Financial Engineering
Lecture 10

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American Options

American Options

- Recall that European options give to the owner the right, but not the obligation, for buying or selling a security at **a fixed time** $T > 0$ (known as the maturity date) at some fixed price $\mathcal{K} > 0$ (called the strike price).

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- That is, the owner of the option has the opportunity (but no the obligation) for buying or selling the asset at price $\mathcal{K} > 0$ from the time the option is written ($t = 0$) and up the maturity time T.
- Also, if you acquire the option after it was written ($0 < t < T$) you can exercise it at any point in time between (and including) t and T.

American Options

- Denote by
$$\tau = \text{Option's Exercise Time}.$$
- Within our discrete-time framework, it is clear that $\tau \in \{0, 1, 2, \ldots, T\}$ and depends on agent's preferences.

American Options: Exercise Time

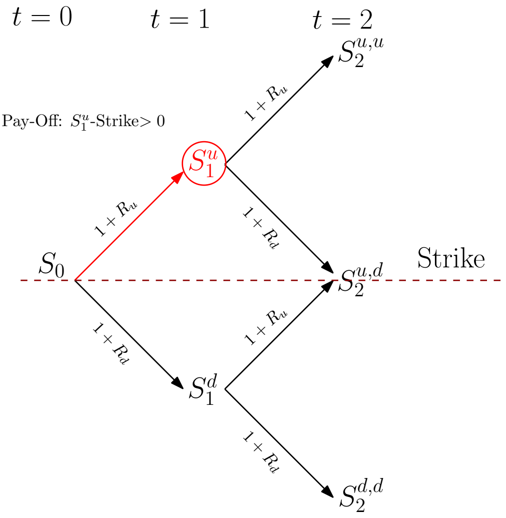

Figure: Scenario 1: Risk Averse$\Rightarrow \tau = 1$. Otherwise$\Rightarrow \tau = 2$.

American Options: Exercise Time

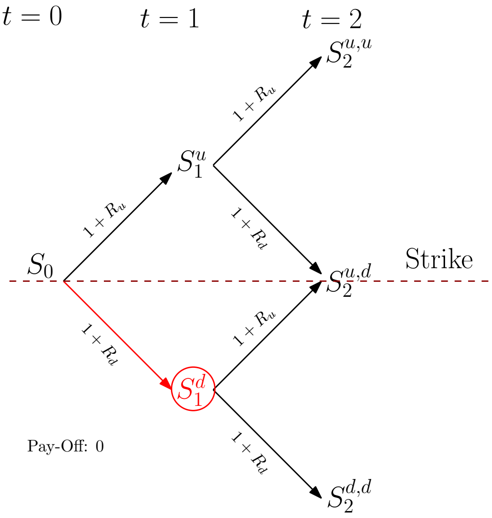

Figure: Scenario 2: Pay-Off is always 0 $\Rightarrow \tau = 1$ or 2.

American Options: Exercise Time

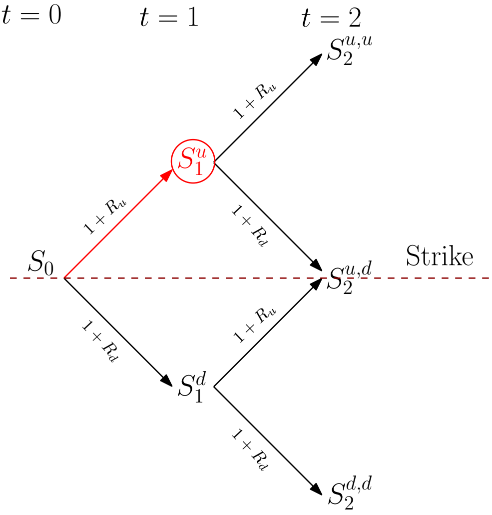

Figure: Scenario 1 under privileged information: We were told that "if the price goes up then in the next period it will again go up" $\Rightarrow \tau = 2$.

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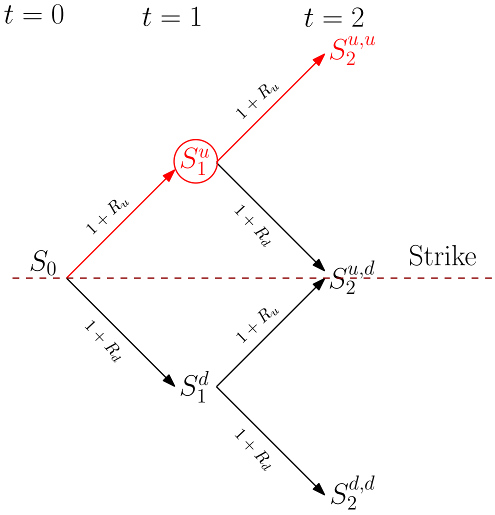

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$$\text{Exercising the Option at time } t = \{\tau = t\} \in \mathscr{F}_t, \;\; t = 0, 1, \ldots, T.$$

- These type of random variables are known as stopping times.

American Call Options: Pay-Off

- We have seen several times that the pay-off of a European Call Option with delivery time $T > 0$ and strike price $\mathcal{K} > 0$ is given by

$$\text{Pay-off or earnings} = (S_T - \mathcal{K})^+ := \begin{cases} S_T - \mathcal{K} & \text{if } S_T > \mathcal{K}; \\ 0 & \text{otherwise.} \end{cases}$$

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- Analogously, the pay-off of an American Call Option with delivery time $T > 0$ and strike price $\mathcal{K} > 0$

$$\text{Pay-off or earnings} = (S_\tau - \mathcal{K})^+ := \begin{cases} S_\tau - \mathcal{K} & \text{if } S_\tau > \mathcal{K}; \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\tau = \text{Option's Exercise Time}.$$

American Call Options: Pay-Off

- Thus, if $\tau = 0$, the pay-off is

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while for $\tau = T$ the earnings of the agent become

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- Thus, if $\tau = 0$, the pay-off is
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while for $\tau = T$ the earnings of the agent become
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- In general, the pay-off or earnings for an American Call Option is given by
$$\sum_{t=0}^{T} (S_t - \mathcal{K})^+ \mathbf{1}_{\tau=t}$$

American Call Options: Pay-Off

- Therefore, it is **impossible to find a fixed** function $\Phi : \prod_{t=0}^{T} \mathbb{R}^{d+1} \to \mathbb{R}$ such that

$$(S_\tau - \mathcal{K})^+ = \sum_{t=0}^{T} (S_t - \mathcal{K})^+ \mathbf{1}_{\tau=t} = \Phi(S_0, S_1, \ldots, S_T).$$

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- We conclude that American Call Options are not European derivatives.
- In particular, the valuation formulas presented in previous lectures breakdown.

Pricing American Call Options

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- Fix an arbitrage-free and complete market (there is a unique martingale measure $\mathbb{Q}$)

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P}), P = (B_t, S_t, \underbrace{\xi_t^{(1)}, \xi_t^{(2)}, \dots, \xi_t^{(d)}}_{\text{Extra Assets}})_{0\le t\le T} \right\},$$

such that $B_{t+1} \geq B_t$, and let

$$C_t^a := \text{Price of the Amer. Opt at time } t.$$

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such that $B_{t+1} \ge B_t$, and let

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- Then C_t^a is chosen such that the augmented market

$$\mathfrak{M}' = \left\{ (\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P}), P = (B_t, S_t, \xi_t^{(1)}, \xi_t^{(2)}, \ldots, \xi_t^{(d)}, C_t^a)_{0 \le t \le T} \right\},$$

is arbitrage-free.

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- To do this, we recall/introduce the concept of rational financial agents:

Definition (Rational Financial Agents)

Rational financial agents are those investors who pursuit the maximization of their pay-off.

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- From now on, we will assume that any investor in the market $\mathfrak{M}'$ is rational.

Rational Agents and Exercise Time

Theorem (Rational Agents and Exercise Time)

Let $0 \leq t \leq T$. If the extended market

$$\mathfrak{M}' = \left\{ (\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leq t\leq T}, \mathbb{P}), P = (B_t, S_t, \xi_t^{(1)}, \xi_t^{(2)}, \ldots, \xi_t^{(d)}, C_t^a)_{0\leq t\leq T} \right\},$$

is arbitrage free, then for all $0 \leq t \leq T$

$$C_t^a \geq (S_t - \mathcal{K})^+, \tag{1}$$

In this context, any rational agent which owns an American Call Option will exercise it at time $\tau = t$ if and only if

$$C_t^a = (S_t - \mathcal{K})^+ > 0. \tag{2}$$

Proof: $C_t^a \geq (S_t - \mathcal{K})^+$

- Let $\xi_t^{(1)} = C_t^e$ be the unique arbitrage-free price at time $0 \leq t \leq T$ of a European Call Option with delivery time $T > 0$ and strike price $\mathcal{K} > 0$.

Proof: $C_t^a \geq (S_t - \mathcal{K})^+$

- Let $\xi_t^{(1)} = C_t^e$ be the unique arbitrage-free price at time $0 \leq t \leq T$ of a European Call Option with delivery time $T > 0$ and strike price $\mathcal{K} > 0$.
- From the Exercise Set, we have have that

$$C_t^a \geq C_t^e, \ \forall\, 0 \leq t \leq T,$$

otherwise the extended market

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- Also from the Exercise set, by the non-arbitrage principle, we have that

$$C_t^e \geq (S_t - \mathcal{K})^+, \ \ \forall\, 0 \leq t \leq T.$$

Proof: $C_t^a = (S_t - \mathcal{K})^+ \leftrightarrow \tau = t$

- Finally, let $\tau =$ Option's Exercise Time, and observe that since the agent is rational, it will maximize its pay-off.

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- Therefore, the option will be exercised at time $\tau = t$ iff

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- Therefore, the option will be exercised at time $\tau = t$ iff

$$\text{Pay-off at time } t = (S_t - \mathcal{K})^+ \geq C_t^a > 0.$$

- From the first part of the theorem,

$$(S_t - \mathcal{K})^+ > C_t^a,$$

cannot occur, which concludes the proof. ∎

Pricing American Call Options

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- Assume that at time $t = T - 1$ the agent has not exercised the option yet, but it wants to get rid of the contract.
- Thus, the investor faces two possible decisions:
  1. Exercise the option. $\iff C_{T-1}^a = (S_{T-1} - \mathcal{K})^+$.
  2. Sell the contract. $\iff C_{T-1}^a > (S_{T-1} - \mathcal{K})^+$
- Since

$$C_{T-1}^a \geq (S_{T-1} - \mathcal{K})^+. \text{ (Previous Theorem)}$$

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- In the latter situation, from the perspective of any agent, at time $t = T-1$, the American Call Option would give the right but, not the obligation, for buying the asset at time $T > 0$ for $\mathcal{K} > 0$.
- In other words, if $C^a_{T-1} > (S_{T-1} - \mathcal{K})^+$ then the American Call Option gives the same rights as a European Call Option written at time $t = T-1$ with delivery time T and strike price $\mathcal{K} > 0$.

Pricing American Call Options

- We conclude that if $C^a_{T-1} > (S_{T-1} - \mathcal{K})^+$, in order to avoid arbitrage, C^a_{T-1} must agree with the value of such European Call Option written at time $t = T - 1$ with delivery time T and strike price $\mathcal{K} > 0$.

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- By the First and the Second FTAP the unique arbitrage free price of such European Option equals

$$\mathbb{E}_* \left[\frac{B_{T-1}}{B_T} (S_T - \mathcal{K})^+ \,\middle|\, \mathscr{F}_{T-1} \right].$$

- Hence, we conclude that if $C^a_{T-1} > (S_{T-1} - \mathcal{K})^+$, then necessarily

$$C^a_{T-1} = \mathbb{E}_* \left[\frac{B_{T-1}}{B_T} (S_T - \mathcal{K})^+ \,\middle|\, \mathscr{F}_{T-1} \right],$$

otherwise the market would allow for an arbitrage.

Pricing American Call Options

- Summarizing:
  1. Either
  $$C^a_{T-1} = (S_{T-1} - \mathcal{K})^+ \quad (\Longleftrightarrow \tau = t).$$
  2. Or
  $$C^a_{T-1} = \mathbb{E}_* \left[\frac{B_{T-1}}{B_T} (S_T - \mathcal{K})^+ \middle| \mathscr{F}_{T-1} \right] > (S_{T-1} - \mathcal{K})^+.$$
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- Else the market would not be arbitrage free.
- Note that we can also write
$$C^a_{T-1} = \max \left\{ (S_{T-1} - \mathcal{K})^+, \mathbb{E}_* \left[\frac{B_{T-1}}{B_T}(S_T - \mathcal{K})^+ \middle| \mathscr{F}_{T-1} \right] \right\}.$$

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- Thus, the investor faces two possible decisions at time $t = T - 2$ (recall that $C^a_{T-2} \geq (S_{T-2} - \mathcal{K})^+$):
  1. Exercise the option $\iff C^a_{T-2} = (S_{T-2} - \mathcal{K})^+$.
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- If 2. occurs, then the American Call Option becomes a European contract written at time $t = T - 2$ that pay-off at time $t = T - 1$, C^a_{T-1}.

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- If 2. occurs, then the American Call Option becomes a European contract written at time $t = T - 2$ that pay-off at time $t = T - 1$, C^a_{T-1}.
- By the Second FTAP the arbitrage-free value of such a derivative is

$$C^a_{T-2} = \mathbb{E}_* \left[\frac{B_{T-2}}{B_{T-1}} C^a_{T-1} \middle| \mathscr{F}_{T-2} \right] > (S_{T-2} - \mathcal{K})^+.$$

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    1. Exercise the option $\iff$ $C^a_{T-2} = (S_{T-2} - \mathcal{K})^+$.
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- If 2. occurs, then the American Call Option becomes a European contract written at time $t = T-2$ that pay-off at time $t = T-1$, C^a_{T-1}.
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- As for the case $t = T-1$, we have

$$C^a_{T-2} = \max \left\{ (S_{T-2} - \mathcal{K})^+, \mathbb{E}_* \left[\frac{B_{T-2}}{B_{T-1}} C^a_{T-1} \middle| \mathscr{F}_{T-2} \right] \right\},$$

otherwise the market would not be arbitrage free.

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- Iterating the previous reasoning give us that for $t = 1, \ldots, T$

$$C^a_{T-t} = \max \left\{ \underbrace{(S_{T-t} - \mathcal{K})^+}_{\text{We Exercise}}, \underbrace{\mathbb{E}_* \left[\frac{B_{T-t}}{B_{T-t+1}} C^a_{T-t+1} \middle| \mathscr{F}_{T-t} \right]}_{\text{Sell the Option}} \right\},$$

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where

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- Replacing $(S_{T-1} - \mathcal{K})^+$ by $(\mathcal{K} - S_{T-1})^+$ in our previous reasoning we conclude that the price of an American Put Option with delivery time $T > 0$ and strike price $\mathcal{K}$ is given by

$$P^a_{T-t} = \max\left\{ (\mathcal{K} - S_{T-t})^+, \mathbb{E}_*\left[\frac{B_{T-t}}{B_{T-t+1}} P^a_{T-t+1} \middle| \mathscr{F}_{T-t} \right] \right\}.$$

where

$$P^a_T := (\mathcal{K} - S_T)^+.$$

Theorem (American Options)

Suppose that the market

$$\mathfrak{M} = \left\{(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P}), P = (B_t, S_t, \xi_t^{(1)}, \xi_t^{(2)}, \dots, \xi_t^{(d)})_{0\le t\le T}\right\},$$

is arbitrage-free and complete market. The unique arbitrage-free prices of Call and Put Options of the American type with delivery time $T > 0$ and strike price $\mathcal{K} > 0$ are respectively given by $C_T^a := (S_{T-t} - \mathcal{K})^+$, and $P_T^a := (\mathcal{K} - S_T)^+$ while for $t = 1, \dots, T$

$$C_{T-t}^a = \max\left\{(S_{T-t} - \mathcal{K})^+, \mathbb{E}_*\left[\frac{B_{T-t}}{B_{T-t+1}} C_{T-t+1}^a \,\middle|\, \mathscr{F}_{T-t}\right]\right\};$$

$$P_{T-t}^a = \max\left\{(\mathcal{K} - S_{T-t})^+, \mathbb{E}_*\left[\frac{B_{T-t}}{B_{T-t+1}} P_{T-t+1}^a \,\middle|\, \mathscr{F}_{T-t}\right]\right\}.$$

The price of American Options: Another Interpretation

For $0 \leq t \leq T$, let

$$\mathscr{T}_{[t,T]} := \{\tau : \Omega \to \{t, t+1, \ldots, T\} \mid \tau \text{ is an exercise time (stopping time)}\}$$

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Theorem (American Options and Expected Utility)

Under the assumptions of the previous theorem, we have that $0 \leq t \leq T$

$$C_t^a = \max_{\tau \in \mathscr{T}_{[t,T]}} \left\{ \mathbb{E}_* \left[\frac{B_t}{B_\tau} (S_\tau - \mathcal{K})^+ \middle| \mathscr{F}_t \right] \right\}$$

$$P_t^a = \max_{\tau \in \mathscr{T}_{[t,T]}} \left\{ \mathbb{E}_* \left[\frac{B_t}{B_\tau} (\mathcal{K} - S_\tau)^+ \middle| \mathscr{F}_t \right] \right\}.$$

respectively.

Theorem (American Options and Expected Utility cont'd)

Moreover,

$$\tau_t^{*,c} := \min\{u \in \{t, t+1, \dots, T\} : (S_u - \mathcal{K})^+ \geq C_u^a\};$$
$$\tau_t^{*,p} := \min\{u \in \{t, t+1, \dots, T\} : (\mathcal{K} - S_u)^+ \geq P_u^a\};$$

satisfies that

$$C_t^a = \mathbb{E}_* \left[\frac{B_t}{B_{\tau_t^{*,c}}} (S_{\tau_t^{*,c}} - \mathcal{K})^+ \middle| \mathscr{F}_t \right];$$
$$P_t^a = \mathbb{E}_* \left[\frac{B_t}{B_{\tau_t^{*,p}}} (\mathcal{K} - S_{\tau_t^{*,p}})^+ \middle| \mathscr{F}_t \right].$$