

Financial Engineering

Exam

Kasper Rosenkrands

Aalborg University

S20

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2. L^p spaces
3. Decomposition of measures
4. Generation of measures and product measures
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Theorem (Monotone Convergence Theorem)

If (f_n) is a monotone increasing sequence of functions in $M^+(X, m)$ which converges to f , then

$$\int f \, d\mu = \lim \int f_n \, d\mu. \quad (1.1)$$

Proof of Monotone Convergence Theorem

The strategy of the proof is to first show that

$$\lim \int f_n d\mu \leq \int f d\mu, \quad (1.2)$$

then afterwards to show that also

$$\lim \int f_n d\mu \geq \int f d\mu, \quad (1.3)$$

in order to conclude that

$$\lim \int f_n d\mu = \int f d\mu \quad (1.4)$$

According to Corollary 2.10

Corollary (2.10)

If (f_n) is a sequence in $M(X, m)$ which converges to f on X , the f is in $M(X, m)$.

the function f is measurable.

Lebesgue integration theory

Proof of Monotone Convergence Theorem

From Lemma 4.5(1.)

Lemma

1. If f and g belong to $M^+(X, m)$ and $f \leq g$, then

$$\int f \, d\mu \leq \int g \, d\mu. \quad (1.5)$$

2. If f belongs to $M^+(X, m)$, if E, F belong to m , and if $E \subseteq F$, then

$$\int_E f \, d\mu \leq \int_F f \, d\mu. \quad (1.6)$$

we have that

$$\int f_n \, d\mu \leq \int f_{n+1} \, d\mu \leq \int f \, d\mu, \quad \forall n \in \mathbb{N}. \quad (1.7)$$

Therefore we must also have that

$$\lim \int f_n d\mu \leq \int f d\mu. \quad (1.8)$$

So this was the first step of our strategy, now we proceed to the second step.

Proof of Monotone Convergence Theorem

Let $\alpha \in \mathbb{R}$ be such that $0 < \alpha < 1$ and let φ be a simple measurable function such that $0 \leq \varphi \leq f$.

Let

$$A_n = \{x \in X : f_n(x) \geq \alpha \varphi(x)\}, \quad (1.9)$$

such that

1. $A_n \in \mathcal{m}$
2. $A_n \subseteq A_{n+1}$
3. $X = \bigcup A_n$

Lebesgue integration theory

Proof of Monotone Convergence Theorem

According to Lemma 4.5

Lemma

1. If f and g belong to $M^+(X, m)$ and $f \leq g$, then

$$\int f \, d\mu \leq \int g \, d\mu. \quad (1.10)$$

2. If f belongs to $M^+(X, m)$, if E, F belong to m , and if $E \subseteq F$, then

$$\int_E f \, d\mu \leq \int_F f \, d\mu. \quad (1.11)$$

it must be that

$$\int_{A_n} \alpha \varphi \, d\mu \leq \int_{A_n} f_n \, d\mu \leq \int f_n \, d\mu. \quad (1.12)$$

Lebesgue integration theory

Proof of Monotone Convergence Theorem

Since the sequence A is monotone increasing and has union X , it follows from Lemma 4.3(2.) and Lemma 3.4(1.),

Lemma (4.3)

1. If φ and ψ are simple functions in $M^+(X, m)$ and $c \geq 0$, then

$$\int c\varphi d\mu = c \int \varphi d\mu, \quad (1.13)$$

$$\int (\varphi + \psi) d\mu = \int \varphi d\mu + \int \psi d\mu. \quad (1.14)$$

2. If λ is defined for E in m by

$$\lambda(E) = \int \varphi \chi_E d\mu, \quad (1.15)$$

then λ is a measure on m .

Lemma (3.4)

Let μ be a measure defined on a σ -algebra m .

1. If (E_n) is an increasing sequence in m , then

$$\mu \left(\bigcup_{n=1}^{\infty} E_n \right) = \lim \mu(E_n). \quad (1.16)$$

2. If (F_n) is a decreasing sequence in m and if $\mu(F_1) < +\infty$, then

$$\mu \left(\bigcap_{n=1}^{\infty} F_n \right) = \lim \mu(F_n). \quad (1.17)$$

Proof of Monotone Convergence Theorem

that

$$\int \varphi d\mu = \lim \int_{A_n} \varphi d\mu. \quad (1.18)$$

Taking the limit as n tends to infinity in (1.12) therefore gives that

$$\alpha \int \varphi d\mu \leq \lim \int f_n d\mu. \quad (1.19)$$

Since this holds for all $0 < \alpha < 1$, by taking the limit as α tends to 1 we obtain

$$\int \varphi d\mu \leq \lim \int f_n d\mu. \quad (1.20)$$

As φ is any simple function in M^+ such that $0 \leq \varphi \leq f$, we can conclude that

$$\int f \, d\mu = \sup_{\varphi} \int \varphi \, d\mu \leq \lim \int f_n \, d\mu, \quad (1.21)$$

which concludes the proof. □

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Minkowski's Inequality

Theorem (Minkowski's Inequality)

If f and h belong to L_p , $p \geq 1$, then $f + h$ belongs to L_p and

$$\|f + h\|_p \leq \|f\|_p + \|h\|_p. \quad (2.1)$$

Proof of Minkowski's Inequality

- ▶ The case of $p = 1$ is equivalent to the triangular inequality and will not be considered here.
- ▶ We suppose $p > 1$.

Proof of Minkowski's Inequality

As both f and h are measurable functions, so is the sum

$$f + h, \tag{2.2}$$

as the sum of measurable functions are also measurable.

L^p spaces

Proof of Minkowski's Inequality

From Corollary 5.4, Theorem 5.5 and the fact that

$$|f + h|^p \leq [2 \sup \{|f|, |h|\}]^p \leq 2^p \{|f|^p + |h|^p\}, \quad (2.3)$$

it follows that $f + h \in L^p$.

Theorem (5.5)

A constant multiple αf and a sum $f + g$ of functions in L belongs to L and

$$\int \alpha f \, d\mu, \quad (2.5)$$

Corollary (5.4)

If f is measurable, g is integrable, and $|f| \leq |g|$, then f is integrable, and

$$\int |f| \, d\mu \leq \int |g| \, d\mu. \quad (2.4)$$

$$\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu. \quad (2.6)$$

Proof of Minkowski's Inequality

Using the triangular inequality we can see that

$$|f + h|^p = |f + h||f + h|^{p-1} \leq |f||f + h|^{p-1} + |h||f + h|^{p-1}. \quad (2.7)$$

Since $f + h \in L^p$ it implies that $|f + h|^p \in L^1$.

Assuming that $\frac{1}{p} + \frac{1}{q} = 1$, it implies that

$$\frac{1}{p} + \frac{1}{q} = 1 \Leftrightarrow 1 + \frac{p}{q} = p \Leftrightarrow \frac{p}{q} = p - 1 \Leftrightarrow p = (p - 1)q, \quad (2.8)$$

and thereby that $|f + h|^{p-1} \in L^q$.

L^p spaces

Proof of Minkowski's Inequality

We can now apply Hölder's Inequality,

Theorem (Hölder's Inequality)

Let $f \in L_p$ and $g \in L_q$ where $p > 1$ and $(1/p) + (1/q) = 1$. Then $fg \in L_1$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$.

to infer that

$$\int |f| |f + h|^{p-1} d\mu \leq \|f\|_p \left\{ \int |f + h|^{(p-1)q} d\mu \right\}^{1/q} \quad (2.9)$$

$$= \|f\|_p \|f + h\|_p^{p/q}. \quad (2.10)$$

Note that exactly the same can be said for the other term

$$\int |h| |f + h|^{p-1} d\mu \leq \|h\|_p \left\{ \int |f + h|^{(p-1)q} d\mu \right\}^{1/q} \quad (2.11)$$

$$= \|h\|_p \|f + h\|_p^{p/q}. \quad (2.12)$$

Proof of Minkowski's Inequality

This tells us that

$$\|f + h\|_p^p \leq \|f\|_p \|f + h\|_p^{p/q} + \|h\|_p \|f + h\|_p^{p/q} \quad (2.13)$$

$$= \{\|f\|_p + \|h\|_p\} \|f + h\|_p^{p/q}. \quad (2.14)$$

If we let $A = \|f + h\|_p = 0$, then the result becomes trivial as a norm by definition is greater than or equal to zero.

Proof of Minkowski's Inequality

Suppose now that $A \neq 0$ then we can divide (2.14) by $A^{p/q}$

$$\frac{A^p}{A^{p/q}} \leq \{\|f\|_p + \|h\|_p\} \frac{A^{p/q}}{A^{p/q}} \quad (2.15)$$

$$A^{p-p/q} \leq \|f\|_p + \|h\|_p, \quad (2.16)$$

by noting that $p - p/q = 1$, we obtain

$$\|f + h\|_p \leq \|f\|_p + \|h\|_p, \quad (2.17)$$

which concludes the proof. □

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Decomposition of measures

Definition of Absolute Continuity

Definition (Absolutely Continuous)

A measure λ on m is said to be absolutely continuous with respect to a measure μ on m if $E \in m$ and $\mu(E) = 0$ imply that $\lambda(E) = 0$. In this case we write $\lambda \ll \mu$.

Note that μ can send more sets to 0 than λ , but not the other way around.

Theorem (Radón-Nikodym Theorem)

Let λ and μ be σ -finite measures defined on m and suppose that λ is absolutely continuous with respects to μ . Then there exists a function f in $M^+(X, m)$, such that,

$$\lambda(E) = \int_E f d\mu, \quad E \in m. \quad (3.1)$$

Moreover, the function f is uniquely determined μ -almost everywhere.

Decomposition of measures

Definition of Singular Measure

Definition (A Singular Measure)

Two measures λ, μ on m are said to be mutually singular if there are disjoint sets A, B in m , such that, $X = A \cup B$ and $\lambda(A) = \mu(B) = 0$. In this case we write $\lambda \perp \mu$.

Decomposition of measures

Lebesgue Decomposition Theorem

Theorem (Lebesgue Decomposition Theorem)

Let λ and μ be sigma-finite measures defined on a sigma-algebra m . Then there exists a measure λ_1 which is singular with respect to μ and a measure λ_2 which is absolutely continuous with respect to μ such that $\lambda = \lambda_1 + \lambda_2$. Moreover, the measures λ_1 and λ_2 are unique.

The Theorem is a consequence of the Radon-Nikodým Theorem.

Decomposition of measures

Proof of Lebesgue Decomposition Theorem

It can be shown that a measure ν can be decomposed

$$\nu = \lambda + \mu \tag{3.2}$$

such that ν is σ -finite, $\lambda \ll \nu$ and $\mu \ll \nu$.

A consequence of the Radon-Nikodým Theorem is that

$$\exists f, g \in M^+(X, m), \tag{3.3}$$

where

$$\lambda(E) = \int_E f \, d\nu, \quad \mu(E) = \int_E g \, d\nu, \quad \forall E \in m. \tag{3.4}$$

Decomposition of measures

Proof of Lebesgue Decomposition Theorem

Define the following sets

$$A := \{x \in X \mid g(x) = 0\}, \quad B := \{x \in X \mid g(x) > 0\}. \quad (3.5)$$

As $g \in M^+$ we have that

$$A \cup B = X, \quad A \cap B = \emptyset. \quad (3.6)$$

Define now $\lambda_1, \lambda_2 : m \longrightarrow \overline{\mathbb{R}}$ by

$$\lambda_1(E) := \lambda(E \cap A), \quad (3.7)$$

$$\lambda_2(E) := \lambda(E \cap B). \quad (3.8)$$

Decomposition of measures

Proof of Lebesgue Decomposition Theorem

What we now want is to prove that

1. $\lambda_1 \perp \mu$,
2. $\lambda_2 \ll \mu$.

We start by proving $\lambda_1 \perp \mu$. From the definition of A we get

$$\left. \begin{array}{l} \mu(A) \underbrace{=}_{\text{def. of } \mu} \int_A g \, d\nu \underbrace{=}_{\text{def. of } A} 0 \\ \lambda_1(B) \underbrace{=}_{\text{def. of } \lambda_1} \lambda(B \cap A) \underbrace{=}_{\text{def. of } B} \lambda(\emptyset) = 0 \end{array} \right\} \lambda_1 \perp \mu. \quad (3.9)$$

This proves the first point.

Decomposition of measures

Proof of Lebesgue Decomposition Theorem

Now we want to show that $\lambda_2 \ll \mu$.

If $E \in \mathcal{m}$ such that $\mu(E) = 0$ then

$$\int_E g \, d\nu = 0. \quad (3.10)$$

By Corollary 4.10,

Corollary (4.10)

Suppose that f belongs to M^+ . Then $f(x) = 0$ μ -almost everywhere on X if and only if

$$\int f \, d\mu = 0. \quad (3.11)$$

this means that $g(x) = 0$ ν -almost everywhere on E .

Decomposition of measures

Proof of Lebesgue Decomposition Theorem

Recall that the terminology μ -almost everywhere means that there exists a subset $N \in \mathcal{m}$ with $\mu(N) = 0$ such that the statement holds on the complement of N . Thus we have that

$$\left. \begin{array}{l} B := \{x \in X \mid g(x) > 0\} \\ \exists N \text{ s.t. } \nu(N) = 0 \\ g(x) = 0 \text{ on } E \setminus N \end{array} \right\} \Rightarrow E \cap B \underbrace{\subseteq}_{\text{outside } N, g(x) = 0} N \quad (3.12)$$

$$\underbrace{\Rightarrow}_{\text{Lemma 3.3}} 0 \leq \nu(E \cap B) \leq \nu(N) = 0 \Rightarrow \nu(E \cap B) = 0. \quad (3.13)$$

Decomposition of measures

Proof of Lebesgue Decomposition Theorem

But remembering that $\lambda \ll \nu$ this implies that

$$\lambda(E \cap B) = 0 \implies \lambda_2(E) = 0. \quad (3.14)$$

Thus we have shown that $\lambda_2 \ll \mu$.

Decomposition of measures

Proof of Lebesgue Decomposition Theorem

To show that $\lambda = \lambda_1 + \lambda_2$ remember that they were constructed from the sets A and B . We have that

$$X = A \cup B \quad \text{and} \quad A \cap B = \emptyset. \quad (3.15)$$

For every measurable set E we have that

$$E = (E \cap A) \cup (E \cap B) \quad \text{and} \quad (E \cap A) \cap (E \cap B) = \emptyset. \quad (3.16)$$

Therefore we get that from the countable additivity of a measure that

$$\lambda(E) = \lambda(E \cap A) + \lambda(E \cap B) \quad (3.17)$$

$$= \lambda_1(E) + \lambda_2(E) \quad (3.18)$$

Decomposition of measures

Proof of Lebesgue Decomposition Theorem

Because this is true for every measurable set we can say that in general

$$\lambda = \lambda_1 + \lambda_2. \quad (3.19)$$

We have now shown that there existence part of the theorem. To finish the proof we would have to also verify that the λ_1 and λ_2 , derived here, are unique.

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Generation of measures and product measures

Tonelli's Theorem

Theorem (Tonelli's Theorem)

Let (X, m, μ) and (Y, n, ν) be a σ -finite measure space and let F be a nonnegative measurable function on $Z = X \times Y$ to $\overline{\mathbb{R}}$. Then the functions defined on X and Y by

$$f(x) = \int_Y F_x d\nu, \quad g(y) = \int_X F^y d\mu, \quad (4.1)$$

are measurable and

$$\int_X f d\mu = \int_Z F d\pi = \int_Y g d\nu. \quad (4.2)$$

In other symbols,

$$\int_X \left(\int_Y F d\nu \right) d\mu = \int_Z F d\pi = \int_Y \left(\int_X F d\mu \right) d\nu. \quad (4.3)$$

Generation of measures and product measures

Proof of Tonelli's Theorem

Suppose first that $F = \chi_E$ where $E \in \mathcal{Z}$, the result is then a consequence of Lemma 10.8.

Lemma (10.8)

Let (X, m, μ) and (Y, n, ν) be σ -finite measure spaces. If $E \in \mathcal{Z} = X \times Y$, then the functions defined by

$$f(x) = \nu(E_x)$$

$$g(y) = \mu(E^y)$$

are measurable and

$$\int_X f \, d\mu = \pi(E) = \int_Y g \, d\nu.$$

Generation of measures and product measures

Proof of Tonelli's Theorem

Furthermore we have that by the linearity of the integral, that the theorem holds true for every φ positive simple function.

Now we would like to extend the result to any nonnegative measurable function.

Let $F \geq 0$, $F : Z \rightarrow \overline{\mathbb{R}}$ be measurable.

Generation of measures and product measures

Proof of Tonelli's Theorem

Lemma 2.11 thus gives that there exists (Φ_n) such that,

Lemma (2.11)

If f is a nonnegative function in $M(X, m)$, then there exists a sequence (φ_n) in $M(X, m)$ such that

1. $0 \leq \varphi_n(x) \leq \varphi_{n+1}$ for $x \in m$, $n \in \mathbb{N}$.
2. $f(x) = \lim \varphi_n(x)$ for each $x \in m$.
3. Each φ_n has only a finite number of real values.

Furthermore Lemma 10.6 gives that every section of Φ_n are measurable

Lemma (10.6)

1. *If E is a measurable subset of Z , then every section of E is measurable.*
2. *If f is a measurable function on Z to $\overline{\mathbb{R}}$, then every section of f is measurable.*

Generation of measures and product measures

Proof of Tonelli's Theorem

We can now define

$$\varphi_n : X \rightarrow \overline{\mathbb{R}}, \quad (4.4)$$

$$\varphi_n(x) := \int_Y (\Phi_n)_x d\nu, \quad (4.5)$$

As well as

$$\psi_n : Y \rightarrow \overline{\mathbb{R}}, \quad (4.6)$$

$$\psi_n(x) := \int_X (\Phi_n)^y d\mu. \quad (4.7)$$

As Φ_n is monotone it implies that both φ_n and ψ_n are monotone.

Generation of measures and product measures

Proof of Tonelli's Theorem

Taking the limit of φ_n gives

$$\lim_n \varphi_n = \lim_n \int_Y (\Phi_n)_x d\nu \quad (4.8)$$

$$\text{(MCT)} \quad = \int_Y \lim_n (\Phi_n)_x d\nu \quad (4.9)$$

$$\text{(Lemma 2.11)} \quad = \int_Y F_x d\nu \quad (4.10)$$

$$\text{(def)} \quad = f, \quad (4.11)$$

the same strategy can be used for ψ_n to obtain

$$\lim_n \psi_n = g. \quad (4.12)$$

Generation of measures and product measures

Proof of Tonelli's Theorem

Another application of the Monotone Convergence Theorem gives that

$$\lim_n \int_X \varphi_n d\mu = \int_X f d\mu, \quad (4.13)$$

and that

$$\lim_n \int_Y \psi_n d\mu = \int_Y g d\nu. \quad (4.14)$$

The thing left now is to show that these two expressions are in fact equal.

Generation of measures and product measures

Proof of Tonelli's Theorem

$$\lim_n \int_X \varphi_n d\mu = \lim_n \int_X \left(\int_Y (\Phi_n)_x d\nu \right) d\mu \quad (4.15)$$

$$= \lim_n \int_X \int_Y \Phi_n d\nu d\mu \quad (4.16)$$

$$\text{(Result for simple functions)} \quad = \lim_n \int_Z \Phi_n d\pi \quad (4.17)$$

$$= \lim_n \int_Y \left(\int_X \Phi_n d\mu \right) d\nu \quad (4.18)$$

$$= \lim_n \int_Y \psi_n d\nu \quad (4.19)$$

The Monotone Convergence Theorem gives that

$$\lim_n \int_Z \Phi_n d\pi = \int_Z F d\pi. \quad (4.20)$$

Altogether we can thus conclude that

$$\int_X f d\mu = \int_Z F d\pi = \int_Y g d\nu,$$

which concludes the proof. □

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Approximation by nice functions

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Fourier transform

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