Financial Engineering
Lecture 7

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Outline

- Martingale measures and Risk Neutral Measures.
- First Fundamental Theorem of Asset Pricing: Statement, consequences and its proof.

First Fundamental Theorem of Asset Pricing

Set-Up

- The sample space is **finite**:

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- We only need to specify the probabilities for each ω_i: There are $0 < p_i < 1$, $i = 1, \dots, N$ such that $\sum_{i=1}^N p_i = 1$ and

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- Thus, if $A \in \mathscr{F}$, then there is $I \subseteq \{1, \ldots, N\}$ such that

$$A = \cup_{i \in I} \{\omega_i\} \Longrightarrow \mathbb{P}(A) = \sum_{i \in I} p_i \ \text{(See Lecture 1).}$$

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- Note that in this framework, any random variable or vector ξ in $(\Omega, \mathscr{F}, \mathbb{P})$ has only N possible outcomes, namely

$$\xi(\omega_i), \ \ i = 1, \ldots, N \Longrightarrow \text{Always integrable!}$$

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- This means that our price process

$$P_t = (B_t, S_t^{(1)} \ldots, S_t^{(d)}),$$

can take **up to** N values.

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but the stock price has only two possible outcomes.

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- Recall that in this model returns are i.i.d. and satisfy that

$$K_S(t) = \begin{cases} R_u & \text{with probability } p; \\ R_d & \text{with probability } 1-p. \end{cases}$$

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- Thus,

$$\mathbb{P}(\{udu\}) = \mathbb{P}(K_S(1) = R_u, K_S(2) = R_d, K_S(3) = R_u) = p(1-p)p.$$

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- The market with information is given:

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P}), P = (B_t, S_t^{(1)} \ldots, S_t^{(d)})_{0 \le t \le T} \right\}. \quad (1)$$

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- A typical example corresponds to the price of a zero-coupon bond which pays a yearly interest rate $r > 0$, i.e.

$$B_t = (1 + r)^t, \ \forall 0 \le t \le T.$$

Admissible Strategies

Definition (Admissible Strategies)

Let $\Theta = (\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)})_{0\le t\le T}$ be a strategy on the market $\mathfrak{M}$. We will say that Θ is **admissible** if:

1. It is self-financed: It only requires an initial capital. In symbols

$$\varphi_t B_t + \sum_{j=1}^{d} \theta_t^{(j)} S_t^{(j)} =: V_t^{\Theta} = \varphi_{t+1} B_t + \sum_{j=1}^{d} \theta_{t+1}^{(j)} S_t^{(j)}, \; \forall\, 0 \le t \le T-1.$$

2. Non-anticipative: It is build up only on current market information, i.e. Θ_{t+1}, the portfolio created at time t, is $\mathscr{F}_t$-measurable.
3. It has a limited credit line: There is a non-random constant $C > 0$, such that

$$V_t^{\Theta} \ge -C, \; \forall\, 0 \le t \le T.$$

Arbitrage

Definition (Arbitrage)

An arbitrage on $\mathfrak{M}$ **is an admissible strategy** $\Theta = (\varphi_t, \theta_t^{(1)}, \dots, \theta_t^{(d)})_{0 \le t \le T}$ such that

1. It has zero initial capital:

$$V_0^{\Theta} = \varphi_0 B_0 + \sum_{j=1}^{d} \theta_0^{(j)} \cdot S_0^{(j)} = 0.$$

2. At time $t = T$, we are out of debts with 100% certainty: Almost surely

$$V_T^{\Theta} = \varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} \ge 0.$$

3. We have a chance to make a profit:

$$\mathbb{P}\left(\varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} > 0\right) > 0.$$

The Main Assumptions

Assumption (A.3)

The market $\mathfrak{M}$ given by (1) does not admit arbitrage opportunities.

Risk-Neutral and Martingale Measures

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$$\mathbb{E}_*(K_S(1)) := R_u q^* + R_d(1 - q^*) = r.$$

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- Such a $0 < q^* < 1$ exists (and it is unique) provided that $R_d < r < R_u$. In such situation

$$q^* = \frac{R_u - r}{R_u - R_d}.$$

Risk-Neutral and Martingale Measures

- Moreover, in the previous lecture we checked that in the Multi-Step Binomial model if we let the returns of S be i.i.d. with

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- Then the discounted price

$$\tilde{S}_t = \frac{S_t}{(1+r)^t}, \quad 0 \leq t \leq T,$$

is a martingale on the filtration

$$\mathscr{F}_0 = \{\emptyset, \Omega\}, \quad \mathscr{F}_t = \sigma\left(K_S(1), \ldots, K_S(t)\right), \quad 1 \leq t \leq T.$$

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     into a risk-neutral world in which the "new" distribution for the returns

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- In this model these two situations are equivalent.

Discounted prices

- Let

$$\mathfrak{M} = \left\{(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leq t\leq T}, \mathbb{P}), P = (B_t, S_t^{(1)} \ldots, S_t^{(d)})_{0\leq t\leq T}\right\}$$

  be as in (1).
- Given a process X, we will define and denote the discounted version of X as

$$\tilde{X}_t := \frac{X_t}{B_t}, \;\; 0 \leq t \leq T.$$

- This notation will be mainly applied to price processes as well as wealth processes.

Martingale Measures

Definition (Martingale Measures)

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be the market described in (1). Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathscr{F})$, that is, there are $0 < q_i^* < 1$, $i = 1, \ldots, N$ such that $\sum_{i=1}^{N} q_i^* = 1$ and

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We will say that $\mathbb{Q}$ is **a martingale measure** if under $\mathbb{Q}$ **the discounted price** $(\tilde{P}_t^{(j)})_{0 \le t \le T}$ **is a martingale** w.r.t. $(\mathscr{F}_t)_{0 \le t \le T}$, for every $j = 1, 2, \ldots, d+1$.

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Remark: $\tilde{P}_t^{(1)} = \tilde{B}_t = 1$, so it is always a martingale under any filtered probability space.

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Notation: Expectations and conditional expectations w.r.t. $\mathbb{Q}$ are gonna be denoted as $\mathbb{E}_*$ and $\mathbb{E}_*(\,|\,\mathscr{H})$, respectively.

Martingale Measures are Risk-Neutral

- By the martingale property, for all $j = 1, 2, \ldots, d$

$$\mathbb{E}_* \left(\frac{S_{t+1}^{(j)}}{\underbrace{B_{t+1}}_{\text{Determ.}}} \middle| \mathscr{F}_t \right) = \frac{\overbrace{S_t^{(j)}}^{\mathscr{F}_t - meas.}}{B_t} \Leftrightarrow \mathbb{E}_* \left(\frac{S_{t+1}^{(j)}}{S_t^{(j)}} \middle| \mathscr{F}_t \right) = \frac{B_{t+1}}{B_t}.$$

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- Thus,

$$\mathbb{E}_* \left(\underbrace{\frac{S_{t+1}^{(j)}}{S_t^{(j)}} - 1}_{K_S(t+1)} \middle| \mathscr{F}_t \right) = \underbrace{\frac{B_{t+1}}{B_t} - 1}_{K_B(t+1)}.$$

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- Taking expectations in the previous equation, we get that

$$\mathbb{E}_*(K_S(t+1)) = K_B(t+1), \; 0 \leq t \leq T-1.$$

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- Also conditional expectations will take different values in $\mathbb{P}$ compared to $\mathbb{Q}$.
- In the exercise set you will encounter some examples of such situations.

The First Fundamental Theorem of Asset Pricing

Theorem (FFTAP)

Let

$$\mathfrak{M} = \left\{(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P}), P = (B_t, S_t^{(1)} \ldots, S_t^{(d)})_{0 \leq t \leq T}\right\}$$

be the market described in (1). $\mathfrak{M}$ *is arbitrage free if and only if there is a martingale measure.*

Corollary (Pricing Financial Derivatives)

Let $\mathfrak{M}$ be

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P}), P = (B_t, S_t^{(1)} \dots, S_t^{(d)})_{0 \le t \le T} \right\}$$

be the market described in (1) and suppose that there exists a martingale measure $\mathbb{Q}$. Let $\xi \ge 0$ be the pay-off of a financial derivative with delivery time $T > 0$. If we put

$$\xi_t := \mathbb{E}_* \left(\frac{B_t}{B_T} \xi \,\middle|\, \mathscr{F}_t \right), \quad 0 \le t \le T,$$

then the extended market

$$\tilde{\mathfrak{M}} = \left\{ (\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P}), \tilde{P} = (B_t, S_t^{(1)} \dots, S_t^{(d)}, \xi_t)_{0 \le t \le T} \right\},$$

is arbitrage free.

Proof: See the Exercise Set 7. ■

Proof of FFTAP: Lemmas

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Lemma (Lemma 1)

Let $\Theta = (\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)})_{0 \leq t \leq T}$ be a self-financed strategy. Then

$$\tilde{V}_t^{\Theta} = V_0^{\Theta} + \sum_{u=1}^{t} \sum_{j=1}^{d} \theta_u^{(j)} \Delta \tilde{S}_u^{(j)}, \quad 1 \leq t \leq T,$$

where $\Delta \tilde{S}_t^{(j)} := \tilde{S}_t^{(j)} - \tilde{S}_{t-1}^{(j)}$. In particular, if Θ is admissible and $\mathbb{Q}$ is a martingale measure, then $(\tilde{V}_t^{\Theta}, \mathscr{F}_t)_{0 \leq t \leq T}$ is a martingale under $\mathbb{Q}$.

Proof: In the "digital blackboard". ∎

Proof of FFTAP: Lemmas

Lemma (Lemma 2)

Let $(X_t, \mathscr{F}_t)_{0 \le t \le T}$ *be a martingale. Then*

$$\mathbb{E}(X_t) = \mathbb{E}(X_0), \ \forall\, 0 \le t \le T.$$

Proof: In the “digital blackboard”. ■

Proof of FFTAP: Lemmas

Lemma (Lemma 3)

If $\xi \geq 0$ *is a random variable and* $\mathbb{E}(\xi) = 0$*, then* $\xi \equiv 0$ *almost surely.*

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$$V_T^{\Theta} = \varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} \ge 0.$$
3. We have a chance to make a profit:
$$\mathbb{P}\left(\varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} > 0\right) > 0.$$

Proof of FFTAP ($\Leftarrow$): $\exists$ MM $\Rightarrow$ NA

The previous lemmas imply:

- By Lemma 1 the discounted wealth process $(\tilde{V}_t^{\Theta}, \mathscr{F}_t)_{0\le t\le T}$ is $\mathbb{Q}$-martingale such that

$$\frac{V_T^{\Theta}}{B_T} = \tilde{V}_T^{\Theta} \ge 0, \quad V_0^{\Theta} = 0.$$

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- Lemma 3 now implies that $V_T^{\Theta} = 0$ $\mathbb{Q}$-almost surely. We will use this to show that

$$V_T^{\Theta}(\omega_i) = 0, \quad i = 1, \ldots, N.$$

Proof of FFTAP ($\Leftarrow$): $\exists$ MM $\Rightarrow$ NA

- Recall that if $A \in \mathscr{F}$, then there is $I \subseteq \{1, \dots, N\}$ such that

$$A = \cup_{i \in I} \{\omega_i\} \Longrightarrow \mathbb{Q}(A) = \sum_{i \in I} q_i^*, \;\; 0 < q_i^* < 1.$$

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- Since V_T^Θ is a random variable there is $I \subseteq \{1, \ldots, N\}$ such that

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$$0 = \mathbb{Q}(V_T^{\Theta} > 0) = \sum_{i \in I} q_i^* > 0.$$

- Therefore $\{\omega \in \Omega : V_T^{\Theta}(\omega) > 0\} = \emptyset$, which contradicts the fact that

$$\mathbb{P}\left(V_T^{\Theta} = \varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} > 0\right) > 0.$$

Proof of FFTAP: More lemmas

For the other direction of the proof (which is the hardest conceptually speaking) we need more lemmas.

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Lemma (Lemma 4)

The following sets are vector spaces

$$\mathcal{V} := \left\{ V_T^{\Theta} \mid \Theta = (\varphi_t, \theta_t^{(1)}, \dots, \theta_t^{(d)})_{0 \le t \le T} \text{ is admissible and } V_0^{\Theta} = 0 \right\}.$$

$$\mathcal{X} := \left\{ (V(\omega_1), V(\omega_2), \dots, V(\omega_N)) \in \mathbb{R}^N \mid V \in \mathcal{V} \right\}.$$

Moreover, if no arbitrage is allowed, then $\mathcal{X} \cap \mathcal{Y} = \emptyset$ *where*

$$\mathcal{Y} = \left\{ (V(\omega_1), V(\omega_2), \dots, V(\omega_N)) \in \mathbb{R}^N \mid V \text{ r.v., } V \ge 0, \mathbb{P}(V > 0) > 0 \right\}.$$

Proof: See the Exercise Set 7. ■

Proof of FFTAP: More lemmas

Theorem (The Separating Hyperplane Theorem)

Let $A \subseteq \mathbb{R}^N$ be convex and compact and $\mathcal{X}$ a vector subspace of $\mathbb{R}^n$. If $A \cap \mathcal{X} = \emptyset$ then there exists $y \in \mathbb{R}^N$ such that:

1. *For all $a \in A$*
$$\sum_{i=1}^{N} y_i a_i > 0.$$
2. *For all $x \in \mathcal{X}$*
$$\sum_{i=1}^{N} y_i x_i = 0.$$

Proof: See Lemma 7.22 in Capinski's book. ■

Proof of FFTAP ($\Rightarrow$): NA $\Rightarrow \exists$ MM

Proof ($\Rightarrow$): We will only check the case $T = 1$. Suppose that there is no arbitrage.

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- By Lemma 4

$$A := \left\{ (q_1, \ldots, q_N) \in \mathcal{Y} \mid 0 \leq q_i \leq 1, \ \sum_{i=1}^{N} q_i = 1 \right\} \Rightarrow A \cap \mathcal{X} = \emptyset.$$

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- Moreover:

1. A is compact: This is obvious because A is closed and bounded.

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- Moreover:

1. *A is compact:* This is obvious because A is closed and bounded.
2. *A is convex:* Proof in In the "digital blackboard".

Proof of FFTAP ($\Rightarrow$): NA $\Rightarrow \exists$ MM

- By The Separating Hyperplane Theorem, there exists $y \in \mathbb{R}^N$ such that:
  1. For all $(q_1, \ldots, q_N) \in \mathcal{Y}$ with $0 \leq q_i \leq 1$ and $\sum_{i=1}^N q_i = 1$
  $$\sum_{i=1}^N y_i q_i > 0.$$
  2. For all $\Theta = (\varphi, \theta^{(1)}, \ldots, \theta^{(d)})_{0 \leq t \leq T}$ portfolios with $V_0^\Theta = 0$
  $$\sum_{i=1}^N y_i V_1^\Theta(\omega_i) = 0, \quad \left(\left(V_1^\Theta(\omega_1), V_1^\Theta(\omega_2), \ldots, V_1^\Theta(\omega_N)\right) \in \mathcal{X}\right)$$

Proof of FFTAP ($\Rightarrow$): NA $\Rightarrow \exists$ MM

- Define
$$q_i^* = \frac{y_i}{\sum_{i=1}^N y_i}, \quad i = 1, \ldots, N,$$
and
$$\mathbb{Q}(\{\omega_i\}) = q_i^*, \quad i = 1, \ldots, N.$$

Proof of FFTAP ($\Rightarrow$): NA $\Rightarrow \exists$ MM

- Define
$$q_i^* = \frac{y_i}{\sum_{i=1}^N y_i}, \quad i = 1, \ldots, N,$$
and
$$\mathbb{Q}\left(\{\omega_i\}\right) = q_i^*, \quad i = 1, \ldots, N.$$

- To conclude the proof we need to show that $\mathbb{Q}$ is a martingale measure, i.e.
$$0 < q_i^* < 1$$
and
$$\mathbb{E}_*\left(\frac{S_1^{(j)}}{B_1}\middle| \mathscr{F}_0 = \{\emptyset, \Omega\}\right) = \frac{S_0^{(j)}}{B_0} = S_0^{(j)}, \; j = 1, \ldots, d.$$

Proof of FFTAP ($\Rightarrow$): NA $\Rightarrow \exists$ MM

- Since for all $i = 1, \ldots, N$

$$v = (\underbrace{0, 0, \ldots, 0}_{i-1}, 1, 0, \ldots, 0) \in A,$$

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$$0 < \sum_{k=1}^{N} y_k v_k = y_i, \quad i = 1, \ldots, N.$$

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- We conclude that

$$0 < \sum_{k=1}^{N} y_k v_k = y_i, \quad i = 1, \ldots, N.$$

- Which trivially implies that

$$0 < q_i^* = \frac{y_i}{\sum_{i=1}^{N} y_i} < 1.$$

Proof of FFTAP ($\Rightarrow$): NA $\Rightarrow \exists$ MM

- Recall that by the Separating Hyperplane Theorem, for any portfolio $(\varphi, \theta^{(1)}, \dots, \theta^{(d)}) \in \mathbb{R}^{d+1}$ satisfying that

$$V_0^{\Theta} = 0, \quad V_1^{\Theta} \geq 0,$$

we have that

$$\begin{aligned}
\mathbb{E}_* \left(V_1^{\Theta} \right) &= \sum_{i=1}^{N} q_i^* V_T^{\Theta}(\omega_i) \\
&= \frac{1}{\sum_{i=1}^{N} y_i} \sum_{i=1}^{N} y_i V_T^{\Theta}(\omega_i) = 0.
\end{aligned}$$

Proof of FFTAP ($\Rightarrow$): NA $\Rightarrow \exists$ MM

- Since B_1 is deterministic we have that from Lemma 1, for these type of portfolios

$$0 = \mathbb{E}_* \left(\frac{V_1^{\Theta}}{B_1} \right) = \mathbb{E}_* \left(V_0^{\Theta} + \sum_{j=1}^{d} \theta^{(j)} \left[\tilde{S}_1^{(j)} - \tilde{S}_0^{(j)} \right] \right)$$

$$= \mathbb{E}_* \left(\sum_{j=1}^{d} \theta^{(j)} \left[\tilde{S}_1^{(j)} - \tilde{S}_0^{(j)} \right] \right).$$

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$$= \mathbb{E}_* \left(\sum_{j=1}^{d} \theta^{(j)} \left[\tilde{S}_1^{(j)} - \tilde{S}_0^{(j)} \right] \right).$$

- In particular for the portfolio

$$\varphi = -S_0^{(j)}, \ \theta^{(j)} = \begin{cases} 1 & \text{if } j = i, \\ 0 & otherwise, \end{cases}$$

we have that

$$0 = \mathbb{E}_* \left(\sum_{j=1}^{d} \theta^{(j)} \left[\tilde{S}_1^{(j)} - \tilde{S}_0^{(j)} \right] \right) = \mathbb{E}_* \left(\left[\tilde{S}_1^{(i)} - \tilde{S}_0^{(i)} \right] \right).$$