Financial Engineering
Lecture 1

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AAU
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General comments

- Mit Dansk er ok ... nok.... men jeg er ikke flydende i det... so I will give the lectures in English.
- Imidlertid, hvis i snakker laaaangsoooom til mig, må i godt formulere jeres spørgsmål på dansk.
- My email is osauri@math.aau.dk and my office number is 5.315.
- I have planned 10 lectures and 1 selfstudy session.
- However, if needed, an extra selfstudy will be programmed.

Structure of the lectures

- As usual, the sessions will be divided between a lecture and an exercise part.
- If needed, a "lecturer+students" exercise session may be arranged.
- Slides will be available before the lectures.
- However, many details will be addressed in the blackboard.
- Along the semester, we will have 3 anonymous (5 min) mini quizzes before the lecture.
- It will not count for your final grade.

Literature

- Most of the lectures will be based on:

  "Mathematics for Finance: An Introduction to Financial Engineering",

  by Capiński, M. and Zastawniak, T.
- It is not mandatory to get this book.
- The structure and some of the content between the first and the second edition differ.
- I will follow the organization of the second edition.

Literature

- Furthermore, the following books will be used as complementary material:
  - "The Mathematics of Arbitrage" by F. Delbaen and W. Schachermayer.
  - "Stochastic Calculus for Finance I: The Binomial Asset Pricing Model" by Steven E. Shreve.

Evaluation

- You will be evaluated by a pass/fail oral exam.
- The exam will have a total duration of 40 min per student.
- 20 minutes preparation of one randomly chosen topic.
- The second part is divided as: 10 minutes presentation + 10 minutes round of questions.
- I will let you know in advance the topics that will be assessed.

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  - Investigating some mathematical techniques used in risk management.
  - Providing the basis for the computational methods used in modern financial analysis.
- The course can be seen as the mathematical foundation as well as an extension of the elective course "Finansielle Markeder".

Content of the course

- Risky and Risk-free financial assets.
- Portfolio Management and Risk measures.
- The Binomial and GARCH models.
- Fundamental Theorems of Asset Pricing.
- Pricing Financial Derivatives via Arbitrage principle.
- Hedging with Financial Derivatives.
- American Options.
- Interest Rate Markets (If we have time).

Outline for today

- Elementary probabilistic concepts.
- Basic Market Assumptions.
- Interests and interest rate.
- Bonds and the value of money over time.

Preliminaries

Notation and basic definitions

- One of our main goals in this course is to model the price of financial securities, e.g. stocks and bonds.
- We start with the premise that we cannot predict with absolute certainty the price movements of every financial asset.
- But if you can, I recommend you to start using it NOW and don't tell anybody!
- Hence, the dynamics (properties) of prices will be formulated from a probabilistic point of view.

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- $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a set function satisfying that $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$ and if $(A_n)_{n\geq 1} \subseteq \mathcal{F}$ are pairwise disjoints, i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$, then

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- We will referred to $\mathbb{P}$ as a probability measure.

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- One typically refers to $\mathcal{F}$ as a σ-algebra.
- In your course of integration theory you will study in more detail the concepts of σ-algebras and general measurable spaces.

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  2. $\mathcal{F} = 2^\Omega$, where 2^Ω denotes the power set: The collection of all subsets of Ω.
  3. For $i = 1, \ldots, N$, there are $0 \leq p_i \leq 1$ such that $\sum_{i=1}^N p_i = 1$ and

$$\mathbb{P}(\{\omega_i\}) = p_i.$$

Notation and basic definitions

- Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a mapping $X : \Omega \to \mathbb{R}$ is a random variable if for every $x \in \mathbb{R}$,

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- Note that in this situation, $F_X(x) := \mathbb{P}(X \leq x)$ makes sense.
- Let $d \in \mathbb{N}$. A random vector on $\mathbb{R}^d$ is a d-dimensional vector whose components are random variables: $(X_1, X_2, \ldots, X_d)$ is a random vector if and only if X_i is a random variable for $i = 1, 2, \ldots, d$.

Examples

- Bernoulli.
- Binomial.
- Normal distribution.

Notation and basic definitions

- Let $\mathbb{T} \neq \emptyset$ be an arbitrary index set.
- The collection $X = (X_t)_{t \in \mathbb{T}}$ is a real-valued ($\mathbb{R}^d$-valued) stochastic process if X_t is a random variable (random vector) for very $t \in \mathbb{T}$.
- The set $\mathbb{T}$ usually denotes points in time.
- The core examples are $[0, +\infty)$ and $\mathbb{N} \cup \{0\}$.

Time set-up

- Discrete structure of the time: Financial markets (agents) trade only at determined points in time with a final horizon $T > 0$:

$$0 = t_0 < t_1 < \cdots < t_n := T.$$

- t_i = Time of the ith trading period.
- This means that our time variable, say t, takes values in

$$\{t_0 < t_1 < \cdots < t_n\}.$$

Time set-up

- For instance, suppose that we are interested on describing the daily behavior of a given market during a year.
- If our unit of time is a day, then we may write

$$t \in \{0, 1, 2, \ldots, 365\}. \;\; (\text{or} \;\; t = 0, 1, 2, \ldots, 365)$$

- In contrast, if we require the unit of time being a year, we instead have that

$$t \in \{0, 1/365, 2/365, \ldots, 1\}. \;\; (\text{or} \;\; t = 0, 1/365, 2/365, \ldots, 1)$$

Time set-up

- More generally, if we want to describe $n \in \mathbb{N}$ trading periods, each of them taking place at a distance $h > 0$ (years, months, days, hours, minutes, etc).
- Then, we may let $T = n$ and

$$t \in \{0, 1, 2, \ldots, T\},$$

or

$$t \in \{0, h, 2h, \ldots, T\}, \quad T := hn.$$

- We will mainly use the first type and write $0 \leq t \leq T$ instead of $t \in \{0, 1, 2, \ldots, T\}$.
- The book typically uses years as unit of time.

Market Assumptions

What shall we ignore?...for the moment

- In order to develop a concise and systematic mathematical theory for financial markets we must impose certain restrictions in the way agents and prices behave.
- Must of the assumptions that we will discuss here can be easily extended in a latter stage.

Dividends

- The market value of a company is determined by the number of shares available in the market and its price.
- Thus, an owner of a share is entitled to a portion of the company's profit.
- Such a profit is known as a dividend and it is paid periodically.
- Nevertheless, we will assume that there are no dividends.

Transaction costs

- In reality, agents buy or sell assets through an intermediary.
- The broker will charge a fee which is proportional to the size of the transaction.
- However, such a fee is often relatively small.
- Hence, we will neglect transaction costs.
- When such a situation occurs, we say that the market is frictionless.

Short-sellings

- Taking a short (long) position on an asset means selling (buying) it.
- A short-selling refers to the situation when the investor borrows an asset and sells it.
- However, the rightful owner of the stock keeps all the rights to it and can reclaim the asset back at any time.
- For instance borrowing money from a bank is considered short-selling.
- Brokers would typically put constraints on the amount of shares an agent can borrow.
- Nonetheless, we will assume that there are no short-selling restrictions.

Divisibility and liquidity

- In reality, the minimum amount of shares that a trader can buy is one.
- This means that we can't buy half of a share.
- However, mathematically this is not a problem: If a stock costs $S > 0$ then buying one of it is equivalent to buy half share of an asset that costs $2S$.
- Hence, we may and do assume that financial assets are infinite divisible.
- Liquidity: Moreover, there is no restriction on the number of assets that an agent can buy or sell.

Basic Market Assumptions

Assumption (A.1)

The market is frictionless and shares do not pay dividends. Moreover, an investor can buy, sell and hold any number $x \in \mathbb{R}$ of financial assets.

Risk-free assets: The value of the money through time

Introduction

- As mentioned previously, the type of markets we are going to consider are those that consist of stocks and bonds.
- A bond is a financial security (piece of paper) that pays the owner a chain of predetermined payments.
- In this way, it can be seen as an asset that has no risk: We are 100% certain of what our profit/loss will be.
- The concept of risk-free rate is key to describe this type of financial assets.

Wealth process

- For most of the course we will be interested on analyzing the investment of a financial agent.
- The evolution over time of such investment will be called the wealth process and will be denoted by

$$V = (V_t)_{0 \leq t \leq T}.$$

- V_0 will be termed as the initial capital or initial investment.
- Our basic assumption is that V is a real-valued stochastic process on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

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- Such a premium is described by an interest rate, which in our context will be represented by a positive number $r > 0$.
- We are mainly going to consider three forms of interest: simple, compounded and continuously compounded.

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- This is indeed a stochastic process!
- This type of investment will be implemented by a passive agent: Nothing happens in between.

Be active!

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- Is this *strategy* better?...see the exercise set :).

Compounded interest

- We will say that an interest on $v_0 \in \mathbb{R}$ is **compounded** over $m \in \mathbb{N}$ periods at rate $r > 0$, if it pays

$$V_t = \left(1 + \frac{r}{m}\right)^{mt} v_0, \quad 0 \leq t \leq T.$$

- When, $m = 1, 12, 365$ we will refer to such interests as annually, monthly and daily compounded, respectively.

Compounded interest

- We have the following properties: For all $0 \leq t \leq T$

$$i) \;\; V_{t+1} = \left(1 + \frac{r}{m}\right)^{m} V_t$$

$$ii) \;\; \text{If } m_1 > m_2, v_0 > 0 \Rightarrow \left(1 + \frac{r}{m_1}\right)^{tm_1} v_0 > \left(1 + \frac{r}{m_2}\right)^{tm_2} v_0.$$

$$iii) \;\; \text{If } m_1 > m_2, v_0 < 0 \Rightarrow \left(1 + \frac{r}{m_1}\right)^{tm_1} v_0 < \left(1 + \frac{r}{m_2}\right)^{tm_2} v_0.$$

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- Compounded interest generates larger size of payments:
    - As investor, its more attractive cause it pays more.
    - As debtor, it is less convenient, you pay more interests on your debt.

Continuously compounded interest

- Note now that

$$V_t = \left[\left(1 + \frac{r}{m} \right)^{\frac{m}{r}} \right]^{rt} v_0, \;\; 0 \leq t \leq T.$$

- Since

$$\lim_{x \to \infty} \left(1 + \frac{1}{x} \right)^{x} = e.$$

- Letting $x = m/r$, we get for every $0 \leq t \leq T$ that

$$\left[\left(1 + \frac{r}{m} \right)^{\frac{m}{r}} \right]^{rt} v_0 \to (e)^{rt} v_0, \;\; \text{as } m \to \infty.$$

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- We get that

$$\underbrace{V_{t+1/m}^m - V_t^m}_{\approx dV_t} = V_t r(1/m) = rV_t \underbrace{(t + 1/m - t)}_{\approx dt}.$$

How valuable 1 DKK will be after a year?

- At a simple interest:

$$(1 + r) \text{ DKK}.$$

- When the interest is compounded over m periods:

$$(1 + \frac{r}{m})^m \text{ DKK}.$$

- Continuously compounded interest:

$$e^r \text{ DKK}.$$

- We have the relation

$$(1 + r) \leq (1 + \frac{r}{m})^m < e^r.$$

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- The pay-off F is called the face value and T the maturity time.
- How much will you be willing to pay/sell for such a financial asset?
- Well, it depends in which way we measure the time value of the money.

Pricing Zero-Coupon Bonds

- Let $B_0 \geq 0$ be the value of the zero-coupon bond with face value $F > 0$ and maturity time $T > 0$.

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- In the exercise set of today you must show that this formula changes depending on the type of interest that is paid.