Financial Engineering
Lecture 6

Orimar Sauri

Department of Mathematics
Aalborg University

AAU
March 20, 2020

Review of the First Half of the Course

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- **Market Assumption 1:** It is is frictionless and shares do not pay dividends. Moreover, an investor can buy, sell and hold any number $x \in \mathbb{R}$ of financial assets.

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  1. Risk-Free Assets: Are those assets whose price is completely deterministic. Our key example are bonds with price process satisfy that

$$\frac{B_t}{B_0} = \begin{cases} (1+rt) & \text{Simple interest;} \\ (1+r/m)^{tm} & \text{Conpounded interest;} \\ e^{rt} & \text{Continuously compounded interest.} \end{cases}$$

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  2. Risky Assets: Any financial asset whose future value is random, e.g. an asset whose price tomorrow satisfy that

$$S_1 = \begin{cases} s_u & \text{with probability } 0 < p < 1; \\ s_d & \text{with probability } 1-p. \end{cases}$$

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- **Portfolio:** It is a $(d+1)$-dimensional vector

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  Optimization Problem:

$$\arg\min_{\mathbf{w}\in\mathbb{R}^{d+1}} \mathcal{R}(\mathbf{w}\cdot\mathbf{K}_P).$$

  Subject to:

$$\sum_{j=0}^{d} w_j = 1, \;\; \mathbb{E}\left[U(\mathbf{w}\cdot\mathbf{K}_P)\right] = \mu, \;\; \mu\in\mathbb{R},$$

  where

$$\mathcal{R} \text{ is a measure of risk.}$$

$$U(\mathbf{w}\cdot\mathbf{K}_P) \text{ is the utility of such strategy.}$$

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- We expressed them as a function

$$\rho(X) \in \mathbb{R};$$

whose input X is a random variable and with output a real number.

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  3. **Expected Shortfall**: The expected shortfall measures the expected losses given that $\mathrm{VaR}_\alpha(X)$ was not enough to cover our position on X:
     $$\mathrm{CVaR}_\alpha(X) = -\mathbb{E}\left[X \mid X + \mathrm{VaR}_\alpha(X) \leq 0\right], \text{ if } X \text{ is continuous.}$$

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- All these three components are encompassed on the pay-off function $\Phi : \prod_{t=0}^{T} \mathbb{R}^{d+1} \to \mathbb{R}$

  Pay-off or earnins $= \Phi\left(P_0, P_1, \ldots, P_T\right)$, where $P_t = \left(B_t, S_t^{(1)} \ldots, S_t^{(d)}\right)$.

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- In the One-Step Binomial Model the following holds:
  1. **First Fundamental Theorem:** The market is arbitrage free if and only if there is a risk-neutral measure.
  2. **Second Fundamental Theorem:** Every financial derivative can be replicated.
  3. **Pricing:** Let ξ_1 be a derivative. The extended market with prices

     $$\tilde{P} = (B_t, S_t, \xi_t)_{t=0,1},$$

     is arbitrage free if and only if there exists a risk neutral probability $(q^*, 1-q^*)$ and the price is the expected value w.r.t. the risk neutral measure of the discounted pay-off, that is

     $$\xi_0 = \mathbb{E}_* \left[\frac{\xi_1}{B_1} \right].$$

Outline

- Financial Markets with Information.
- Admissible strategies and arbitrage.
- Martingales.

Financial Markets with Information and Investment Strategies

Information and σ-fields

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- For instance, in the Multi-Step Binomial model with $T = 3$, the sets

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express the possible outcomes in the situation in which the price either went up or down at time $t = 1$.

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  express the possible outcomes in the situation in which the price either went up or down at time $t = 1$.
- We have in this situation that

$$B_u, B_d \subseteq \sigma(\underbrace{K_S(1)}_{\text{Return at time } t=1}).$$

Information and σ-fields

- Repeating this heuristics, we construct a family of σ-fields defined as

$$
\begin{aligned}
\mathscr{F}_0 &= \{\Omega, \emptyset\}\,; \\
\mathscr{F}_1 &= \sigma(K_S(1)) = \{B_u, B_d, \Omega, \emptyset\}\,; \\
\mathscr{F}_2 &= \sigma(K_S(1), K_S(2)); \\
\mathscr{F}_3 &= \sigma(K_S(1), K_S(2), K_S(3)) = \mathscr{F}\,.
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- Thus, $\mathscr{F}_t$ contains the available information up to time t of the price movements.
- We also mentioned that such a collection is nested in the sense that

$$\mathscr{F}_{t-1} \subseteq \mathscr{F}_t, \;\; t = 1, 2, 3,$$

which we termed as a filtration.

Filtrations and Adapted Processes

Definition (Filtrations and Adapted Process)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. A collection of σ-algebras $\mathbb{F} = (\mathscr{F}_t)_{t=0,1,\ldots,T}$ is called **a filtration** if for all $1 \leq t \leq T$,

$$\mathscr{F}_{t-1} \subseteq \mathscr{F}_t \subseteq \mathscr{F}.$$

The quadruplet

$$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P}),$$

is termed as **a filtered probability space**. Furthermore, a stochastic process $(X_t)_{0 \leq t \leq T}$ is said to be **adapted to the filtration** $\mathbb{F}$ if

$$\sigma(X_t) \subseteq \mathscr{F}_t, \ \forall\, 0 \leq t \leq T.$$

Financial Markets with Information

Definition (Financial Markets with Information)

- Fix $T, d \in \mathbb{N}$. **A finite-horizon financial market with information** is the pair

$$\mathfrak{M} = \left\{(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P}), P = (B_t, S_t^{(1)} \ldots, S_t^{(d)})_{0 \leq t \leq T}\right\},$$

consisting of

1. A filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P})$.
2. P_t is adapted.
3. $S^{(j)}$ is the price process of the jth asset traded in the market
4. B is a numéraire (e.g. bonds), i.e.

$$\mathbb{P}(B_t > 0) = 1, \;\; 0 \leq t \leq T.$$

Financial Markets with Information

- **The Price Process** $P = (B_t, S_t^{(1)} \dots, S_t^{(d)})_{0 \le t \le T}$: How we model the evolution of prices. The asset B, typically risk-free, determines the value of money through time.

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- Since we require adaptedness on the price process, $\mathscr{F}_t$ contains the information generated by the price movements up to time t, that is

$$\sigma(P_0, P_1, \dots P_t) \subseteq \mathscr{F}_t.$$

Admissible Strategies

- Recall that our main goal in this part of the course is to assign a price to financial derivatives in a given market, say

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- In One-Step Financial Markets the concept of arbitrage was fundamental for providing a pricing method: If the pay-off of a derivative is ξ_1 then its price at time $t = 0$, ξ_0, must be such that **the augmented market**

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  does not allow arbitrage.
- Let us now try to derive a notion of arbitrage that takes into account the available information.

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- In Multi-Step Financial Markets agents will re-design their strategies based on the information available.
- Therefore, if we want to create an arbitrage, we need to update our portfolio every time that prices change, so we must to consider a collection of portfolios $(\Theta_t)_{0 \leq t \leq T}$.

Portfolio and Financial Strategies

Recall that in Lecture 2 we introduced the concept of strategies as:

Definition (Portfolio and Strategies)

A portfolio in $\mathfrak{M}$ is a $(d+1)$-dimensional vector $\Theta_t = (\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)})$, in which

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The wealth process associated to $\Theta = (\Theta_t)_{0 \le t \le T}$ is defined and denoted by

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$$\Rightarrow V_0^{\Theta} = \varphi_0 B_0 + \sum_{j=1}^{d} \theta_0^{(j)} S_0^{(j)} = \varphi_1 B_0 + \sum_{j=1}^{d} \theta_1^{(j)} S_0^{(j)}$$

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Admissible Strategies

- The previous approach does not put any restrictions on the type of strategies that can be considered an arbitrage.
- By doing this, many strategies that from an intuitive point of view should not be thought as an arbitrage opportunity, will be believed to be an arbitrage.
- As an example consider the situations:
    - Injection of capital at time $t \geq 1$.
    - Under privileged information.
    - Unlimited credit line.

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- Suppose that we are allow to trade in the market for two periods, i.e. $T = 2$.
- One risky asset is traded and its price is assumed to be strictly positive, i.e. $S_t > 0$ for all $t = 0, 1, 2$.

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- One risky asset is traded and its price is assumed to be strictly positive, i.e. $S_t > 0$ for all $t = 0, 1, 2$.
- In addition to this, the bank offers a yearly compound interest rate $r > 0$.

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- Our bank account can be thought as a risk-free asset satisfying that $B_0 = 1$ and
$$B_t = (1 + r)^t, \quad t = 0, 1, 2.$$
- We will create an strategy $(\Theta_t = (\varphi_t, \theta_t))_{t=1,2}$ that has zero initial capital and at time $t = T = 2$ it satisfies
$$V_T^{\Theta} > 0, \text{ almost surely.}$$

Injection of Capital ($B_t = (1+r)^t, S_t > 0$)

- At time $t = 0$ we do the following:
  1. We borrow $S_0 > 0$ from the bank, i.e. put

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$$\begin{aligned} V_0^{\Theta} &= \varphi_0 \underbrace{B_0}_{=1} + \theta_0 S_0 \\ &= -S_0 + S_0 = 0 \end{aligned}$$

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- Since we have injected money to our original investment, this cannot be considered an arbitrage.

Self-financed strategies

- Proper arbitrage opportunities are those who do not require us to put any money out of our pocket.
- Instead, arbitrage strategies must be able to be construct exclusively with the wealth generated by themself, i.e. they must be self-financed.

Self-financed strategies

Definition (Self-financed strategies)

Let $\Theta = (\varphi_t, \theta_t^{(1)}, \dots, \theta_t^{(d)})_{0 \le t \le T}$ be a strategy on the market

$$\mathfrak{M} = \left\{(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P}), P = (B_t, S_t^{(1)} \dots, S_t^{(d)})_{0 \le t \le T}\right\}.$$

We will say that Θ is **self-financed** if

$$V_t^{\Theta} = \varphi_{t+1} B_t + \sum_{j=1}^{d} \theta_{t+1}^{(j)} S_t^{(j)}, \quad 0 \le t \le T-1.$$

Strategies ($V_t^{\Theta} = \varphi_{t+1} B_t + \sum_{j=1}^{d} \theta_{t+1}^{(j)} S_t^{(j)}$)

- A self-financed strategy constructs the portfolio to be held between period t and $t+1$ using the wealth generated by itself at the end of period t.

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but

$$\varphi_2 B_1 + \theta_2 S_1 = S_1 \neq \underbrace{-(1+r)S_0}_{\neq 0} + S_1 = V_1^\Theta.$$

Strategies with priviliged information

- If for some reason we managed to get future information about price movements, it is then clear that we can make a risk-less profit out of this.
- If you get this type of information you are in a privileged position, or perhaps you paid for having it... quite illegal.

Strategies with priviliged information

- To see this, let us consider the Two-Step Binomial model, i.e. $T = 2$ in which the bonds satisfy that

$$B_t = (1+r)^t, \quad t = 0, 1, 2.$$

- The price of a stock is given by $S_0 > 0$ (non-random)

$$S_t = S_{t-1}(1 + K_S(t)), \; t = 1, 2,$$

where

$$K_S(t) = \begin{cases} R_u & \text{with probability } p; \\ R_d & \text{with probability } 1-p, \end{cases}$$

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  with the relation $R_d < r < R_u$.

- Suppose that we know that whenever the stock prices goes up it will go down in the next period of time.

Strategies with priviliged information

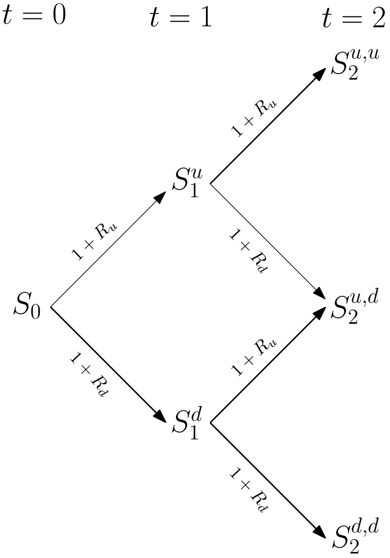

Figure: Possible outcomes without privileged information.

Strategies with priviliged information

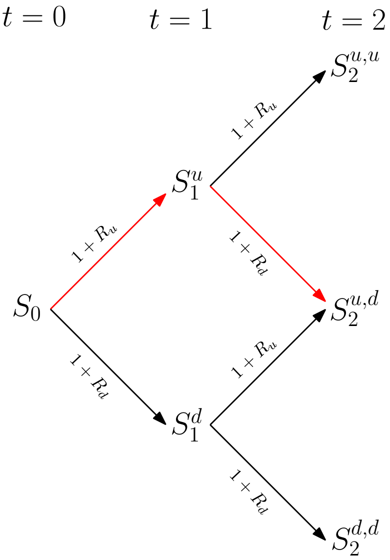

Figure: If the price goes up, we now that it will immediately goes down.

Strategies with priviliged information

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  2. At time $t = 1$, if the price went down do not buy any asset. But if the price went up then immediately short sell the risky asset and invest such amount on bonds. More precisely, let
  $$\theta_2 = \begin{cases} -1 & \text{if } S_1 = S_0(1+R_u); \\ 0 & \text{otherwise} \end{cases}, \quad \varphi_2 = \begin{cases} \frac{S_0(1+R_u)}{B_1} & \text{if } S_1 = S_0(1+R_u); \\ 0 & \text{otherwise} \end{cases}$$

Strategies with priviliged information

- Therefore, the strategy $\varphi_1 = \theta_1 = 0$ and

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  has clearly zero initial capital.
- Moreover, it is self-financed

$$\varphi_2 B_1 + \theta_2 S_1 = \begin{cases} S_1 - S_1 & \text{if } S_1 = S_0(1+R_u) \\ 0 & \text{otherwise,} \end{cases} = 0 = V_1^{\Theta}.$$

Strategies with priviliged information

- If the price went up at time $t = 1$, we know that necessarily

$$S_2 = S_0(1 + R_u)(1 + R_d),$$

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- Otherwise $V_2^{\Theta} = 0$.

Non-anticipative strategies

- Recall that

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Non-anticipative strategies

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$\Theta_{t+1}^{(j)} =$ **Number of shares** of the jth asset held between time $\mathbf{t}$ **and** $\mathbf{t+1}$

Definition (Non-anticipative strategies)

Let $\Theta = (\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)})_{0\le t\le T}$ be a strategy on the market

$$\mathfrak{M} = \left\{(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P}), P = (B_t, S_t^{(1)} \ldots, S_t^{(d)})_{0\le t\le T}\right\}.$$

We will say that Θ is **non-anticipative or predictable** if Θ_{t+1} depends only on the information at time t, that is Θ_{t+1} is $\mathscr{F}_t$-measurable.

Strategies and credit

- Another type of situation that was not considered before is when an agent has access to unlimited credit from a bank.
- If T is large, then we just keep "beating" that the price will in certain point will go high enough to make a profit.
- Let us see how can we generate a risk-less profit if we have access to unlimited credit.
- Suppose for simplicity one risky asset is traded with strictly positive price, i.e. $S_t > 0$.
- In addition to this, the bank offers a yearly compound interest rate $r > 0$.

Strategies and credit

1. At time $t = 0$ we borrow S_0 from the bank, i.e.

$$\varphi_1 = -S_0, \;\; \theta_1 = 1.$$

2. If at time $t = 1$, $V_1 > 0$ then we stop the strategy and we had won V_1.
3. Otherwise, borrow $2V_1$ from the bank to pay V_1 and invest the rest in bonds and stocks.
4. If at time $t = 2$ the result of this rebalanced give us that $V_2 > 0$, then we stop the strategy and we had won $V_2 > 0$, otherwise we repeat the procedure in 3 by borrowing from the bank.
5. Financed by the bank, you keep investing in this way and you stop as soon as $V_t > 0$.
6. Eventually you will win V_t if T is very large.

Strategies and credit

Definition (Admissible Strategies)

Let $\Theta = (\varphi_t, \theta_t^{(1)}, \dots, \theta_t^{(d)})_{0\le t\le T}$ be a strategy on the market

$$\mathfrak{M} = \left\{(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P}), P = (B_t, S_t^{(1)} \dots, S_t^{(d)})_{0\le t\le T}\right\}.$$

We will say that Θ is **admissible** if:

1. It is self-financed: It only requires an initial capital.
2. Non-anticipative: It is build up only on current market information.
3. It has a limited credit line: There is a non-random constant $C > 0$, such that
$$V_t^{\Theta} \ge -C, \ \forall\, 0 \le t \le T.$$

Strategies and credit

Definition (Arbitrage)

Consider the market

$$\mathfrak{M} = \left\{(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leq t\leq T}, \mathbb{P}), P = (B_t, S_t^{(1)} \ldots, S_t^{(d)})_{0\leq t\leq T}\right\}.$$

An admissible strategy $\Theta = (\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)})_{0\leq t\leq T}$ is said to be **an arbitrage** if:

1. It has zero initial capital:

$$V_0^{\Theta} = \varphi_0 B_0 + \sum_{j=1}^{d} \theta_0^{(j)} \cdot S_0^{(j)} = 0.$$

2. At time $t = T$, we are out of debts with 100% certainty: Almost surely

$$V_T^{\Theta} = \varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} \geq 0.$$

3. We have a chance to make a profit:

$$\mathbb{P}\left(\varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} > 0\right) > 0.$$

Discrete-time Martingales

Quick Review of the Selfstudy Session

- During the selfstudy session, we studied the concept of conditional expectation of X (with finite first moment!) given a sub-σ-algebra $\mathscr{G}$ in a given probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
- More precisely, we say that a random variable (vector) Z is the conditional expectation of X, and we write

$$Z := \mathbb{E}\left(X \middle| \mathscr{G}\right),$$

if and only if the following two condition hold:

1. Z is $\mathscr{G}$-measurable: The possible outcomes of Z are registered in $\mathscr{G}$.
2. For all $G \in \mathscr{G}$

$$\mathbb{E}\left(Z 1_G\right) = \mathbb{E}\left(X 1_G\right).$$

Quick Review of the Selfstudy Session

- Thus, if we want to verify that a given r.v. Z is the conditional expectation of X given $\mathscr{G}$ we must check the previous two conditions hold for this particular Z.
- However, we checked that in certain circumstances, we can compute $\mathbb{E}(X|\mathscr{G})$ without checking directly from the definition.

Quick Review of the Selfstudy Session

1. **Finite Partitions:** If $\{A_1, A_2, \ldots, A_N\} \subseteq \mathscr{F}$ is a partition of Ω, with $\mathbb{P}(A_n) > 0$ for all $n = 1, \ldots, N$, then for $\mathscr{G} = \sigma(A_1, A_2, \ldots, A_n)$

$$\mathbb{E}(X|\mathscr{G}) = \sum_{n=1}^{N} \mathbf{1}_{A_n} \mathbb{E}(X|A_n),$$

with

$$\mathbb{E}(X|A_n) := \frac{1}{\mathbb{P}(A_n)} \mathbb{E}(X\mathbf{1}_{A_n}).$$

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2. **Product Rule:** If X is $\mathscr{G}$-measurable, then

$$\mathbb{E}(X|\mathscr{G}) = X.$$
$$\mathbb{E}(XY|\mathscr{G}) = X\mathbb{E}(Y|\mathscr{G}),$$

provided that XY makes sense and it has finite first moment.

Quick Review of the Selfstudy Session

3. **Tower Property:** If $\mathscr{H} \subseteq \mathscr{G} \subseteq \mathscr{F}$ are two sub-σ-algebras of $\mathscr{F}$, then

$$\mathbb{E}\left[\mathbb{E}\left(X \middle| \mathscr{H}\right) \middle| \mathscr{G}\right] = \mathbb{E}\left[\mathbb{E}\left(X \middle| \mathscr{G}\right) \middle| \mathscr{H}\right] = \mathbb{E}\left(X \middle| \mathscr{H}\right).$$

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4. **Independence:** If X and Y are independent, then

$$\mathbb{E}\left(X \mid Y\right) := \mathbb{E}\left(X \mid \sigma(Y)\right) = \mathbb{E}\left(X\right).$$

Set-Up

- Fix a filtered probability space

$$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P}).$$

- Recall that this means that $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space and $(\mathscr{F}_t)_{0 \le t \le T}$ is a filtration, i.e. a nested collection of sub-σ-algebras of $\mathscr{F}$: In symbols

$$\mathscr{F}_{t-1} \subseteq \mathscr{F}_t \subseteq \mathscr{F}, \ \forall 1 \le t \le T.$$

- We are given a discrete-time stochastic process

$$X = (X_t)_{0 \le t \le T}.$$

Definition of Martingales

Definition (Martingales)

The collection $\{(X_t, \mathscr{F}_t) : t = 0, 1, \dots, T\}$ is said to be a **martingale** if

1. X is adapted, i.e.
$$\sigma(X_t) \subseteq \mathscr{F}_t,\ \forall\, 0 \le t \le T.$$
2. X_t has finite first moment, that is
$$\mathbb{E}\left(|X_t|\right) < \infty.$$
3. For all $0 \le t \le T - 1$, we have that almost surely
$$\mathbb{E}\left(X_{t+1} \middle| \mathscr{F}_t\right) = X_t.$$

Definition of Martingales

Definition (Submartingales and Supermartingales)

The collection $\{(X_t, \mathscr{F}_t) : t = 0, 1, \ldots, T\}$. Will say that such a collection is

1. A **submartingale** if conditions 1. and 2. of the previous definition holds, and almost surely

   $$\mathbb{E}\left(X_{t+1} | \mathscr{F}_t\right) \geq X_t, \ \forall\, 0 \leq t \leq T-1. \quad \text{(Things are getting better on aver}$$

2. A **supermartingale** if condition 1. and 2. of the previous definition holds, and almost surely

   $$\mathbb{E}\left(X_{t+1} | \mathscr{F}_t\right) \leq X_t, \ \forall\, 0 \leq t \leq T-1. \quad \text{(Things are getting worse on aver}$$

Example: Random Walks

- Let $(\xi_n)_{n\geq 1}$ be a family of independent and identically distributed random variables with mean zero, i.e.

$$\mathbb{E}(\xi_n) = 0, \ \forall\, n \geq 1.$$

- Put $X_0 = 0$ and for every $t = 1, 2, \ldots, T$ define

$$X_t := \sum_{n=1}^{t} \xi_n.$$

- If we let $\mathscr{F}_0 := \{\emptyset, \Omega\}$ as well as

$$\mathscr{F}_t := \sigma\left(\xi_1, \ldots, \xi_t\right), \ \ 1 \leq t \leq T,$$

then the family $\{(X_t.\mathscr{F}_t) : t = 0, 1, \ldots, T\}$ is a martingale.

Example: Martingales with given final value

- Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a given probability space and χ a random variable with finite first moment.
- Define
$$X_t := \mathbb{E}\left(\chi \middle| \mathscr{F}_t\right), \quad t = 0, 1, \ldots, T.$$
- Then the family $\{(X_t.\mathscr{F}_t) : t = 0, 1, \ldots, T\}$ is a martingale.

Example: Binomial Model under Risk-Neutrality

- Consider the *risk-neutral* Multi-Step Binomial model: $B_0 = 1$, $S_0 > 0$ is a non-random constant and

$$B_t = (1+r)^t; \;\; S_t = S_{t-1}(1 + K_S(t)), \;\; t = 1, \ldots, T.$$

- $(K_S(t))_{t=1,\ldots,T}$ are i.i.d. with

$$K_S(t) = \begin{cases} R_u & \text{with probability } q^*; \\ R_d & \text{with probability } 1 - q^*, \end{cases}$$

with the relation $R_d < r < R_u$ and

$$q^* = \frac{r - R_d}{R_u - R_d}.$$

- The discounted price

$$X_t := \frac{S_t}{B_t}, \;\; 0 \le t \le T,$$

is a martingale under the filtration

$$\mathscr{F}_0 = \{\emptyset, \Omega\}\,, \;\; \mathscr{F}_t = \sigma\left(K_S(1), \ldots, K_S(t)\right), \;\; 1 \le t \le T.$$