Financial Engineering

Exam

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S20

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Monotone Convergence Theorem

Theorem (Monotone Convergence Thereom)

If (f_n) is a monotone increasing sequence of functions in $M^+(X, m)$ which converges to f, then

$$\int f \, d\mu = \lim \int f_n \, d\mu. \tag{1.1}$$

Proof of Monotone Convergence Theorem

The strategy of the proof is to first show that

$$\lim \int f_n \, d\mu \leq \int f \, d\mu, \tag{1.2}$$

then afterwards to show that also

$$\lim \int f_n \, d\mu \geq \int f \, d\mu, \tag{1.3}$$

in order to conclude that

$$\lim \int f_n \, d\mu = \int f \, d\mu \tag{1.4}$$

Proof of Monotone Convergence Theorem

According to Corollary 2.10

Corollary (2.10)

If (f_n) *is a sequence in* $M(X, m)$ *which converges to* f *on* X*, the* f *is in* $M(X, m)$*.*

the function f is measurable.

Proof of Monotone Convergence Theorem

From Lemma 4.5(1.)

Lemma

1. *If f and g belong to $M^+(X, m)$ and $f \leq g$, then*

$$\int f \, d\mu \leq \int g \, d\mu. \tag{1.5}$$

2. *If f belongs to $M^+(X, m)$, if E, F belong to m, and if $E \subseteq F$, then*

$$\int_E f \, d\mu \leq \int_F f \, d\mu. \tag{1.6}$$

we have that

$$\int f_n \, d\mu \leq \int f_{n+1} \, d\mu \leq \int f \, d\mu, \quad \forall\, n \in \mathbb{N}. \tag{1.7}$$

Proof of Monotone Convergence Theorem

Therefore we must also have that

$$\lim \int f_n \, d\mu \leq \int f \, d\mu. \tag{1.8}$$

So this was the first step of our strategy, now we proceed to the second step.

Proof of Monotone Convergence Theorem

Let $\alpha \in \mathbb{R}$ be such that $0 < \alpha < 1$ and let φ be a simple measurable function such that $0 \leq \varphi \leq f$.
Let

$$A_n = \{x \in X \,:\, f_n(x) \geq \alpha\varphi(x)\}\,, \tag{1.9}$$

such that

1. $A_n \in m$
2. $A_n \subseteq A_{n+1}$
3. $X = \bigcup A_n$

Proof of Monotone Convergence Theorem

According to Lemma 4.5

Lemma

1. *If f and g belong to $M^{+}(X, m)$ and $f \leq g$, then*

$$\int f \, d\mu \leq \int g \, d\mu. \tag{1.10}$$

2. *If f belongs to $M^{+}(X, m)$, if E, F belong to m, and if $E \subseteq F$, then*

$$\int_E f \, d\mu \leq \int_F f \, d\mu. \tag{1.11}$$

it must be that

$$\int_{A_n} \alpha\varphi \, d\mu \leq \int_{A_n} f_n \, d\mu \leq \int f_n \, d\mu. \tag{1.12}$$

Proof of Monotone Convergence Theorem

Since the sequence A is monotone increasing and has union X, it follows from Lemma 4.3(2.) and Lemma 3.4(1.),

Lemma (4.3)

1. *If φ and ψ are simple functions in $M^+(X, m)$ and $c \geq 0$, then*
$$\int c\varphi \, d\mu = c \int \varphi \, d\mu, \tag{1.13}$$
$$\int (\varphi + \psi) \, d\mu = \int \varphi \, d\mu + \int \psi \, d\mu. \tag{1.14}$$
2. *If λ is defined for E in m by*
$$\lambda(E) = \int \varphi \chi_E \, d\mu, \tag{1.15}$$
then λ is a measure on m.

Lemma (3.4)

Let μ be a measure defined on a σ-algebra m.

1. *If (E_n) is an increasing sequence in m, then*
$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu\left(E_n\right). \tag{1.16}$$
2. *If (F_n) is a decreasing sequence in m and if $\mu\left(F_1\right) < +\infty$, then*
$$\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim \mu\left(F_n\right). \tag{1.17}$$

Proof of Monotone Convergence Theorem

that

$$\int \varphi \, d\mu = \lim \int_{A_n} \varphi \, d\mu. \tag{1.18}$$

Taking the limit as n tends to infinity in (1.12) therefore gives that

$$\alpha \int \varphi \, d\mu \leq \lim \int f_n \, d\mu. \tag{1.19}$$

Since this holds for all $0 < \alpha < 1$, by taking the limit as α tends to 1 we obtain

$$\int \varphi \, d\mu \leq \lim \int f_n \, d\mu. \tag{1.20}$$

Proof of Monotone Convergence Theorem

As φ is any simple function in M^+ such that $0 \leq \varphi \leq f$, we can conclude that

$$\int f \, d\mu = \sup_{\varphi} \int \varphi \, d\mu \leq \lim \int f_n \, d\mu, \tag{1.21}$$

which concludes the proof. $\square$

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Minkowski's Inequality

Theorem (Minkowski's Inequality)

If f and h belong to L_p, $p \geq 1$, then $f + h$ belongs to L_p and

$$\|f + h\|_p \leq \|f\|_p + \|h\|_p. \tag{2.1}$$

Proof of Minkowski's Inequality

- The case of $p = 1$ is equivalent to the triangular inequality and will not be considered here.
- We suppose $p > 1$.

Proof of Minkowski's Inequality

As both f and h are measurable functions, so is the sum

$$f + h, \tag{2.2}$$

as the sum of measurable functions are also measurable.

Proof of Minkowski's Inequality

From Corollary 5.4, Theorem 5.5 and the fact that

$$|f+h|^p \leq [2\sup\{|f|,|h|\}]^p \leq 2^p\{|f|^p+|h|^p\}, \qquad (2.3)$$

it follows that $f+h \in L^p$.

Corollary (5.4)

If f is measurable, g is integrable, and $|f| \leq |g|$, then f is integrable, and

$$\int |f|\,d\mu \leq \int |g|\,d\mu. \quad (2.4)$$

Theorem (5.5)

A constant multiple αf and a sum $f+g$ of functions in L belongs to L and

$$\int \alpha f\,d\mu, \qquad (2.5)$$

$$\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu. \quad (2.6)$$

Proof of Minkowski's Inequality

Using the triangular inequality we can see that

$$|f+h|^p = |f+h||f+h|^{p-1} \le |f||f+h|^{p-1} + |h||f+h|^{p-1}. \tag{2.7}$$

Since $f+h \in L^p$ it implies that $|f+h|^p \in L^1$.

Assuming that $\frac{1}{p} + \frac{1}{q} = 1$, it implies that

$$\frac{1}{p} + \frac{1}{q} = 1 \Leftrightarrow 1 + \frac{p}{q} = p \Leftrightarrow \frac{p}{q} = p - 1 \Leftrightarrow p = (p-1)q, \tag{2.8}$$

and thereby that $|f+h|^{p-1} \in L^q$.

Proof of Minkowski's Inequality

We can now apply Hölder's Inequality,

Theorem (Hölder's Inequality)

Let $f \in L_p$ and $g \in L_q$ where $p > 1$ and $(1/p) + (1/q) = 1$. Then $fg \in L_1$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$.

to infer that

$$\int |f||f+h|^{p-1}\, d\mu \leq \|f\|_p \left\{ \int |f+h|^{(p-1)q}\, d\mu \right\}^{1/q} \tag{2.9}$$

$$= \|f\|_p \|f+h\|_p^{p/q}. \tag{2.10}$$

Note that exactly the same can be said for the other term

$$\int |h||f+h|^{p-1}\, d\mu \leq \|h\|_p \left\{ \int |f+h|^{(p-1)q}\, d\mu \right\}^{1/q} \tag{2.11}$$

$$= \|h\|_p \|f+h\|_p^{p/q}. \tag{2.12}$$

Proof of Minkowski's Inequality

This tells us that

$$\|f+h\|_p^p \leq \|f\|_p \|f+h\|_p^{p/q} + \|h\|_p \|f+h\|_p^{p/q} \tag{2.13}$$

$$= \{\|f\|_p + \|h\|_p\} \|f+h\|_p^{p/q}. \tag{2.14}$$

If we let $A = \|f+h\|_p = 0$, then the result becomes trivial as a norm by definition is greater than or equal to zero.

Proof of Minkowski's Inequality

Suppose now that $A \neq 0$ then we can divide (2.14) by $A^{p/q}$

$$\frac{A^p}{A^{p/q}} \leq \{\|f\|_p + \|h\|_p\} \frac{A^{p/q}}{A^{p/q}} \tag{2.15}$$

$$A^{p-p/q} \leq \|f\|_p + \|h\|_p, \tag{2.16}$$

by noting that $p - p/q = 1$, we obtain

$$\|f + h\|_p \leq \|f\|_p + \|h\|_p, \tag{2.17}$$

which concludes the proof. $\square$

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Definition of Absolute Continuity

Definition (Absolutely Continuous)

A measure λ on m is said to be absolutely continuous with respect to a measure μ on m if $E \in m$ and $\mu(E) = 0$ imply that $\lambda(E) = 0$. In this case we write $\lambda << \mu$.

Note that μ can send more sets to 0 than λ, but not the other way around.

Theorem (Radón-Nikodym Theorem)

Let λ and μ be σ-finite measures defined on m and suppose that λ is absolutely continuous with respects to μ. Then there exists a function f in $M^+(X, m)$, such that,

$$\lambda(E) = \int_E f d\mu, \quad E \in m. \tag{3.1}$$

Moreover, the function f is uniquely determined μ-almost everywhere.

Definition of Singular Measure

Definition (A Singular Measure)

Two measures λ, μ on m are said to be mutually singular if there are disjoint sets A, B in m, such that, $X = A \cup B$ and $\lambda(A) = \mu(B) = 0$. In this case we write $\lambda \perp \mu$.

Lebesgue Decomposition Theorem

Theorem (Lebesgue Decomposition Theorem)

Let λ and μ be sigma-finite measures defined on a sigma-algebra m. Then there exists a measure λ_1 which is singular with respect to μ and a measure λ_2 which is absolutely continuous with respect to μ such that $\lambda = \lambda_1 + \lambda_2$. Moreover, the measures λ_1 and λ_2 are unique.

The Theorem is a consequence of the Radon-Nikodým Theorem.

Proof of Lebesgue Decomposition Theorem

It can be shown that a measure ν can be decomposed

$$\nu = \lambda + \mu \tag{3.2}$$

such that ν is σ-finite, $\lambda << \nu$ and $\mu << \nu$.

A consequence of the Radon-Nikodým Theorem is that

$$\exists\, f, g \in M^+(X, m), \tag{3.3}$$

where

$$\lambda(E) = \int_E f \, d\nu, \quad \mu(E) = \int_E g \, d\nu, \qquad \forall\, E \in m. \tag{3.4}$$

Proof of Lebesgue Decomposition Theorem

Define the following sets

$$A := \{x \in X \,|\, g(x) = 0\}, \quad B := \{x \in X \,|\, g(x) > 0\}. \tag{3.5}$$

As $g \in M^+$ we have that

$$A \cup B = X, \quad A \cap B = \emptyset. \tag{3.6}$$

Define now $\lambda_1, \lambda_2 : m \longrightarrow \overline{\mathbb{R}}$ by

$$\lambda_1(E) := \lambda(E \cap A), \tag{3.7}$$
$$\lambda_2(E) := \lambda(E \cap B). \tag{3.8}$$

Proof of Lebesgue Decomposition Theorem

What we now want is to prove that

1. $\lambda_1 \perp \mu$,
2. $\lambda_2 << \mu$.

We start by proving $\lambda_1 \perp \mu$. From the definition of A we get

$$\left.\begin{array}{r} \mu(A) \underbrace{=}_{\text{def. of } \mu} \int_A g \, d\nu \underbrace{=}_{\text{def. of } A} 0 \\ \lambda_1(B) \underbrace{=}_{\text{def. of } \lambda_1} \lambda(B \cap A) \underbrace{=}_{\text{def. of } B} \lambda(\emptyset) = 0 \end{array}\right\} \quad \lambda_1 \perp \mu. \tag{3.9}$$

This proves the first point.

Proof of Lebesgue Decomposition Theorem

Now we want to show that $\lambda_2 << \mu$.

If $E \in m$ such that $\mu(E) = 0$ then

$$\int_E g \, d\nu = 0. \tag{3.10}$$

By Corollary 4.10,

Corollary (4.10)

Suppose that f belongs to M^+. Then $f(x) = 0$ μ-almost everywhere on X if and only if

$$\int f \, d\mu. \tag{3.11}$$

this means that $g(x) = 0$ ν-almost everywhere on E.

Proof of Lebesgue Decomposition Theorem

Recall that the terminology μ-almost everywhere means that there exists a subset $N \in m$ with $\mu(N) = 0$ such that the statement holds on the complement of N. Thus we have that

$$\left.\begin{array}{l} B := \{x \in X \mid g(x) > 0\} \\ \exists\, N \text{ s.t. } \nu(N) = 0 \\ g(x) = 0 \text{ on } E \backslash N \end{array}\right\} \Rightarrow E \cap B \underbrace{\subseteq}_{\text{outside } N,\ g(x)=0} N \quad (3.12)$$

$$\underbrace{\Rightarrow}_{\text{Lemma 3.3}} 0 \leq \nu(E \cap B) \leq \nu(N) = 0 \Rightarrow \nu(E \cap B) = 0. \quad (3.13)$$

Proof of Lebesgue Decomposition Theorem

But remembering that $\lambda << \nu$ this implies that

$$\lambda(E \cap B) = 0 \implies \lambda_2(E) = 0. \tag{3.14}$$

Thus we have shown that $\lambda_2 << \mu$.

Proof of Lebesgue Decomposition Theorem

To show that $\lambda = \lambda_1 + \lambda_2$ remeber that they were constructed from the sets A and B. We have that

$$X = A \cup B \quad \text{and} \quad A \cap B = \emptyset. \tag{3.15}$$

For every measurable set E we have that

$$E = (E \cap A) \cup (E \cap B) \quad \text{and} \quad (E \cap A) \cap (E \cap B) = \emptyset. \tag{3.16}$$

Therefore we het that from the countable additivity of a measure that

$$\lambda(E) = \lambda(E \cap A) + \lambda(E \cap B) \tag{3.17}$$
$$= \lambda_1(E) + \lambda_2(E) \tag{3.18}$$

Proof of Lebesgue Decomposition Theorem

Because this is true for every measurable set we can say that in general

$$\lambda = \lambda_1 + \lambda_2. \tag{3.19}$$

We have now shown that there existence part of the theorem. To finish the proof we would have to also verify that the λ_1 and λ_2, derived here, are unique.

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