

Measure Theory

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MATØK7

Fall 2020

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Notation

Measure with densities

Theorem 0.1 (Measures with densities). *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $f : \Omega \rightarrow [0, \infty]$ be measurable. Then*

$$(f \odot \mu)(A) := \int_A f d\mu = \int_{\Omega} \chi_A f d\mu, \quad A \in \mathcal{A},$$

defines a new measure $f \odot \mu : (A) \rightarrow [0, \infty]$, called the measure with density f with respect to μ . For every $N \in \mathcal{A}$, the following implication holds,

$$\mu(N) = 0 \Rightarrow (f \odot \mu)(N) = 0.$$

Product σ -algebras

Definition 0.1. *Let $n \in \mathbb{N}$, $n \geq 2$, and suppose that, for every $i \in \{1, \dots, n\}$, we are given a measurable space $(\Omega_i, \mathcal{A}_i)$.*

- *The smallest σ -algebra on $\times_{i=1}^n \Omega_i$ containing*

$$\mathcal{A}_1 * \dots * \mathcal{A}_n := \{A_1 \times \dots \times A_n \mid A_1 \in \mathcal{A}_1, \dots, A_n \in \mathcal{A}_n\} \subset \mathcal{P}\left(\bigtimes_{i=1}^n \Omega_i\right)$$

is called the product σ -algebra defined by measns of $\mathcal{A}_1, \dots, \mathcal{A}_n$. It is denoted by

$$\bigotimes_{i=1}^n \mathcal{A}_i := \sigma(\mathcal{A}_1 * \dots * \mathcal{A}_n).$$

- *Let Γ be a set and let $f_i : \Gamma \rightarrow \Omega_i$ be an arbitrary map for every $i \in \{1, \dots, n\}$. Then the smallest σ -algebra on Γ turning all maps f_1, \dots, f_n into measurable maps, i.e.,*

$$\sigma(f_1, \dots, f_n) := \sigma(\{f_i^{-1}(A_i) \mid A_i \in \mathcal{A}_i, i \in 1, \dots, n\}),$$

is called the initial σ -algebra generated by f_1, \dots, f_n .

1 Betingede forventningsværdier

2 Processer med uafhængige og stationære tilvækst, specielt standard brownske bevægelser

3 Martingaler og kvadratisk variation

relevante dele til forelæsning 9

- afsnit 5.8.1
 - def 5.101
 - eks 5.103
 - def 5.104
 - eks 5.105
 - sæt 5.110
- afsnit 5.8.2
 - def 5.112
 - bem 5.114
 - bem 5.115
 - sæt 5.118
 - sæt 5.120
- afsnit 5.8.3
 - def 5.122
 - bem 5.123
 - sæt 5.125

4 Itô-formlen

5 Girsanov-transformationen