Integration Theory

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MATØK6

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1 Lebesque Integration Theory

(Chapter 2-5 of Bartle's book, the most important results are: monotone convergence theorem, Fatou's lemma and Lebesgue dominated convergence theorem)

4.6 - Monotone Convergence Thereom

Theorem 1.1 (Monotone Convergence Thereom)**.** *If (f_n) is a monotone increasing sequence of functions in $M^+(X, m)$ which converges to f, then*

$$\int f\, d\mu = \lim \int f_n\, d\mu. \tag{1.1}$$

4.8 - Fatou's Lemma

Theorem 1.2 (Fatou's Lemma)**.** *If (f_n) belongs to $M^+(X, m)$, then*

$$\int (\liminf f_n)\, d\mu \leq \liminf \int f_n\, d\mu. \tag{1.2}$$

5.6 - Lebesgue Dominated Convergence Theorem

Theorem 1.3 (Lebesgue Dominated Convergence Theorem)**.** *Let (f_n) be a sequence of integrable function which converges almost everywhere to a real-valued measureable function f. If there exists an integrable function g such that $|f_n| \leq g$ for all n, then f is integrable and*

$$\int f\, d\mu = \lim \int f_n\, d\mu. \tag{1.3}$$

2 L^p Spaces

(Chapter 6 of Bartle's book, the most important results are: Hölder's inequality, Minkowski's inequality and Riesz-Fischer Theorem)

6.9 - Hölder's Inequality

Theorem 2.1 (Hölder's Inequality). *Let $f \in L_p$ and $g \in L_q$ where $p > 1$ and $(1/p) + (1/q) = 1$. Then $fg \in L_1$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$.*

6.11 - Minkowski's Inequality

Theorem 2.2 (Minkowski's Inequality). *If f and h belong to L_p, $p \geq 1$, then $f + h$ belongs to L_p and*

$$\|f + h\|_p \leq \|f\|_p + \|h\|_p. \tag{2.1}$$

6.14 - Completeness Theorem (Riesz-Fischer Theorem)

Theorem 2.3 (Completeness Theorem (Riesz-Fischer Theorem)). *If $1 \leq p < \infty$, then the space L_p is a complete normed linear space under the norm*

$$\|f\|_p = \left\{ \int |f|^p \, d\mu \right\}^{1/p}. \tag{2.2}$$

3 Decomposition of Measures

(Chapter 8 of Bartle's book, the most important results are: Radon-Nikodym theorem, Lebesgue decomposition theorem and Riesz representation theorem)

8.9 - Radon-Nikodým Theorem

Theorem 3.1 (Radon-Nikodým Theorem). *Let λ and μ be σ-finite measures defined on m and suppose that λ is absolutely continuous with respect to μ. Then there exists a function f in $M^+(X, m)$ such that*

$$\lambda(E) = \int_E f \, d\mu, \quad E \in m. \tag{3.1}$$

Moreover, the functino f is uniquely determined μ-almost everywhere.

8.11 - Lebesgue Decomposition Theorem

Theorem 3.2 (Lebesgue Decomposition Theorem). *Let λ and μ be sigma-finite measures defined on a sigma-algebra m. Then there exists a measure λ_1 which is singular with respect to μ and a measure λ_2 which is absolutely continuous with respect to μ such that $\lambda = \lambda_1 + \lambda_2$. Moreover, the measures λ_1 and λ_2 are unique.*

8.14 - Riesz Representation Theorem

Theorem 3.3 (Riesz Representation Theorem). *If (X, m, μ) is a sigma-finite measure space and G is a bounded linear functional on $L_1(X, m, \mu)$, then there exists a g in $L_\infty(X, m, \mu)$ such that*

$$G(f) = \int f g \, d\mu \tag{3.2}$$

holds for all f in L_1. Moreover, $\|G\| = \|g\|_\infty$ and $g \geq 0$ if G is a positive linear functional.

4 Generation of Measures and Product Measures

(Chapter 9 and 10 of Bartle's book, the most important results are: Carathéodory extension theorem, Hahn extension theorem, Product measure theorem, Fubini's theorem and Tonelli's theorem)

9.7 - Carathéodory Extension Theorem

Theorem 4.1 (Carathéodory Extension Theorem)**.** *the collection A^* of all μ^*-measureable sets is a σ-algebra containing A. Moreover, if (E_n) is a disjoint sequence in A^*, then*

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu^*(E_n). \tag{4.1}$$

9.8 - Hahn Extension Theorem

Theorem 4.2 (Hahn Extension Theorem)**.** *Suppose that μ is a σ-finite measure on an algebra A. Then there exists a unique extension of μ to a measure on A^*.*

10.4 - Product Measure Thereom

Theorem 4.3 (Product Measure Thereom)**.** *If (X, m, μ) and (Y, n, ν) are measure spaces, then there exists a measure π defined $Z_0 = m \times n$ such that*

$$\pi(A \times B) = \mu(A)\nu(B) \tag{4.2}$$

for all $A \in m$ and $B \in n$. If these measure spaces are σ-fintie, then there is a unique measure π with property (4.2)*.*

10.9 - Tonelli's Theorem

Theorem 4.4 (Tonelli's Theorem)**.** *Let (X, m, μ) and (Y, n, ν) be a σ-finite measure space and let F be a nonnegative measureable function on $Z = X \times Y$ to $\overline{\mathbb{R}}$. Then the functions defined on X and Y by*

$$f(x) = \int_Y F_x \, d\nu, \quad g(y) = \int_X F^y \, d\mu, \tag{4.3}$$

are measureable and

$$\int_X f \, d\mu = \int_Z F \, d\pi = \int_Y g \, d\nu. \tag{4.4}$$

In other symbols,

$$\int_X \left(\int_Y F \, d\nu\right) d\mu = \int_Z F \, d\pi = \int_Y \left(\int_X F \, d\mu\right) d\nu. \tag{4.5}$$

10.10 - Fubini's Theorem

Theorem 4.5 (Fubini's Theorem)**.** *Let (X, m, μ) and (Y, n, ν) be σ-finite spaces and let the measure π on $Z_0 = m \times n$ be the product measure of μ and ν. If the function F on $Z = X \times Y$ to $\mathbb{R}$ is integrable with respect to π, then the extended real-valued functions defined almost everywhere by*

$$f(x) = \int_Y F_x \, d\nu, \quad g(y) = \int_X F^y \, d\mu \tag{4.6}$$

have finite integrals and

$$\int_x f \, d\mu = \int_Z F \, d\pi = \int_Y g \, d\nu. \tag{4.7}$$

In other symbols,

$$\int_X \left[\int_Y F \, d\nu \right] d\mu = \int_Z F \, d\pi = \int_Y \left[\int_X F \, d\mu \right] d\nu. \tag{4.8}$$

5 Approximation by Nice Functions

(Lecture 11-12, the main results are: density in L^p of continuous function with constant support, density in L^p of smooth functions with constant support using the approximate identity).

6 Fourier Transform