Financial Engineering

Exam

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S20

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Monotone Convergence Theorem

Theorem (Monotone Convergence Thereom)

If (f_n) is a monotone increasing sequence of functions in $M^+(X, m)$ which converges to f, then

$$\int f \, d\mu = \lim \int f_n \, d\mu. \tag{1.1}$$

Proof of Monotone Convergence Theorem

The strategy of the proof is to first show that

$$\lim \int f_n \, d\mu \leq \int f \, d\mu, \tag{1.2}$$

then afterwards to show that also

$$\lim \int f_n \, d\mu \geq \int f \, d\mu, \tag{1.3}$$

in order to conclude that

$$\lim \int f_n \, d\mu = \int f \, d\mu \tag{1.4}$$

Proof of Monotone Convergence Theorem

According to Corollary 2.10

Corollary

If (f_n) is a sequence in $M(X, m)$ which converges to f on X, the f is in $M(X, m)$.

the function f is measurable.

Proof of Monotone Convergence Theorem

From Lemma 4.5(1.)

Lemma

1. *If f and g belong to $M^+(X, m)$ and $f \leq g$, then*

$$\int f \, d\mu \leq \int g \, d\mu. \tag{1.5}$$

2. *If f belongs to $M^+(X, m)$, if E, F belong to m, and if $E \subseteq F$, then*

$$\int_E f \, d\mu \leq \int_F f \, d\mu. \tag{1.6}$$

we have that

$$\int f_n \, d\mu \leq \int f_{n+1} \, d\mu \leq \int f \, d\mu, \quad \forall \, n \in \mathbb{N}. \tag{1.7}$$

Proof of Monotone Convergence Theorem

Therefore we must also have that

$$\lim \int f_n \, d\mu \leq \int f \, d\mu. \tag{1.8}$$

So this was the first step of our strategy, now we proceed to the second step.

Proof of Monotone Convergence Theorem

Let $\alpha \in \mathbb{R}$ be such that $0 < \alpha < 1$ and let φ be a simple measurable function such that $0 \leq \varphi \leq f$.
Let

$$A_n = \{x \in X \,:\, f_n(x) \geq \alpha\varphi(x)\}\,, \tag{1.9}$$

such that

1. $A_n \in m$
2. $A_n \subseteq A_{n+1}$
3. $X = \bigcup A_n$

Proof of Monotone Convergence Theorem

According to Lemma 4.5

Lemma

1. *If f and g belong to $M^+(X, m)$ and $f \leq g$, then*

$$\int f \, d\mu \leq \int g \, d\mu. \tag{1.10}$$

2. *If f belongs to $M^+(X, m)$, if E, F belong to m, and if $E \subseteq F$, then*

$$\int_E f \, d\mu \leq \int_F f \, d\mu. \tag{1.11}$$

it must be that

$$\int_{A_n} \alpha\varphi \, d\mu \leq \int_{A_n} f_n \, d\mu \leq \int f_n \, d\mu. \tag{1.12}$$

Proof of Monotone Convergence Theorem

Since the sequence A is monotone increasing and has union X, it follows from Lemma 4.3(2.) and Lemma 3.4(1.),

Lemma (4.3)

1. *If φ and ψ are simple functions in $M^+(X, m)$ and $c \geq 0$, then*
$$\int c\varphi \, d\mu = c \int \varphi \, d\mu, \tag{1.13}$$
$$\int (\varphi + \psi) \, d\mu = \int \varphi \, d\mu + \int \psi \, d\mu. \tag{1.14}$$
2. *If λ is defined for E in m by*
$$\lambda(E) = \int \varphi \chi_E \, d\mu, \tag{1.15}$$
then λ is a measure on m.

Lemma (3.4)

Let μ be a measure defined on a σ-algebra m.

1. *If (E_n) is an increasing sequence in m, then*
$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu\left(E_n\right). \tag{1.16}$$
2. *If (F_n) is a decreasing sequence in m and if $\mu\left(F_1\right) < +\infty$, then*
$$\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim \mu\left(F_n\right). \tag{1.17}$$

Proof of Monotone Convergence Theorem

that

$$\int \varphi \, d\mu = \lim \int_{A_n} \varphi \, d\mu. \tag{1.18}$$

Taking the limit as n tends to infinity in (1.12) therefore gives that

$$\alpha \int \varphi \, d\mu \leq \lim \int f_n \, d\mu. \tag{1.19}$$

Since this holds for all $0 < \alpha < 1$, by taking the limit as α tends to 1 we obtain

$$\int \varphi \, d\mu \leq \lim \int f_n \, d\mu. \tag{1.20}$$

Proof of Monotone Convergence Theorem

As φ is any simple function in M^+ such that $0 \leq \varphi \leq f$, we can conclude that

$$\int f \, d\mu = \sup_{\varphi} \int \varphi \, d\mu \leq \lim \int f_n \, d\mu, \tag{1.21}$$

which concludes the proof. $\square$

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