

Integration Theory

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MATØK6

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1 Lebesgue Integration Theory

(Chapter 2-5 of Bartle's book, the most important results are: monotone convergence theorem, Fatou's lemma and Lebesgue dominated convergence theorem)

4.6 - Monotone Convergence Theorem

Theorem 1.1 (Monotone Convergence Theorem). *If (f_n) is a monotone increasing sequence of functions in $M^+(X, m)$ which converges to f , then*

$$\int f \, d\mu = \lim \int f_n \, d\mu. \quad (1.1)$$

4.8 - Fatou's Lemma

Theorem 1.2 (Fatou's Lemma). *If (f_n) belongs to $M^+(X, m)$, then*

$$\int (\liminf f_n) \, d\mu \leq \liminf \int f_n \, d\mu. \quad (1.2)$$

5.6 - Lebesgue Dominated Convergence Theorem

Theorem 1.3 (Lebesgue Dominated Convergence Theorem). *Let (f_n) be a sequence of integrable function which converges almost everywhere to a real-valued measurable function f . If there exists an integrable function g such that $|f_n| \leq g$ for all n , then f is integrable and*

$$\int f \, d\mu = \lim \int f_n \, d\mu. \quad (1.3)$$

2 L^p Spaces

(Chapter 6 of Bartle's book, the most important results are: Hölder's inequality, Minkowski's inequality and Riesz-Fischer Theorem)

6.9 - Hölder's Inequality

Theorem 2.1 (Hölder's Inequality). *Let $f \in L_p$ and $g \in L_q$ where $p > 1$ and $(1/p) + (1/q) = 1$. Then $fg \in L_1$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$.*

6.11 - Minkowski's Inequality

Theorem 2.2 (Minkowski's Inequality). *If f and h belong to L_p , $p \geq 1$, then $f + h$ belongs to L_p and*

$$\|f + h\|_p \leq \|f\|_p + \|h\|_p. \quad (2.1)$$

6.14 - Completeness Theorem (Riesz-Fischer Theorem)

Theorem 2.3 (Completeness Theorem (Riesz-Fischer Theorem)). *If $1 \leq p < \infty$, then the space L_p is a complete normed linear space under the norm*

$$\|f\|_p = \left\{ \int |f|^p d\mu \right\}^{1/p}. \quad (2.2)$$

3 Decomposition of Measures

(Chapter 8 of Bartle's book, the most important results are: Radon-Nikodym theorem, Lebesgue decomposition theorem and Riesz representation theorem)

8.9 - Radon-Nikodým Theorem

Theorem 3.1 (Radon-Nikodým Theorem). *Let λ and μ be σ -finite measures defined on m and suppose that λ is absolutely continuous with respect to μ . Then there exists a function f in $M^+(X, m)$ such that*

$$\lambda(E) = \int_E f d\mu, \quad E \in m. \quad (3.1)$$

Moreover, the function f is uniquely determined μ -almost everywhere.

8.11 - Lebesgue Decomposition Theorem

Theorem 3.2 (Lebesgue Decomposition Theorem). *Let λ and μ be sigma-finite measures defined on a sigma-algebra m . Then there exists a measure λ_1 which is singular with respect to μ and a measure λ_2 which is absolutely continuous with respect to μ such that $\lambda = \lambda_1 + \lambda_2$. Moreover, the measures λ_1 and λ_2 are unique.*

8.14 - Riesz Representation Theorem

Theorem 3.3 (Riesz Representation Theorem). *If (X, m, μ) is a sigma-finite measure space and G is a bounded linear functional on $L_1(X, m, \mu)$, then there exists a g in $L_\infty(X, m, \mu)$ such that*

$$G(f) = \int f g d\mu \quad (3.2)$$

holds for all f in L_1 . Moreover, $\|G\| = \|g\|_\infty$ and $g \geq 0$ if G is a positive linear functional.

4 Generation of Measures and Product Measures

(Chapter 9 and 10 of Bartle's book, the most important results are: Carathéodory extension theorem, Hahn extension theorem, Product measure theorem, Fubini's theorem and Tonelli's theorem)

9.7 - Carathéodory Extension Theorem

9.8 - Hahn Extension Theorem

10.4 - Product Measure Theorem

10.9 - Tonelli's Theorem

10.10 - Fubini's Theorem

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