# Financial Engineering

## Exam

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S20

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## Monotone Convergence Theorem

### Theorem (Monotone Convergence Thereom)

*If $(f_n)$ is a monotone increasing sequence of functions in $M^+(X, m)$ which converges to $f$, then*

$$\int f \, d\mu = \lim \int f_n \, d\mu. \tag{1.1}$$

## Proof of Monotone Convergence Theorem

The strategy of the proof is to first show that

$$\lim \int f_n \, d\mu \leq \int f \, d\mu, \tag{1.2}$$

then afterwards to show that also

$$\lim \int f_n \, d\mu \geq \int f \, d\mu, \tag{1.3}$$

in order to conclude that

$$\lim \int f_n \, d\mu = \int f \, d\mu \tag{1.4}$$

## Proof of Monotone Convergence Theorem

According to Corollary 2.10

### Corollary (2.10)

*If $(f_n)$ is a sequence in $M(X, m)$ which converges to $f$ on $X$, the $f$ is in $M(X, m)$.*

the function $f$ is measurable.

## Proof of Monotone Convergence Theorem

From Lemma 4.5(1.)

### Lemma

1. *If $f$ and $g$ belong to $M^+(X, m)$ and $f \leq g$, then*

$$\int f \, d\mu \leq \int g \, d\mu. \tag{1.5}$$

2. *If $f$ belongs to $M^+(X, m)$, if $E$, $F$ belong to $m$, and if $E \subseteq F$, then*

$$\int_E f \, d\mu \leq \int_F f \, d\mu. \tag{1.6}$$

we have that

$$\int f_n \, d\mu \leq \int f_{n+1} \, d\mu \leq \int f \, d\mu, \quad \forall\, n \in \mathbb{N}. \tag{1.7}$$

## Proof of Monotone Convergence Theorem

Therefore we must also have that

$$\lim \int f_n \, d\mu \leq \int f \, d\mu. \tag{1.8}$$

So this was the first step of our strategy, now we proceed to the second step.

## Proof of Monotone Convergence Theorem

Let $\alpha \in \mathbb{R}$ be such that $0 < \alpha < 1$ and let $\varphi$ be a simple measurable function such that $0 \leq \varphi \leq f$.
Let

$$A_n = \{x \in X \,:\, f_n(x) \geq \alpha\varphi(x)\}\,, \tag{1.9}$$

such that

1. $A_n \in m$
2. $A_n \subseteq A_{n+1}$
3. $X = \bigcup A_n$

## Proof of Monotone Convergence Theorem

According to Lemma 4.5

### Lemma

1. *If $f$ and $g$ belong to $M^+(X, m)$ and $f \leq g$, then*
$$\int f \, d\mu \leq \int g \, d\mu. \tag{1.10}$$
2. *If $f$ belongs to $M^+(X, m)$, if $E$, $F$ belong to $m$, and if $E \subseteq F$, then*
$$\int_E f \, d\mu \leq \int_F f \, d\mu. \tag{1.11}$$

it must be that

$$\int_{A_n} \alpha\varphi \, d\mu \leq \int_{A_n} f_n \, d\mu \leq \int f_n \, d\mu. \tag{1.12}$$

## Proof of Monotone Convergence Theorem

Since the sequence $A$ is monotone increasing and has union $X$, it follows from Lemma 4.3(2.) and Lemma 3.4(1.),

### Lemma (4.3)

1. *If $\varphi$ and $\psi$ are simple functions in $M^+(X, m)$ and $c \geq 0$, then*

$$\int c\varphi \, d\mu = c \int \varphi \, d\mu, \tag{1.13}$$

$$\int (\varphi + \psi) \, d\mu = \int \varphi \, d\mu + \int \psi \, d\mu. \tag{1.14}$$

2. *If $\lambda$ is defined for $E$ in $m$ by*

$$\lambda(E) = \int \varphi \chi_E \, d\mu, \tag{1.15}$$

*then $\lambda$ is a measure on $m$.*

### Lemma (3.4)

*Let $\mu$ be a measure defined on a $\sigma$-algebra $m$.*

1. *If $(E_n)$ is an increasing sequence in $m$, then*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu\left(E_n\right). \tag{1.16}$$

2. *If $(F_n)$ is a decreasing sequence in $m$ and if $\mu\left(F_1\right) < +\infty$, then*

$$\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim \mu\left(F_n\right). \tag{1.17}$$

## Proof of Monotone Convergence Theorem

that

$$\int \varphi \, d\mu = \lim \int_{A_n} \varphi \, d\mu. \tag{1.18}$$

Taking the limit as $n$ tends to infinity in (1.12) therefore gives that

$$\alpha \int \varphi \, d\mu \leq \lim \int f_n \, d\mu. \tag{1.19}$$

Since this holds for all $0 < \alpha < 1$, by taking the limit as $\alpha$ tends to 1 we obtain

$$\int \varphi \, d\mu \leq \lim \int f_n \, d\mu. \tag{1.20}$$

## Proof of Monotone Convergence Theorem

As $\varphi$ is any simple function in $M^+$ such that $0 \le \varphi \le f$, we can conclude that

$$\int f \, d\mu = \sup_{\varphi} \int \varphi \, d\mu \le \lim \int f_n \, d\mu, \tag{1.21}$$

which concludes the proof. $\square$

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## Minkowski's Inequality

### Theorem (Minkowski's Inequality)

*If $f$ and $h$ belong to $L_p$, $p \geq 1$, then $f + h$ belongs to $L_p$ and*

$$\|f + h\|_p \leq \|f\|_p + \|h\|_p. \tag{2.1}$$

## Proof of Minkowski's Inequality

- The case of $p = 1$ is equivalent to the triangular inequality and will not be considered here.
- We suppose $p > 1$.

## Proof of Minkowski's Inequality

As both $f$ and $h$ are measurable functions, so is the sum

$$f + h, \tag{2.2}$$

as the sum of measurable functions are also measurable.

## Proof of Minkowski's Inequality

From Corollary 5.4, Theorem 5.5 and the fact that

$$|f+h|^p \leq [2\sup\{|f|,|h|\}]^p \leq 2^p\{|f|^p + |h|^p\}, \qquad (2.3)$$

it follows that $f + h \in L^p$.

### Corollary (5.4)

*If $f$ is measurable, $g$ is integrable, and $|f| \leq |g|$, then $f$ is integrable, and*

$$\int |f|\,d\mu \leq \int |g|\,d\mu. \quad (2.4)$$

### Theorem (5.5)

*A constant multiple $\alpha f$ and a sum $f + g$ of functions in $L$ belongs to $L$ and*

$$\int \alpha f\,d\mu, \qquad (2.5)$$

$$\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu. \quad (2.6)$$

## Proof of Minkowski's Inequality

Using the triangular inequality we can see that

$$|f+h|^p = |f+h||f+h|^{p-1} \leq |f||f+h|^{p-1} + |h||f+h|^{p-1}. \tag{2.7}$$

Since $f+h \in L^p$ it implies that $|f+h|^p \in L^1$

## Proof of Minkowski's Inequality

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