Financial Engineering
Lecture 9

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Outline

- Second Fundamental Theorem of Asset Pricing.
- The Monte Carlo Method.

Second Fundamental Theorem of Asset Pricing

Motivation

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where the returns $K_S(t)$ are i.i.d. with

$$K_S(t) = \begin{cases} R_u & \text{with probability } p_1; \\ R_m & \text{with probability } p_2; \\ R_d & \text{with probability } p_3. \end{cases}$$

there are infinitely many martingale measures iff $R_d < r < R_u$ iif the market is arbitrage free.

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$\Longrightarrow$ Infinitely Many Arbitrage-Free Prices!!!

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- Otherwise we could create an arbitrage.

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- This is the price given by the FFTAP.
- However, the One-Step Trinomial Model is arbitrage-free but there are some derivatives that are impossible to replicate!
- Therefore, there are models that are arbitrage free but it is impossible to assign a unique price to every single derivative in any meaningful way.
- These type of markets are called incomplete.

Complete Markets

Definition

The market

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P}), P = (B_t, S_t^{(1)}, \ldots, S_t^{(d)})_{0 \le t \le T} \right\},$$

is said to be **complete** if every European contingent claim ξ (i.e. a random variable that depends on the information up to time T) can be replicated, that is, there exists an admissible strategy $\Theta = (\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)})_{0 \le t \le T}$ such that almost surely

$$\xi = V_T^{\Theta} = \varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} S_T^{(j)}.$$

Complete Markets and Arbitrage

- Every Financial Derivative is a contingent claim: The pay-off of a derivative is

$$\xi = \Phi\left(P_0, P_1, \ldots, P_T\right),$$

for some function $\Phi : \prod_{t=0}^{T} \mathbb{R}^{d+1} \to \mathbb{R}$.

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- If the market is complete, then there is an admissible strategy $\Theta = (\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)})_{0 \le t \le T}$ such that almost surely

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$$\xi = V_T^{\Theta} = \varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} S_T^{(j)}.$$

- Thus, if the market is arbitrage free and $\mathbb{P}(\xi > 0) > 0$, then necessarily

$$V_0^{\Theta} > 0.$$

Pricing Methods

1. FFTAP: Find a martingale measure (risk-neutral measure) and let

$$\xi_0 := \text{Initial Arb-Free Price of a der. with pay-off } \xi = \mathbb{E}_* \left(\frac{\xi}{B_T} \right).$$

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$$\xi_0 := \text{Initial Arb-Free Price of a der. with pay-off } \xi = \mathbb{E}_* \left(\frac{\xi}{B_T} \right).$$

2. Completeness: Find a replicating strategy, i.e. an admissible strategy $\Theta = (\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)})_{0 \leq t \leq T}$ such that

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Set-Up

- The market with information is given:

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P}), P = (B_t, S_t^{(1)}, \ldots, S_t^{(d)})_{0 \le t \le T} \right\}. \quad (1)$$

- The filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P})$ satisfies that
  1. The sample space is **finite** and our σ-algebra is **the power set**:

  $$\Omega = \{\omega_1, \ldots, \omega_N\}, \quad N \in \mathbb{N}, \quad \mathscr{F} = 2^{\Omega}.$$

  2. There are $0 < p_i < 1$, $i = 1, \ldots, N$ such that $\sum_{i=1}^{N} p_i = 1$ and

  $$\mathbb{P}(\{\omega_i\}) = p_i, \quad i = 1, \ldots, N.$$

  3. The set of information satisfies that $\mathscr{F}_0 = \{\emptyset, \Omega\}$, $\mathscr{F}_T = \mathscr{F}$, and

  $$\mathscr{F}_{t-1} \subseteq \mathscr{F}_t, \quad 1 \le t \le T - 1.$$

- We will assume that B is deterministic and such that $B_0 = 1$ and

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- We will assume that B is deterministic and such that $B_0 = 1$ and
$$B_t > 0, \;\; \forall\, 1 \le t \le T.$$

- Since $\mathscr{F}_T = \mathscr{F} = 2^\Omega$, any function of $\omega \in \Omega$ is a European contingent claim.

The Second Fundamental Theorem of Asset Pricing

Theorem (SFTAP)

Let the market

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P}), P = (B_t, S_t^{(1)} \ldots, S_t^{(d)})_{0 \leq t \leq T} \right\}$$

*be as in (1). Then $\mathfrak{M}$ **is complete and arbitrage free** if and only if there is a **unique** martingale measure.*

Remarks on the SFTAP

- If the market is arbitrage free and complete, then we have a unique arbitrage-free price for any derivative with pay-off ξ which is given by

$$\mathbb{E}_* \left(\frac{B_t}{B_T} \xi \,\middle|\, \mathscr{F}_t \right).$$

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- The assumption "the market is arbitrage free" is crucial.
- There are markets that are complete but they might allow for an arbitrage.

Remarks on the SFTAP

- In the One-Step Binomial Model, the portfolio

$$\theta = \frac{\zeta^u - \zeta^d}{S_0(R_u - R_d)}; \quad \varphi = \frac{1}{B_1}\left\{\frac{\zeta^d(1+R_u) - \zeta^u(1+R_d)}{R_u - R_d}\right\}.$$

- Replicates the contingent claim

$$\xi = \phi(S_1) = \begin{cases} \zeta^u & \text{if } K_S(1) = R_u; \\ \zeta^d & \text{if } K_S(1) = R_d, \end{cases}$$

and

$$V_0^{\Theta} = \frac{1}{(1+r)}\left\{\zeta^u q^* + \zeta^d(1-q^*)\right\}, \quad q^* = \frac{r - R_d}{R_u - R_d}.$$

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$$V_0^{\Theta} = \frac{1}{(1+r)}\left\{\zeta^u q^* + \zeta^d(1-q^*)\right\}, \;\; q^* = \frac{r - R_d}{R_u - R_d}.$$

- This is independent of whether the market is arbitrage free or not, being the former equivalent to

$$R_d < r < R_u, \;\; \text{(In general we only have that } R_d < R_u).$$

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- Suppose that $r < R_d < R_u$ such that the market allows for an arbitrage and let

$$\xi = \begin{cases} 1 & \text{if } K_S(1) = R_u; \\ 0 & \text{if } K_S(1) = R_d. \end{cases}$$

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- The replicating portfolio is

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- However, it has initial investment

$$V_0^{\Theta} = \frac{1}{B_1}\left\{\zeta^u \underbrace{q^*}_{(r-R_d)/(R_u-R_d)} + \zeta^d(1-q^*)\right\} = \frac{1}{B_1}\frac{r - R_d}{R_u - R_d} < 0.$$

Proof of the SFTAP

We will use the following two lemmas that were verified during the proof of the FFTAP:

Lemma (Lemma 1)

If Θ is admissible and $\mathbb{Q}$ is a martingale measure, then the discounted wealth process $(\tilde{V}_t^\Theta = V_t^\Theta / B_t, \mathscr{F}_t)_{0 \le t \le T}$ *is a martingale under $\mathbb{Q}$.*

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Lemma (Lemma 2)

Let $(X_t, \mathscr{F}_t)_{0 \leq t \leq T}$ *be a martingale. Then*

$$\mathbb{E}(X_t) = \mathbb{E}(X_0), \ \forall\, 0 \leq t \leq T.$$

Proof (⇒): "There is only one martingale measure"

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- We will proceed by contradiction.
- Specifically, we will assume that the market is arbitrage free but there are two different martingale measures, say $\mathbb{Q}$ and $\mathbb{Q}'$.

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Proof (⇒): "There is only one martingale measure"

- Thus, suppose that the market is complete and arbitrage free.
- By the FFTAP there is a martingale measure $\mathbb{Q}$.
- Suppose that there is another martingale measure $\mathbb{Q}'$ such that for some $\omega_j \in \Omega = \{\omega_1, \ldots, \omega_N\}$

$$\mathbb{Q}'\left(\{\omega_j\}\right) \neq \mathbb{Q}\left(\{\omega_j\}\right).$$

Proof ($\Rightarrow$) ($\exists \omega_j, \mathbb{Q}(\{\omega_j\}) \neq \mathbb{Q}'(\{\omega_j\})$)

- Denote by $\mathbb{E}_*(\cdot)$ and $\mathbb{E}'_*(\cdot)$ the expectations w.r.t. $\mathbb{Q}$ and $\mathbb{Q}'$, respectively.

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- Denote by $\mathbb{E}_*(\cdot)$ and $\mathbb{E}'_*(\cdot)$ the expectations w.r.t. $\mathbb{Q}$ and $\mathbb{Q}'$, respectively.
- By Lemmas 1 and 2, for **every admissible strategy** $\Theta = (\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)})_{0 \leq t \leq T}$, the discounted wealth process is a martingale under $\mathbb{Q}$ and

$$\mathbb{E}_*\left(\frac{V_T^\Theta}{B_T}\right) = \mathbb{E}_*\left(\tilde{V}_T^\Theta\right) = \mathbb{E}_*\left(\tilde{V}_0^\Theta\right) = V_0^\Theta / B_0, \; 0 \leq t \leq T.$$

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- Similarly for $\mathbb{Q}'$

$$\mathbb{E}'_*\left(\frac{V_T^\Theta}{B_T}\right) = \mathbb{E}'_*\left(\tilde{V}_T^\Theta\right) = \mathbb{E}'_*\left(\tilde{V}_0^\Theta\right) = V_0^\Theta / B_0, \; 0 \le t \le T.$$

Proof ($\Rightarrow$) ($\exists\omega_j, \mathbb{Q}(\{\omega_j\}) \neq \mathbb{Q}'(\{\omega_j\})$)

- Therefore, **for any admissible strategy** $\Theta = (\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)})_{0 \leq t \leq T}$,

$$\mathbb{E}_* \left(V_T^{\Theta} \right) = \mathbb{E}'_* \left(V_T^{\Theta} \right).$$

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- In view that the market is complete, for every European contingent claim ξ, there exists an **admissible strategy** $\Theta = (\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)})_{0 \leq t \leq T}$, such that

$$\xi = V_T^{\Theta}.$$

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$$\xi = V_T^{\Theta}.$$

- Using the previous relation, we deduce that for **every European contingent claim** ξ.

$$\mathbb{E}_*(\xi) = \mathbb{E}'_*(\xi).$$

Proof $(\Rightarrow)$ $(\exists \omega_j, \mathbb{Q}(\{\omega_j\}) \neq \mathbb{Q}'(\{\omega_j\}))$

- Recall that in view of $\mathscr{F}_T = \mathscr{F} = 2^{\Omega}$, any mapping $\omega \mapsto \xi(\omega)$ is a European contingent claim and from above, it holds that

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- In particular, if we let

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- In particular, if we let

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- We conclude that

$$\mathbb{Q}(\{\omega_j\}) := \mathbb{E}_*\left(\mathbf{1}_{\{\omega_j\}}\right) = \mathbb{E}'_*\left(\mathbf{1}_{\{\omega_j\}}\right) =: \mathbb{Q}'(\{\omega_j\}),$$

which is absurd. ■

Completeness of the Binomial Model

- In this part we will consider Multi-Step Binomial model, i.e. only a bond and a risky asset are traded.
- The bond price is given by

$$B_t = (1+r)^t, \quad t = 0, 1, \ldots, T.$$

- The price of the risky asset is given by $S_0 > 0$ (non-random) and

$$S_t = S_{t-1}(1 + K_S(t)), \; t = 1, \ldots, T.$$

where

$$K_S(t) = \begin{cases} R_u & \text{with probability } p; \\ R_d & \text{with probability } 1-p, \end{cases}$$

with the relation $R_d < R_u$ as well as

$$\mathscr{F}_0 = \{\emptyset, \Omega\}, \; \mathscr{F}_t = \sigma(K_S(1), \ldots, K_S(t)), \; 1 \le t \le T.$$

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- Recall that a strategy $\Theta = (\varphi_t, \theta_t)_{0 \le t \le T}$ is admissible if:
  1. It is self-financed: ($\theta_{t+1} = \#$ of shares held between t and $t+1$)

$$\varphi_t B_t + \theta_t S_t = V_t^{\Theta} = \varphi_{t+1} B_t + \theta_{t+1} S_t.$$

  2. Non-anticipative: Θ_t depends on the market information up to time t.
  3. It has a limited credit line: There is a non-random constant $C > 0$, such that

$$V_t^{\Theta} \ge -C, \ \forall\, 0 \le t \le T.$$

Completeness of the Binomial Model

- When $T = 1$ we have seen (Lecture 5) that

$$\phi(S_1) = V_1^{\Theta} = \varphi B_1 + \theta S_1. \tag{3}$$

holds if and only if

$$\begin{aligned} \zeta^u &:= \phi\left[S_0(1+R_u)\right] = \varphi B_1 + \theta S_0(1+R_u); \\ \zeta^d &:= \phi\left[S_0(1+R_d)\right] = \varphi B_1 + \theta S_0(1+R_d). \end{aligned}$$

- Which can be written as

$$\begin{bmatrix} B_1 & S_0(1+R_u) \\ B_1 & S_0(1+R_d) \end{bmatrix} \begin{bmatrix} \varphi \\ \theta \end{bmatrix} = \begin{bmatrix} \zeta^u \\ \zeta^d \end{bmatrix}$$

Completeness of the Binomial Model

- The previous system has a unique solution given by

$$\theta = \frac{\zeta^u - \zeta^d}{S_0(R_u - R_d)};$$

$$\varphi = \frac{1}{B_1}\left\{\frac{\zeta^d(1 + R_u) - \zeta^u(1 + R_d)}{R_u - R_d}\right\}.$$

- Moreover, for $q^* = \frac{r - R_d}{R_u - R_d}$

$$\theta S_0 + \varphi B_0 = \frac{1}{1 + r}\left\{\zeta^u q^* + \zeta^d(1 - q^*)\right\}.$$

Completeness of the BM $\left(\begin{bmatrix} B_1 & S_0(1+R_u) \\ B_1 & S_0(1+R_d) \end{bmatrix} \begin{bmatrix} \varphi \\ \theta \end{bmatrix} = \begin{bmatrix} \zeta^u \\ \zeta^d \end{bmatrix}\right)$

- Suppose that we have found the replicating strategy up to time $t = T - 1$, i.e. $(\varphi_t, \theta_t)_{0 \le t \le T-1}$ is given.

Completeness of the BM $\left(\begin{bmatrix} B_1 & S_0(1+R_u) \\ B_1 & S_0(1+R_d) \end{bmatrix} \begin{bmatrix} \varphi \\ \theta \end{bmatrix} = \begin{bmatrix} \zeta^u \\ \zeta^d \end{bmatrix}\right)$

- Suppose that we have found the replicating strategy up to time $t = T - 1$, i.e. $(\varphi_t, \theta_t)_{0 \le t \le T-1}$ is given.
- Thus, it is left to find (φ_T, θ_T) such that

$$\phi(S_T) = V_T^{\Theta} = \varphi_T B_T + \theta_T S_T. \tag{4}$$

Completeness of the BM $\left(\begin{bmatrix} B_1 & S_0(1+R_u) \\ B_1 & S_0(1+R_d) \end{bmatrix}\begin{bmatrix}\varphi \\ \theta\end{bmatrix} = \begin{bmatrix}\zeta^u \\ \zeta^d\end{bmatrix}\right)$

- Suppose that we have found the replicating strategy up to time $t = T-1$, i.e. $(\varphi_t, \theta_t)_{0\le t\le T-1}$ is given.
- Thus, it is left to find (φ_T, θ_T) such that

$$\phi(S_T) = V_T^{\Theta} = \varphi_T B_T + \theta_T S_T. \tag{4}$$

- Using that $S_T = S_{T-1}(1+K_S(T))$, we have that (4) holds if and only if

$$\zeta_1^u(S_{T-1}) = \varphi_T B_T + \theta_T S_{T-1}(1+R_u);$$
$$\zeta_1^d(S_{T-1}) = \varphi_T B_T + \theta_T S_{T-1}(1+R_d).$$

where

$$\zeta_1^u(y) := \phi\left[y(1+R_u)\right],\ \zeta_1^d(y) := \phi\left[y(1+R_d)\right].$$

Completeness of the BM $\left(\begin{bmatrix} B_1 & S_0(1+R_u) \\ B_1 & S_0(1+R_d) \end{bmatrix} \begin{bmatrix} \varphi \\ \theta \end{bmatrix} = \begin{bmatrix} \zeta^u \\ \zeta^d \end{bmatrix}\right)$

- Therefore,

$$\phi(S_T) = V_T^{\ominus} = \varphi_T B_T + \theta_T S_T. \tag{5}$$

  holds if and only if

$$\zeta_1^u(S_{T-1}) = \varphi_T B_T + \theta_T S_{T-1}(1+R_u);$$
$$\zeta_1^d(S_{T-1}) = \varphi_T B_T + \theta_T S_{T-1}(1+R_d).$$

- In matrix notation

$$\begin{bmatrix} B_T & S_{T-1}(1+R_u) \\ B_T & S_{T-1}(1+R_d) \end{bmatrix} \begin{bmatrix} \varphi_T \\ \theta_T \end{bmatrix} = \begin{bmatrix} \zeta_1^u(S_{T-1}) \\ \zeta_1^d(S_{T-1}) \end{bmatrix}$$

Completeness of the BM $\left(\theta = \frac{\zeta^u - \zeta^d}{S_0(R_u - R_d)};\ \varphi = \frac{1}{B_1}\left\{\frac{\zeta^d(1+R_u) - \zeta^u(1+R_d)}{R_u - R_d}\right\}\right)$

- Replacing S_0, ζ^u, ζ^d and B_1 by S_{T-1}, $\zeta_1^u(S_{T-1})$, $\zeta_1^d(S_{T-1})$, and B_T, respectively, in the one-step case, we get that

$$\theta_T = \frac{\zeta_1^u(S_{T-1}) - \zeta_1^d(S_{T-1})}{S_{T-1}(R_u - R_d)};$$

$$\varphi_T = \frac{1}{B_T}\left\{\frac{\zeta_1^d(S_{T-1})(1 + R_u) - \zeta_1^u(S_{T-1})(1 + R_d)}{R_u - R_d}\right\}.$$

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- In a similar way, we obtain

$$\varphi_T B_{T-1} + \theta_T S_{T-1} = \frac{1}{1+r}\left[\zeta_1^u(S_{T-1})q^* + \zeta_1^d(S_{T-1})(1-q^*)\right].$$

Completeness of the BM $\left(\theta = \frac{\zeta^u - \zeta^d}{S_0(R_u - R_d)};\ \varphi = \frac{1}{B_1}\left\{\frac{\zeta^d(1+R_u) - \zeta^u(1+R_d)}{R_u - R_d}\right\}\right)$

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$$\begin{aligned}\theta_T =& \frac{\zeta_1^u(S_{T-1}) - \zeta_1^d(S_{T-1})}{S_{T-1}(R_u - R_d)};\\ \varphi_T =& \frac{1}{B_T}\left\{\frac{\zeta_1^d(S_{T-1})(1+R_u) - \zeta_1^u(S_{T-1})(1+R_d)}{R_u - R_d}\right\}.\end{aligned}$$

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$$\varphi_T B_{T-1} + \theta_T S_{T-1} = \frac{1}{1+r}\left[\zeta_1^u(S_{T-1})q^* + \zeta_1^d(S_{T-1})(1-q^*)\right].$$

- (φ_T, θ_T) are functions exclusively of S_{T-1} and do not rely on $(\varphi_t, \theta_t)_{0 \le t \le T-1}$.

Completeness of the BM

- Since $\Theta = (\varphi_t, \theta_t)_{0 \leq t \leq T}$ must be self-financed, necessarily

$$\begin{aligned}
\varphi_{T-1} B_{T-1} + \theta_{T-1} S_{T-1} =& V^{\Theta}_{T-1} \\
=& \varphi_T B_{T-1} + \theta_T S_{T-1} \\
=& \frac{1}{1+r} \left[\zeta_1^u(S_{T-1}) q^* + \zeta_1^d(S_{T-1})(1-q^*) \right].
\end{aligned}$$

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- Which, in view that $S_{T-1} = S_{T-2}(1 + K_S(T-2))$, holds if and only if

$$\begin{aligned}\zeta_2^u(S_{T-2}) &= \varphi_{T-1} B_{T-1} + \theta_{T-1} S_{T-2}(1 + R_u); \\ \zeta_2^d(S_{T-1}) &= \varphi_{T-1} B_{T-1} + \theta_{T-1} S_{T-2}(1 + R_d).\end{aligned}$$

- Where

$$\begin{aligned}\zeta_2^u(y) &:= \frac{1}{1+r}\left[\zeta_1^u\left[y(1+R_u)\right] q^* + \zeta_1^d\left[y(1+R_u)\right](1-q^*)\right]; \\ \zeta_2^d(y) &:= \frac{1}{1+r}\left[\zeta_1^u\left[y(1+R_d)\right] q^* + \zeta_1^d\left[y(1+R_d)\right](1-q^*)\right].\end{aligned}$$

Completeness of the BM $\left(\begin{bmatrix} B_1 & S_0(1+R_u) \\ B_1 & S_0(1+R_d) \end{bmatrix} \begin{bmatrix} \varphi \\ \theta \end{bmatrix} = \begin{bmatrix} \zeta^u \\ \zeta^d \end{bmatrix}\right)$

- Therefore

$$V^{\ominus}_{T-1} = \varphi_T B_{T-1} + \theta_T S_{T-1},$$

if and only if

$$\begin{bmatrix} B_{T-1} & S_{T-2}(1+R_u) \\ B_{T-1} & S_{T-2}(1+R_d) \end{bmatrix} \begin{bmatrix} \varphi_{T-1} \\ \theta_{T-1} \end{bmatrix} = \begin{bmatrix} \zeta_2^u(S_{T-2}) \\ \zeta_2^d(S_{T-2}) \end{bmatrix}.$$

- As before

$$\begin{aligned} \theta_{T-1} &= \frac{\zeta_2^u(S_{T-2}) - \zeta_1^d(S_{T-2})}{S_{T-2}(R_u - R_d)}; \\ \varphi_{T-1} &= \frac{1}{B_{T-1}} \left\{ \frac{\zeta_2^d(S_{T-2})(1+R_u) - \zeta_2^u(S_{T-2})(1+R_d)}{R_u - R_d} \right\}; \\ V^{\ominus}_{T-2} &= \frac{1}{1+r} \left[\zeta_2^u(S_{T-2})q^* + \zeta_2^d(S_{T-2})(1-q^*) \right]. \end{aligned}$$

Completeness of the Binomial Model

- Iterating the previous reasoning give us that for $2 \leq i \leq T-1$

$$V^{\Theta}_{T-i} = \varphi_{T-i+1} B_{T-i} + \theta_{T-i+1} S_{T-i},$$

which holds iff

$$\varphi_{T-i} B_{T-i} + \theta_{T-i} S_{T-i} = \frac{1}{1+r} \underbrace{\left[\zeta_i^u(S_{T-i}) q^* + \zeta_i^d(S_{T-i})(1-q^*)\right]}_{\text{Given from the previous step.}}.$$

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- Where

$$\zeta_1^u(y) := \phi\left[y(1+R_u)\right], \; \zeta_1^d(y) := \phi\left[y(1+R_d)\right],$$

and for $i = 2, \ldots, T-1$

$$\zeta_i^u(y) := \frac{1}{1+r}\left[\zeta_{i-1}^u\left[y(1+R_u)\right]q^* + \zeta_i^d\left[y(1+R_u)\right](1-q^*)\right];$$

$$\zeta_i^d(y) := \frac{1}{1+r}\left[\zeta_{i-1}^u\left[y(1+R_d)\right]q^* + \zeta_i^d\left[y(1+R_d)\right](1-q^*)\right].$$

Completeness of the Binomial Model

- Which means that for $2 \leq i \leq T-1$

$$V^{\ominus}_{T-i} = \varphi_{T-i+1} B_{T-i} + \theta_{T-i+1} S_{T-i},$$

which holds iff

$$\begin{bmatrix} B_{T-i} & S_{T-i-1}(1+R_u) \\ B_{T-i} & S_{T-i-1}(1+R_d) \end{bmatrix} \begin{bmatrix} \varphi_{T-i} \\ \theta_{T-i} \end{bmatrix} = \begin{bmatrix} \zeta^u_{i+1}(S_{T-i-1}) \\ \zeta^d_{i+1}(S_{T-i-1}) \end{bmatrix}.$$

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- Therefore, as before

$$\begin{aligned} \varphi_{T-i} =& \frac{\zeta^u_{i+1}(S_{T-i-1}) - \zeta^d_{i+1}(S_{T-i-1})}{S_{T-i-1}(R_u - R_d)}; \\ \theta_{T-i} =& \frac{1}{B_{T-i}} \left\{ \frac{\zeta^d_{i+1}(S_{T-i-1})(1+R_u) - \zeta^u_{i+1}(S_{T-i-1})(1+R_d)}{R_u - R_d} \right\}; \\ V^{\ominus}_{T-i} =& \frac{1}{1+r} \left[\zeta^u_{i+1}(S_{T-i-1}) q^* + \zeta^d_{i+1}(S_{T-i-1})(1-q^*) \right]. \end{aligned}$$

Completeness of the Binomial Model

- A very similar method can be used to find a replicating strategy $\Theta = (\varphi_t, \theta_t)_{0 \le t \le T}$ for a general contingent claim of the form

$$\xi = \Phi(K_S(1), \ldots, K_S(T)), \;\; (\mathscr{F}_T = \sigma\left(K_S(1), \ldots, K_S(T)\right)).$$

- The main difference is that now θ_{T-t} and φ_{T-t} are functions of

$$(S_0, \ldots, S_{T-t-1}),$$

and not only of S_{T-t-1}.

Completeness of the Binomial Model

- In any case, the market described by the Multi-Step Binomial is always complete.
- By the First and the Second FTA, then the market described by this model is arbitrage free and admits a unique martingale measure whenever $R_u < r < R_d$.
- Such a martingale measure satisfies that, under $\mathbb{Q}$, the returns $K_S(t)$ are i.i.d. such that

$$\mathbb{Q}(K_S(t) = R_u) = q^*; \; \mathbb{Q}(K_S(t) = R_d) = 1 - q^*,$$

  where $q^* = \frac{r - R_d}{R_u - R_d}$.
- Therefore, **the unique arbitrage free price** at time $0 \leq t \leq T$ of a derivative with pay-off ξ is given by

$$\xi_t = \frac{1}{(1+r)^{T-t}} \mathbb{E}_* \left(\xi | \mathscr{F}_t \right).$$

The Monte Carlo Method

Motivation

- We have seen that in the Multi-Step Binomial model, that **the** arbitrage-free price of the derivative $\xi = \phi(S_T)$ at time $0 \leq t \leq T$, is given by

$$\xi_t = F(t, S_t).$$

where

$$F(t,y) := \frac{1}{(1+r)^{T-t}} \mathbb{E}_* \left\{ \phi\left(S^y_{T-t}\right) \right\}, \quad y \geq 0, 0 \leq t \leq T,$$

- Moreover $(S^y_t)_{0 \leq t \leq T}$ is a process satisfying that $S^y_0 = y$ and

$$S^y_t = S^y_{t-1}(1 + K_S(t)), \; t = 1, \ldots, T.$$

Motivation

- We have also shown that

$$S_t^y = y(1+R_u)^{N_t}(1+R_d)^{t-N_t}, \quad N_t \sim \mathrm{Bin}(t, q^*).$$

- Therefore

$$\begin{aligned}
F(t,y) &= \frac{1}{(1+r)^{T-t}} \mathbb{E}_* \left\{ \phi\left(S_{T-t}^y\right) \right\}, \\
&= \frac{1}{(1+r)^{T-t}} \mathbb{E}_* \left\{ \phi\left[y(1+R_u)^{N_{T-t}}(1+R_d)^{T-t-N_{T-t}} \right] \right\} \\
&= \frac{1}{(1+r)^{T-t}} \sum_{x=0}^{T-t} \phi\left[y(1+R_u)^x(1+R_d)^x\right] p_T(x;q^*).
\end{aligned}$$

- Where

$$p_{T-t}(x;q^*) = \binom{T-t}{x} (q^*)^x (1-q^*)^{T-t-x}.$$

The method

- Suppose that we want to estimate

$$\mu = \mathbb{E}(X).$$

- The Law of Large Numbers dictates that if $(X_n)_{n\geq 1}$ is a sequence of i.i.d. random variables such that $X_n \sim X$, then

$$\hat{\mu}_n := \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\mathbb{P}} \mu, \text{ as } n \to \infty.$$

- That is, for all $\varepsilon > 0$

$$\mathbb{P}\left(|\hat{\mu}_n - \mu| > \varepsilon\right) \to 0 \text{ as } n \to \infty.$$

The method

- Moreover, by the Central Limit Theorem, if $\mathrm{Var}(X) = \sigma^2 < \infty$, then

$$\sqrt{n}\,(\hat{\mu}_n - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2), \text{ as } n \to \infty.$$

- That is for all $x \in \mathbb{R}$

$$\mathbb{P}\left(\sqrt{n}\,(\hat{\mu}_n - \mu) \leq x\right) \to \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{1}{2\sigma^2} y^2} dy.$$

- This can be written as

$$\text{Error of estimation} = \hat{\mu}_n - \mu = \mathrm{O}_{\mathbb{P}}(1/\sqrt{n}).$$

The method

- In particular, if

$$\mu = \mathbb{E}_* \left\{ \phi\left(S^y_{T-t} \right) \right\}, \quad S^y_{T-t} = y(1+R_u)^{N_{T-t}}(1+R_d)^{T-t-N_{T-t}}$$

- Then

$$\hat{\mu}_n := \frac{1}{n} \sum_{i=1}^{n} \phi\left[y(1+R_u)^{X_i}(1+R_d)^{T-t-X_i} \right] \xrightarrow{\mathbb{P}} \mu, \quad \text{as } n \to \infty.$$

- Where

$$X_i \sim \mathrm{Bin}(T-t, q^*).$$

An Algoritm to Approximate $F(t, y)$

Algorithm 1: Approximation of the price function associated to the simple derivative $\phi(S_T)$

Input : $n, T \in \mathbb{N}, R_u, R_d, r$ and $\phi : \mathbb{R} \to \mathbb{R}, t, y$.

Output: Approx. of the price function

$$F(t, y) = \frac{1}{(1+r)^{T-t}} \mathbb{E}_* \left\{ \phi \left(S^y_{T-t} \right) \right\}$$

Define Risk-Neutral Measure: $q^* = \frac{r - R_d}{R_d - R_u}$;

Initialize Sum: $Z \leftarrow 0$;

for $i = 1$ **to** n **do**

  Generate: $X_i \sim \mathrm{Bin}(T - t, q^*)$;

  $\xi_i \leftarrow \phi \left[y(1 + R_u)^{X_i} (1 + R_d)^{T-t-X_i} \right]$

  Update: $Z \leftarrow Z + X_i$;

end

Return: $F(t, y) = \frac{1}{n} Z \frac{1}{(1+r)^{T-t}}$.