) = \left\{ X^{-1}(A) \,:\, A \in \mathscr{B}(\mathbb{R}). \right\} \tag{3.14}$$

This tells me that, in the event that $K_S(1) = R_d$ it follows that

$$\left\{ \left(\omega^{(1)}, \omega^{(2)} \right) \, : \, K_S(1) = R_d \right\} = \{du, dd\} = B_d. \qquad (3.15)$$

On the other hand if $K_S(1) = R_u$ it follows that

$$\left\{ \left(\omega^{(1)}, \omega^{(2)} \right) \, : \, K_S(1) = R_u \right\} = \{uu, ud\} = B_u. \qquad (3.16)$$

Therefore

$$B_u, B_d \subseteq \sigma \left(K_S(1) \right). \qquad (3.17)$$

It is in fact so, that because $K_S(1)$ only takes two values

$$\sigma(K_S(1)) = \{B_u, B_d, \Omega, \emptyset\}. \qquad (3.18)$$

The Multi-Step Binomial Model
Market Information

At the end of the trading period, in this case $t = T = 2$, the outcome of the price movements are known and stored in

$$\mathscr{F} = \sigma\left(K_S(1), K_S(2)\right). \tag{3.19}$$

Furthermore there has been constructed a family of sub-σ-algebras of $\mathscr{F}$;

$$\mathscr{F}_1 = \sigma(K_S(1)) = \{B_u, B_d, \Omega, \emptyset\}. \tag{3.20}$$
$$\mathscr{F}_2 = \sigma(K_S(1), K_S(2)) = \mathscr{F}. \tag{3.21}$$

Note that by construction, $\mathscr{F}$ contains the whole information available on the price movements up until time t.

Mathematically this means that the price process S_t is adapted to $\mathscr{F}$;

$$\sigma(S_t) \subseteq \mathscr{F}_t, \quad t = 0, 1, 2 \text{ where } \mathscr{F}_0 = \{\Omega, \emptyset\}, \tag{3.22}$$

and moreover

$$\mathscr{F}_{t-1} \subseteq \mathscr{F}_t, \quad t = 1, 2. \tag{3.23}$$

The Multi-Step Binomial Model

Market Information

Definition (Filtrations and Adapted Process)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{R})$. A collection of σ-algebras $\mathbb{F} = (\mathscr{F}_t)_{t=0,1,\dots,T}$ is called a filtration if for all $1 \leq t \leq T$,

$$\mathscr{F}_{t-1} \subseteq \mathscr{F}_t \subseteq \mathscr{F}. \qquad (3.24)$$

The quadruplet

$$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P}), \qquad (3.25)$$

is termed a filtered probability space.

Furthermore, a stochastic process $(X_t)_{0 \leq t \leq T}$ is said to be adapted to a filtration $\mathbb{F}$ if

$$\sigma(X_t) \subseteq \mathscr{F}_t, \quad \forall 0 \leq t \leq T. \qquad (3.26)$$

Market Information

Definition (Financial Markets with Information)

Fix $T, d \in \mathbb{N}$. A finite-horizon financial market with information is the pair

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P} \right), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)} \right)_{0 \leq t \leq T} \right\}, \tag{3.27}$$

consisting of

1. A filtered probability space $\left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P} \right)$

2. P_t is adapted.

3. $S^{(j)}$ is the price process of the j'th asset traded in the market

4. B is a numéraire (e.g. a bond), i.e.

$$\mathbb{P}(B_t > 0) = 1, \quad \forall\, 0 \leq t \leq T. \tag{3.28}$$

The Multi-Step Binomial Model
Absence of Arbitrage

An arbitrage is understood as a way to generate positive wealth with zero investment and zero risk. In the one-step binomial model it is only allowed to allocate wealth ones, namely at time $t = 0$. Returns are then measured at time $t = T = 1$.

In the multi-step binomial model agents will redesign their strategies based on the available information. In order to create an arbitrage the portfolio need to be updated every time the price changes. Therefore, instead of considering a single portfolio, an arbitrage now consists of a collection of portfolios $(\Theta_t)_{0 \le t \le T}$.

Definition (Portfolio and Strategy)

A portfolio in $\mathfrak{M}$ is a $(d+1)$-dimensional vector $\Theta_t = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)}\right)$, in which

$$\theta_t^{(j)} = \text{ Number of shares of the } j\text{'th asset held between time } t-1 \text{ and } t. \tag{3.29}$$

The collection $\Theta = (\Theta_t)_{0 \leq t \leq T}$, with the convention that $\Theta_0 = \Theta_1$ is termed a strategy.

The wealth process associated to $\Theta = (\Theta_t)_{0 \leq t \leq T}$ is defined and denoted by

$$V_t^\Theta := \varphi_t B_t + \sum_{j=1}^d \theta_t^{(j)} S_t^{(j)} = \Theta_t \cdot P_t, \quad 0 \leq t \leq T \tag{3.30}$$

$$\Rightarrow V_0^\Theta = \varphi_0 B_0 + \sum_{j=1}^d \theta_0^{(j)} S_0^{(j)} = \varphi_1 B_0 + \sum_{j=1}^d \theta_1^{(j)} S_0^{(j)} \tag{3.31}$$

The Multi-Step Binomial Model
Absence of Arbitrage

The arbitrage approach from the one-step binomial model can not be used when considering the multi-step binomial model as many strategies that from an intuitive point of view should not be thought of as an arbitrage, will be believed to be an arbitrage. Consider for example the following situation

1. Injection of capital at $t \geq 1$.
2. Priviliged information.
3. Unlimited credit.

The Multi-Step Binomial Model
Absence of Arbitrage

Definition (Admissible Strategy)

Let $\Theta = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)} \right)_{0 \leq t \leq T}$ be strategy on the market

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P} \right), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)} \right)_{0 \leq t \leq T} \right\}.$$
(3.32)

I will say that Θ is admissible if:

The Multi-Step Binomial Model
Absence of Arbitrage

1. **It is self-financed**: It only requires initial capital. I will say that Θ is self financed if

$$V_t^\Theta = \varphi_{t+1} B_t + \sum_{j=1}^d \theta_{t+1}^{(j)} S_t^{(j)}, \quad 0 \le t \le T-1. \quad (3.33)$$

2. **Non-anticipative**: It is build up only on current information. I will say that Θ is non-anticipative or predictable if Θ_{t+1} depends only on the information at time t, that is Θ_{t+1} is $\mathscr{F}_t$-measurable.

3. **It has a limited credit line**: There is a non-random constant $C > 0$, such that

$$V_t^\Theta \ge -C, \ \forall\, 0 \le t \le T. \quad (3.34)$$

The Multi-Step Binomial Model
Absence of Arbitrage

Definition (Arbitrage)

Consider the market

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P} \right), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)} \right)_{0 \le t \le T} \right\}. \tag{3.35}$$

An admissible strategy $\Theta = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)} \right)_{0 \le t \le T}$ is said to be an arbitrage if:

The Multi-Step Binomial Model

Absence of Arbitrage

1. It has zero initial capital:

$$V_0^\Theta = \varphi_0 B_0 + \sum_{j=1}^{d} \theta_0^{(j)} \cdot S_0^{(j)} = 0. \tag{3.36}$$

2. At time $t = T$, i am out of debts with 100% certainty: Almost surely

$$V_T^\Theta = \varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} \geq 0. \tag{3.37}$$

3. I have a chance to make a profit:

$$\mathbb{P}\left(\varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} > 0 \right) > 0. \tag{3.38}$$

The Multi-Step Binomial Model
Risk-Neutral Measure

To begin this section i will define the notion of a martingale. Fix a filtered probability space

$$\left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P}\right). \tag{3.39}$$

I am given a discrete time stochastic process

$$X = (X_t)_{0 \le t \le T}. \tag{3.40}$$

Under this framework i give the definition of a martingale.

The Multi-Step Binomial Model

Risk-Neutral Measure

Definition (Martingale)

The collection $\{(X_t, \mathscr{F}_t) : t = 0, 1, \ldots, T\}$ is said to be a martingale if

1.

2.

3.

The Multi-Step Binomial Model
Risk-Neutral Measure

Consider the risk neutral multi-step binomial model: $B_0 = 1$, $S_0 > 0$ is a non-random constant and

$$B_t = (1 + r)^t; \quad S_t = S_{t-1}(1 + K_S(t)), \quad t = 1, \dots, T. \quad (3.41)$$

$(K_S(t))_{t=1,\dots,T}$ are i.i.d. with

$$K_S(t) = \begin{cases} R_u & \text{with probability } q^* \\ R_d & \text{with probability } 1 - q* \end{cases} \quad (3.42)$$

with the relation $R_d < r < R_u$ and

The risk neutral measure is the following:

I will now show that the discounted price

$$X_t := \frac{S_t}{B_t}, \quad 0 \le t \le T, \tag{3.43}$$

is a martingale under the filtration

$$\mathscr{F}_0 = \{\emptyset, \Omega\} \tag{3.44}$$

$$\mathscr{F}_t = \sigma(K_S(1), \ldots, K_S(t)), \quad 1 \le t \le T. \tag{3.45}$$

In order to show this, i need to verify that the discounted price satisfies all three conditions in definition 15 (Martingale)

The Multi-Step Binomial Model

Risk-Neutral Measure

By definition

$$\mathbb{E}\left(X_{t+1}|\mathscr{F}_t\right) = \mathbb{E}\left(\left.\frac{S_{t+1}}{B_{t+1}}\right|\mathscr{F}_t\right) \tag{3.46}$$

$$= \mathbb{E}\left(\left.\frac{S_t(1+K_S(t+1))}{B_{t+1}}\right|\mathscr{F}_t\right) \tag{3.47}$$

$$= \mathbb{E}\left(\left.\frac{S_t(1+K_S(t+1))}{(1+r)^{t+1}}\right|\mathscr{F}_t\right) \tag{3.48}$$

$$= \mathbb{E}\left(\left.\frac{S_t}{(1+r)^{t+1}}(1+K_S(t+1))\right|\mathscr{F}_t\right) \tag{3.49}$$

$$\text{(By the Product Rule)} = \frac{S_t}{(1+r)^{t+1}}\mathbb{E}\left(\left.(1+K_S(t+1))\right|\mathscr{F}_t\right) \tag{3.50}$$

$$= \frac{S_t}{(1+r)^t}\frac{1}{(1+r)}\mathbb{E}\left(\left.(1+K_S(t+1))\right|\mathscr{F}_t\right) \tag{3.51}$$

$$= X_t\frac{1}{(1+r)}\mathbb{E}\left(\left.(1+K_S(t+1))\right|\mathscr{F}_t\right). \tag{3.52}$$

I am now left to show that

$$\mathbb{E}\left(1 + K_S(t+1)\,\middle|\,\mathscr{F}_t\right) = 1 + r, \tag{3.53}$$

to do this i will use the independence of the returns but let me start by considering $t = 0$.

The Multi-Step Binomial Model
Risk-Neutral Measure

In this case

$$\mathbb{E}\left((1 + K_S(1) \,|\, \mathscr{F}_0\right) = \mathbb{E}\left((1 + K_S(1) \,|\, \{\emptyset, \Omega\}\right) \qquad (3.54)$$

$$\text{(Conditioning on the trivial } \sigma\text{-algebra)} = \mathbb{E}\left((1 + K_S(1)\right) \qquad (3.55)$$

$$\text{(From Risk-Neutrality)} = (1 + r). \qquad (3.56)$$

Let me now consider the case when $t \geq 1$

$$\mathbb{E}\left((1 + K_S(t+1) \,|\, \mathscr{F}_t\right) = \mathbb{E}\left((1 + K_S(t+1) \,|\, K_S(1), \ldots, K_S(t)\right) \qquad (3.57)$$

$$\text{(Indepence prop. of the cond. expectation)} = \mathbb{E}\left((1 + K_S(1)\right) \qquad (3.58)$$

$$= (1 + r). \qquad (3.59)$$

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The First Fundamental Theorem of Asset Pricing

The Theorem

The Fundamental theorem of asset pricing gives necessary and sufficient conditions for the absence of arbitrage in the market with information, given by

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P} \right), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)} \right)_{0 \le t \le T} \right\}. \qquad (4.1)$$

Theorem (FFTAP)

The market given by (4.1) is arbitrage free if and only if there is a martingale measure.

The First Fundamental Theorem of Asset Pricing

Explaining the Hypothesis

To understand the hypothesis presented in the theorem lets first remember the concept of an admissible strategy.

Definition (Admissible Strategy)

Let $\Theta = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)} \right)_{0 \leq t \leq T}$ be strategy on the market

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P} \right), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)} \right)_{0 \leq t \leq T} \right\}. \quad (4.2)$$

I will say that Θ is admissible if:

Explaining the Hypothesis

1. **It is self-financed**: It only requires initial capital. I will say that Θ is self financed if

$$V_t^{\Theta} = \varphi_{t+1} B_t + \sum_{j=1}^{d} \theta_{t+1}^{(j)} S_t^{(j)}, \quad 0 \leq t \leq T - 1. \quad (4.3)$$

2. **Non-anticipative**: It is build up only on current information. I will say that Θ is non-anticipative or predictable if Θ_{t+1} depends only on the information at time t, that is Θ_{t+1} is $\mathscr{F}_t$-measurable.

3. **It has a limited credit line**: There is a non-random constant $C > 0$, such that

$$V_t^{\Theta} \geq -C, \ \forall \, 0 \leq t \leq T. \quad (4.4)$$

Next let me recall the notion of arbitrage.

Definition (Arbitrage)

Consider the market

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F})_{0 \leq t \leq T}, \mathbb{P} \right), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)} \right)_{0 \leq t \leq T} \right\}. \qquad (4.5)$$

An admissible strategy $\Theta = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)} \right)_{0 \leq t \leq T}$ is said to be an arbitrage if:

The First Fundamental Theorem of Asset Pricing

Explaining the Hypothesis

1. It has zero initial capital:

$$V_0^{\Theta} = \varphi_0 B_0 + \sum_{j=1}^{d} \theta_0^{(j)} \cdot S_0^{(j)} = 0. \qquad (4.6)$$

2. At time $t = T$, i am out of debts with 100 % certainty: Almost surely

$$V_T^{\Theta} = \varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} \geq 0. \qquad (4.7)$$

3. I have a chance to make a profit:

$$\mathbb{P}\left(\varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} > 0\right) > 0. \qquad (4.8)$$

The First Fundamental Theorem of Asset Pricing
Explaining the Hypothesis

Lastly i will define a martingale measure.

Definition (Martingale Measure)

Let $\mathbb{Q}$ be probability measure on $(\Omega, \mathscr{F})$, that is there are $0 < q_i^* < 1, i = 1, \ldots, N$ such that $\sum_{i=1}^{N} q_i^* = 1$ and

$$\mathbb{Q}(\{\omega_i\}) = q_i^* \quad \text{for } i = 1, \ldots, N. \tag{4.9}$$

We will say that $\mathbb{Q}$ is a martingale measure if under $\mathbb{Q}$ the discounted price $(\tilde{P}_j^{(j)})_{0 \leq t \leq T}$ is a martingale w.r.t $(\mathscr{F}_t)_{0 \leq t \leq T}$, for every $j = 1, 2, \ldots, d + 1$.

The First Fundamental Theorem of Asset Pricing

Risk-Neutral and Martingale Measures

In the binomial model q^* plays two roles

1. It transform the real-world probabilities (the one observed in the market) $(p, 1 - p)$ satisfying that

$$K_S(t) = \begin{cases} R_u & \text{with probability } p; \\ R_d & \text{with probability } 1 - p, \end{cases} \tag{4.10}$$

into a risk-neutral world in which the "new" distribution for the returns

$$K_S(t) = \begin{cases} R_u & \text{with probability } q^*; \\ R_d & \text{with probability } 1 - q^*, \end{cases} \tag{4.11}$$

is such that

$$\mathbb{E}_*(K_S(t)) = r, \quad t = 1, 2, \ldots, T. \tag{4.12}$$

2. It transforms the discounted prices into a martingale.

The First Fundamental Theorem of Asset Pricing
Risk-Neutral and Martingale Measures

In this model the two are actually equivalent. Consider the martingale property

$$\mathbb{E}_* \left(\frac{S_{t+1}^{(j)}}{B_{t+1}} \;\middle|\; \mathscr{F}_t \right) = \frac{S_t^{(j)}}{B_t} \tag{4.13}$$

$$\frac{1}{B_{t+1}} \mathbb{E}_* \left(S_{t+1}^{(j)} \;\middle|\; \mathscr{F}_t \right) = \frac{S_t^{(j)}}{B_t} \tag{4.14}$$

$$\frac{1}{S_t^{(j)}} \mathbb{E}_* \left(S_{t+1}^{(j)} \;\middle|\; \mathscr{F}_t \right) = \frac{B_{t+1}}{B_t} \tag{4.15}$$

$$\mathbb{E}_* \left(\frac{S_{t+1}^{(j)}}{S_t^{(j)}} \;\middle|\; \mathscr{F}_t \right) = \frac{B_{t+1}}{B_t}, \tag{4.16}$$

subtracting 1 from both sides i get:

The First Fundamental Theorem of Asset Pricing

Risk-Neutral and Martingale Measures

$$\mathbb{E}_* \left(\frac{S_{t+1}^{(j)}}{S_t^{(j)}} - 1 \,\middle|\, \mathscr{F}_t \right) = \frac{B_{t+1}}{B_t} - 1, \tag{4.17}$$

taking expectations gives

$$\mathbb{E}_* \left[\mathbb{E}_* \left(\frac{S_{t+1}^{(j)}}{S_t^{(j)}} - 1 \,\middle|\, \mathscr{F}_t \right) \right] = \mathbb{E}_* \left[\frac{B_{t+1}}{B_t} - 1 \right] \tag{4.18}$$

$$\mathbb{E}_* \left(\frac{S_{t+1}^{(j)}}{S_t^{(j)}} - 1 \right) = \frac{B_{t+1}}{B_t} - 1 \tag{4.19}$$

$$\mathbb{E}_* \left(K_S(t+1) \right) = K_B(t+1). \tag{4.20}$$

This is exactly the definition of risk-neutral probabilities.

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Pricing in the Binomial Model

Methodology for Pricing

I will start by outlining the methodology used for pricing derivatives in the binomial model.

1. **Define a model for the prices**: Specify the dynamics for

$$P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)} \right)_{0 \leq t \leq T}, \tag{5.1}$$

   i.e. describe how the distribution, evolution and behavior of the prices should be. . . according to some empirical facts observed in the data.

2. **Indicate a set of information**: Determine what information is available at each point in time, i.e. specify $(\mathscr{F}_t)_{0 \leq t \leq T}$.

3. **Check for arbitrage opportunities**: Establish under which circumstances the proprosed model does not admit an arbitrage, i.e. by the First Fundamental Theorem of Asset Pricing find a martingale measure $\mathbb{Q}$ for the model.

4. **Dynamics in the risk-neutral world**: Investigate what the distribution of the price process is under the martingale measure. This is necessary as the $\xi = \Phi(P_0, \ldots, P_T)$ is a function of the prices.

5. **Price the derivative**: Compute the prices of the derivative by

$$\xi_t := \mathbb{E}_* \left(\frac{B_t}{B_T} \xi \,\middle|\, \mathscr{F}_t \right) \tag{5.2}$$

Pricing in the Binomial Model

Pricing a Call Option

> Let us now find the inital price of a call option. Recall that such a
> contract gives the owner the right but not the obligation to buy an
> asset at a given value, say $C > 0$ at a predetermined point in time,
> say $T > 0$.

Pricing in the Binomial Model
Pricing a Call Option

The pay-off is then given by the random variable

$$\xi = \max\{S_T - C, 0\} = (S_T - C)^+. \qquad (5.3)$$

Independently of the model, we know by theorem 16 (FFTAP) that the arbitrage free initial price of such an option is

$$\xi_0 = \mathbb{E}_* \left(\frac{B_0}{B_T} (S_T - C)^+ \right). \qquad (5.4)$$

In the multi-step binomial model we have

$$\xi_0 = \mathbb{E}_* \left(\frac{B_0}{B_T} (S_T - C)^+ \right) = \frac{1}{(1+r)^T} \mathbb{E}_* \left((S_T - C)^+ \right). \qquad (5.5)$$

Pricing in the Binomial Model

Pricing a Call Option

Using that under the probability measure $\mathbb{Q}$ we have that for all $x = 0, \ldots, T$

$$p_T(x; q^*) = \mathbb{Q}\left[S_T := S_0(1 + R_u)^x(1 + R_d)^{T-x}\right] \quad (5.6)$$

$$= \binom{T}{x}(q^*)^x(1 - q^*)^{T-x} \quad (5.7)$$

it follows that

$$\mathbb{E}_*\left((S_T - C)^+\right) = \sum_{x=0}^{T}\left(S_0(1 + R_u)^x(1 + R_d)^{T-x} - C\right)^+ p_T(x; q^*). \quad (5.8)$$

Observe that

$$\left(S_0(1 + R_u)^x(1 + R_d)^{T-x} - C\right)^+ \neq 0 \quad (5.9)$$

if and only if

$$S_0(1 + R_u)^x(1 + R_d)^{T-x} > C. \quad (5.10)$$

Pricing in the Binomial Model
Pricing a Call Option

Let (with the convention that $\min\{\emptyset\} = +\infty$)

$$x_0(S_0, R_u, R_d, T) = \min\{0 \leq x \leq T : S_0(1 + R_u)^x(1 + R_d)^{T-x} > C\}. \quad (5.11)$$

Then

$$\mathbb{E}_* \left((S_T - C)^+\right) = \sum_{x=0}^{T} \left(S_0(1 + R_u)^x(1 + R_d)^{T-x} - C\right)^+ p_T(x; q^*) \quad (5.12)$$

$$= \sum_{x=x_0}^{T} \left(S_0(1 + R_u)^x(1 + R_d)^{T-x} - C\right)^+ p_T(x; q^*) \quad (5.13)$$

$$= S_0 \sum_{x=x_0}^{T} (1 + R_u)^x(1 + R_d)^{T-x} p_T(x; q^*) \quad (5.14)$$

$$- C \sum_{x=x_0}^{T} p_T(x; q^*). \quad (5.15)$$

Since

$$p_T(x; q^*) = \mathbb{Q}\left[S_T = S_0(1 + R_u)^x(1 + R_d)^{t-x}\right] = \binom{T}{x}(q^*)^x(1 - q^*)^{T-x}$$

(5.16)

Then

$$(1 + R_u)^x(1 + R_d)^{T-x}p_T(x; q^*) = (1 + R_u)^x(1 + R_d)^{T-x} \times \binom{T}{x}(q^*)^x(1 - q^*)^{T-x}$$

(5.17)

$$= \binom{T}{x}[(1 + R_u)q^*]^x \times [(1 + R_d)(1 - q^*)]^{T-x}$$

(5.18)

Pricing a Call Option

All in all this implies that the arbitrage free initial price of a call option in the multi step binomial model is given by the formula

$$\xi_0 = \mathbb{E}_* \left(\frac{(S_T - C)^+}{(1+r)^T} \right) = S_0 \Psi_1(T, S_0, C, q^*) - \frac{C}{(1+r)^T} \Psi_2(T, S_0, C, q^*) \tag{5.19}$$

where

$$\Psi_1(T, S_0, C, q^*) := \sum_{x=x_0}^{T} \binom{T}{x} \left[\frac{1+R_u}{1+r} q^* \right]^x \left[\frac{1+R_d}{1+r} (1-q^*) \right]^{T-x} \tag{5.20}$$

$$\Psi_2(T, S_0, C, q^*) := \sum_{x=x_0}^{T} p_T(x; q^*) \tag{5.21}$$

The expression in (5.19) is known as the **Cox-Ross-Rubenstein** formula.

Pricing in the Binomial Model

Price function for simple derivatives

Theorem (Price Function)

Within the framework of the multi-step binomial model, let ξ be a simple derivative, that is $\xi = \varphi(S_T)$. Put

$$F(t, y) := \frac{1}{(1+r)^{T-t}} \mathbb{E}_* \{\varphi(S_{T-t}^y)\}, \quad y \geq 0, \ 0 \leq t \leq T, \quad (5.22)$$

where $(S_t^y)_{0 \leq t \leq T}$ is a process satisfying that $S_0^y = y$ and

$$S_t^y = S_{t-1}^y (1 + K_S(t)), t = 1, \ldots, T. \quad (5.23)$$

Then, almost surely

$$F(t, S_t) = \frac{1}{(1+r)^{T-t}} \mathbb{E}_* (\varphi(S_T)|\mathscr{F}_t), 0 \leq t \leq T. \quad (5.24)$$

In other words the arbitrage free price of the simple derivative $\xi = \varphi(S_T)$ at time $0 \leq t \leq T$ is given by $F(t, S_t)$.

Remark: The function F is known as the price function associated to the pay-off function φ.

Interpretation: The function $F(t, y)$ can be thought as the initial price of a simple derivative with maturity time $T - t$ and pay-off function φ under the circumstances that the initial price of the risky asset is $y \geq 0$.

According to the previous theorem the price of a call option in the multi step binomial model at time $t = 0, 1, \ldots, T$ is given by

$$\xi_t = F(t, S_t) = \frac{1}{(1+r)^{T-t}} \mathbb{E}_* \left((S_T - C)^+ | \mathscr{F}_t \right) \qquad (5.25)$$

where

$$F(t, y) = \mathbb{E}_* \left[\frac{(S_{T-t}^y - C)^+}{(1+r)^{T-t}} \right], y \geq 0, 0 \leq t \leq T \qquad (5.26)$$

in which $(S_t^y)_{0 \leq t \leq T}$ is a process satisfying that $S_0^y = y$ and

$$S_t^y = S_{t-1}^y (1 + K_S(t)), t = 1, \ldots, T. \qquad (5.27)$$

The Cox-Ross-Rubunstein formula dictates that

$$F(t, S_0) = \mathbb{E}_* \left(\frac{(S_{T-t} - C)^+}{(1+r)^{T-t}} \right) \tag{5.28}$$

$$= S_0 \Psi_1(T - t, S_0, C, q^*) - \frac{C}{(1+r)^{T_0}} \Psi_2(T - t, S_0, C, q^*). \tag{5.29}$$

Replacing S_0 by y the Cox-Ross-Rubenstein formula we get that

$$F(t, y) = y\Psi_1(T - t, y, C, q^*) - \frac{C}{(1 + r)^{T-t}}\Psi_2(T - t, y, C, q^*).$$
(5.30)

Hence the price call option in the multi-step binomial model is

$$\xi_t = F(t, S_t) = S_t\Psi_1(T - t, S_t, C, q^*) - \frac{C}{(1 + r)^{T-t}}\Psi_2(T - t, S_t, C, q^*).$$
(5.31)

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The Second Fundamental Theorem of Asset Pricing

The Theorem

Theorem (Second Fundamental Theorem of Asset Pricing)

Let the market

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P}), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)} \right)_{0 \le t \le T} \right\}.$$

be as described. Then $\mathfrak{M}$ is complete and arbitrage free if and only if there is a unique martingale measure.

Definition (Martingale)

The collection $\{(X_t, \mathscr{F}_t) : t = 0, 1, \ldots, T\}$ is said to be a martingale if

1.

2.

3.

The Second Fundamental Theorem of Asset Pricing
Definitions

Definition (Martingale Measure)

Let $\mathbb{Q}$ be probability measure on $(\Omega, \mathscr{F})$, that is there are $0 < q_i^* < 1, i = 1, \ldots, N$ such that $\sum_{i=1}^{N} q_i^* = 1$ and

$$\mathbb{Q}(\{\omega_i\}) = q_i^* \quad \text{for } i = 1, \ldots, N. \tag{6.1}$$

We will say that $\mathbb{Q}$ is a martingale measure if under $\mathbb{Q}$ the discounted price $(\tilde{P}_j^{(j)})_{0 \leq t \leq T}$ is a martingale w.r.t $(\mathscr{F}_t)_{0 \leq t \leq T}$, for every $j = 1, 2, \ldots, d + 1$.

The Second Fundamental Theorem of Asset Pricing
Definitions

Definition (Admissible Strategy)

Let $\Theta = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)} \right)_{0 \leq t \leq T}$ be strategy on the market

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F})_{0 \leq t \leq T}, \mathbb{P} \right), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)} \right)_{0 \leq t \leq T} \right\}. \tag{6.2}$$

I will say that Θ is admissible if:

The Second Fundamental Theorem of Asset Pricing

Definitions

1. **It is self-financed**: It only requires initial capital. I will say that Θ is self financed if

$$V_t^\Theta = \varphi_{t+1} B_t + \sum_{j=1}^{d} \theta_{t+1}^{(j)} S_t^{(j)}, \quad 0 \leq t \leq T-1. \quad (6.3)$$

2. **Non-anticipative**: It is build up only on current information. I will say that Θ is non-anticipative or predictable if Θ_{t+1} depends only on the information at time t, that is Θ_{t+1} is $\mathscr{F}_t$-measurable.

3. **It has a limited credit line**: There is a non-random constant $C > 0$, such that

$$V_t^\Theta \geq -C, \ \forall\, 0 \leq t \leq T. \quad (6.4)$$

The Second Fundamental Theorem of Asset Pricing
Definitions

Definition (Arbitrage)

Consider the market

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P} \right), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)} \right)_{0 \le t \le T} \right\}.$$
(6.5)

An admissible strategy $\Theta = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)} \right)_{0 \le t \le T}$ is said to be an arbitrage if:

The Second Fundamental Theorem of Asset Pricing

Definitions

1. It has zero initial capital:

$$V_0^\Theta = \varphi_0 B_0 + \sum_{j=1}^{d} \theta_0^{(j)} \cdot S_0^{(j)} = 0. \qquad (6.6)$$

2. At time $t = T$, i am out of debts with 100 % certainty:
Almost surely

$$V_T^\Theta = \varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} \geq 0. \qquad (6.7)$$

3. I have a chance to make a profit:

$$\mathbb{P}\left(\varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} > 0\right) > 0. \qquad (6.8)$$

The Second Fundamental Theorem of Asset Pricing

Definitions

Definition (Complete Financial Market)

The market is said to be complete if every European contingent claim ξ (i.e. a random variable that depends on the information up to time T) can be replicated, that is, there exists an admissible strategy

$$\Theta = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)} \right)_{0 \leq t \leq T} \tag{6.9}$$

such that almost surely

$$\xi = V_T^\Theta = \varphi B_T + \sum_{j=1}^{d} \theta_T^{(j)} S_T^{(j)}. \tag{6.10}$$

This means that, if the market is arbitrage free and $\mathbb{P}\left(\xi > 0 \right) > 0$, then necessarily

$$V_0^\Theta > 0. \tag{6.11}$$

Let ϕ be the pay-off function of a simple derivative. We want to find an admissible strategy $\Theta = (\varphi_t, \theta_t)_{0 \leq t \leq T}$ such that

$$\xi = \phi(S_T) = V_T^\Theta = \varphi_T B_T + \theta_T S_T.$$

The Second Fundamental Theorem of Asset Pricing
Completeness

▶ When we have $T = 1$ we have seen that
$\phi(S_1) = V_1^\Theta = \varphi B_1 + \theta S_1$ holds if and only if

$$\zeta^u := \phi[S_0(1 + R_u)] = \varphi B_1 + \theta S_0(1 + R_u)$$
$$\zeta^d := \phi[S_0(1 + R_d)] = \varphi B_1 + \theta S_0(1 + R_d)$$
$$\Leftrightarrow \begin{bmatrix} B_1 & S_0(1 + R_u) \\ B_1 & S_0(1 + R_d) \end{bmatrix} \begin{bmatrix} \varphi \\ \theta \end{bmatrix} = \begin{bmatrix} \zeta^u \\ \zeta^d \end{bmatrix}$$

▶ This system has a unique solution given by

$$\theta = \frac{\zeta^u - \zeta^d}{S_0(R_u - R_d)}, \varphi = \frac{1}{B_1} \left\{ \frac{\zeta^d(1 + R_u) - \zeta^u(1 + R_d)}{R_u - R_d} \right\}.$$

The Second Fundamental Theorem of Asset Pricing

Completeness

▶ Suppose that we have found the replicating strategy up to time $t = T - 1$, i.e. $(\varphi_t, \theta_t)_{0 \leq t \leq T-1}$ is given . Thus it is left to find (φ_T, θ_T) such that

$$\phi(S_T) = V_T^\Theta = \varphi_T B_T + \theta_T S_T \qquad (6.12)$$

▶ Using that $S_T = S_{T-1}(1 + K_S(T))$, we have that (6.12) holds if and only if

$$\zeta_1^u(S_{T-1}) = \varphi_T B_T + \theta_T S_{T-1}(1 + R_u)$$
$$\zeta_1^d(S_{T-1}) = \varphi_T B_T + \theta_T S_{T-1}(1 + R_d)$$

where

$$\zeta_1^u(y) := \phi[y(1 + R_u)], \zeta_1^d(y) := \phi[y(1 + R_d)].$$

The Second Fundamental Theorem of Asset Pricing

Completeness

▶ In matrix notation:

$$\begin{bmatrix} B_T & S_{T-1}(1+R_u) \\ B_T & S_{T-1}(1+R_d) \end{bmatrix} \begin{bmatrix} \varphi_T \\ \theta_T \end{bmatrix} = \begin{bmatrix} \zeta_1^u(S_{T-1}) \\ \zeta_1^d(S_{T-1}) \end{bmatrix}.$$

The Second Fundamental Theorem of Asset Pricing

Completeness

$$\theta = \frac{\zeta^u - \zeta^d}{S_0(R_u - R_d)}, \quad \varphi = \frac{1}{B_1}\left\{ \frac{\zeta^d(1+R_u) - \zeta^u(1+R_d)}{R_u - R_d} \right\}$$

▶ Replacing S_0, ζ^u, ζ^d and B_1 by $S_{T-1}, \zeta_1^u(S_{T-1}), \zeta_1^d(S_{T-1})$ and B_T respectively in the one step case we get that

$$\theta_T = \frac{\zeta_1^u(S_{T-1}) - \zeta_1^d(S_{T-1})}{S_{T-1}(R_u - R_d)}$$

$$\varphi_T = \frac{1}{B_T}\left\{ \frac{\zeta_1^d(S_{T-1})(1 + R_u) - \zeta_1^u(S_{T-1})(1 + R_d)}{R_u - R_d} \right\}.$$

▶ In a similar way we obtain

$$\varphi_T B_{T-1} + \theta_T S_{T-1} = \frac{1}{1+r}\left[\zeta_1^u(S_{T-1})q^* + \zeta_1^d(S_{T-1})(1 - q^*) \right]$$

▶ (φ_T, θ_T) are functions exclusively of S_{T-1} and do not rely on $(\varphi_t, \theta_t)_{0 \le t \le T-1}$.

The Second Fundamental Theorem of Asset Pricing

Completeness

- ▶ I have now found one portfolio, but to show completeness we need the whole strategy
- ▶ Using similar arguments as well as the self-financed property, one can find the remaining portfolios to form the complete strategy

The Second Fundamental Theorem of Asset Pricing

Completeness

- ▶ Binomial model is complete, however this is not true for the Trinomial model