Integration Theory

Exam

Kasper Rosenkrands

Aalborg University

S20

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Definition (σ-algebra)

A family m of subsets of a set X is said to be a σ-algebra if

1. $\emptyset, X$ belongs to m
2. If A belongs to X, then the complement $\mathcal{C}(A) = X \setminus A$ belongs to m.
3. If (A_n) is a sequence of sets in in X then the infinte union belongs to X.

An ordered pair (X, m) consisting of a set X and a σ-algebra m of subsets of X is called measurable space.

Definition (Measurable Function)

A function $f : X \to \mathbb{R}$ is said to be *m*-measurable if $\forall\, \alpha \in \mathbb{R}$

$$\{x \in X \,:\, f(x) > \alpha\} \in m. \tag{1.1}$$

Monotone Convergence Theorem

Theorem (Monotone Convergence Thereom)

If (f_n) is a monotone increasing sequence of functions in $M^+(X, m)$ which converges to f, then

$$\int f\, d\mu = \lim \int f_n\, d\mu. \tag{1.2}$$

Proof of Monotone Convergence Theorem

The strategy of the proof is to first show that

$$\lim \int f_n \, d\mu \leq \int f \, d\mu, \tag{1.3}$$

then afterwards to show that also

$$\lim \int f_n \, d\mu \geq \int f \, d\mu, \tag{1.4}$$

in order to conclude that

$$\lim \int f_n \, d\mu = \int f \, d\mu \tag{1.5}$$

Proof of Monotone Convergence Theorem

According to Corollary 2.10

Corollary (2.10)

If (f_n) is a sequence in $M(X, m)$ which converges to f on X, the f is in $M(X, m)$.

the function f is measurable.

Proof of Monotone Convergence Theorem

From Lemma 4.5(1.)

Lemma

1. *If f and g belong to $M^+(X, m)$ and $f \leq g$, then*

$$\int f \, d\mu \leq \int g \, d\mu. \tag{1.6}$$

2. *If f belongs to $M^+(X, m)$, if E, F belong to m, and if $E \subseteq F$, then*

$$\int_E f \, d\mu \leq \int_F f \, d\mu. \tag{1.7}$$

we have that

$$\int f_n \, d\mu \leq \int f_{n+1} \, d\mu \leq \int f \, d\mu, \quad \forall \, n \in \mathbb{N}. \tag{1.8}$$

Proof of Monotone Convergence Theorem

Therefore we must also have that

$$\lim \int f_n \, d\mu \leq \int f \, d\mu. \tag{1.9}$$

So this was the first step of our strategy, now we proceed to the second step.

Proof of Monotone Convergence Theorem

Let $\alpha \in \mathbb{R}$ be such that $0 < \alpha < 1$ and let φ be a simple measurable function such that $0 \leq \varphi \leq f$.
Let

$$A_n = \{x \in X \,:\, f_n(x) \geq \alpha\varphi(x)\}\,, \tag{1.10}$$

such that

1. $A_n \in m$
2. $A_n \subseteq A_{n+1}$
3. $X = \bigcup A_n$

Proof of Monotone Convergence Theorem

According to Lemma 4.5

Lemma

1. *If f and g belong to $M^+(X, m)$ and $f \leq g$, then*

$$\int f \, d\mu \leq \int g \, d\mu. \tag{1.11}$$

2. *If f belongs to $M^+(X, m)$, if E, F belong to m, and if $E \subseteq F$, then*

$$\int_E f \, d\mu \leq \int_F f \, d\mu. \tag{1.12}$$

it must be that

$$\int_{A_n} \alpha\varphi \, d\mu \leq \int_{A_n} f_n \, d\mu \leq \int f_n \, d\mu. \tag{1.13}$$

Proof of Monotone Convergence Theorem

Since the sequence A is monotone increasing and has union X, it follows from Lemma 4.3(2.) and Lemma 3.4(1.),

Lemma (4.3)

1. *If φ and ψ are simple functions in $M^+(X,m)$ and $c \geq 0$, then*

$$\int c\varphi \, d\mu = c \int \varphi \, d\mu, \tag{1.14}$$

$$\int (\varphi + \psi) \, d\mu = \int \varphi \, d\mu + \int \psi \, d\mu. \tag{1.15}$$

2. *If λ is defined for E in m by*

$$\lambda(E) = \int \varphi \chi_E \, d\mu, \tag{1.16}$$

then λ is a measure on m.

Lemma (3.4)

Let μ be a measure defined on a σ-algebra m.

1. *If (E_n) is an increasing sequence in m, then*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu\left(E_n\right). \tag{1.17}$$

2. *If (F_n) is a decreasing sequence in m and if $\mu(F_1) < +\infty$, then*

$$\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim \mu\left(F_n\right). \tag{1.18}$$

Proof of Monotone Convergence Theorem

that

$$\int \varphi \, d\mu = \lim \int_{A_n} \varphi \, d\mu. \tag{1.19}$$

Taking the limit as n tends to infinity in (1.13) therefore gives that

$$\alpha \int \varphi \, d\mu \leq \lim \int f_n \, d\mu. \tag{1.20}$$

Since this holds for all $0 < \alpha < 1$, by taking the limit as α tends to 1 we obtain

$$\int \varphi \, d\mu \leq \lim \int f_n \, d\mu. \tag{1.21}$$

Proof of Monotone Convergence Theorem

As φ is any simple function in M^+ such that $0 \leq \varphi \leq f$, we can conclude that

$$\int f \, d\mu = \sup_{\varphi} \int \varphi \, d\mu \leq \lim \int f_n \, d\mu, \tag{1.22}$$

which concludes the proof. □

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Theorem (Minkowski's Inequality)

If f and h belong to L_p, $p \geq 1$, then $f + h$ belongs to L_p and

$$\|f + h\|_p \leq \|f\|_p + \|h\|_p. \tag{2.1}$$

Proof of Minkowski's Inequality

- The case of $p = 1$ is equivalent to the triangular inequality and will not be considered here.
- We suppose $p > 1$.

Proof of Minkowski's Inequality

As both f and h are measurable functions, so is the sum

$$f + h, \tag{2.2}$$

as the sum of measurable functions are also measurable.

Proof of Minkowski's Inequality

From Corollary 5.4, Theorem 5.5 and the fact that

$$|f+h|^p \leq [2\sup\{|f|,|h|\}]^p \leq 2^p\{|f|^p+|h|^p\}, \tag{2.3}$$

it follows that $f+h \in L^p$.

Corollary (5.4)

If f is measurable, g is integrable, and $|f| \leq |g|$, then f is integrable, and

$$\int |f|\,d\mu \leq \int |g|\,d\mu. \tag{2.4}$$

Theorem (5.5)

A constant multiple αf and a sum $f+g$ of functions in L belongs to L and

$$\int \alpha f\,d\mu, \tag{2.5}$$

$$\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu. \tag{2.6}$$

Proof of Minkowski's Inequality

Using the triangular inequality we can see that

$$|f+h|^p = |f+h||f+h|^{p-1} \leq |f||f+h|^{p-1} + |h||f+h|^{p-1}. \tag{2.7}$$

Since $f+h \in L^p$ it implies that $|f+h|^p \in L^1$.

Assuming that $\frac{1}{p}+\frac{1}{q}=1$, it implies that

$$\frac{1}{p}+\frac{1}{q}=1 \Leftrightarrow 1+\frac{p}{q}=p \Leftrightarrow \frac{p}{q}=p-1 \Leftrightarrow p=(p-1)q, \tag{2.8}$$

and thereby that $|f+h|^{p-1} \in L^q$.

Proof of Minkowski's Inequality

We can now apply Hölder's Inequality,

Theorem (Hölder's Inequality)

Let $f \in L_p$ and $g \in L_q$ where $p > 1$ and $(1/p) + (1/q) = 1$. Then $fg \in L_1$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$.

to infer that

$$\int |f||f+h|^{p-1}\, d\mu \leq \|f\|_p \left\{ \int |f+h|^{(p-1)q}\, d\mu \right\}^{1/q} \tag{2.9}$$

$$= \|f\|_p \|f+h\|_p^{p/q}. \tag{2.10}$$

Note that exactly the same can be said for the other term

$$\int |h||f+h|^{p-1}\, d\mu \leq \|h\|_p \left\{ \int |f+h|^{(p-1)q}\, d\mu \right\}^{1/q} \tag{2.11}$$

$$= \|h\|_p \|f+h\|_p^{p/q}. \tag{2.12}$$

Proof of Minkowski's Inequality

This tells us that

$$\|f+h\|_p^p \leq \|f\|_p \|f+h\|_p^{p/q} + \|h\|_p \|f+h\|_p^{p/q} \tag{2.13}$$

$$= \{\|f\|_p + \|h\|_p\} \|f+h\|_p^{p/q}. \tag{2.14}$$

If we let $A = \|f+h\|_p = 0$, then the result becomes trivial as a norm by definition is greater than or equal to zero.

Proof of Minkowski's Inequality

Suppose now that $A \neq 0$ then we can divide (2.14) by $A^{p/q}$

$$\frac{A^p}{A^{p/q}} \leq \{\|f\|_p + \|h\|_p\} \frac{A^{p/q}}{A^{p/q}} \tag{2.15}$$

$$A^{p-p/q} \leq \|f\|_p + \|h\|_p, \tag{2.16}$$

by noting that $p - p/q = 1$, we obtain

$$\|f + h\|_p \leq \|f\|_p + \|h\|_p, \tag{2.17}$$

which concludes the proof. $\square$

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Definition of Absolute Continuity

Definition (Absolutely Continuous)

A measure λ on m is said to be absolutely continuous with respect to a measure μ on m if $E \in m$ and $\mu(E) = 0$ imply that $\lambda(E) = 0$. In this case we write $\lambda << \mu$.

Note that μ can send more sets to 0 than λ, but not the other way around.

Theorem (Radón-Nikodym Theorem)

Let λ and μ be σ-finite measures defined on m and suppose that λ is absolutely continuous with respects to μ. Then there exists a function f in $M^+(X, m)$, such that,

$$\lambda(E) = \int_E f d\mu, \quad E \in m. \tag{3.1}$$

Moreover, the function f is uniquely determined μ-almost everywhere.

Definition of Singular Measure

Definition (A Singular Measure)

Two measures λ, μ on m are said to be mutually singular if there are disjoint sets A, B in m, such that, $X = A \cup B$ and $\lambda(A) = \mu(B) = 0$. In this case we write $\lambda \perp \mu$.

Lebesgue Decomposition Theorem

Theorem (Lebesgue Decomposition Theorem)

Let λ and μ be sigma-finite measures defined on a sigma-algebra m. Then there exists a measure λ_1 which is singular with respect to μ and a measure λ_2 which is absolutely continuous with respect to μ such that $\lambda = \lambda_1 + \lambda_2$. Moreover, the measures λ_1 and λ_2 are unique.

The Theorem is a consequence of the Radon-Nikodým Theorem.

Proof of Lebesgue Decomposition Theorem

It can be shown that a measure ν can be decomposed

$$\nu = \lambda + \mu \tag{3.2}$$

such that ν is σ-finite, $\lambda << \nu$ and $\mu << \nu$.

A consequence of the Radon-Nikodým Theorem is that

$$\exists\, f, g \in M^{+}(X, m), \tag{3.3}$$

where

$$\lambda(E) = \int_E f\, d\nu, \quad \mu(E) = \int_E g\, d\nu, \qquad \forall\, E \in m. \tag{3.4}$$

Proof of Lebesgue Decomposition Theorem

Define the following sets

$$A := \{x \in X \,|\, g(x) = 0\}, \quad B := \{x \in X \,|\, g(x) > 0\}. \tag{3.5}$$

As $g \in M^+$ we have that

$$A \cup B = X, \quad A \cap B = \emptyset. \tag{3.6}$$

Define now $\lambda_1, \lambda_2 \,:\, m \longrightarrow \overline{\mathbb{R}}$ by

$$\lambda_1(E) := \lambda(E \cap A), \tag{3.7}$$
$$\lambda_2(E) := \lambda(E \cap B). \tag{3.8}$$

Proof of Lebesgue Decomposition Theorem

What we now want is to prove that

1. $\lambda_1 \perp \mu$,
2. $\lambda_2 << \mu$.

We start by proving $\lambda_1 \perp \mu$. From the definition of A we get

$$\left.\begin{array}{r} \mu(A) \underbrace{=}_{\text{def. of } \mu} \int_A g\, d\nu \underbrace{=}_{\text{def. of } A} 0 \\ \lambda_1(B) \underbrace{=}_{\text{def. of } \lambda_1} \lambda(B \cap A) \underbrace{=}_{\text{def. of } B} \lambda(\emptyset) = 0 \end{array}\right\} \quad \lambda_1 \perp \mu. \tag{3.9}$$

This proves the first point.

Proof of Lebesgue Decomposition Theorem

Now we want to show that $\lambda_2 << \mu$.

If $E \in m$ such that $\mu(E) = 0$ then

$$\int_E g \, d\nu = 0. \tag{3.10}$$

By Corollary 4.10,

Corollary (4.10)

Suppose that f belongs to M^+. Then $f(x) = 0$ μ-almost everywhere on X if and only if

$$\int f \, d\mu = 0. \tag{3.11}$$

this means that $g(x) = 0$ ν-almost everywhere on E.

Proof of Lebesgue Decomposition Theorem

Recall that the terminology μ-almost everywhere means that there exists a subset $N \in m$ with $\mu(N) = 0$ such that the statement holds on the complement of N. Thus we have that

$$\left.\begin{array}{l} B := \{x \in X \mid g(x) > 0\} \\ \exists\, N \text{ s.t. } \nu(N) = 0 \\ g(x) = 0 \text{ on } E \backslash N \end{array}\right\} \Rightarrow E \cap B \underbrace{\subseteq}_{\text{outside } N,\ g(x)=0} N \quad (3.12)$$

$$\underbrace{\Rightarrow}_{\text{Lemma 3.3}} 0 \leq \nu(E \cap B) \leq \nu(N) = 0 \Rightarrow \nu(E \cap B) = 0. \quad (3.13)$$

Proof of Lebesgue Decomposition Theorem

But remembering that $\lambda << \nu$ this implies that

$$\lambda(E \cap B) = 0 \implies \lambda_2(E) = 0. \tag{3.14}$$

Thus we have shown that $\lambda_2 << \mu$.

Proof of Lebesgue Decomposition Theorem

To show that $\lambda = \lambda_1 + \lambda_2$ remeber that they were constructed from the sets A and B. We have that

$$X = A \cup B \quad \text{and} \quad A \cap B = \emptyset. \tag{3.15}$$

For every measurable set E we have that

$$E = (E \cap A) \cup (E \cap B) \quad \text{and} \quad (E \cap A) \cap (E \cap B) = \emptyset. \tag{3.16}$$

Therefore we het that from the countable additivity of a measure that

$$\begin{aligned} \lambda(E) &= \lambda(E \cap A) + \lambda(E \cap B) && (3.17) \\ &= \lambda_1(E) + \lambda_2(E) && (3.18) \end{aligned}$$

Proof of Lebesgue Decomposition Theorem

Because this is true for every measurable set we can say that in general

$$\lambda = \lambda_1 + \lambda_2. \tag{3.19}$$

We have now shown that there existence part of the theorem. To finish the proof we would have to also verify that the λ_1 and λ_2, derived here, are unique.

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Tonelli's Theorem

Theorem (Tonelli's Theorem)

Let (X, m, μ) and (Y, n, ν) be a σ-finite measure space and let F be a nonnegative measureable function on $Z = X \times Y$ to $\overline{\mathbb{R}}$. Then the functions defined on X and Y by

$$f(x) = \int_Y F_x \, d\nu, \quad g(y) = \int_X F^y \, d\mu, \tag{4.1}$$

are measureable and

$$\int_X f \, d\mu = \int_Z F \, d\pi = \int_Y g \, d\nu. \tag{4.2}$$

In other symbols,

$$\int_X \left(\int_Y F \, d\nu \right) d\mu = \int_Z F \, d\pi = \int_Y \left(\int_X F \, d\mu \right) d\nu. \tag{4.3}$$

Proof of Tonelli's Theorem

Suppose first that $F = \chi_E$ where $E \in Z$, the result is then a consequence of Lemma 10.8.

Lemma (10.8)

Let (X, m, μ) and (Y, n, ν) be σ-finite measure spaces. If $E \in Z = X \times Y$, then the functions defined by

$$f(x) = \nu(E_x)$$
$$g(y) = \mu(E^y)$$

are measurable and

$$\int_X f \, d\mu = \pi(E) = \int_Y g \, d\nu.$$

Proof of Tonelli's Theorem

Furthermore we have that by the linearity of the integral, that the theorem holds true for every φ positive simple function.

Now we would like to extend the result to any nonnegative measurable function.

Let $F \geq 0$, $F : Z \to \overline{\mathbb{R}}$ be measurable.

Proof of Tonelli's Theorem

Lemma 2.11 thus gives that there exists (Φ_n) such that,

Lemma (2.11)

If f is a nonnegative function in $M(X, m)$, then there exists a sequence (φ_n) in $M(X, m)$ such that

1. $0 \leq \varphi_n(x) \leq \varphi_{n+1}$ *for* $x \in m$, $n \in \mathbb{N}$.
2. $f(x) = \lim \varphi_n(x)$ *for each* $x \in m$.
3. *Each* φ_n *has only a finite number of real values.*

Furthermore Lemma 10.6 gives that every section of Φ_n are measurable

Lemma (10.6)

1. *If E is a measurable subset of Z, then every section of E is measurable.*
2. *If f is a measurable function on Z to $\overline{\mathbb{R}}$, then every section of f is measurable.*

Proof of Tonelli's Theorem

We can now define

$$\varphi_n : X \to \overline{\mathbb{R}}, \tag{4.4}$$
$$\varphi_n(x) := \int_Y (\Phi_n)_x \, d\nu, \tag{4.5}$$

As well as

$$\psi_n : Y \to \overline{\mathbb{R}}, \tag{4.6}$$
$$\psi_n(x) := \int_X (\Phi_n)^y \, d\mu. \tag{4.7}$$

As Φ_n is monotone it implies that both φ_n and ψ_n are monotone.

Proof of Tonelli's Theorem

Taking the limit of φ_n gives

$$
\begin{aligned}
\lim_n \varphi_n &= \lim_n \int_Y (\Phi_n)_x \, d\nu && (4.8)\\
\text{(MCT)} \quad &= \int_Y \lim_n (\Phi_n)_x \, d\nu && (4.9)\\
\text{(Lemma 2.11)} \quad &= \int_Y F_x \, d\nu && (4.10)\\
\text{(def)} \quad &= f, && (4.11)
\end{aligned}
$$

the same strategy can be used for ψ_n to obtain

$$\lim_n \psi_n = g. \tag{4.12}$$

Proof of Tonelli's Theorem

Another application of the Monotone Convergence Theorem gives that

$$\lim_n \int_X \varphi_n \, d\mu = \int_X f \, d\mu, \tag{4.13}$$

and that

$$\lim_n \int_Y \psi_n \, d\mu = \int_Y g \, d\nu. \tag{4.14}$$

The thing left now is to show that these two expressions are infact equal.

Proof of Tonelli's Theorem

$$\lim_n \int_X \varphi_n \, d\mu = \lim_n \int_X \left(\int_Y (\Phi_n)_x \, d\nu \right) d\mu \tag{4.15}$$

$$= \lim_n \int_X \int_Y \Phi_n \, d\nu \, d\mu \tag{4.16}$$

(Result for simple functions)

$$= \lim_n \int_Z \Phi_n \, d\pi \tag{4.17}$$

$$= \lim_n \int_Y \left(\int_X \Phi_n \, d\mu \right) d\nu \tag{4.18}$$

$$= \lim_n \int_Y \psi_n \, d\nu \tag{4.19}$$

Proof of Tonelli's Theorem

The Monotone Convergence Theorem gives that

$$\lim_n \int_Z \Phi_n \, d\pi = \int_Z F \, d\pi. \tag{4.20}$$

Altogether we can thus conclude that

$$\int_X f \, d\mu = \int_Z F \, d\pi = \int_Y g \, d\nu,$$

which concludes the proof. $\square$

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Theorem 3.16

Theorem (3.16)

Let a measure space be given by $(\mathbb{R}, \mathcal{B}, \lambda)$*. The set* $\mathcal{C}_c(\mathbb{R})$ *of continuous functions with compact support is dense in* $L^p(\mathbb{R}, \mathcal{B}, \lambda)$*, for* $1 \leq p < +\infty$*.*

For a function f to have compact support, or $f \in \mathcal{C}_c(\mathbb{R})$, it means that f is continuous and that $\text{supp}(f) \subseteq K$ where K is compact, where supp is defined as

$$\text{supp}(f) = \{x \in X \mid f(x) \neq 0\}. \tag{5.1}$$

Proof of Theorem 3.16

Using the fact that simple function on compact sets are dense i L^p it is enough to show that we can "approximate" every simple function on compact sets, $\tilde{\varphi}$, with a continuous function.

First we fix $\tilde{\varphi}$ and $\varepsilon > 0$, where $\tilde{\varphi} = \sum a_i \chi_{K_i}$ with K_i being a compact set.

Using the fact that we can "approximate" any Lebesgue measurable set by an open set, that is $\forall\, A \in \mathcal{L} \supseteq \mathcal{B}$

$$l^*(A) = \inf \{ l^*(\mathcal{O}) \,|\, A \subseteq \mathcal{O},\ \mathcal{O} \text{ open} \}, \tag{5.2}$$

we obtain that $\forall\, K$ compact, $\forall\, \varepsilon' > 0$ there exists an open set $\mathcal{O}$ such that $K \subset \mathcal{O}$ and

$$l^*(\mathcal{O} \backslash K) \leq \varepsilon'. \tag{5.3}$$

Proof of Theorem 3.16

No matter what ε' we choose we are able to find an open set so that the previous equation is satisfied. Lets defines ε' as

$$\varepsilon' = \left(\frac{\varepsilon}{\sum |a_i|}\right)^p. \tag{5.4}$$

Now we need to find a continuous function $f : \mathbb{R} \to [0,1]$ suhc that

1. $\text{supp}(f)$ is compact.
2. $f(x) = 1$ for $x \in K$.
3. $f(x) = 0$ for $x \in \mathbb{R} \backslash \mathcal{O}$.

As we are in $\mathbb{R}$ the function can be written explicitly as;

Proof of Theorem 3.16

$$f_i(x) := \frac{\mathrm{dist}(x, C(\mathcal{O}_i))}{\mathrm{dist}(x, C(\mathcal{O}_i)) + \mathrm{dist}(x, K_i)}. \tag{5.5}$$

where $\mathrm{dist}(x, E) = \inf\{|x - z| : z \in E\}$ for $E \in \mathcal{B}$.

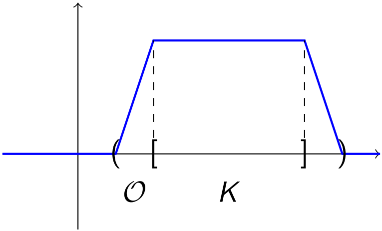

Figure: Function given in (5.5).

Proof of Theorem 3.16

The first two properties follows immediately, so we are left to show continuity. We have that

$$\forall z \in E \,:\, \operatorname{dist}(x, E) \leq |x - z| \leq |x - y| + |y - z|. \tag{5.6}$$

Taking the inf over $E \in Z$ yields

$$\operatorname{dist}(x, E) \leq |x - y| + \operatorname{dist}(y, E), \tag{5.7}$$

the reverse inequality is obtained in the same way

$$\operatorname{dist}(y, E) \leq |x - y| + \operatorname{dist}(x, E). \tag{5.8}$$

Proof of Theorem 3.16

Thus we see that

$$|\text{dist}(x, E) - \text{dist}(y, E)| \le |x - y|,$$

which gives the continuity of the dist function, which makes f_i a composition of continuous function and thus it is continuous. Furthermore the denominator cannot be zero.

We can now $\forall K_i$ construct f_i. Lets now define the function

$$F = \sum a_i f_i. \tag{5.9}$$

Using the triangular inequality on the p-norm of the difference between $\tilde{\varphi}$ and F gives

$$||\tilde{\varphi} - F||_p \ \leq \sum |a_i| \, ||\chi_{K_i} - f_i||_p. \tag{5.10}$$

Proof of Theorem 3.16

The left part of the inequality (raised to the p-th power) satisfies

$$
\begin{aligned}
||\chi_{K_i} - f_i||_p^p &= \int_{\mathbb{R}} |\chi_{K_i}(x) - f_i(x)|^p dx && (5.11)\\
&= \int_{\mathcal{O}_i \backslash K_i} |f_i(x)|^p dx && (5.12)\\
&\leq l^*(\mathcal{O}_i \backslash K_i) && (5.13)\\
&\leq \left(\frac{\varepsilon}{\sum |a_i|} \right)^p, && (5.14)
\end{aligned}
$$

this implies that

$$||\tilde{\varphi} - F||_p \leq \varepsilon. \qquad (5.15)$$

Proof of Theorem 3.16

We have now shown that we can "approximate" every simple function with compact support by a continuous function, which concludes the proof. □

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Definitions

Definition (Fourier Transform)

For $f \in L^1(\mathbb{R})$ we define its Fourier Transform as

$$F(f)(p) \equiv \hat{f}(p) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ipx} f(x)\, dx. \tag{6.1}$$

Theorem 7.7

Theorem (7.7)

The Fourier transform is a linear bounded injective map from L^1 into C_0. Its inverse is given by

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ipx - \varepsilon |p|^2/2} \hat{f}(p)\, dp, \tag{6.2}$$

where the limit has to be understood in L^1. Moreover, this holds at every point of continuity.

Proof of Theorem 7.7

We know that the Fourier Transform is a function satisfying $F : L^1(\mathbb{R}) \to C_0(\mathbb{R}) \subset C_b(\mathbb{R})$. F is bounded since $\forall f \in L^1$

$$\|F(f)\|_\infty \le \frac{1}{\sqrt{2\pi}} \|f\|_1. \tag{6.3}$$

The first thing we will prove is that the Fourier Transform is injective, suppose therefore that we have already proved (6.2).

Proof of Theorem 7.7

For F to be injective, it means that

$$f_1 \neq f_2 \;\Rightarrow\; F(f_1) \neq F(f_2). \tag{6.4}$$

We will use proof by contradiction, so assume that

$$F(f_1) = F(f_2), \tag{6.5}$$

by linearity we have that

$$F(f_1 - f_2) = F(f_1) - F(f_2) = 0 \tag{6.6}$$

but this is also equal to

$$= (\widehat{f_1 - f_2})(p), \quad \forall\, p \in \hat{\mathbb{R}}. \tag{6.7}$$

Proof of Theorem 7.7

As $f_1 - f_2 \in L^1(\mathbb{R})$, (6.2) gives that

$$(f_1 - f_2)(x) = 0 \implies f_1 = f_2, \tag{6.8}$$

which is a contradiction. Thus we conclude that the Fourier Transform is an injective map.

Next we can prove (6.2).

Proof of Theorem 7.7

Let

$$\phi_\varepsilon(x) := \frac{1}{\sqrt{2\pi}} e^{-\varepsilon \frac{|x|^2}{2}} \tag{6.9}$$

then we can rewrite the right hand side of (6.2) as

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ipx} e^{-\varepsilon|p|^2/2} \hat{f}(p)\, dp &= \int_{\mathbb{R}} e^{ipx} \phi_\varepsilon(p) \hat{f}(p)\, dp & (6.10) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ipx} \phi_\varepsilon(p) e^{-ipy} f(y)\, dy\, dp. & (6.11) \end{aligned}$$

Proof of Theorem 7.7

By Tonelli's Theorem we have that the following inequality holds

$$|e^{ipx}e^{-ipy}\phi_\varepsilon(p)f(y)| \leq \underbrace{|\phi_\varepsilon(p)|}_{\in L^1(\hat{\mathbb{R}})}\underbrace{|f(y)|}_{\in L^1(\mathbb{R})} \overbrace{\in}^{\text{Tonelli}} L^1(\mathbb{R}\times\mathbb{R}). \tag{6.12}$$

Applying now Fubini's Theorem gives

$$\int_{\mathbb{R}} (\widehat{e^{ipx}\phi_\varepsilon})(y)f(y)\,dy \overbrace{=}^{\text{Lemma 7.2 (2.)}} \int_{\mathbb{R}} \hat{\phi}_\varepsilon(y-x)f(y)\,dy \tag{6.13}$$

$$\overbrace{=}^{\text{Gaussian Lemma}} \int_{\mathbb{R}} \frac{1}{\sqrt{\varepsilon}}\phi_{\frac{1}{\varepsilon}}(y-x)f(y) \tag{6.14}$$

$$= (\Phi_\varepsilon * f)(x), \tag{6.15}$$

Proof of Theorem 7.7

Lemma (Gaussian Lemma)

We have that

$$F\left(e^{-z|x|^2/2}\right)(p) = \frac{1}{\sqrt{z}} e^{-|p|^2/2z}. \tag{6.16}$$

Definition (Convolution)

The convolution of two functions is defined as

$$(\phi * f)(x) := \int_{\mathbb{R}} \phi(x-y) f(y)\, dy = \int_{\mathbb{R}} \phi(y) f(x-y)\, dy. \tag{6.17}$$

Proof of Theorem 7.7

Where $\Phi_\varepsilon(x)$ is defined as

$$\Phi_\varepsilon(x) := \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\varepsilon}}e^{-\frac{|x|^2}{2\varepsilon}}, \tag{6.18}$$

where we have that $\Phi_\varepsilon(x)$ is an approximate identity as we have that

$$\Phi_\varepsilon \geq 0,\ \Phi_\varepsilon \in C^\infty \text{ and } \int_{\mathbb{R}} \Phi_\varepsilon(x)dx = \underbrace{\frac{1}{\sqrt{2\pi}}\int e^{-\frac{|x|^2}{2}}dx}_{\text{Gaussian integral}} = 1 \tag{6.19}$$

Proof of Theorem 7.7

Taking now the limit as ε goes to zero and using linear change of variable and the Dominated Convergence Theorem yields

$$\forall r > 0 \;\lim_{\varepsilon \to 0} \int_{|x| \geq r} \Phi_\varepsilon(x)\, dx = \lim_{\varepsilon \to 0} \int_{|x| \geq \frac{r}{\sqrt{\varepsilon}}} e^{-\frac{|x|^2}{2}}\, dx = 0, \quad (6.20)$$

telling us that $\Phi_\varepsilon(x)$ is infact an approximate identity. The right hand side thus becomes

$$\lim_{\varepsilon \to 0} \Phi_\varepsilon * f = f, \quad (6.21)$$

by Lemma 3.19. This concludes the proof. $\square$