Financial Engineering
Lecture 2

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General comments

- Jeg vil gerne snakke lidt Dansk i første del af forelæsningen.. Jeg håber at I er tålmodig med mig.
- Questions or comments about the previous lecture and/or exercise set?
- Typos in the book/slides?
- I suggest to discuss your questions in the room where there lecture takes place.

Review of the previous lecture

What did we do in the previous lecture?

Review of the previous lecture

- Basics of Probability Theory: Probability spaces, Random Variables, Stochastic Processes.
- Market Assumptions: It is frictionless, shares do not pay dividends, and divisibility as well as liquidity of assets.
- Interest: Simple, Compounded and Continuously Compounded.
- Zero-Coupon Bonds: Definition and how we price them.

Outline for today

- Coupon Bonds.
- Risky Assets.
- Returns.
- One-Step Binomial model.
- Portfolios in discrete-time financial markets.

Coupon bonds

- Recall that a zero-coupon bond is a financial contract that promises a single payment $F > 0$ at time $T > 0$.
- When interest is compounded annually at rate $r > 0$, we show that the initial value of the bond must be

$$B_0 = \frac{F}{(1+r)^T}.$$

- Otherwise the investor and issuer won't agree on the transaction.

Coupon Bonds

- A **coupon bond** also guarantees to the owner a payment $F > 0$ at time $T > 0$.
- However, in addition to this, at times $t = 1, \ldots, T$ the owner will receive a fixed amount $C > 0$.
- The value C is known as the coupon.
- What is a fair value B_0 for such an asset?

Pricing Coupon Bonds

- How did we price the zero-coupon bond?
- As buyer:

We compare the costs of generating $F > 0$ at time $T > 0$ via a bank account.

- As seller:

The minimum amount required in order to pay $F > 0$ at time $T > 0$.

Pricing Coupon Bonds

- Let us start assuming that $T = 2$ and that annually compound interest is available at rate $r > 0$.
- As before, the buyer will compare the price $B_0 > 0$ against the cost of receiving the same benefits from a bank account, i.e.
  1. Withdraw $C > 0$ after a year.
  2. At the end of the second year our bank account must have $F + C$ left.

Pricing Coupon Bonds

- In order to be able to get $C > 0$ after a year, we must put in the bank
$$\frac{C}{1+r}.$$

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$$\frac{C+F}{(1+r)^2}.$$

Pricing Coupon Bonds

- In order to be able to get $C > 0$ after a year, we must put in the bank
$$\frac{C}{1+r}.$$

- Similarly, to have $F + C$ in our account after 2 years, we must make a deposit of
$$\frac{C+F}{(1+r)^2}.$$

- Thus, in order to get the same payments as the coupon bond, we must put in our bank
$$\frac{C}{1+r} + \frac{C+F}{(1+r)^2}.$$

Pricing Coupon Bonds

- Therefore, as buyer we expect

$$B_0 \leq \frac{C}{1+r} + \frac{C+F}{(1+r)^2}.$$

- Interchanging roles between buyer and seller gives us that the minimum amount we will accept for the bond is

$$B_0 \geq \frac{C}{1+r} + \frac{C+F}{(1+r)^2}.$$

Pricing Coupon Bonds

- Hence, the fair price of the coupon is

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Pricing Coupon Bonds

- Hence, the fair price of the coupon is

$$B_0 = \frac{C}{1+r} + \frac{C+F}{(1+r)^2}.$$

- Observe that

$$B_0 = \frac{\xi_T}{(1+r)^2},$$

where

$$\xi_T := C(1+r) + C + F,$$

is the total pay-off at the maturity time $T = 2$.

Pricing Coupon Bonds

- Repeating this argument for general T gives us that

$$B_0 = \sum_{i=1}^{T} \frac{C}{(1+r)^i} + \frac{F}{(1+r)^T} = \frac{\xi_T}{(1+r)^T},$$

where the pay-off ξ_T at time T is given by

$$\xi_T := \sum_{i=1}^{T} C(1+r)^{T-i} + F.$$

- The "fair" price of the coupon bond can be written as the discounted price of the total pay-off!
- This is just a particular case of the First Fundamental Theorem of Asset Pricing which will be discussed in a later stage of the course.

Pricing Coupon Bonds

- What about the value of the bond at time $1 \leq t \leq T$?

Pricing Coupon Bonds

- What about the value of the bond at time $1 \leq t \leq T$?

$$B_t = \sum_{i=1}^{T-t} \frac{C}{(1+r)^i} + \frac{F}{(1+r)^{T-t}}.$$

Bonds and the time value of money

- The interest rates that banks usually offer are based on the type of bonds they can acquire.
- Therefore, by putting our savings in a bank account, we indirectly are buying bonds.

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- The interest rates that banks usually offer are based on the type of bonds they can acquire.
- Therefore, by putting our savings in a bank account, we indirectly are buying bonds.
- Suppose that we make a deposit of $v_0 > 0$ at time 0.
- The bank will then use your money to buy v_0/B_0 zero-coupon bonds.

Bonds and the time value of money

- Thus, at time $t = 1, \ldots, T$, the investment v_0/B_0 becomes

$$V_t = \underbrace{\frac{v_0}{B_0}}_{\text{Number of Bonds}} \times \underbrace{B_t}_{\text{The value of the bond at time } t.}$$

Bonds and the time value of money

- Thus, at time $t = 1, \dots, T$, the investment v_0/B_0 becomes

$$V_t = \underbrace{\frac{v_0}{B_0}}_{\text{Number of Bonds}} \times \underbrace{B_t}_{\text{The value of the bond at time } t.}$$

- Hence, if interest is compounded annually, we get that

$$V_t = v_0 \times \frac{B_t}{B_0} = v_0 \times (1 + r)^t, \;\; 0 \le t \le T.$$

- This result also holds if instead of buying zero-coupon bonds, the bank buys coupon bonds (See Exercise Set 2 :)).

Bonds and the time value of money

- In general

$$\frac{B_t}{B_0} = \begin{cases} (1+rt) & \text{simple interest;} \\ (1+r/m)^{tm} & \text{Conpounded interest;} \\ e^{rt} & \text{Continuously compounded interest.} \end{cases}$$

Bonds and the time value of money

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- The interest that is paid on v_0 only depends on the bond price and not on the type of interest that the bank pays off!

Bonds and the time value of money

- In particular, if $B_0 = 1$, then the price $(B_t)_{0 \leq t \leq T}$ describes the evolution of the value of money through time.
- This method is completely independent of the interest rate and the type of interest that is paid.
- A bond is a particular type of numéraire:

$$B_t > 0, \ \forall\, 0 \leq t \leq T.$$

- Numéraires are used to represent the unit in which prices are measured.
- The US treasury bonds are typically used as numéraire in international trading.

Risky assets

How stock prices look like

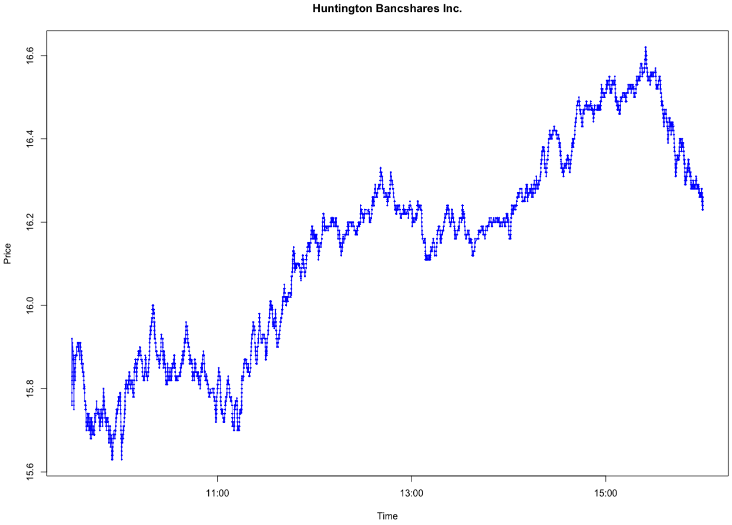

Figure: HBAN intraday prices (November 2007). Around 10,000 price movements.

How stock prices look like

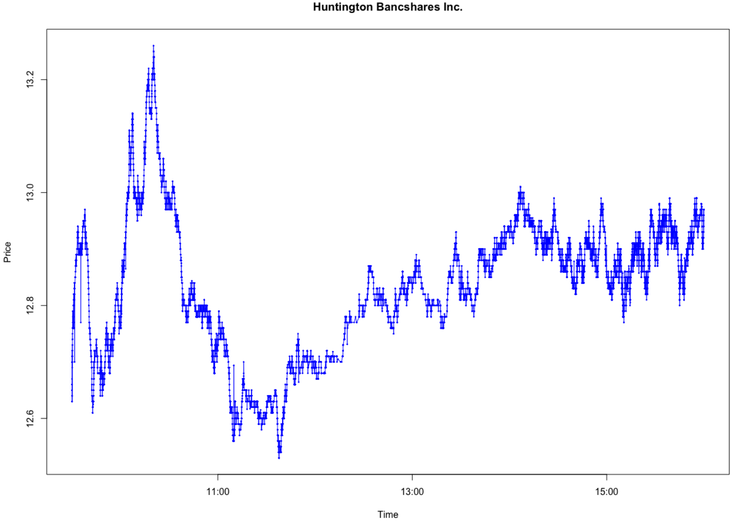

Figure: HBAN intraday prices (February 2008). Around 10,000 price movements.

How stock prices look like

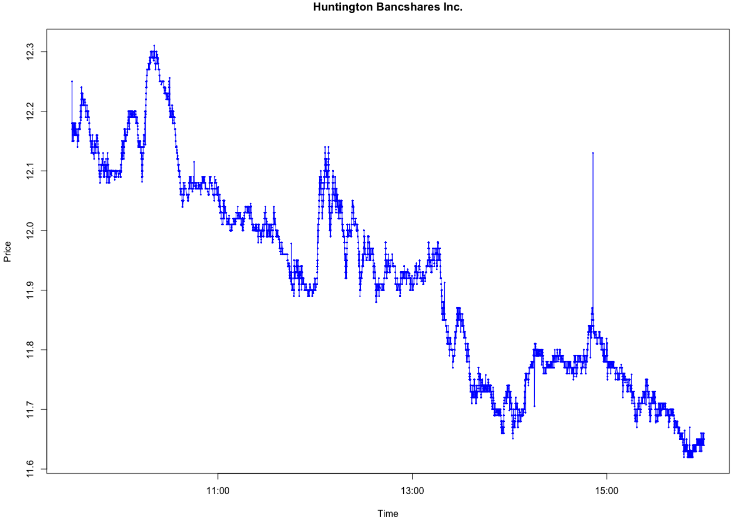

Figure: HBAN intraday prices (March 2008). Around 10,000 price movements.

How stock prices look like

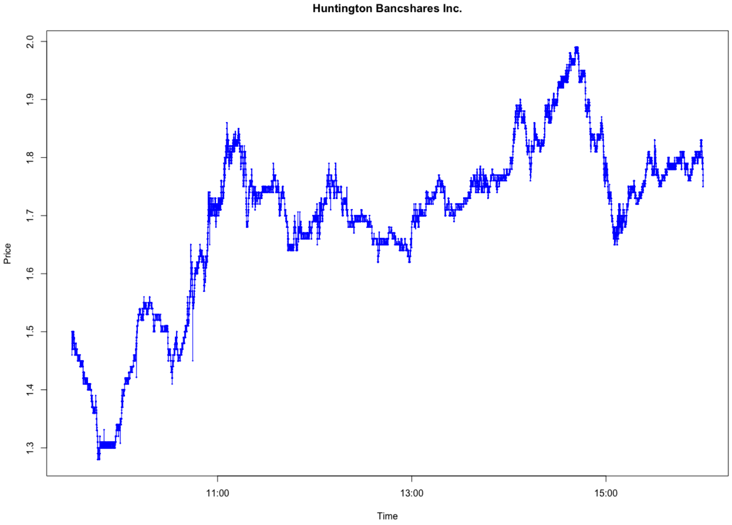

Figure: HBAN intraday prices (February 2009). Around 10,000 prices movements.

Facts about stock prices

- Prices are erratic: Random movements.
- High degree of variability (Volatility).
- Spikes or jumps.

Price process

- Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and suppose that in the market $d \in \mathbb{N}$ assets are traded as well as a bond with price B_t.
- For $j = 1, \ldots .d$, we will define and denote

$$S_t^{(j)} := \text{Price of the } jth \text{ asset at time } t, \ \ 0 \leq t \leq T.$$

- To match with our previous observations, we will assume that $S_t^{(j)}$ is a strictly positive random variable on $(\Omega, \mathscr{F}, \mathbb{P})$.
- The stochastic process $S^{(j)} = (S_t^{(j)})_{0 \leq t \leq T}$ will be called the price process of the jth asset.

Price process

- In the previous exercise set you show that

$$V_t = (1 + r/m)^{mt} v_0, \;\; 0 \leq t \leq T, v_0 \in \mathbb{R},$$

  is a random variable in every probability space.
- This is due to the fact V_t does not depend on ω.

Price process

- For instance, if $r = m = v_0 = 1$, then

$$V_t = 2^t, \;\; 0 \leq t \leq T.$$

- Therefore, if $\Omega \neq \emptyset$, then

$$X(\omega) = V_t = 2^t, \;\; \forall\, \omega \in \Omega.$$

- The random variable X is deterministic because we always know what the outcome will be.
- More formally, we will say that a random variable X is deterministic if there exists $x \in \mathbb{R}$ such that

$$X(\omega) = x, \;\; \forall\, \omega \in \Omega.$$

Price process

Definition (Risky and non-risky assets)

A financial asset is said to be risk-less if its price process is deterministic. Otherwise, we will refer to the asset as risky.

Price process

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Remark

The notation $S_t^{(j)}$ will be used exclusively to represents stock prices, which are typically risky securities. However, they may also be deterministic.

Warning!

- Be aware that there are random variables that are "almost surely constant (deterministic)".
- This means that there exists $N \in \mathscr{F}$ and $x \in \mathbb{R}$, such that $\mathbb{P}(N) = 0$ and
$$X(\omega) = x, \ \forall \omega \in N^c \ \ (\mathbb{P}(X = x) = \mathbb{P}(N^c) = 1).$$

Warning!

- For instance, if $\mathbb{P}$ is the uniform distribution on $[0,1]$, i.e. $\Omega = [0,1]$ with
$$\mathbb{P}(A) = \int_{A\cap[0,1]} dx,$$
and for all $0 \leq \omega \leq 1$ we let
$$X(\omega) := \mathbf{1}_{\{0\}}(\omega) = \begin{cases} 1 & \text{if } \omega = 0 \\ 0 & \text{otherwise} \end{cases}.$$

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$$X(\omega) := \mathbf{1}_{\{0\}}(\omega) = \begin{cases} 1 & \text{if } \omega = 0 \\ 0 & \text{otherwise} \end{cases}.$$
- X truly depends on ω, i.e. it is not deterministic. However,
$$\mathbb{P}(X \neq 0) = \mathbb{P}(X = 1) = \mathbb{P}(\{0\}) = \int_{\{0\}} dx = 0.$$
- Thus
$$\mathbb{P}(X = 0) = 1.$$

Returns

Definition (Asset returns)

Let $X = (X_t)_{0 \le t \le T}$ be an stochastic process such that

$$\mathbb{P}(X_t > 0) = 1, \;\; 0 \le t \le T.$$

The return process associated to X is defined and denoted as

$$K_X(t) := \frac{X_t - X_{t-1}}{X_{t-1}}, \;\; 1 \le t \le T,$$

and $K_X(0) = 0$.

Returns

Definition (Asset returns)

Within the same framework of the previous definition, the stochastic process defined

$$k_X(t) := \begin{cases} 0 & \text{if } t = 0; \\ \log\left(\frac{X_t}{X_{t-1}}\right) & 1 \leq t \leq T, \end{cases}$$

is known as the log-return process of X.

Returns vs Prices

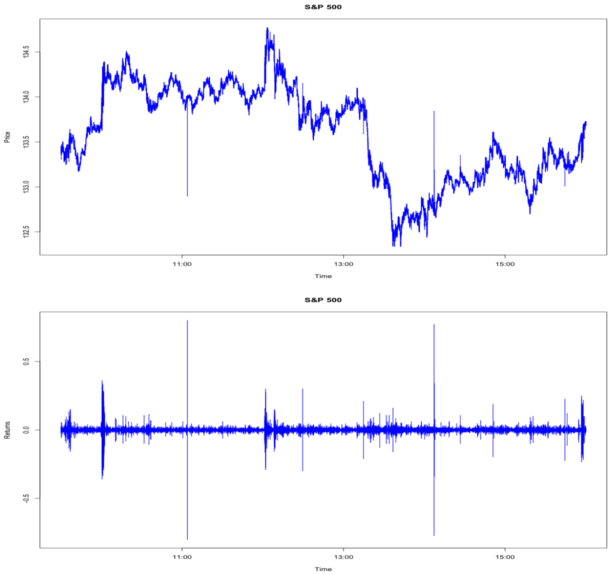

Figure: Prices and Returns of the S&P 500.

Returns vs Prices

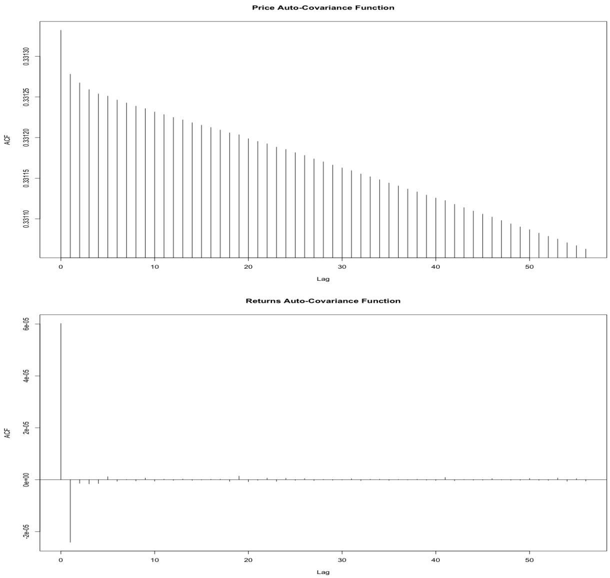

Figure: Prices and Returns of the S&P 500.

Basic return properties

1. Returns are dimensionless, i.e.

$$K_X(t) = \frac{X_t - X_{t-1}}{X_{t-1}} \frac{\text{DKK}}{\text{DKK}}, \quad k_{S^{(j)}}(t) = \log\left(\frac{X_t}{X_{t-1}} \frac{\text{DKK}}{\text{DKK}}\right)$$

2. For all $1 \leq t \leq T$

   i. $X_t = [1 + K_X(t)]\, X_{t-1} = \prod_{i=1}^{t} (1 + K_X(i)) \times X_0.$

   ii. $X_t = \exp(k_X(t)) X_{t-1} = \exp\left\{\sum_{i=1}^{t} k_X(i)\right\} \times X_0.$

One-Step Binomial model

- In this model, only one bond and one risky asset are traded, so $d = 1$.
- Additionally, we are only allow to trade once, i.e. $T = 1$.
- Furthermore, the traded bond satisfies that $B_0 = 1$, so

$$B_1 = (1 + K_B(1)).$$

One-Step Binomial model

- The main assumption here is that the stock price only takes two possible values at time T: $S_0 > 0$ and

$$S_1 = \begin{cases} s_u & \text{with probability } p; \\ s_d & \text{with probability } 1 - p. \end{cases}$$

- Where

$$0 < p < 1, \; 0 < s_d < s_u.$$

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- We can rewrite the model in terms of returns:

$$K_S(1) = \begin{cases} R_u & \text{with probability } p; \\ R_d & \text{with probability } 1 - p, \end{cases}$$

in which

$$R_d < R_u.$$

One-Step Binomial model

- What an agent would do if the bond pays more than the risky asset, i.e.
$$\mathbb{P}(K_B(1) \geq K_S(1)) = 1 \iff K_B(1) \geq R_u?$$

One-Step Binomial model

- What an agent would do if the bond pays more than the risky asset, i.e.
$$\mathbb{P}(K_B(1) \geq K_S(1)) = 1 \iff K_B(1) \geq R_u?$$

- There are two possible outcomes:
  1. The risky asset is not attractive at all, so it won't buy it.
  2. It cheats the system by short-selling the risky asset.

One-Step Binomial model

- If nobody buys the risky asset, then the demand for it decreases while the supply remains the same.

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One-Step Binomial model

- If nobody buys the risky asset, then the demand for it decreases while the supply remains the same.
- Eventually the price will drop to 0.
- Hence, there is no need to include such a risky asset as part of the market in the first place.
- Another option, perhaps more harmful, is that the agent will have the chance of using the risky asset to generate as much profit as it wants with zero initial capital.

One-Step Binomial model

- At time $t = 0$ we short-sell the risky asset and we buy

$$y = S_0 \text{ bonds } \ (B_0 = 1).$$

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- At time $t = T = 1$ we will receive

$$yB_1 = S_0(1 + K_B(1)).$$

One-Step Binomial model

- At time $t = 0$ we short-sell the risky asset and we buy
$$y = S_0 \text{ bonds } \ (B_0 = 1).$$
- In terms of our initial capital, this means that
$$V_0 = -S_0 + S_0 \times B_0 = 0.$$
- At time $t = T = 1$ we will receive
$$yB_1 = S_0(1 + K_B(1)).$$
- Using this money we buy the risky asset and give it back, so our wealth becomes
$$\begin{aligned} V_1 &= yB_1 - S_1 \\ &= S_0(1 + K_B(1)) - S_0(1 + K_S(1)) \\ &= S_0(K_B(1) - K_S(1)) \geq 0. \end{aligned}$$

One-Step Binomial model

- Therefore, if $K_B(1) > R_u$ then

$$V_1 = S_0(K_B(1) - K_S(1)) > 0.$$

- On the other hand, if $K_B(1) = R_u$, then

$$V_1 = \begin{cases} 0 & \text{with probability } p; \\ S_0 \underbrace{(R_u - R_d)}_{>0} & \text{with probability } 1 - p. \end{cases}$$

- Thus,

$$\mathbb{P}(V_1 > 0) = \mathbb{P}(V_1 = S_0(R_u - R_d)) = 1 - p > 0.$$

Arbitrage in One-Step Binomial model

- Note that we were able to generate an investment with zero initial capital that can generate a profit with positive probability:

$$
\begin{aligned}
i)&\ V_0 = 0;\\
ii)&\ \mathbb{P}(V_1 \geq 0) = 1;\\
iii)&\ \mathbb{P}(V_1 > 0) > 0.
\end{aligned}
$$

- Such investment (strategy) is called an arbitrage.

Arbitrage in One-Step Binomial model

- If a substantially amount of agents are speculators, then the demand for the cheap risky asset will increase.
- Therefore, in the long term, the price of the stock will necessarily rise so that the relation

$$K_B(1) \geq R_u,$$

won't hold anymore.

- Hence, we deduce that in the steady state of the economy it must hold that

$$K_B(1) < R_u.$$

Arbitrage in One-Step Binomial model

- In the exercise set of today you must show that if

$$K_B(1) \leq R_d,$$

we can create an arbitrage.

- Hence, we see that in a "equilibrated economy", necessarily

$$R_u < K_B(1) < R_d.$$

Portfolios and Risk Management

Introduction

- In the previous example we saw that the wealth process can be written as

$$V_0 = yB_0 - 1 \times S_0 = (y, -1) \cdot (B_0, S_0),$$

and

$$V_1 = yB_1 - 1 \times S_1 = (y, -1) \cdot (B_1, S_1),$$

in which the notation $u \cdot v$ represents the standard inner product on $\mathbb{R}^2$.

- The vector $(y, -1)$ stands for the number of shares and bonds held during the trading period.
- This is an example of a portfolio.

Financial Markets without time information

- Fix $T > 0$ and $d \in \mathbb{N}$. **For the moment**, we will say that a finite-horizon financial market is a pair

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, \mathbb{P}), S = (B_t, S_t^{(1)} \dots, S_t^{(d)})_{0 \le t \le T} \right\},$$

consisting of

1. A probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
2. $S^{(j)}$ is the price process of the jth asset traded in the market.
3. B is a numéraire (e.g. bonds), i.e.

$$\mathbb{P}(B_t > 0) = 1, \;\; 0 \le t \le T.$$

Portfolio and Financial Strategies

Definition (Portfolio and Strategies)

Let

$$\mathfrak{M} = \left\{(\Omega, \mathscr{F}, \mathbb{P}), P = (B_t, S_t^{(1)} \dots, S_t^{(d)})_{0 \le t \le T}\right\},$$

be a finite-horizon financial market. A portfolio in $\mathfrak{M}$ is a $(d+1)$-dimensional vector

$$\Theta_t = (\varphi_t, \theta_t^{(1)}, \dots, \theta_t^{(d)}),$$

in which

$$\Theta_t^j = \text{Number of shares of the } j\text{th asset held between time } t-1 \text{ and } t.$$

for $j = 1, \dots, d+1$. The collection $\Theta = (\Theta_t)_{0 \le t \le T}$, with the convention that $\Theta_0 = \Theta_1$ is termed as a strategy.

Wealth process associated to a portfolio

Let $(\Theta_t = (\varphi_t, \theta_t^1, \dots, \theta_t^j))_{0 \le t \le T}$ be a strategy on the market

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, \mathbb{P}), S = (B_t, S_t^{(1)} \dots, S_t^{(d)})_{0 \le t \le T} \right\}.$$

The wealth process associated to Θ is defined and denoted by

$$V_t^{\Theta} := \varphi_t B_t + \sum_{j=1}^{d} \theta_t^{(j)} S_t^{(j)} = \Theta_t \cdot P_t, \ \ 0 \le t \le T.$$

Measuring the Risk

- Unless the market allows arbitrage, it is clear that any strategy carries a risk: it's random so we cannot predict our profit or losses with certainty.
- How do we measure the risk?
- It depends on the agent!
- Some agents wants to maximizes profit while reducing the variation of such investment: Mean-Variance approach.
- Some agents would prefer to minimizes their chances of losing money: Value At Risk approach.
- In the next lecture we will see how these two ways of measuring the risk relate to each other.