Financial Engineering

Exam

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F20

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Bonds

Definition (Bond)

A bond is a financial security that pays the owner a chain of predetermined payments.

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A bond is a financial security that pays the owner a chain of predetermined payments.

- Financial asset with no risk

Interest Rate

- Predetermined payments are also known as interest
- Fraction of an investment paid either ones for several periods
- Different types of interest
  1. Simple
  2. Compounded
  3. Continuously compounded

Definition (Wealth Process)

The evolution of an investment over time is called the wealth process of that investment and is denoted by

$$V = (V_t)_{0 \leq t \leq T}. \tag{1.1}$$

The initial capital is denoted by v_0, and we assume that V is a real-valued stochastic process on a given probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Simple Interest)

Let $v_0 \in \mathbb{R}$ be our initial capital. An interest on v_0 is said to be simple if it follows the wealth process

$$V_t = (1 + rt)v_0, \quad 0 \leq t \leq T. \tag{1.2}$$

Interest Rate

I will now show that the wealth process in (1.2) is indeed a stochastic process in any probability space. Any stochastic process X on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ satisfies

$$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathscr{F}, \quad \forall x \in \mathbb{R}. \tag{1.3}$$

Suppose $v_0 > 0$ and $x \geq (1 + rt)v_0$ then

$$\{\omega \in \Omega : (1 + rt)v_0 \leq x\} = \{\Omega\} \in \mathscr{F}, \tag{1.4}$$

on the other hand if $x < (1 + rt)v_0$

$$\{\omega \in \Omega : (1 + rt)v_0 \leq x\} = \{\emptyset\} \in \mathscr{F}. \tag{1.5}$$

As both Ω and $\emptyset$ is contained in any σ-algebra we have shown that the wealth process in (1.2) is a stochastic process in any probability space.

Definition (Compounded Interest)

Let $v_0 \in \mathbb{R}$ be our initial capital. An interest on v_0 is said to be compunded over $m \in \mathbb{N}$ periods if it follows the wealth process

$$V_t = \left(1 + \frac{r}{m}\right)^{mt} v_0, \quad 0 \leq t \leq T. \tag{1.6}$$

Note that we have the following properties $\forall\, 0 \leq t \leq T$

1. $V_{t+1} = \left(1 + \frac{r}{m}\right)^m V_t,$
2. If $m_1 > m_2, v_0 > 0 \Rightarrow \left(1 + \frac{r}{m_1}\right)^{m_1 t} v_0 > \left(1 + \frac{r}{m_2}\right)^{m_2 t} v_0,$
3. If $m_1 > m_2, v_0 < 0 \Rightarrow \left(1 + \frac{r}{m_1}\right)^{m_1 t} v_0 < \left(1 + \frac{r}{m_2}\right)^{m_2 t} v_0.$

From this is follows that for an *investor* compund interest is more attractive as it pays more, however as a *debtor* it is less attractive as he or she will have to pay more on his or hers debt.

At last i can turn to continuously compounded interest which i will present as the limit of (1.6) as $m \to \infty$. Note that by the following definition of e

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e, \tag{1.7}$$

by letting $x = r/m$ in the above the limit of the wealth process of compounded interest can be seen as

$$\left[\left(1 + \frac{r}{m}\right)^{\frac{m}{r}}\right]^{rt} v_0 \to (e)^{rt} v_0, \quad \text{as } m \to \infty. \tag{1.8}$$

This leads to the definition of continuously compounded interest.

Definition (Continuously Compounded Interest)

Let v_0 be our initial capital. An interest on v_0 is said to be continuously compounded at rate $r > 0$ if the wealth process

$$V_t = e^{rt} v_0, \quad 0 \leq t \leq T. \tag{1.9}$$

There exists the following relation between the different types of interest

$$(1+r) \leq \left(1+\frac{r}{m}\right)^m < e^r. \tag{1.10}$$

To show that the relation indeed holds i will show that the sequence

$$a_m = \left(1+\frac{r}{m}\right)^m, \tag{1.11}$$

is increasing.

Using the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

we have

$$\begin{aligned} \left(1+\frac{r}{m}\right)^m &= \sum_{k=0}^{m} \binom{m}{k} 1^{m-k} \left(\tfrac{r}{m}\right)^k \\ &= \sum_{k=0}^{m} \binom{m}{k} \left(\tfrac{r}{m}\right)^k := \clubsuit \end{aligned}$$

Each term is of the form

$$\binom{m}{k}\left(\tfrac{r}{m}\right)^k = \prod_{l=0}^{k-1} \frac{m-l}{k-l}\left(\tfrac{r}{m}\right)$$

Each term is of the form

$$\begin{aligned}\frac{m-l}{k-l}\frac{r}{m} &= \frac{rm-lr}{m(k-l)}\\ &= \frac{m(r-lr/m)}{m(k-l)}\\ &= \frac{r-lr/m}{k-l} := \bigstar\end{aligned}$$

The term $\bigstar$ increases with m and thus the product increases with m and thus the sum $\clubsuit$ increases with m and therefore it is an increasing sequence.

Types of Bonds

I will discuss the following two types here

1. zero-coupon bonds,
2. coupon bonds.

A **zero-coupon bond** is a bond with a single payment $F > 0$ at time $T > 0$. The pay-off F is called the face value and T the maturity time. The next question i will answer is how much i will be willing to pay for such a financial assest. This depends on the way the time value of money is measured. Consider for example the following setup; let $B_0 \geq 0$ be the value of the zero-coupon bond with face value $F > 0$ and maturity time $T > 0$. Suppose that only annual compound interest at rate $r > 0$ is available.

From a buyers perspective what if

$$B_0 > \frac{F}{(1+r)^T}, \tag{1.12}$$

would i buy the bond?

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Suppose now that we flip the inequality and look from a sellers perspective, that is if

$$B_0 < \frac{F}{(1+r)^T}, \tag{1.13}$$

would i sell the bond?

Now i will consider the situatuion where at time $1 \leq t \leq T$ i want to get rid of a bond, but i what to determine what price i should sell it to. At this time the bond can be considered a new zero-coupon bond with face value $F > 0$ and maturity time $T - t$. Thus we have from the previous argumentation that

$$B_t = \frac{F}{(1+r)^{T-t}}, \quad 0 \leq t \leq T. \tag{1.14}$$

The chain of arguments holds also when the time value of money is different, if a compounded interest over m periods where considered then the fair price of a zero-coupon bond at time t would be

$$B_t = \frac{F}{\left(1 + \frac{r}{m}\right)^{m(T-t)}}. \tag{1.15}$$

If we consider the continuously compounded case the fair price would be

$$B_t = \frac{F}{e^{r(T-t)}}. \tag{1.16}$$

how much money will i have to deposit in my bank account today if i want to

$$
\begin{aligned}
1. &\quad \text{withdraw } C > 0 \text{ after } 1 \text{ year} && (1.17)\\
2. &\quad \text{withdraw } C > 0 \text{ after } 2 \text{ years} && (1.18)\\
&\quad \vdots \\
T-1. &\quad \text{withdraw } C > 0 \text{ after } T-1 \text{ years} && (1.19)\\
T. &\quad \text{withdraw } F + C \text{ after } T \text{ years} && (1.20)
\end{aligned}
$$

and have nothing left in the bank account afterwards.

In order to be able to get $C > 0$ after one year i have to put

$$\frac{C}{1+r} \tag{1.21}$$

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In order to be able to get $C > 0$ after two years i have to put

$$\frac{C}{(1+r)^2} \tag{1.22}$$

in the bank.

Generalizing this argument tells me that in order to recieve $C > 0$ after t years i have to put

$$\frac{C}{(1+r)^t} \tag{1.23}$$

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Generalizing this argument tells me that in order to recieve $C > 0$ after t years i have to put

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in the bank.
Lastly in order to get $F + C$ after T years i have to put

$$\frac{F+C}{(1+r)^T} = \frac{F}{(1+r)^T} + \frac{C}{(1+r)^T} \tag{1.24}$$

in the bank.

Types of Bonds

Adding up all these amounts it is concluded that i have to make a deposit of

$$\sum_{i=1}^{T} \frac{C}{(1+r)^i} + \frac{F}{(1+r)^T}. \tag{1.25}$$

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The agreeable price of a coupon bond is thus given by

$$B_0 = \sum_{i=1}^{T} \frac{C}{(1+r)^i} + \frac{F}{(1+r)^T} = \frac{\xi_T}{(1+r)^T}. \qquad (1.26)$$

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Where the pay-off ξ_T at time T is given by

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Where the pay-off ξ_T at time T is given by

$$\xi_T := \sum_{i=1}^{T} C(1+r)^{T-i} + F, \tag{1.27}$$

in other words the fair price of the coupon bond (as well as the zero-coupon bond) can be written as the discounted price of the total pay-off.

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Portfolio

Let

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, \mathbb{P}), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)} \right)_{0 \le t \le T} \right\}, \tag{2.1}$$

be a finite-horizon financial market.

Definition (Portfolio and Strategies)

A portfolio in $\mathfrak{M}$ is a $(d+1)$-dimensional vector

$$\Theta_t = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)}\right), \tag{2.2}$$

in which

$$\Theta_t^j = \text{Number of shares of the } j\text{'th asset held between time } t-1 \text{ and } t. \tag{2.3}$$

for $j = 1, \ldots, d+1$. The collection $\Theta = (\Theta_t)_{0 \le t \le T}$, with the convention that $\Theta_0 = \Theta_1$, is termed a strategy.

For every strategy on market there is an associated wealth process. The wealth process for Θ is defined and denoted by

$$V_t^{\Theta} = \varphi_t B_t + \sum_{j=1}^{d} \theta_t^{(j)} S_t^{(j)} = \Theta_t \cdot P_t, \quad 0 \leq t \leq T. \tag{2.4}$$

Risk Measures

Any strategy on a given market inherently carries a risk because the return is random, there is in other words no way to predict our profit or losses with certainty. There is no one way to measure the risk associated with a strategy, however in the next two sections i will explore two approaches. Both of these is based on portfolio allocation as an optimization problem.

The problem is given in this way; solve

$$\underset{\mathbf{w}\in\mathbb{R}^{d+1}}{\operatorname{argmin}}\,\mathcal{R}\left(\mathbf{w}\cdot\mathbf{K}_P\right), \tag{2.5}$$

subject to:

$$\sum_{j=0}^{d} w_j = 1,\ \mathbb{E}\left[U(\mathbf{w}\cdot\mathbf{K}_P)\right] = \mu,\ \mu\in\mathbb{R}, \tag{2.6}$$

where $\mathcal{R}$ is a measure of risk and $U(\mathbf{w}\cdot\mathbf{K}_P)$ is the utility of the strategy.

The mean variance approach assumes the uitility function as the identity, that is

$$U(x) = x. \tag{2.7}$$

By letting

$$\mu_{\mathsf{K}} := \mathbb{E}\left[\mathsf{K}_P\right], \tag{2.8}$$

it follows that

$$\mathbb{E}\left[U(\mathsf{w} \cdot \mathsf{K}_P)\right] = \mathsf{w} \cdot \mu_{\mathsf{K}}. \tag{2.9}$$

Thus the optimization problem, in the mean-variance approach becomes

Problem (Optimization Problem Mean-Variance)

$$\underset{\mathrm{w}\in\mathbb{R}^{d+1}}{\operatorname{argmin}} \sqrt{\mathrm{w}^\top C \mathrm{w}} \tag{2.10}$$

Subject to:

$$\textit{1.)}\ \sum_{j=0}^{d} w_j = 1, \tag{2.11}$$

$$\textit{2.)}\ \mathrm{w}\cdot\mu_{\mathsf{K}} = \mu, \quad \mu\in\mathbb{R}. \tag{2.12}$$

Risk Measures

An illustrative example to demonstrate why one might consider a different risk measure than the standard deviation is the following. Consider two portfolios that generate the following wealth

$$V_1^{(1)} = \begin{cases} 1 & \text{with probability } 1/2 \\ -9 & \text{with probability } 1/2 \end{cases} \tag{2.13}$$

and

$$V_1^{(2)} = \begin{cases} 5 & \text{with probability } 1/2 \\ -5 & \text{with probability } 1/2 \end{cases} \tag{2.14}$$

According to their standard deviation these two portfolios carries the same risk, but one could argue that the first is riskier that the latter. The next risk measure i will consider i concerned with controlling losses rather than the variation of the return.

If i denote the outcome of an investment with V_1 then my potential losses will be given by $-V_1$. Suppose that for some $x \in \mathbb{R}$

$$-V_1 \leq x. \tag{2.15}$$

Then to cover my risk of bankruptcy i must keep at least the amount x in my bank account. In reality the only thing i can quantify is the chance of that happening, which is denoted by

$$\mathbb{P}\left(-V_1 \leq x\right). \tag{2.16}$$

This is the motivation behind the risk measure Value at Risk.

Definition (Value at Risk)

Let $0 < \alpha < 1$ and X be a random variable. The Value at Risk (VaR) of X is defined and denoted by

$$\mathrm{VaR}_\alpha(X) := \inf\left\{x \in \mathbb{R} \,:\, \mathbb{P}(X + x \geq 0) \geq 1 - \alpha\right\}. \tag{2.17}$$

In other words the $\mathrm{VaR}_\alpha(X)$ represents the amount of extra capital i need to hold in order to reduce my risk of bankruptcy to $1 - \alpha$.

An alternative representation of VaR can be formulated using the fact that

$$\mathbb{P}(-X \leq x) \geq 1 - \alpha \iff \mathbb{P}(X + x < 0) \leq \alpha, \tag{2.18}$$

this lets us formulate an equivalent representation given by

$$\mathrm{VaR}_\alpha(X) := \inf \left\{ x \in \mathbb{R} \,:\, \mathbb{P}(X < -x) \leq \alpha \right\}. \tag{2.19}$$

Proposition (Properties of VaR)

Let X, Y be arbitrary random variables. Then, the following holds

1. *If $X \geq 0$ almost surely, then $\mathrm{VaR}_\alpha(X) \leq 0$.*
2. *For all $y \in \mathbb{R}$ we have that $\mathrm{VaR}_\alpha(X + y) = \mathrm{VaR}_\alpha(X) - y$. In particular $\mathrm{VaR}_\alpha(X + \mathrm{VaR}_\alpha(X)) = 0$.*
3. *If $\lambda \geq 0$, then $\mathrm{VaR}_\alpha(\lambda X) = \lambda \mathrm{VaR}_\alpha(X)$.*
4. *If $X \geq Y$ almost surely, then $\mathrm{VaR}_\alpha(X) \leq \mathrm{VaR}_\alpha(Y)$.*

Risk Measures

Proof:

Coherent Risk Measures

Note that when using standard deviation as a risk measure we get the following

$$\sigma(X+Y)^2 = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\rho_{X,Y}\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}, \tag{2.20}$$

where $\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} \leq 1$. Then

$$\begin{aligned} \sigma(X+Y)^2 &= \sigma(X)^2 + \sigma(Y)^2 + 2\rho_{X,Y}\sigma(X)\sigma(Y) & (2.21)\\ &\leq \sigma(X)^2 + \sigma(Y)^2 + 2\sigma(Y)\sigma(X) = [\sigma(X)+\sigma(Y)]^2, & (2.22) \end{aligned}$$

which would imply that

$$\sigma(X+Y) \leq \sigma(X) + \sigma(Y). \tag{2.23}$$

However VaR as a risk measure is not able to reproduce this, that is in general we do not have that

$$\mathrm{VaR}_\alpha(X + Y) \leq \mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y). \tag{2.24}$$

As diversification is a desired i will know introduce the concept of Coherent Risk Measures.

Definition (Coherent Risk Measure)

A function $\rho : L^1 \to \mathbb{R}$ is said to be a Coherent Risk Measure if

1. If $X \geq 0$ almost surely, then $\rho(X) \leq 0$.
2. For all $y \in \mathbb{R}$ we have that $\rho(X + y) = \rho(X) - y$.
3. If $\lambda \geq 0$, then $\rho(\lambda X) = \lambda \rho(X)$.
4. We have that $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

Coherent Risk Measures

The Conditional Value at Risk (CVAR) is a common example of a coherent risk measure.
Given a random variable X, we will write

$$q_\alpha(X) := \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq \alpha\}. \tag{2.25}$$

With this notation now introduced i can present the definition of CVaR.

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With this notation now introduced i can present the definition of CVaR.

Definition (Condition Value at Risk)

Let $0 < \alpha < 1$ and $X \in L^1$. The Conditional Value at Risk or Expected Shortfall of X is defined and denoted buy

$$\mathrm{CVaR}_\alpha := -\frac{1}{\alpha}\int_0^\alpha q_r(X)\, dr. \tag{2.26}$$

The name Expected Shortfall comes from the fact that if X has a continuous distribution, then

$$\mathrm{CVaR}_\alpha(X) = -\mathbb{E}[X \mid X + \mathrm{VaR}_\alpha(X) \leq 0]. \tag{2.27}$$

Thus, CVaR_α measures the expected losses given that $\mathrm{VaR}_\alpha(X)$ was not enough to cover our position on X.

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Model Setup

Consider a model where only two assets are traded: A bond and a risky asset. The bond satisfies that

$$B_t = (1 + r), \quad t = 0, 1, \dots, T. \tag{3.1}$$

The price process $(S_t)_{t=0,1,\dots,T}$ satisfies that $S_0 > 0$ (non-random) and

$$S_t = S_{t-1}(1 + K_s(t)), \quad t = 1, \dots, T. \tag{3.2}$$

The returns are independent and identically distributed with

$$K_s(t) = \begin{cases} R_u & \text{with probability } p; \\ R_d & \text{with probability } 1 - p, \end{cases} \tag{3.3}$$

with the relation

$$R_d < R_u. \tag{3.4}$$

I am considering the multi-step financial market

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, \mathbb{P}), P = \left(B_t, S_t^{(1)}, S_t^{(2)}, \dots, S_t^{(d)} \right)_{t=0,1,\dots,T} \right\}. \tag{3.5}$$

In order to further explain how information flows in the market i need to be more specific about the definition of the probability space. I will start by letting

$$\Omega = \left\{ \left(\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(T)} \right) \, : \, \omega^{(t)} \in \{u, d\}, \; t = 1, 2, \dots, T \right\}. \tag{3.6}$$

Take for example the two-step binomial model, that is $T = 2$.

I will consider all possible events, which means that the σ-algebra in this case is

$$\mathscr{F} = 2^{\Omega}. \quad \text{(The power set)} \tag{3.7}$$

Using this set-up the return process can be written as

$$K_s(t)(\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(T)}) = \begin{cases} R_u & \text{if } \omega^{(t)} = u; \\ R_d & \text{if } \omega^{(t)} = d. \end{cases} \tag{3.8}$$

To ensure that the returns are independent and can go up and down with probability p and $(1-p)$ respectively, i let

$$\mathbb{P}\left(\left\{\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(d)}\right\}\right) = p^k(1-p)^{T-k}, \tag{3.9}$$

where

$$k = \text{number of } u\text{'s in } \left(\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(d)}\right). \tag{3.10}$$

As an example lets consider the two-step binomial model, that is $T = 2$ and we have that

$$\Omega = \{uu, ud, du, dd\}, \tag{3.11}$$

using the notational convention that

$$\left(\omega^{(1)}, \omega^{(2)}\right) = \omega^{(1)}\omega^{(2)}. \tag{3.12}$$

The Multi-Step Binomial Model

Market Information

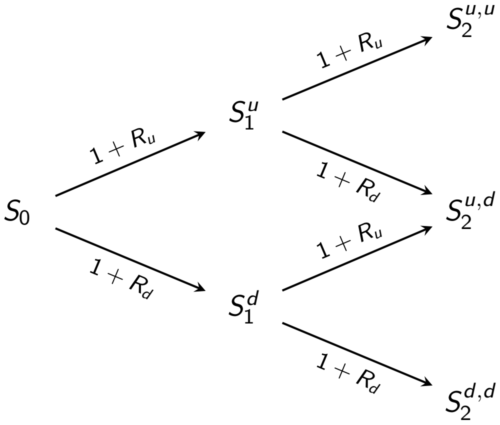

Figure: A graphical representation of the 2-step binomial model.

At time $t = 0$ no information is available but as time goes by, the price movements reveal information about future outcomes. Say for example that at time $t = 1$ the price went down, the future outcomes are restricted to the set

$$B_d := \{du, dd\}, \tag{3.13}$$

as the the events (uu) and (ud) are no longer possible.

The aforementioned set is related to the information generated by the return process. Consider the σ-field generated by a random variable, which is given by

$$\sigma(X) = \left\{X^{-1}(A) \,:\, A \in \mathscr{B}(\mathbb{R}).\right\} \tag{3.14}$$

This tells me that, in the event that $K_S(1) = R_d$ it follows that

$$\left\{ \left(\omega^{(1)}, \omega^{(2)} \right) \, : \, K_S(1) = R_d \right\} = \{du, dd\} = B_d. \tag{3.15}$$

On the other hand if $K_S(1) = R_u$ it follows that

$$\left\{ \left(\omega^{(1)}, \omega^{(2)} \right) \, : \, K_S(1) = R_u \right\} = \{uu, ud\} = B_u. \tag{3.16}$$

Therefore

$$B_u, B_d \subseteq \sigma\left(K_S(1)\right). \tag{3.17}$$

It is in fact so, that because $K_S(1)$ only takes two values

$$\sigma(K_S(1)) = \{B_u, B_d, \Omega, \emptyset\}. \tag{3.18}$$

At the end of the trading period, in this case $t = T = 2$, the outcome of the price movements are known and stored in

$$\mathscr{F} = \sigma\left(K_S(1), K_S(2)\right). \tag{3.19}$$

Furthermore there has been constructed a family of sub-σ-algebras of $\mathscr{F}$;

$$\mathscr{F}_1 = \sigma(K_S(1)) = \{B_u, B_d, \Omega, \emptyset\}. \tag{3.20}$$

$$\mathscr{F}_2 = \sigma(K_S(1), K_S(2)) = \mathscr{F}. \tag{3.21}$$

Note that by construction, $\mathscr{F}$ contains the whole information available on the price movements up until time t.

Mathematically this means that the price process S_t is adapted to $\mathscr{F}$;

$$\sigma(S_t) \subseteq \mathscr{F}_t, \quad t = 0, 1, 2 \text{ where } \mathscr{F}_0 = \{\Omega, \emptyset\}, \tag{3.22}$$

and moreover

$$\mathscr{F}_{t-1} \subseteq \mathscr{F}_t, \quad t = 1, 2. \tag{3.23}$$

Definition (Filtrations and Adapted Process)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{R})$. A collection of σ-algebras $\mathbb{F} = (\mathscr{F}_t)_{t=0,1,\ldots,T}$ is called a filtration if for all $1 \leq t \leq T$,

$$\mathscr{F}_{t-1} \subseteq \mathscr{F}_t \subseteq \mathscr{F}. \tag{3.24}$$

The quadruplet

$$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P}), \tag{3.25}$$

is termed a filtered probability space.
Furthermore, a stochastic process $(X_t)_{0 \leq t \leq T}$ is said to be adapted to a filtration $\mathbb{F}$ if

$$\sigma(X_t) \subseteq \mathscr{F}_t, \quad \forall 0 \leq t \leq T. \tag{3.26}$$

Market Information

Definition (Financial Markets with Information)

Fix $T, d \in \mathbb{N}$. A finite-horizon financial market with information is the pair

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leq t\leq T}, \mathbb{P}\right), P = \left(B_t, S_t^{(1)}, \dots, S_t^{(d)}\right)_{0\leq t\leq T} \right\}, \tag{3.27}$$

consisting of

1. A filtered probability space $\left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leq t\leq T}, \mathbb{P}\right)$
2. P_t is adapted.
3. $S^{(j)}$ is the price process of the j'th asset traded in the market
4. B is a numéraire (e.g. a bond), i.e.

$$\mathbb{P}(B_t > 0) = 1, \quad \forall\, 0 \leq t \leq T. \tag{3.28}$$

Absence of Arbitrage

An arbitrage is understood as a way to generate positive wealth with zero investment and zero risk. In the one-step binomial model it is only allowed to allocate wealth ones, namely at time $t = 0$. Returns are then measured at time $t = T = 1$.
In the multi-step binomial model agents will redesign their strategies based on the available information. In order to create an arbitrage the portfolio need to be updated every time the price changes. Therefore, instead of considering a single portfolio, an arbitrage now consists of a collection of portfolios $(\Theta_t)_{0 \leq t \leq T}$.

Definition (Portfolio and Strategy)

A portfolio in $\mathfrak{M}$ is a $(d+1)$-dimensional vector $\Theta_t = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)}\right)$, in which

$$\Theta_t^{(j)} = \text{ Number of shares of the } j\text{'th asset held between time } t-1 \text{ and } t. \tag{3.29}$$

The collection $\Theta = (\Theta_t)_{0 \le t \le T}$, with the convention that $\Theta_0 = \Theta_1$ is termed a strategy.

The wealth process associated to $\Theta = (\Theta_t)_{0 \le t \le T}$ is defined and denoted by

$$V_t^{\Theta} := \varphi_t B_t + \sum_{j=1}^{d} \theta_t^{(j)} S_t^{(j)} = \Theta_t \cdot P_t, \quad 0 \le t \le T \tag{3.30}$$

$$\Rightarrow V_0^{\Theta} = \varphi_0 B_0 + \sum_{j=1}^{d} \theta_0^{(j)} S_0^{(j)} = \varphi_1 B_0 + \sum_{j=1}^{d} \theta_1^{(j)} S_0^{(j)} \tag{3.31}$$

Absence of Arbitrage

The arbitrage approach from the one-step binomial model can not be used when considering the multi-step binomial model as many strategies that from an intuitive point of view should not be thought of as an arbitrage, will be believed to be an arbitrage. Consider for example the following situation

1. Injection of capital at $t \geq 1$.
2. Priviliged information.
3. Unlimited credit.

Definition (Admissible Strategy)

Let $\Theta = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)}\right)_{0 \leq t \leq T}$ be strategy on the market

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F})_{0 \leq t \leq T}, \mathbb{P}\right), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)}\right)_{0 \leq t \leq T} \right\}. \tag{3.32}$$

I will say that Θ is admissible if:

1. **It is self-financed**: It only requires initial capital. I will say that Θ is self financed if

$$V_t^{\Theta} = \varphi_{t+1} B_t + \sum_{j=1}^{d} \theta_{t+1}^{(j)} S_t^{(j)}, \quad 0 \leq t \leq T-1. \tag{3.33}$$

2. **Non-anticipative**: It is build up only on current information. I will say that Θ is non-anticipative or predictable if Θ_{t+1} depends only on the information at time t, that is Θ_{t+1} is $\mathscr{F}_t$-measurable.
3. **It has a limited credit line**: There is a non-random constant $C > 0$, such that

$$V_t^{\Theta} \geq -C, \; \forall\, 0 \leq t \leq T. \tag{3.34}$$

Definition (Arbitrage)

Consider the market

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F})_{0 \leq t \leq T}, \mathbb{P}\right), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)}\right)_{0 \leq t \leq T} \right\}. \tag{3.35}$$

An admissible strategy $\Theta = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)}\right)_{0 \leq t \leq T}$ is said to be an arbitrage if:

Absence of Arbitrage

1. It has zero initial capital:

$$V_0^{\Theta} = \varphi_0 B_0 + \sum_{j=1}^{d} \theta_0^{(j)} \cdot S_0^{(j)} = 0. \tag{3.36}$$

2. At time $t = T$, i am out of debts with 100% certainty: Almost surely

$$V_T^{\Theta} = \varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} \geq 0. \tag{3.37}$$

3. I have a chance to make a profit:

$$\mathbb{P}\left(\varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} > 0\right) > 0. \tag{3.38}$$

To begin this section i will define the notion of a martingale. Fix a filtered probability space

$$\left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P}\right). \tag{3.39}$$

I am given a discrete time stochastic process

$$X = (X_t)_{0 \leq t \leq T}. \tag{3.40}$$

Under this framework i give the definition of a martingale.

Definition (Martingale)

The collection $\{(X_t, \mathscr{F}_t) : t = 0, 1, \ldots, T\}$ is said to be a martingale if

1. X is adapted, i.e. $\sigma(X_t) \subseteq \mathscr{F}_t, \quad \forall\, 0 \leq t \leq T$.
2. X_t has finite first moment, that is $\mathbb{E}(|X_t|) < \infty$.
3. For all $0 \leq t \leq T - 1$ we have that almost surely $\mathbb{E}(X_{t+1}|\mathscr{F}_t) = X_t$.

Consider the risk neutral multi-step binomial model: $B_0 = 1$, $S_0 > 0$ is a non-random constant and

$$B_t = (1+r)^t; \quad S_t = S_{t-1}(1 + K_S(t)), \quad t = 1, \dots, T. \tag{3.41}$$

$(K_S(t))_{t=1,\dots,T}$ are i.i.d. with

$$K_S(t) = \begin{cases} R_u & \text{with probability } q^* \\ R_d & \text{with probability } 1 - q* \end{cases} \tag{3.42}$$

with the relation $R_d < r < R_u$ and

Risk-Neutral Measure

The risk neutral measure is the following:

I will now show that the discounted price

$$X_t := \frac{S_t}{B_t}, \quad 0 \leq t \leq T, \tag{3.43}$$

is a martingale under the filtration

$$\mathscr{F}_0 = \{\emptyset, \Omega\} \tag{3.44}$$

$$\mathscr{F}_t = \sigma(K_S(1), \ldots, K_S(t)), \quad 1 \leq t \leq T. \tag{3.45}$$

In order to show this, i need to verify that the discounted price satisfies all three conditions in definition 16 (Martingale)

By definition

$$
\begin{aligned}
\mathbb{E}\left(X_{t+1}|\mathscr{F}_t\right) &= \mathbb{E}\left(\frac{S_{t+1}}{B_{t+1}}\,\middle|\,\mathscr{F}_t\right) && (3.46)\\
&= \mathbb{E}\left(\frac{S_t(1+K_S(t+1))}{B_{t+1}}\,\middle|\,\mathscr{F}_t\right) && (3.47)\\
&= \mathbb{E}\left(\frac{S_t(1+K_S(t+1))}{(1+r)^{t+1}}\,\middle|\,\mathscr{F}_t\right) && (3.48)\\
&= \mathbb{E}\left(\frac{S_t}{(1+r)^{t+1}}(1+K_S(t+1))\,\middle|\,\mathscr{F}_t\right) && (3.49)\\
\text{(By the Product Rule)} &= \frac{S_t}{(1+r)^{t+1}}\mathbb{E}\left((1+K_S(t+1))\,\middle|\,\mathscr{F}_t\right) && (3.50)\\
&= \frac{S_t}{(1+r)^t}\frac{1}{(1+r)}\mathbb{E}\left((1+K_S(t+1))\,\middle|\,\mathscr{F}_t\right) && (3.51)\\
&= X_t\frac{1}{(1+r)}\mathbb{E}\left((1+K_S(t+1))\,\middle|\,\mathscr{F}_t\right). && (3.52)
\end{aligned}
$$

I am now left to show that

$$\mathbb{E}\left((1+K_S(t+1)\,\middle|\,\mathscr{F}_t\right)=1, \tag{3.53}$$

to do this i will use the independence of the returns but let me start by considering $t=0$.

In this case

$$\begin{aligned}
\mathbb{E}\left((1+K_S(1)\,|\,\mathscr{F}_0\right) &= \mathbb{E}\left((1+K_S(1)\,|\,\{\emptyset,\Omega\}\right) && (3.54)\\
\text{(Conditioning on the trivial } \sigma\text{-algebra)} &= \mathbb{E}\left((1+K_S(1)\right) && (3.55)\\
\text{(From Risk-Neutrality)} &= (1+r). && (3.56)
\end{aligned}$$

Let me now consider the case when $t \geq 1$

$$\begin{aligned}
\mathbb{E}\left((1+K_S(t+1)\,|\,\mathscr{F}_t\right) &= \mathbb{E}\left((1+K_S(t+1)\,|\,K_S(1),\ldots,K_S(t)\right) && (3.57)\\
\text{(Indepence prop. of the cond. expectation)} &= \mathbb{E}\left((1+K_S(1)\right) && (3.58)\\
&= (1+r). && (3.59)
\end{aligned}$$

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The Theorem

The Fundamental theorem of asset pricing gives necessary and sufficient conditions for the absence of arbitrage in the market with information, given by

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P} \right), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)} \right)_{0 \le t \le T} \right\}. \tag{4.1}$$

Theorem (FFTAP)

The market given by (4.1) *is arbitrage free if and only if there is a martingale measure.*

To understand the hypothesis presented in the theorem lets first remember the concept of an admissible strategy.

Definition (Admissible Strategy)

Let $\Theta = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)}\right)_{0 \leq t \leq T}$ be strategy on the market

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P}\right), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)}\right)_{0 \leq t \leq T} \right\}. \tag{4.2}$$

I will say that Θ is admissible if:

Explaining the Hypothesis

1. **It is self-financed**: It only requires initial capital. I will say that Θ is self financed if

$$V_t^{\Theta} = \varphi_{t+1} B_t + \sum_{j=1}^{d} \theta_{t+1}^{(j)} S_t^{(j)}, \quad 0 \leq t \leq T-1. \tag{4.3}$$

2. **Non-anticipative**: It is build up only on current information. I will say that Θ is non-anticipative or predictable if Θ_{t+1} depends only on the information at time t, that is Θ_{t+1} is $\mathscr{F}_t$-measurable.
3. **It has a limited credit line**: There is a non-random constant $C > 0$, such that

$$V_t^{\Theta} \geq -C, \; \forall \, 0 \leq t \leq T. \tag{4.4}$$

Next let me recall the notion of arbitrage.

Definition (Arbitrage)

Consider the market

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F})_{0 \le t \le T}, \mathbb{P}\right), P = \left(B_t, S_t^{(1)}, \dots, S_t^{(d)}\right)_{0 \le t \le T} \right\}. \tag{4.5}$$

An admissible strategy $\Theta = \left(\varphi_t, \theta_t^{(1)}, \dots, \theta_t^{(d)}\right)_{0 \le t \le T}$ is said to be an arbitrage if:

Explaining the Hypothesis

1. It has zero initial capital:

$$V_0^{\ominus} = \varphi_0 B_0 + \sum_{j=1}^{d} \theta_0^{(j)} \cdot S_0^{(j)} = 0. \tag{4.6}$$

2. At time $t = T$, i am out of debts with 100 % certainty: Almost surely

$$V_T^{\ominus} = \varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} \geq 0. \tag{4.7}$$

3. I have a chance to make a profit:

$$\mathbb{P}\left(\varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} > 0\right) > 0. \tag{4.8}$$

Lastly i will define a martingale measure.

Definition (Martingale Measure)

Let $\mathbb{Q}$ be probability measure on $(\Omega, \mathscr{F})$, that is there are $0 < q_i^* < 1, i = 1, \ldots, N$ such that $\sum_{i=1}^{N} q_i^* = 1$ and

$$\mathbb{Q}(\{\omega_i\}) = q_i^* \quad \text{for } i = 1, \ldots, N. \tag{4.9}$$

We will say that $\mathbb{Q}$ is a martingale measure if under $\mathbb{Q}$ the discounted price $(\tilde{P}_j^{(j)})_{0 \leq t \leq T}$ is a martingale w.r.t $(\mathscr{F}_t)_{0 \leq t \leq T}$, for every $j = 1, 2, \ldots, d + 1$.

Risk-Neutral and Martingale Measures

In the binomial model q^* plays two roles

1. It transform the real-world probabilities (the one observed in the market) $(p, 1-p)$ satisfying that

$$K_S(t) = \begin{cases} R_u & \text{with probability } p; \\ R_d & \text{with probability } 1-p, \end{cases} \tag{4.10}$$

   into a risk-neutral world in which the "new" distribution for the returns

$$K_S(t) = \begin{cases} R_u & \text{with probability } q^*; \\ R_d & \text{with probability } 1-q^*, \end{cases} \tag{4.11}$$

   is such that

$$\mathbb{E}_*(K_S(t)) = r, \quad t = 1, 2, \ldots, T. \tag{4.12}$$

2. It transforms the discounted prices into a martingale.

In this model the two are actually equivalent. Consider the martingale property

$$\mathbb{E}_* \left(\frac{S_{t+1}^{(j)}}{B_{t+1}} \,\middle|\, \mathscr{F}_t \right) = \frac{S_t^{(j)}}{B_t} \tag{4.13}$$

$$\frac{1}{B_{t+1}} \mathbb{E}_* \left(S_{t+1}^{(j)} \,\middle|\, \mathscr{F}_t \right) = \frac{S_t^{(j)}}{B_t} \tag{4.14}$$

$$\frac{1}{S_t^{(j)}} \mathbb{E}_* \left(S_{t+1}^{(j)} \,\middle|\, \mathscr{F}_t \right) = \frac{B_{t+1}}{B_t} \tag{4.15}$$

$$\mathbb{E}_* \left(\frac{S_{t+1}^{(j)}}{S_t^{(j)}} \,\middle|\, \mathscr{F}_t \right) = \frac{B_{t+1}}{B_t}, \tag{4.16}$$

subtracting 1 from both sides i get:

$$\mathbb{E}_*\left(\frac{S_{t+1}^{(j)}}{S_t^{(j)}} - 1 \,\middle|\, \mathscr{F}_t\right) = \frac{B_{t+1}}{B_t} - 1, \tag{4.17}$$

taking expectations gives

$$\mathbb{E}_*\left[\mathbb{E}_*\left(\frac{S_{t+1}^{(j)}}{S_t^{(j)}} - 1 \,\middle|\, \mathscr{F}_t\right)\right] = \mathbb{E}_*\left[\frac{B_{t+1}}{B_t} - 1\right] \tag{4.18}$$

$$\mathbb{E}_*\left(\frac{S_{t+1}^{(j)}}{S_t^{(j)}} - 1\right) = \frac{B_{t+1}}{B_t} - 1 \tag{4.19}$$

$$\mathbb{E}_*\left(K_S(t+1)\right) = K_B(t+1). \tag{4.20}$$

This is exactly the definition of risk-neutral probabilities.

Risk-Neutral and Martingale Measures

Add application to the pricing of financial derivaties.

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Methodology for Pricing

I will start by outlining the methodology used for pricing derivatives in the binomial model.

1. **Define a model for the prices**: Specify the dynamics for
$$P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)}\right)_{0\leq t\leq T}, \tag{5.1}$$
i.e. describe how the distribution, evolution and behavior of the prices should be. . . according to some empirical facts observed in the data.
2. **Indicate a set of information**: Determine what information is available at each point in time, i.e. specify $(\mathscr{F}_t)_{0\leq t\leq T}$.
3. **Check for arbitrage opportunities**: Establish under which circumstances the proprosed model does not admit an arbitrage, i.e. by the First Fundamental Theorem of Asset Pricing find a martingale measure $\mathbb{Q}$ for the model.
4. **Dynamics in the risk-neutral world**: Investigate what the distribution of the price process is under the martingale measure. This is necessary as the $\xi = \Phi(P_0, \ldots, P_T)$ is a function of the prices.
5. **Price the derivative**: Compute the prices of the derivative by
$$\xi_t := \mathbb{E}_* \left(\frac{B_t}{B_T}\xi \,\middle|\, \mathscr{F}_t \right) \tag{5.2}$$

Pricing a Call Option

Let us now find the inital price of a call option. Recall that such a contract gives the owner the right but not the obligation to buy an asset at a given value, say $C > 0$ at a predetermined point in time, say $T > 0$.

Pricing a Call Option

The pay-off is then given by the random variable

$$\xi = \max\{S_T - C, 0\} = (S_T - C)^+. \tag{5.3}$$

Independently of the model, we know by theorem 17 (FFTAP) that the arbitrage free initial price of such an option is

$$\xi_0 = \mathbb{E}_* \left(\frac{B_0}{B_T} (S_T - C)^+ \right). \tag{5.4}$$

In the multi-step binomial model we have

$$\xi_0 = \mathbb{E}_* \left(\frac{B_0}{B_T} (S_T - C)^+ \right) = \frac{1}{(1+r)^T} \mathbb{E}_* \left((S_T - C)^+ \right). \tag{5.5}$$

Pricing a Call Option

Using that under the probability measure $\mathbb{Q}$ we have that for all $x = 0, \ldots, T$

$$p_T(x; q^*) = \mathbb{Q}\left[S_T := S_0(1+R_u)^x(1+R_d)^{T-x}\right] \tag{5.6}$$

$$= \binom{T}{x}(q^*)^x(1-q^*)^{T-x} \tag{5.7}$$

it follows that

$$\mathbb{E}_*\left((S_T - C)^+\right) = \sum_{x=0}^{T}\left(S_0(1+R_u)^x(1+R_d)^{T-x} - C\right)^+ p_T(x; q^*). \tag{5.8}$$

Observe that

$$\left(S_0(1+R_u)^x(1+R_d)^{T-x} - C\right)^+ \neq 0 \tag{5.9}$$

if and only if

$$S_0(1+R_u)^x(1+R_d)^{T-x} > C. \tag{5.10}$$

Pricing a Call Option

Let (with the convention that $\min\{\emptyset\} = +\infty$)

$$x_0(S_0, R_u, R_d, T) = \min\{0 \leq x \leq T : S_0(1+R_u)^x(1+R_d)^{T-x} > C\}. \quad (5.11)$$

Then

$$\mathbb{E}_*\left((S_T - C)^+\right) = \sum_{x=0}^{T} \left(S_0(1+R_u)^x(1+R_d)^{T-x} - C\right)^+ p_T(x; q^*) \quad (5.12)$$

$$= \sum_{x=x_0}^{T} \left(S_0(1+R_u)^x(1+R_d)^{T-x} - C\right)^+ p_T(x; q^*) \quad (5.13)$$

$$= S_0 \sum_{x=x_0}^{T} \left((1+R_u)^x(1+R_d)^{T-x} - C\right)^+ p_T(x; q^*) \quad (5.14)$$

$$- C \sum_{x=x_0}^{T} p_T(x; q^*). \quad (5.15)$$

Pricing a Call Option

Since

$$p_T(x;q^*) = \mathbb{Q}\left[S_T = S_0(1+R_u)^x(1+R_d)^{t-x}\right] = \binom{T}{x}(q^*)^x(1-q^*)^{T-x} \tag{5.16}$$

Then

$$(1+R_u)^x(1+R_d)^{T-x}p_T(x;q^*) = (1+R_u)^x(1+R_d)^{T-x} \times \binom{T}{x}(q^*)^x(1-q^*)^{T-x} \tag{5.17}$$

$$= \binom{T}{x}[(1+R_u)q^*]^x \times [(1+R_d)(1-q^*)]^{T-x} \tag{5.18}$$

Pricing a Call Option

All in all this implies that the arbitrage free initial price of a call option in the multi step binomial model is given by the formula

$$\xi_0 = \mathbb{E}_* \left(\frac{(S_T - C)^+}{(1+r)^T} \right) = S_0 \Psi_1(T, S_0, C, q^*) - \frac{C}{(1+r)^T} \Psi_2(T, S_0, C, q^*) \tag{5.19}$$

where

$$\Psi_1(T, S_0, C, q^*) := \sum_{x=x_0}^{T} \binom{T}{x} \left[\frac{1+R_u}{1+r} q^* \right]^x \left[\frac{1+R_d}{1+r} (1-q^*) \right]^{T-x} \tag{5.20}$$

$$\Psi_2(T, S_0, C, q^*) := \sum_{x=x_0}^{T} p_T(x; q^*) \tag{5.21}$$

The expression in (5.19) is known as the **Cox-Ross-Rubenstein** formula.

Price function for simple derivatives

Theorem (Price Function)

Within the framework of the multi-step binomial model, let ξ be a simple derivative, that is $\xi = \varphi(S_T)$. Put

$$F(t, y) := \frac{1}{(1+r)^{T-t}} \mathbb{E}_* \{\varphi(S^y_{T-t})\}, \quad y \geq 0,\ 0 \leq t \leq T, \tag{5.22}$$

where $(S^y_t)_{0 \leq t \leq T}$ is a process satisfying that $S^y_0 = y$ and

$$S^y_t = S^y_{t-1}(1 + K_S(t)), t = 1, \ldots, T. \tag{5.23}$$

Then, almost surely

$$F(t, S_t) = \frac{1}{(1+r)^{T-t}} \mathbb{E}_* (\varphi(S_T) | \mathscr{F}_t), 0 \leq t \leq T. \tag{5.24}$$

In other words the arbitrage free price of the simple derivative $\xi = \varphi(S_T)$ at time $0 \leq t \leq T$ is given by $F(t, S_t)$.

Remark: The function F is known as the price function associated to the pay-off function φ.

Interpretation: The function $F(t, y)$ can be thought as the initial price of a simple derivative with maturity time $T - t$ and pay-off function φ under the circumstances that the initial price of the risky asset is $y \geq 0$.

General Price of a Call option in the Binomial Model

According to the previous theorem the price of a call option in the multi step binomial model at time $t = 0, 1, \ldots, T$ is given by

$$\xi_t = F(t, S_t) = \frac{1}{(1+r)^{T-t}} \mathbb{E}_* \left((S_T - C)^+ | \mathscr{F}_t \right) \tag{5.25}$$

where

$$F(t, y) = \mathbb{E}_* \left[\frac{(S^y_{T-t} - C)^+}{(1+r)^{T-t}} \right], y \geq 0, 0 \leq t \leq T \tag{5.26}$$

in which $(S^y_t)_{0 \leq t \leq T}$ is a process satisfying that $S^y_0 = y$ and

$$S^y_t = S^y_{t-1}(1 + K_S(t)), t = 1, \ldots, T. \tag{5.27}$$

General Price of a Call option in the Binomial Model

The Cox-Ross-Rubunstein formula dictates that

$$F(t, S_0) = \mathbb{E}_* \left(\frac{(S_{T-t} - C)^+}{(1+r)^{T-t}} \right) \tag{5.28}$$

$$= S_0 \Psi_1(T-t, S_0, C, q^*) - \frac{C}{(1+r)^{T_0}} \Psi_2(T-t, S_0, C, q^*). \tag{5.29}$$

General Price of a Call option in the Binomial Model

Replacing S_0 by y the Cox-Ross-Rubenstein formula we get that

$$F(t, y) = y\Psi_1(T - t, y, C, q^*) - \frac{C}{(1+r)^{T-t}}\Psi_2(T - t, y, C, q^*). \tag{5.30}$$

Hence the price call option in the multi-step binomial model is

$$\xi_t = F(t, S_t) = S_t\Psi_1(T - t, S_t, C, q^*) - \frac{C}{(1+r)^{T-t}}\Psi_2(T - t, S_t, C, q^*). \tag{5.31}$$

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The market with information is given

$$\mathfrak{M} = \{(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P}), P = (B_t, S_t^{(1)}, \dots, S_t^{(d)})_{0\le t\le T}\}.$$

The filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P})$ satisfies that

The Second Fundamental Theorem of Asset Pricing

Market

1. The sample space is finite and our σ-algebra is the power set:

$$\Omega = \{\omega_1, \ldots, \omega_N\}, \ N \in \mathbb{N}, \ \mathscr{F} = 2^{\Omega}.$$

2. There are $0 < p_i < 1$ for $i = 1, \ldots, N$ such that

$$\sum_{i=1}^{N} p_i = 1$$
$$\mathbb{P}(\{\omega_i\}) = p_i, \quad i = 1, \ldots, N$$

3. The set of information satisfies that $\mathscr{F}_0 = \{\emptyset, \Omega\}, \mathscr{F}_T = \mathscr{F}$, and

$$\mathscr{F}_{t-1} \subseteq \mathscr{F}_t, \quad 1 \leq t \leq T-1.$$

We will assume that B is deterministic, $B_0 = 1$ and $B_t > 0$, $\forall\ 1 \leq t \leq T$.

Theorem (Second Fundamental Theorem of Asset Pricing)

Let the market

$$\mathfrak{M} = \left\{ (\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P}), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)} \right)_{0 \le t \le T} \right\}.$$

be as described. Then $\mathfrak{M}$ is complete and arbitrage free if and only if there is a unique martingale measure.

Definition (Martingale)

The collection $\{(X_t, \mathscr{F}_t) : t = 0, 1, \ldots, T\}$ is said to be a martingale if

1. X is adapted, i.e. $\sigma(X_t) \subseteq \mathscr{F}_t, \quad \forall\, 0 \le t \le T$.
2. X_t has finite first moment, that is $\mathbb{E}(|X_t|) < \infty$.
3. For all $0 \le t \le T - 1$ we have that almost surely $\mathbb{E}(X_{t+1}|\mathscr{F}_t) = X_t$.

Definition (Martingale Measure)

Let $\mathbb{Q}$ be probability measure on $(\Omega, \mathscr{F})$, that is there are $0 < q_i^* < 1, i = 1, \ldots, N$ such that $\sum_{i=1}^{N} q_i^* = 1$ and

$$\mathbb{Q}(\{\omega_i\}) = q_i^* \quad \text{for } i = 1, \ldots, N. \tag{6.1}$$

We will say that $\mathbb{Q}$ is a martingale measure if under $\mathbb{Q}$ the discounted price $(\tilde{P}_j^{(j)})_{0 \le t \le T}$ is a martingale w.r.t $(\mathscr{F}_t)_{0 \le t \le T}$, for every $j = 1, 2, \ldots, d+1$.

Definition (Admissible Strategy)

Let $\Theta = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)}\right)_{0 \le t \le T}$ be strategy on the market

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F})_{0 \le t \le T}, \mathbb{P}\right), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)}\right)_{0 \le t \le T} \right\}. \tag{6.2}$$

I will say that Θ is admissible if:

1. **It is self-financed**: It only requires initial capital. I will say that Θ is self financed if

$$V_t^{\Theta} = \varphi_{t+1} B_t + \sum_{j=1}^{d} \theta_{t+1}^{(j)} S_t^{(j)}, \quad 0 \leq t \leq T-1. \tag{6.3}$$

2. **Non-anticipative**: It is build up only on current information. I will say that Θ is non-anticipative or predictable if Θ_{t+1} depends only on the information at time t, that is Θ_{t+1} is $\mathscr{F}_t$-measurable.
3. **It has a limited credit line**: There is a non-random constant $C > 0$, such that

$$V_t^{\Theta} \geq -C, \; \forall \, 0 \leq t \leq T. \tag{6.4}$$

Definition (Arbitrage)

Consider the market

$$\mathfrak{M} = \left\{ \left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leq t\leq T}, \mathbb{P}\right), P = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)}\right)_{0\leq t\leq T} \right\}. \tag{6.5}$$

An admissible strategy $\Theta = \left(\varphi_t, \theta_t^{(1)}, \ldots, \theta_t^{(d)}\right)_{0\leq t\leq T}$ is said to be an arbitrage if:

Definitions

1. It has zero initial capital:

$$V_0^{\Theta} = \varphi_0 B_0 + \sum_{j=1}^{d} \theta_0^{(j)} \cdot S_0^{(j)} = 0. \tag{6.6}$$

2. At time $t = T$, i am out of debts with 100 % certainty: Almost surely

$$V_T^{\Theta} = \varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} \geq 0. \tag{6.7}$$

3. I have a chance to make a profit:

$$\mathbb{P}\left(\varphi_T B_T + \sum_{j=1}^{d} \theta_T^{(j)} \cdot S_T^{(j)} > 0\right) > 0. \tag{6.8}$$

Definition (Complete Financial Market)

The market is said to be complete if every European contingent claim ξ (i.e. a random variable that depends on the information up to time T) can be replicated, that is, there exists an admissible strategy

$$\Theta = \left(\varphi_t, \theta_t^{(1)}, \dots, \theta_t^{(d)}\right)_{0 \leq t \leq T} \tag{6.9}$$

such that almost surely

$$\xi = V_T^\Theta = \varphi B_T + \sum_{j=1}^{d} \theta_T^{(j)} S_T^{(j)}. \tag{6.10}$$

This means that, if the market is arbitrage free and $\mathbb{P}(\xi > 0) > 0$, then necessarily

$$V_0^\Theta > 0. \tag{6.11}$$

Remarks

If the market is arbitrage free and complete, the we have a unique arbitrage-free price for any derivative with pay-off ξ which is given by

$$\mathbb{E}_* \left(\frac{B_t}{B_T} \xi \,\middle|\, \mathscr{F}_t \right). \tag{6.12}$$

Remarks