

Financial Engineering

Exam

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S20

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2. L^p spaces
3. Decomposition of measures
4. Generation of measures and product measures
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Theorem (Monotone Convergence Theorem)

If (f_n) is a monotone increasing sequence of functions in $M^+(X, m)$ which converges to f , then

$$\int f \, d\mu = \lim \int f_n \, d\mu. \quad (1.1)$$

Proof of Monotone Convergence Theorem

The strategy of the proof is to first show that

$$\lim \int f_n d\mu \leq \int f d\mu, \quad (1.2)$$

then afterwards to show that also

$$\lim \int f_n d\mu \geq \int f d\mu, \quad (1.3)$$

in order to conclude that

$$\lim \int f_n d\mu = \int f d\mu \quad (1.4)$$

According to Corollary 2.10

Corollary (2.10)

If (f_n) is a sequence in $M(X, m)$ which converges to f on X , the f is in $M(X, m)$.

the function f is measurable.

Lebesgue integration theory

Proof of Monotone Convergence Theorem

From Lemma 4.5(1.)

Lemma

1. If f and g belong to $M^+(X, m)$ and $f \leq g$, then

$$\int f \, d\mu \leq \int g \, d\mu. \quad (1.5)$$

2. If f belongs to $M^+(X, m)$, if E, F belong to m , and if $E \subseteq F$, then

$$\int_E f \, d\mu \leq \int_F f \, d\mu. \quad (1.6)$$

we have that

$$\int f_n \, d\mu \leq \int f_{n+1} \, d\mu \leq \int f \, d\mu, \quad \forall n \in \mathbb{N}. \quad (1.7)$$

Therefore we must also have that

$$\lim \int f_n d\mu \leq \int f d\mu. \quad (1.8)$$

So this was the first step of our strategy, now we proceed to the second step.

Proof of Monotone Convergence Theorem

Let $\alpha \in \mathbb{R}$ be such that $0 < \alpha < 1$ and let φ be a simple measurable function such that $0 \leq \varphi \leq f$.

Let

$$A_n = \{x \in X : f_n(x) \geq \alpha \varphi(x)\}, \quad (1.9)$$

such that

1. $A_n \in \mathcal{m}$
2. $A_n \subseteq A_{n+1}$
3. $X = \bigcup A_n$

Lebesgue integration theory

Proof of Monotone Convergence Theorem

According to Lemma 4.5

Lemma

1. If f and g belong to $M^+(X, m)$ and $f \leq g$, then

$$\int f \, d\mu \leq \int g \, d\mu. \quad (1.10)$$

2. If f belongs to $M^+(X, m)$, if E, F belong to m , and if $E \subseteq F$, then

$$\int_E f \, d\mu \leq \int_F f \, d\mu. \quad (1.11)$$

it must be that

$$\int_{A_n} \alpha \varphi \, d\mu \leq \int_{A_n} f_n \, d\mu \leq \int f_n \, d\mu. \quad (1.12)$$

Lebesgue integration theory

Proof of Monotone Convergence Theorem

Since the sequence A is monotone increasing and has union X , it follows from Lemma 4.3(2.) and Lemma 3.4(1.),

Lemma (4.3)

1. If φ and ψ are simple functions in $M^+(X, m)$ and $c \geq 0$, then

$$\int c\varphi d\mu = c \int \varphi d\mu, \quad (1.13)$$

$$\int (\varphi + \psi) d\mu = \int \varphi d\mu + \int \psi d\mu. \quad (1.14)$$

2. If λ is defined for E in m by

$$\lambda(E) = \int \varphi \chi_E d\mu, \quad (1.15)$$

then λ is a measure on m .

Lemma (3.4)

Let μ be a measure defined on a σ -algebra m .

1. If (E_n) is an increasing sequence in m , then

$$\mu \left(\bigcup_{n=1}^{\infty} E_n \right) = \lim \mu(E_n). \quad (1.16)$$

2. If (F_n) is a decreasing sequence in m and if $\mu(F_1) < +\infty$, then

$$\mu \left(\bigcap_{n=1}^{\infty} F_n \right) = \lim \mu(F_n). \quad (1.17)$$

Proof of Monotone Convergence Theorem

that

$$\int \varphi d\mu = \lim \int_{A_n} \varphi d\mu. \quad (1.18)$$

Taking the limit as n tends to infinity in (1.12) therefore gives that

$$\alpha \int \varphi d\mu \leq \lim \int f_n d\mu. \quad (1.19)$$

Since this holds for all $0 < \alpha < 1$, by taking the limit as α tends to 1 we obtain

$$\int \varphi d\mu \leq \lim \int f_n d\mu. \quad (1.20)$$

As φ is any simple function in M^+ such that $0 \leq \varphi \leq f$, we can conclude that

$$\int f \, d\mu = \sup_{\varphi} \int \varphi \, d\mu \leq \lim \int f_n \, d\mu, \quad (1.21)$$

which concludes the proof. \square

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Minkowski's Inequality

Theorem (Minkowski's Inequality)

If f and h belong to L_p , $p \geq 1$, then $f + h$ belongs to L_p and

$$\|f + h\|_p \leq \|f\|_p + \|h\|_p. \quad (2.1)$$

Proof of Minkowski's Inequality

- ▶ The case of $p = 1$ is equivalent to the triangular inequality and will not be considered here.
- ▶ We suppose $p > 1$.

Proof of Minkowski's Inequality

As both f and h are measurable functions, so is the sum

$$f + h, \tag{2.2}$$

as the sum of measurable functions are also measurable.

Proof of Minkowski's Inequality

From Corollary 5.4, Theorem 5.5 and the fact that

$$|f + h|^p \leq [2 \sup \{|f|, |h|\}]^p \leq 2^p \{|f|^p + |h|^p\}, \quad (2.3)$$

it follows that $f + h \in L^p$.

Theorem (5.5)

A constant multiple αf and a sum $f + g$ of functions in L belongs to L and

$$\int \alpha f \, d\mu, \quad (2.5)$$

Corollary (5.4)

If f is measurable, g is integrable, and $|f| \leq |g|$, then f is integrable, and

$$\int |f| \, d\mu \leq \int |g| \, d\mu. \quad (2.4)$$

$$\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu. \quad (2.6)$$

Proof of Minkowski's Inequality

Using the triangular inequality we can see that

$$|f + h|^p = |f + h||f + h|^{p-1} \leq |f||f + h|^{p-1} + |h||f + h|^{p-1}. \quad (2.7)$$

Since $f + h \in L^p$ it implies that $|f + h|^p \in L^1$.

Assuming that $\frac{1}{p} + \frac{1}{q} = 1$, it implies that

$$\frac{1}{p} + \frac{1}{q} = 1 \Leftrightarrow 1 + \frac{p}{q} = p \Leftrightarrow \frac{p}{q} = p - 1 \Leftrightarrow p = (p - 1)q, \quad (2.8)$$

and thereby that $|f + h|^{p-1} \in L^q$.

L^p spaces

Proof of Minkowski's Inequality

We can now apply Hölder's Inequality,

Theorem (Hölder's Inequality)

Let $f \in L_p$ and $g \in L_q$ where $p > 1$ and $(1/p) + (1/q) = 1$. Then $fg \in L_1$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$.

to infer that

$$\int |f| |f + h|^{p-1} d\mu \leq \|f\|_p \left\{ \int |f + h|^{(p-1)q} d\mu \right\}^{1/q} \quad (2.9)$$

$$= \|f\|_p \|f + h\|_p^{p/q}. \quad (2.10)$$

Note that exactly the same can be said for the other term

$$\int |h| |f + h|^{p-1} d\mu \leq \|h\|_p \left\{ \int |f + h|^{(p-1)q} d\mu \right\}^{1/q} \quad (2.11)$$

$$= \|h\|_p \|f + h\|_p^{p/q}. \quad (2.12)$$

Proof of Minkowski's Inequality

This tells us that

$$\|f + h\|_p^p \leq \|f\|_p \|f + h\|_p^{p/q} + \|h\|_p \|f + h\|_p^{p/q} \quad (2.13)$$

$$= \{\|f\|_p + \|h\|_p\} \|f + h\|_p^{p/q}. \quad (2.14)$$

If we let $A = \|f + h\|_p = 0$, then the result becomes trivial as a norm by definition is greater than or equal to zero.

Proof of Minkowski's Inequality

Suppose now that $A \neq 0$ then we can divide (2.14) by $A^{p/q}$

$$\frac{A^p}{A^{p/q}} \leq \{\|f\|_p + \|h\|_p\} \frac{A^{p/q}}{A^{p/q}} \quad (2.15)$$

$$A^{p-p/q} \leq \|f\|_p + \|h\|_p, \quad (2.16)$$

by noting that $p - p/q = 1$, we obtain

$$\|f + h\|_p \leq \|f\|_p + \|h\|_p, \quad (2.17)$$

which concludes the proof. □

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Decomposition of measures

Definition of Absolute Continuity

Definition (Absolutely Continuous)

A measure λ on m is said to be absolutely continuous with respect to a measure μ on m if $E \in m$ and $\mu(E) = 0$ imply that $\lambda(E) = 0$. In this case we write $\lambda \ll \mu$.

Note that μ can send more sets to 0 than λ , but not the other way around.

Theorem (Radón-Nikodym Theorem)

Let λ and μ be σ -finite measures defined on m and suppose that λ is absolutely continuous with respects to μ . Then there exists a function f in $M^+(X, m)$, such that,

$$\lambda(E) = \int_E f d\mu, \quad E \in m. \quad (3.1)$$

Moreover, the function f is uniquely determined μ -almost everywhere.

Decomposition of measures

Definition of Singular Measure

Definition (A Singular Measure)

Two measures λ, μ on m are said to be mutually singular if there are disjoint sets A, B in m , such that, $X = A \cup B$ and $\lambda(A) = \mu(B) = 0$. In this case we write $\lambda \perp \mu$.

Decomposition of measures

Lebesgue Decomposition Theorem

Theorem (Lebesgue Decomposition Theorem)

Let λ and μ be sigma-finite measures defined on a sigma-algebra m . Then there exists a measure λ_1 which is singular with respect to μ and a measure λ_2 which is absolutely continuous with respect to μ such that $\lambda = \lambda_1 + \lambda_2$. Moreover, the measures λ_1 and λ_2 are unique.

The Theorem is a consequence of the Radon-Nikodým Theorem.

Decomposition of measures

Proof of Lebesgue Decomposition Theorem

It can be shown that a measure ν can be decomposed

$$\nu = \lambda + \mu \tag{3.2}$$

such that ν is σ -finite, $\lambda \ll \nu$ and $\mu \ll \nu$.

A consequence of the Radon-Nikodým Theorem is that

$$\exists f, g \in M^+(X, m), \tag{3.3}$$

where

$$\lambda(E) = \int_E f \, d\nu, \quad \mu(E) = \int_E g \, d\nu, \quad \forall E \in m. \tag{3.4}$$

Decomposition of measures

Proof of Lebesgue Decomposition Theorem

Define the following sets

$$A := \{x \in X \mid g(x) = 0\}, \quad B := \{x \in X \mid g(x) > 0\}. \quad (3.5)$$

As $g \in M^+$ we have that

$$A \cup B = X, \quad A \cap B = \emptyset. \quad (3.6)$$

Define now $\lambda_1, \lambda_2 : m \longrightarrow \overline{\mathbb{R}}$ by

$$\lambda_1(E) := \lambda(E \cap A), \quad (3.7)$$

$$\lambda_2(E) := \lambda(E \cap B). \quad (3.8)$$

Decomposition of measures

Proof of Lebesgue Decomposition Theorem

What we now want is to prove that

1. $\lambda_1 \perp \mu$,
2. $\lambda_2 \ll \mu$.

We start by proving $\lambda_1 \perp \mu$. From the definition of A we get

$$\left. \begin{array}{l} \mu(A) \underbrace{=}_{\text{def. of } \mu} \int_A g \, d\nu \underbrace{=}_{\text{def. of } A} 0 \\ \lambda_1(B) \underbrace{=}_{\text{def. of } \lambda_1} \lambda(B \cap A) \underbrace{=}_{\text{def. of } B} \lambda(\emptyset) = 0 \end{array} \right\} \lambda_1 \perp \mu. \quad (3.9)$$

This proves the first point.

Decomposition of measures

Proof of Lebesgue Decomposition Theorem

Now we want to show that $\lambda_2 \ll \mu$.

If $E \in \mathcal{m}$ such that $\mu(E) = 0$ then

$$\int_E g \, d\nu = 0. \quad (3.10)$$

By Corollary 4.10,

Corollary (4.10)

Suppose that f belongs to M^+ . Then $f(x) = 0$ μ -almost everywhere on X if and only if

$$\int f \, d\mu = 0. \quad (3.11)$$

this means that $g(x) = 0$ ν -almost everywhere on E .

Decomposition of measures

Proof of Lebesgue Decomposition Theorem

Recall that the terminology μ -almost everywhere means that there exists a subset $N \in \mathcal{m}$ with $\mu(N) = 0$ such that the statement holds on the complement of N . Thus we have that

$$\left. \begin{array}{l} B := \{x \in X \mid g(x) > 0\} \\ \exists N \text{ s.t. } \nu(N) = 0 \\ g(x) = 0 \text{ on } E \setminus N \end{array} \right\} \Rightarrow E \cap B \underbrace{\subseteq}_{\text{outside } N, g(x) = 0} N \quad (3.12)$$

$$\underbrace{\Rightarrow}_{\text{Lemma 3.3}} 0 \leq \nu(E \cap B) \leq \nu(N) = 0 \Rightarrow \nu(E \cap B) = 0. \quad (3.13)$$

Decomposition of measures

Proof of Lebesgue Decomposition Theorem

But remembering that $\lambda \ll \nu$ this implies that

$$\lambda(E \cap B) = 0 \implies \lambda_2(E) = 0. \quad (3.14)$$

Thus we have shown that $\lambda_2 \ll \mu$.

Decomposition of measures

Proof of Lebesgue Decomposition Theorem

To show that $\lambda = \lambda_1 + \lambda_2$ remember that they were constructed from the sets A and B . We have that

$$X = A \cup B \quad \text{and} \quad A \cap B = \emptyset. \quad (3.15)$$

For every measurable set E we have that

$$E = (E \cap A) \cup (E \cap B) \quad \text{and} \quad (E \cap A) \cap (E \cap B) = \emptyset. \quad (3.16)$$

Therefore we get that from the countable additivity of a measure that

$$\lambda(E) = \lambda(E \cap A) + \lambda(E \cap B) \quad (3.17)$$

$$= \lambda_1(E) + \lambda_2(E) \quad (3.18)$$

Decomposition of measures

Proof of Lebesgue Decomposition Theorem

Because this is true for every measurable set we can say that in general

$$\lambda = \lambda_1 + \lambda_2. \quad (3.19)$$

We have now shown that there existence part of the theorem. To finish the proof we would have to also verify that the λ_1 and λ_2 , derived here, are unique.

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Generation of measures and product measures

Tonelli's Theorem

Theorem (Tonelli's Theorem)

Let (X, m, μ) and (Y, n, ν) be σ -finite measure spaces and let F be a nonnegative measurable function on $Z = X \times Y$ to $\overline{\mathbb{R}}$. Then the functions defined on X and Y by

$$f(x) = \int_Y F_x d\nu, \quad g(y) = \int_X F^y d\mu, \quad (4.1)$$

are measurable and

$$\int_X f d\mu = \int_Z F d\pi = \int_Y g d\nu. \quad (4.2)$$

In other symbols,

$$\int_X \left(\int_Y F d\nu \right) d\mu = \int_Z F d\pi = \int_Y \left(\int_X F d\mu \right) d\nu. \quad (4.3)$$

Generation of measures and product measures

Proof of Tonelli's Theorem

Suppose first that $F = \chi_E$ where $E \in \mathcal{Z}$, the result is then a consequence of Lemma 10.8.

Lemma (10.8)

Let (X, m, μ) and (Y, n, ν) be σ -finite measure spaces. If $E \in \mathcal{Z} = X \times Y$, then the functions defined by

$$f(x) = \nu(E_x)$$

$$g(y) = \mu(E^y)$$

are measurable and

$$\int_X f \, d\mu = \pi(E) = \int_Y g \, d\nu.$$

Generation of measures and product measures

Proof of Tonelli's Theorem

Furthermore we have that by the linearity of the integral, that the theorem holds true for every φ positive simple function.

Now we would like to extend the result to any nonnegative measurable function.

Let $F \geq 0$, $F : Z \rightarrow \overline{\mathbb{R}}$ be measurable.

Generation of measures and product measures

Proof of Tonelli's Theorem

Lemma 2.11 thus gives that there exists (Φ_n) such that,

Lemma (2.11)

If f is a nonnegative function in $M(X, m)$, then there exists a sequence (φ_n) in $M(X, m)$ such that

1. $0 \leq \varphi_n(x) \leq \varphi_{n+1}$ for $x \in m$, $n \in \mathbb{N}$.
2. $f(x) = \lim \varphi_n(x)$ for each $x \in m$.
3. Each φ_n has only a finite number of real values.

Furthermore Lemma 10.6 gives that every section of Φ_n are measurable

Lemma (10.6)

1. *If E is a measurable subset of Z , then every section of E is measurable.*
2. *If f is a measurable function on Z to $\overline{\mathbb{R}}$, then every section of f is measurable.*

Generation of measures and product measures

Proof of Tonelli's Theorem

We can now define

$$\varphi_n : X \rightarrow \overline{\mathbb{R}}, \quad (4.4)$$

$$\varphi_n(x) := \int_Y (\Phi_n)_x d\nu, \quad (4.5)$$

As well as

$$\psi_n : Y \rightarrow \overline{\mathbb{R}}, \quad (4.6)$$

$$\psi_n(x) := \int_X (\Phi_n)^y d\mu. \quad (4.7)$$

As Φ_n is monotone it implies that both φ_n and ψ_n are monotone.

Generation of measures and product measures

Proof of Tonelli's Theorem

Taking the limit of φ_n gives

$$\lim_n \varphi_n = \lim_n \int_Y (\Phi_n)_x d\nu \quad (4.8)$$

$$\text{(MCT)} \quad = \int_Y \lim_n (\Phi_n)_x d\nu \quad (4.9)$$

$$\text{(Lemma 2.11)} \quad = \int_Y F_x d\nu \quad (4.10)$$

$$\text{(def)} \quad = f, \quad (4.11)$$

the same strategy can be used for ψ_n to obtain

$$\lim_n \psi_n = g. \quad (4.12)$$

Proof of Tonelli's Theorem

Another application of the Monotone Convergence Theorem gives that

$$\lim_n \int_X \varphi_n d\mu = \int_X f d\mu, \quad (4.13)$$

and that

$$\lim_n \int_Y \psi_n d\mu = \int_Y g d\nu. \quad (4.14)$$

The thing left now is to show that these two expressions are in fact equal.

Generation of measures and product measures

Proof of Tonelli's Theorem

$$\lim_n \int_X \varphi_n d\mu = \lim_n \int_X \left(\int_Y (\Phi_n)_x d\nu \right) d\mu \quad (4.15)$$

$$= \lim_n \int_X \int_Y \Phi_n d\nu d\mu \quad (4.16)$$

$$\text{(Result for simple functions)} \quad = \lim_n \int_Z \Phi_n d\pi \quad (4.17)$$

$$= \lim_n \int_Y \left(\int_X \Phi_n d\mu \right) d\nu \quad (4.18)$$

$$= \lim_n \int_Y \psi_n d\nu \quad (4.19)$$

The Monotone Convergence Theorem gives that

$$\lim_n \int_Z \Phi_n d\pi = \int_Z F d\pi. \quad (4.20)$$

Altogether we can thus conclude that

$$\int_X f d\mu = \int_Z F d\pi = \int_Y g d\nu,$$

which concludes the proof. □

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Approximation by nice functions

Theorem 3.16

Theorem (3.16)

Let a measure space be given by $(\mathbb{R}, \mathcal{B}, \lambda)$. The set $\mathcal{C}_c(\mathbb{R})$ of continuous functions with compact support is dense in $L^p(\mathbb{R}, \mathcal{B}, \lambda)$, for $1 \leq p < +\infty$.

For a function f to have compact support, or $f \in \mathcal{C}_c(\mathbb{R})$, it means that f is continuous and that $\text{supp}(f) \subseteq K$ where K is compact, where supp is defined as

$$\text{supp}(f) = \{x \in X \mid f(x) \neq 0\}. \quad (5.1)$$

Approximation by nice functions

Proof of Theorem 3.16

Using the fact that simple function on compact sets are dense in L^p it is enough to show that we can “approximate” every simple function on compact sets, $\tilde{\varphi}$, with a continuous function.

First we fix $\tilde{\varphi}$ and $\varepsilon > 0$, where $\tilde{\varphi} = \sum a_i \chi_{K_i}$ with K_i being a compact set.

Using the fact that we can “approximate” any Lebesgue measurable set by an open set, that is $\forall A \in \mathcal{L} \supseteq \mathcal{B}$

$$I^*(A) = \inf \{I^*(\mathcal{O}) \mid A \subseteq \mathcal{O}, \mathcal{O} \text{ open}\}, \quad (5.2)$$

we obtain that $\forall K$ compact, $\forall \varepsilon' > 0$ there exists an open set \mathcal{O} such that $K \subset \mathcal{O}$ and

$$I^*(\mathcal{O} \setminus K) \leq \varepsilon'. \quad (5.3)$$

Approximation by nice functions

Proof of Theorem 3.16

No matter what ε' we choose we are able to find an open set so that the previous equation is satisfied. Lets defines ε' as

$$\varepsilon' = \left(\frac{\varepsilon}{\sum |a_i|} \right)^p. \quad (5.4)$$

Now we need to find a continuous function $f : \mathbb{R} \rightarrow [0, 1]$ such that

1. $\text{supp}(f)$ is compact.
2. $f(x) = 1$ for $x \in K$.
3. $f(x) = 0$ for $x \in \mathbb{R} \setminus \mathcal{O}$.

As we are in \mathbb{R} the function can be written explicitly as;

Approximation by nice functions

Proof of Theorem 3.16

$$f_i(x) := \frac{\text{dist}(x, C(\mathcal{O}_i))}{\text{dist}(x, C(\mathcal{O}_i)) + \text{dist}(x, K_i)}. \quad (5.5)$$

where $\text{dist}(x, E) = \inf \{|x - z| : z \in E\}$ for $E \in \mathcal{B}$.

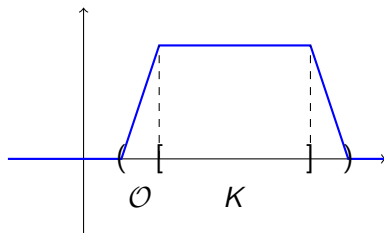


Figure: Function given in (5.5).

Approximation by nice functions

Proof of Theorem 3.16

The first two properties follow immediately, so we are left to show continuity. We have that

$$\forall z \in E : \text{dist}(x, E) \leq |x - z| \leq |x - y| + |y - z|. \quad (5.6)$$

Taking the inf over $E \in Z$ yields

$$\text{dist}(x, E) \leq |x - y| + \text{dist}(y, E), \quad (5.7)$$

the reverse inequality is obtained in the same way

$$\text{dist}(y, E) \leq |x - y| + \text{dist}(x, E). \quad (5.8)$$

Approximation by nice functions

Proof of Theorem 3.16

Thus we see that

$$|\text{dist}(x, E) - \text{dist}(y, E)| \leq |x - y|,$$

which gives the continuity of the dist function, which makes f_i a composition of continuous function and thus it is continuous.

Furthermore the denominator cannot be zero.

Approximation by nice functions

Proof of Theorem 3.16

We can now $\forall K_i$ construct f_i . Lets now define the function

$$F = \sum a_i f_i. \quad (5.9)$$

Using the triangular inequality on the p -norm of the difference between $\tilde{\varphi}$ and F gives

$$\|\tilde{\varphi} - F\|_p \leq \sum |a_i| \|\chi_{K_i} - f_i\|_p. \quad (5.10)$$

Approximation by nice functions

Proof of Theorem 3.16

The left part of the inequality (raised to the p -th power) satisfies

$$\|\chi_{K_i} - f_i\|_p^p = \int_{\mathbb{R}} |\chi_{K_i}(x) - f_i(x)|^p dx \quad (5.11)$$

$$= \int_{\mathcal{O}_i \setminus K_i} |f_i(x)|^p dx \quad (5.12)$$

$$\leq I^*(\mathcal{O}_i \setminus K_i) \quad (5.13)$$

$$\leq \left(\frac{\varepsilon}{\sum |a_i|} \right)^p, \quad (5.14)$$

this implies that

$$\|\tilde{\varphi} - F\|_p \leq \varepsilon. \quad (5.15)$$

Approximation by nice functions

Proof of Theorem 3.16

We have now shown that we can “approximate” every simple function with compact support by a continuous function, which concludes the proof. □

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Fourier transform

Theorem 7.7

Theorem (7.7)

The Fourier transform is a linear bounded injective map from L^1 into C_0 . Its inverse is given by

$$f(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ipx - \varepsilon|p|^2/2} \hat{f}(p) dp, \quad (6.1)$$

where the limit has to be understood in L^1 . Moreover, this holds at every point of continuity.

Fourier transform

Proof of Theorem 7.7

We know that the Fourier Transform is a function satisfying $F : L^1(\mathbb{R}) \rightarrow C_0(\mathbb{R}) \subset C_b(\mathbb{R})$. F is bounded since $\forall f \in L^1$

$$\|F(f)\|_\infty \leq \frac{1}{2\pi} \|f\|_1. \quad (6.2)$$

The first thing we will prove is that the Fourier Transform is injective, suppose therefore that we have already proved (6.1).

Fourier transform

Proof of Theorem 7.7

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