Financial Engineering
Lecture 5

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General comments

- Questions or comments about the previous lecture and/or exercise set?
- Next week you will have a selfstudy session.
- The topic will be "Conditional Expectation".
- I will be available from 12:30 til 16:00 if you have any questions.
- Besides the topics I will also suggest some exercises for you to solve.

Review of the previous lecture

What did we do in the previous lecture?

Review of the previous lecture

- Value at risk does not measure properly diversified strategies.
- Coherent risk measures.
- Conditional Value at Risk.
- Financial Derivatives.

Outline for today

- The One-Step Binomial model: Arbitrage, replication and pricing.
- Filtrations: The flow of information in financial market.

Prelude to Arbitrage Theory: The Binomial model

Arbitrage in the One-Step Binomial Model

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$$B_1 = (1 + K_B(1)). \text{ (Recall that } K_X \text{ is the return of the process } X\text{)}$$

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- The bond satisfies that $B_0 = 1$, so

$$B_1 = (1 + K_B(1)). \text{ (Recall that } K_X \text{ is the return of the process } X)$$

- On the other hand, the stock price $(S_t)_{t=0,1}$ satisfies that $S_0 > 0$ (non-random) and

$$S_1 = S_0(1 + K_S(1)),$$

where

$$K_S(1) = \begin{cases} R_u & \text{with probability } p; \\ R_d & \text{with probability } 1 - p, \end{cases}$$

with the relation

$$0 < p < 1, \; R_d < R_u.$$

Arbitrage in the One-Step Binomial Model

- Furthermore, we conclude that in the "steady state of the economy", i.e. when supply and demand are the same, necessarily it must hold that
$$R_u < K_B(1) < R_d.$$
- We also briefly discussed that this is equivalent to the absence of arbitrage.

Arbitrage in the One-Step Binomial Model

Definition (Arbitrage in one-step financial markets)

We will say that the one-step market $\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, \mathbb{P}), P = (B_t, S_t^{(1)}, S_t^{(2)}, \ldots, S_t^{(d)})_{t=0,1} \right\}$ admits **an arbitrage** if there exists a portfolio $\Theta = (\varphi, \theta^{(1)}, \ldots, \theta^{(d)}) \in \mathbb{R}^{d+1}$ such that

1. It has zero initial capital:
$$V_0^{\Theta} = \Theta \cdot P_0 = 0.$$
2. At the end of the trading time we can pay our debts with 100% certainty:
$$\mathbb{P}\left(V_1^{\Theta} = \Theta \cdot P_1 \geq 0 \right) = 1.$$
3. We have a chance to make a profit:
$$\mathbb{P}\left(V_1^{\Theta} = \Theta \cdot P_1 > 0 \right) > 0.$$

Arbitrage in the One-Step Binomial Model

Theorem (Theorem NAOSBM)

In the One-Step Binomial Model the following statements are equivalent:

1. *The market does not admit an arbitrage.*
2. $R_d < K_B(1) < R_u$.
3. *There exists a unique* $0 < q < 1$ *such that*

$$qR_u + (1-q)R_d = K_B(1).$$

Proof of Theorem NAOSBM

- We have that the equation

$$K_B(1) = qR_u + (1-q)R_d = q(R_u - R_d) + R_d,$$

is equivalent to

$$q = \frac{K_B(1) - R_d}{(R_u - R_d)}.$$

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- From here, it is obvious that

$$R_d < K_B(1) < R_u \iff 0 < q < 1.$$

Proof of Theorem NAOSBM

- In Lecture 2 we already checked that if $K_B(1) \geq R_u$ then we can create an arbitrage by choosing the portfolio

$$\Theta = (\underbrace{S_0}_{\text{We buy } S_0 \text{ bonds.}} , \underbrace{-1}_{\text{We short-sell a risky asset}}).$$

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- By interchanging the roles in that argument, we easily deduce that if $K_B(1) \leq R_d$ then the portfolio

$$\Theta = (\underbrace{-S_0}_{\text{We borrow from the bank} S_0}, \underbrace{1}_{\text{We buy a risky asset}}),$$

is an arbitrage.

Proof of Theorem NAOSBM

- Therefore, in order to finish the proof we only need to show that if $R_d < K_B(1) < R_u$, then the market does not allow arbitrage.

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- Therefore, in order to finish the proof we only need to show that if $R_d < K_B(1) < R_u$, then the market does not allow arbitrage.
- Let us proceed by contradiction: Suppose that there exists a portfolio $\Theta = (\varphi, \theta) \in \mathbb{R}^2$ such that:
  1. It has zero initial capital:
  $$\varphi + \theta S_0 = \underbrace{(\varphi, \theta)}_{\Theta} \cdot \underbrace{(1, S_0)}_{P_0} = 0.$$
  2. At the end of the trading time we can pay our debts with 100% certainty:
  $$\varphi(1 + K_B(1)) + \theta S_1 = \underbrace{(\varphi, \theta)}_{\Theta} \cdot \underbrace{((1 + K_B(1)), S_1)}_{P_1} \geq 0, \text{ almost surely.}$$
  3. We have a chance to make a profit:
  $$\mathbb{P}\left(\varphi(1 + K_B(1)) + \theta S_1 > 0\right) > 0.$$

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- Therefore, in order to finish the proof we only need to show that if $R_d < K_B(1) < R_u$, then the market does not allow arbitrage.
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  3. We have a chance to make a profit:
  $$\mathbb{P}\left(\varphi(1 + K_B(1)) + \theta S_1 > 0\right) > 0.$$
- The proof will be finished in the blackboard.

Risk-Neutral Measure

- We have shown that the One-Step binomial model is arbitrage free if and only if
$$0 < q^* = \frac{K_B(1) - R_d}{(R_u - R_d)} < 1.$$
- By construction $0 < q^* < 1$, so the pair $(q^*, 1 - q^*)$ defines a new probability measure.
- Also by construction, we have imposed that such probabilities satisfies that
$$q^* R_u + (1 - q^*) R_d = K_B(1).$$
- The quantity on the left-hand side of the previous equation is the expected value of the random variable

$$X = \begin{cases} R_u & \text{with probability } q^*; \\ R_d & \text{with probability } 1 - q^*. \end{cases}$$

Risk-Neutral Measure

- In particular, if we change dynamics of the return of S as

$$K_S(1) = \begin{cases} R_u & \text{with probability } q^*; \\ R_d & \text{with probability } 1 - q^*, \end{cases}$$

- Then the expectation of $K_S(1)$ under the probability $(q^*, 1 - q^*)$ is given by

$$\mathbb{E}_* (K_S(1)) := R_u q^* + R_d(1 - q^*) = K_B(1).$$

- Therefore we can re-state the previous theorem as "The One-Step binomial model is arbitrage free if and only if there exists a probability $(q^*, 1 - q^*)$ satisfying that

$$\mathbb{E}_* (K_S(1)) = K_B(1).$$

- Such a q^* is termed as a risk-neutral measure.

Pricing and the Non-Arbitrage principle

- How much shall we pay, for instance, to get the rights that a European option gives you?
- If we want to get rid of such an option by transfer it before the exercise time $t < T$, in how much shall we sell it?

$$
\begin{aligned}
&\text{Original Market:} && P = (B_t, S_t^{(1)}, S_t^{(2)}, \ldots, S_t^{(d)})_{0 \leq t \leq T} \\
\xrightarrow[\xi]{\text{Derivative}}\ &\text{Extended Market:} && \tilde{P} = (B_t, S_t^{(1)}, S_t^{(2)}, \ldots, S_t^{(d)}, \xi_t)_{0 \leq t \leq T}
\end{aligned}
$$

Pricing and the Non-Arbitrage principle

- If the original market is in the equilibrium then it is natural to expect that the extended market would also be equilibrated.
- This means that if a-priori our market is arbitrage free, there is no reason to believe that by adding a derivative this would change, at least in a short term.

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Assumption (A.2.0)

One-step financial markets do not admit arbitrage.

Pricing and the Non-Arbitrage principle

- Consider the arbitrage free One-Step Binomial model:
  1. $B_0 = 1$ and $B_1 = (1 + K_B(1))$.
  2. $S_0 > 0$ (non-random) and
  $$S_1 = S_0(1 + K_S(1)),$$
  where for $0 < p < 1$
  $$K_S(1) = \begin{cases} R_u & \text{with probability } p; \\ R_d & \text{with probability } 1 - p, \end{cases}$$
  3. We have the relation
  $$R_d < K_B(1) < R_u.$$

Pricing and the Non-Arbitrage principle

- Suppose now that in this model we want to assign a price to a call option that pays off
$$(S_1 - K)^+, \quad K > 0.$$
- Denote by $C_0 \geq 0$ such a price.

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- The quantity C_0 must be enough to generate at least the wealth $(S_1 - K)^+$, i.e. being able to pay to the owner of the option its rights.

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- The quantity C_0 must be enough to generate at least the wealth $(S_1 - K)^+$, i.e. being able to pay to the owner of the option its rights.
- If you cannot cover yourself in this way you wouldn't sell the option.
- We can only invest our capital on a bond and a risky-asset.

Pricing and the Non-Arbitrage principle

- Thus, we aim at creating a portfolio $\Theta = (\varphi, \theta) \in \mathbb{R}^2$ satisfying that:

  1. The initial capital is paid-out by C_0:

  $$C_0 \geq V_0^{\Theta}.$$

  2. At time $t = 1$, it must generate the pay-off of the call option:

  $$V_1^{\Theta} = (S_1 - K)^+.$$

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- Suppose that such portfolio exists and the seller set the call price above V_0^Θ, i.e.

  $$C_0 > V_0^\Theta.$$

- What would happen in this situation?

Pricing and the Non-Arbitrage principle

- If $C_0 > V_0^{\theta}$, then we short-sell the option (remember that we do not need to own the asset to write the option).
- We buy $C_0 - V_0^{\Theta} > 0$ bonds whose value at time $t = 1$ would become
$$(C_0 - V_0)B_1 > 0.$$
- This is an arbitrage in the extended market
$$\text{Extended Market: } \tilde{P} = (B_t, S_t, C_t)_{t=0,1},$$
where
$$C_1 = (S_1 - K)^+.$$
- Therefore, to avoid arbitrage, the call price must be
$$C_0 = V_0^{\Theta}.$$

Pricing and the Non-Arbitrage principle

- Let us find such a portfolio, i.e. a portfolio $\Theta = (\varphi, \theta) \in \mathbb{R}^2$ satisfying that:
  1. The initial capital is paid-out by C_0:
  $$C_0 = V_0^\Theta = \varphi + \theta S_0, \ \ (B_0 = 1).$$
  2. At time $t = 1$, it must generate the pay-off of the call option:
  $$\begin{aligned} (S_0(1 + K_S(1)) - K)^+ =& (S_1 - K)^+ \\ =& V_1^\Theta \\ =& \varphi B_1 + \theta S_0(1 + K_S(1)) \end{aligned}$$

Pricing and the Non-Arbitrage principle

- Since $K_S(1)$ has only two possible outcomes, the equation

$$(S_0(1+K_S(1))-K)^+ = \varphi B_1 + \theta S_0(1+K_S(1)),$$

can be rewritten as

$$\begin{aligned} \theta S_0(1+R_u)+\varphi B_1 &= \zeta^u; \\ \theta S_0(1+R_d)+\varphi B_1 &= \zeta^d. \end{aligned} \tag{1}$$

where

$$\zeta^u = (S_0(1+R_u)-K)^+ \text{ and } \zeta^d = (S_0(1+R_d)-K)^+.$$

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$$\zeta^u = (S_0(1+R_u)-K)^+ \text{ and } \zeta^d = (S_0(1+R_d)-K)^+.$$

- Solving (1) (in the blackboard), we obtain that the desired portfolio is given by

$$\theta = \frac{\zeta^u - \zeta^d}{S_0(R_u - R_d)}; \quad \varphi = \frac{1}{B_1}\left\{\frac{\zeta^d(1+R_u)-\zeta^u(1+R_d)}{R_u - R_d}\right\}.$$

Pricing and the Non-Arbitrage principle

- To find the call price that is arbitrage free we only need to use the relation

$$C_0 = V_0^{\Theta} = \varphi + \theta S_0,$$

for the portfolio

$$\theta = \frac{\zeta^u - \zeta^d}{S_0(R_u - R_d)}; \quad \varphi = \frac{1}{B_1}\left\{\frac{\zeta^d(1+R_u) - \zeta^u(1+R_d)}{R_u - R_d}\right\}.$$

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- However, if we let $q^* = \frac{K_B(1) - R_d}{R_u - R_d}$, then we deduce (on the blackboard) that
$$C_0 = \frac{1}{B_1}\left\{\zeta^u q^* + \zeta^d(1-q^*)\right\} = \mathbb{E}_*\left[\frac{(S_1 - K)^+}{B_1}\right].$$

Replication (cf $\theta = \frac{\zeta^u - \zeta^d}{S_0(R_u - R_d)}$; $\varphi = \frac{1}{B_1}\left\{\frac{\zeta^d(1+R_u) - \zeta^u(1+R_d)}{R_u - R_d}\right\}$)

- If we replace $(S_1 - K)^+$ with a general derivative with pay-off

$$\xi_1 = \Phi(\underbrace{B_0, B_1, S_0}_{\text{Deterministic}}, S_1) = \begin{cases} \xi_1^u & \text{if } K_S(1) = u \\ \xi_1^u & \text{if } K_S(1) = d \end{cases} \quad \text{(Only two outcomes)}$$

in the previous argument, then the portfolio given by

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generates the wealth $V_1^\Theta = \xi_1$.

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generates the wealth $V_1^\Theta = \xi_1$.

- Moreover, to avoid arbitrage the initial price of such derivative must be

$$\xi_0 = V_0^\Theta = \mathbb{E}_*\left[\frac{\xi_1}{B_1}\right] = \frac{1}{B_1}\{\xi_1^u q^* + \xi_1^u(1 - q^*)\}$$

- Such a strategy is called a replicating portfolio for ξ_1.

Replication

Definition (Replicable derivatives)

Let

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, \mathbb{P}), P = (B_t, S_t^{(1)}, S_t^{(2)}, \ldots, S_t^{(d)})_{t=0,1} \right\},$$

be a one-step financial market. We will say that **a financial derivative** (see Definition DefFD) with pay-off ξ_1 **is replicable** if there exists a portfolio $\Theta = (\varphi, \theta^{(1)}, \ldots, \theta^{(d)}) \in \mathbb{R}^{d+1}$ such that almost surely

$$\xi_1 = V_1^{\Theta} = \varphi B_1 + \sum_{j=1}^{d} \theta^{(j)} S_1^{(j)}.$$

Pricing and the Non-Arbitrage principle

We have shown that in the One-Step Binomial model:

1. The model is arbitrage free if and only if a risk-neutral measure exists: There is $0 < q^* < 1$ such that

$$\mathbb{E}_* \left(K_S(1) \right) = R_u q^* + R_d (1 - q^*) = K_B(1).$$

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$$\mathbb{E}_*\left(K_S(1)\right) = R_u q^* + R_d(1 - q^*) = K_B(1).$$

2. Every derivative ξ_1 can be replicated by the portfolio

$$\theta = \frac{\xi_1^u - \xi_1^d}{S_0(R_u - R_d)};\ \varphi = \frac{1}{B_1}\left\{\frac{\xi_1^d(1 + R_u) - \xi_1^u(1 + R_d)}{R_u - R_d}\right\}.$$

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2. Every derivative ξ_1 can be replicated by the portfolio
$$\theta = \frac{\xi_1^u - \xi_1^d}{S_0(R_u - R_d)}; \; \varphi = \frac{1}{B_1} \left\{ \frac{\xi_1^d(1+R_u) - \xi_1^u(1+R_d)}{R_u - R_d} \right\}.$$
3. In order to avoid arbitrage, the price of such derivative at time $t = 0$, say ξ_0, must be
$$\xi_0 = \mathbb{E}_* \left[\frac{\xi_1}{B_1} \right].$$

Fundamental Theorems of Asset Princing in the OSBM

Theorem (Fundamental Theorems of Asset Princing in the OSBM)

In the One-Step Binomial Model the following holds:

1. ***First Fundamental Theorem:*** *The market is arbitrage free if and only if there is a risk-neutral measure.*
2. ***Second Fundamental Theorem:*** *Every financial derivative can be replicated.*
3. ***Pricing:*** *Let ξ_1 be a* ***derivative****. The extended market with prices*

$$\tilde{P} = (B_t, S_t, \xi_t)_{t=0,1},$$

is arbitrage free if and only if there exists a risk neutral probability $(q^*, 1-q^*)$ and ***the price is the expected value w.r.t. the risk neutral measure of the discounted pay-off,*** *that is*

$$\xi_0 = \mathbb{E}_* \left[\frac{\xi_1}{B_1} \right].$$

Filtrations: The flow of information

Information in financial markets

- We want now to consider multi-step financial markets

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, \mathbb{P}), P = (B_t, S_t^{(1)}, S_t^{(2)}, \dots, S_t^{(d)})_{t=0,1,\dots,T} \right\}.$$

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- Information about the market movements plays a key role in the design of strategies.

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- Information about the market movements plays a key role in the design of strategies.
- For instance: If we own a call option with delivery time $T = 3$ with strike price $K = 100$ and we at time $t = 1$ we observe that

$$S_0 = 100,\ S_1 = 90.$$

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- We "predict" that our chances of exercising the option are small.

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- Information about the market movements plays a key role in the design of strategies.
- For instance: If we own a call option with delivery time $T = 3$ with strike price $K = 100$ and we at time $t = 1$ we observe that

$$S_0 = 100, \; S_1 = 90.$$

- We "predict" that our chances of exercising the option are small.
- On the other, if at time $t = 2$, the price of the asset rises to 120, we predict, based on this information, that it is very likely that we will exercise the option.

The Multi-Step Binomial Model

- To understand how can we quantified the information, let us now consider the Multi-Step Binomial Model (MSBM from now on).

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- As before, in this model only two assets are traded: A bond and a risky asset.
- The bonds satisfies that

$$B_t = (1 + r)^t, \quad t = 0, 1, \ldots, T.$$

The Multi-Step Binomial Model

- On the other hand, the price process $(S_t)_{t=0,1,\ldots,T}$ satisfies that $S_0 > 0$ (non-random) and

$$S_t = S_{t-1}(1 + K_S(t)), \ t = 1, \ldots, T.$$

- The returns are independent and identically distributed with

$$K_S(t) = \begin{cases} R_u & \text{with probability } p; \\ R_d & \text{with probability } 1 - p, \end{cases}$$

with the relation

$$R_u < R_d.$$

The Multi-Step Binomial Model

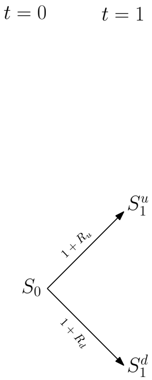

Figure: Graphical representation of the model.

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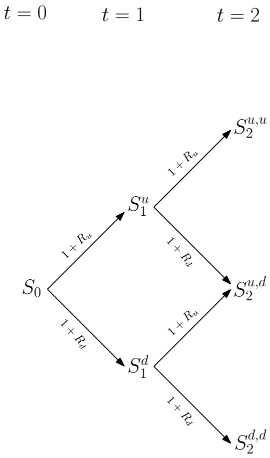

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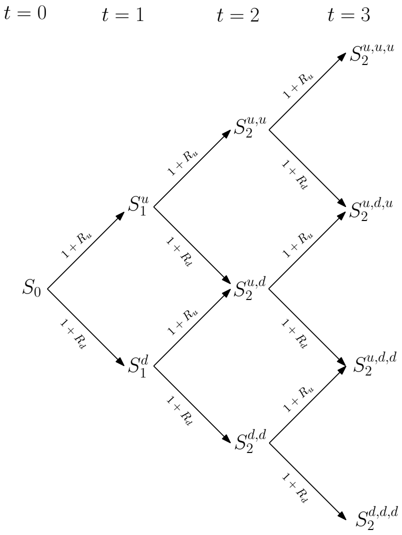

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- For instance, if $T = 2$, the vector (u, u) represents the event in which the price went **first up and then down**.
- As usual, we will consider all the possible events, that is the σ-algebra in this case is

$$\mathscr{F} = 2^{\Omega}.\ \text{(The power set)}$$

The probability space in the MSBM

- In this set-up the return process can be written as

$$K_S(t)(\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(T)}) = \begin{cases} R_u & \text{if } \omega^{(t)} = u; \\ R_d & \text{if } \omega^{(t)} = d. \end{cases}$$

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- To assure that the returns are independent and can go up (down) with probability p $(1-p)$, we let

$$\mathbb{P}(\{\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(T)}\}) = p^k(1-p)^{T-k},$$

where

$$k = \text{number of } u's \text{ in } (\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(T)}).$$

Example: $T = 3$

- If $T = 3$, we have that

$$\Omega = \{uuu, uud, udu, udd, duu, dud, ddu, ddd\},$$

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$$(\omega^{(1)}, \omega^{(2)}, \omega^{(3)}) = \omega^{(1)}\omega^{(2)}\omega^{(2)}.$$

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- As time goes by, the price movements reveal information of which of all these outcomes will actually happen.

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- If at time $t = 1$ we observe that the price went up, then the future outcomes are restricted to the set

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- In contrast, if the price goes down, the possible outcomes in this situation as

$$B_d := \{duu, dud, ddu, ddd\} .$$

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- How are these sets related to the information generated by the return process?
- Can you remember how do we model the information that random variables provide?
- This is done by the σ-field generated by the random variable (vector) $X : \Omega \to \mathbb{R}$, i.e.

$$\sigma(X) = \left\{X^{-1}(A) : A \in A \in \mathscr{B}(\mathbb{R})\right\}.$$

Example: $T = 3$ $\left(\Omega = \{uuu, uud, udu, udd, duu, dud, ddu, ddd\}\right)$

- Recall that for $t = 1, 2, 3$

$$K_S(t)(\omega^{(1)}, \omega^{(2)}, \omega^{(3)}) = \begin{cases} R_u & \text{if } \omega^{(t)} = u; \\ R_d & \text{if } \omega^{(t)} = d. \end{cases}$$

- Then,

$$\{(\omega^{(1)}, \omega^{(2)}, \omega^{(3)}) : K_S(1) = R_u\} = \{uuu, uud, udu, udd\} = B_u,$$

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- Then,

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- Similarly, we get that

$$\{(\omega^{(1)}, \omega^{(2)}, \omega^{(2)}) : K_S(1) = R_d\} = \{duu, dud, ddu, ddd\} = B_d.$$

Example: $T = 3$

- Therefore
$$B_u, B_d \subseteq \sigma(K_S(1)).$$
- Actually, since $K_S(1)$ only takes two values, it follows that
$$\sigma(K_S(1)) = \{B_u, B_d, \Omega, \emptyset\}.$$

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3. The price first went down and then up: The future outcomes are

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4. The price went down twice: The future outcomes are

$$B_{dd} = \{ddu, ddd\}.$$

Example: $T = 3$

- In this situation we have

$$\{(\omega^{(1)}, \omega^{(2)}, \omega^{(2)}) : K_S(1) = R_u, K_S(2) = R_u\} = \{uuu, uud\} = B_{uu};$$

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- Hence,

$$B_{uu}, B_{ud}, B_{du}, B_{dd} \subseteq \sigma(K_S(1), K_S(2))$$

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- Hence,

$$B_{uu}, B_{ud}, B_{du}, B_{dd} \subseteq \sigma(K_S(1), K_S(2))$$

- In fact

$$\sigma(K_S(1), K_S(2)) = \{\emptyset \text{ and arbitrary unions of } B_{uu}, B_{ud}, B_{du}, \text{ and } B_{dd}\}.$$

Example: $T = 3$

- At the end of the trading period, i.e. $t = 3$, we have the precise outcome for the price movements, so the possible outcomes are storage in
$$\mathscr{F} = \sigma(K_S(1), K_S(2), K_S(3)).$$
- Note that we have constructed a family of sub-σ-algebras of $\mathscr{F}$:

$$\begin{aligned}
\mathscr{F}_1 &= \sigma(K_S(1)) = \{B_u, B_d, \Omega, \emptyset\}. \\
\mathscr{F}_2 &= \sigma(K_S(1), K_S(2)) \\
&= \{\emptyset \text{ and arbitrary unions of } B_{uu}, B_{ud}, B_{du}, \text{ and } B_{dd}\}. \\
\mathscr{F}_3 &= \sigma(K_S(1), K_S(2), K_S(3)) = \mathscr{F}.
\end{aligned}$$

Example: $T = 3$

- By construction, $\mathscr{F}_t$ contains the whole information available on the movements of our price process up to time t.
- Mathematically, this means that the price process S_t is adapted to $\mathscr{F}_t$:

$$\sigma(S_t) \subseteq \mathscr{F}_t, \quad t = 0, 1, 2, 3 \ \text{where} \ \mathscr{F}_0 = \{\Omega, \emptyset\}.$$

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- Moreover

$$\mathscr{F}_{t-1} \subseteq \mathscr{F}_t, \;\; t = 1, 2, 3.$$

- A collection of sub-σ-algebras satisfying the previous relation is termed as a filtration.

Filtrations and Adapted Processes

Definition (Filtrations and Adapted Process)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. A collection of σ-algebras $\mathbb{F} = (\mathscr{F}_t)_{t=0,1,\dots,T}$ is called **a filtration** if for all $1 \le t \le T$,

$$\mathscr{F}_{t-1} \subseteq \mathscr{F}_t \subseteq \mathscr{F}.$$

The quadruplet

$$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P}),$$

is termed as **a filtered probability space**. Furthermore, a stochastic process $(X_t)_{0 \le t \le T}$ is said to be **adapted to the filtration $\mathbb{F}$** if

$$\sigma(X_t) \subseteq \mathscr{F}_t, \ \ \forall\, 0 \le t \le T.$$

Financial Markets with Information

Definition (Financial Markets with Information)

- Fix $T, d \in \mathbb{N}$. **A finite-horizon financial market with information** is the pair

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P}), P = (B_t, S_t^{(1)} \dots, S_t^{(d)})_{0 \le t \le T} \right\},$$

consisting of

1. A filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P})$.
2. P_t is adapted.
3. $S^{(j)}$ is the price process of the jth asset traded in the market
4. B is a numéraire (e.g. bonds), i.e.

$$\mathbb{P}(B_t > 0) = 1, \;\; 0 \le t \le T.$$