

Financial Engineering

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1 Bonds and the Value of Money Through Time

Here you can explain what type of interests we studied and how the fair-price of bonds depends on these types of interest and viceversa.

1.1 Bonds

I will begin by giving the definition of a bond.

Definition 1.1 (Bond). A bond is a financial security that pays the owner a chain of predetermined payments.

Using definition 1.1 we can interpret a bond as a financial asset which have no risk. In other words we know for a fact how much money we will make on this investment, disregarding the inherent risk of default.

1.2 Value of Money Through Time

The value of money changes through time. Assume you were offered an amount of money with value M , furthermore you would have the choice of receiving the money today or 1 year from now. Choosing to receive the money now you would be able to spend them straightaway which could be preferable compared to waiting a year. Prices could also have risen during the waiting time, further diminishing the value. Last but not least, receiving the money today would enable you to put the money in the bank and thereby receive a premium. All these arguments serve the purpose of showing that the value of money changes through time, and that the value M of the money would decrease by waiting a year before receiving the money.

1.3 Interest Rate

The premium received by depositing money at bank mentioned earlier is described by an interest rate. This interest rate will in the following be described by a positive number

$$r > 0.$$

Different types of interest exist and I will describe the following three types

1. simple interest,
2. compounded interest,
3. continuously compounded interest.

Common for all three types is that they represent a fraction of an investment that is paid either once or through several periods.

However to further dive into the different types of interest we will first define the notion of a wealth process.

Definition 1.2 (Wealth Process). The evolution of an investment over time is called the wealth process of that investment and is denoted by

$$V = (V_t)_{0 \leq t \leq T}. \quad (1.1)$$

The initial capital is denoted by v_0 , and we assume that V is a real-valued stochastic process on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

With this definition in place we are now ready to define the three different types of interest starting with simple interest.

Definition 1.3 (Simple Interest). Let $v_0 \in \mathbb{R}$ be our initial capital. An interest on v_0 is said to be simple if it follows the wealth process

$$V_t = (1 + rt)v_0, \quad 0 \leq t \leq T. \quad (1.2)$$

I will now show that the wealth process in (1.2) is indeed a stochastic process in any probability space. Any stochastic process X on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfies

$$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}, \quad \forall x \in \mathbb{R}. \quad (1.3)$$

Suppose $v_0 > 0$ and $x \geq (1 + rt)v_0$ then

$$\{\omega \in \Omega : (1 + rt)v_0 \leq x\} = \{\Omega\} \in \mathcal{F}, \quad (1.4)$$

on the other hand if $x < (1 + rt)v_0$

$$\{\omega \in \Omega : (1 + rt)v_0 \leq x\} = \{\emptyset\} \in \mathcal{F}. \quad (1.5)$$

As both Ω and \emptyset is contained in any σ -algebra we have shown that the wealth process in (1.2) is a stochastic process in any probability space. Next i will define the notion of compound interest.

Definition 1.4 (Compounded Interest). Let $v_0 \in \mathbb{R}$ be our initial capital. An interest on v_0 is said to be compounded over $m \in \mathbb{N}$ periods if it follows the wealth process

$$V_t = \left(1 + \frac{r}{m}\right)^{mt} v_0, \quad 0 \leq t \leq T. \quad (1.6)$$

As stated in the definition m can be any natural number, however if $m = 1, 12, 365$ the interest is referred to as annually, monthly and daily compounded, respectively. Note that we have the following properties $\forall 0 \leq t \leq T$

$$i) \quad V_{t+1} = \left(1 + \frac{r}{m}\right)^m V_t,$$

$$ii) \quad \text{If } m_1 > m_2, v_0 > 0 \Rightarrow \left(1 + \frac{r}{m_1}\right)^{m_1 t} v_0 > \left(1 + \frac{r}{m_2}\right)^{m_2 t} v_0,$$

$$iii) \text{ If } m_1 > m_2, v_0 < 0 \Rightarrow \left(1 + \frac{r}{m_1}\right)^{m_1 t} v_0 < \left(1 + \frac{r}{m_2}\right)^{m_2 t} v_0.$$

From this it follows that for an *investor* compound interest is more attractive as it pays more, however as a *debtor* it is less attractive as he or she will have to pay more on his or her debt.

At last I can turn to continuously compounded interest which I will present as the limit of (1.6) as $m \rightarrow \infty$. Note that by the following definition of e

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e, \quad (1.7)$$

by letting $x = r/m$ in the above the limit of the wealth process of compounded interest can be seen as

$$\left[\left(1 + \frac{r}{m}\right)^{\frac{m}{r}}\right]^{rt} v_0 \rightarrow (e)^{rt} v_0, \quad \text{as } m \rightarrow \infty. \quad (1.8)$$

This leads to the definition of continuously compounded interest

Definition 1.5 (Continuously Compounded Interest). Let v_0 be our initial capital. An interest on v_0 is said to be continuously compounded at rate $r > 0$ if the wealth process

$$V_t = e^{rt} v_0, \quad 0 \leq t \leq T. \quad (1.9)$$

There exists the following relation between the different types of interest

$$(1 + r) \leq \left(1 + \frac{r}{m}\right)^m < e^r. \quad (1.10)$$

To show that the relation indeed holds I will show that the sequence

$$a_m = \left(1 + \frac{r}{m}\right)^m, \quad (1.11)$$

is increasing.

1.4 Bonds

A bond is a financial security that pays to the owner a chain of predetermined payments. However there are different types of bonds. I will discuss the following two types here

1. zero-coupon bonds,
2. coupon bonds.

2 Portfolio Allocation and Risk Measures

Try to explain what a portfolio is and how we can create (allocate) portfolio based on risk measures. In your discussion about the latter, remember to include the differences/benefits/drawbacks of considering the standard deviation, VaR and CVaR as risk measures.

3 The Multi-Step Binomial Model

Describe the dynamics of the model. Try to discuss, among other things, the way that the market information is described through time, its absence of arbitrage, and how to find a risk-neutral measure in such model.

4 The First Fundamental Theorem of Asset Pricing

Focus on explaining the hypothesis in the theorem (martingale measure, admissible strategies, arbitrage, etc.) and its applications to the pricing of financial derivatives.

5 Pricing in the Binomial Model

Explain what is the typical methodology for pricing derivatives and how such a methodology works in the Binomial model. Remember to include an example (e.g. the price of call-option).

6 The Second Fundamental Theorem of Asset Pricing

Focus on explaining the hypothesis in the theorem (martingale measure, admissible strategies, arbitrage, completeness, etc.) and its consequences on the pricing of financial derivatives. You can, in particular, show that the Binomial Model is complete while the Trinomial Model is not.