Measure Theory

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Notation

Measure with densities

Theorem 0.1 (Measures with densities). *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $f : \Omega \to [0, \infty]$ be measureable. Then*

$$(f \odot \mu)(A) := \int_A f \, d\mu = \int_\Omega \chi_A f \, d\mu, \quad A \in \mathcal{A},$$

defines a new measure $f \odot \mu : (A) \to [0, \infty]$, called the measure with density f with respect to μ. For every $N \in \mathcal{A}$, the following implication holds,

$$\mu(N) = 0 \;\Rightarrow\; (f \odot \mu)(N) = 0.$$

Product $\boldsymbol{\sigma}$-algrebras

Definition 0.1. *Let $n \in \mathbb{N}$, $n \geqslant 2$, and suppose that, for every $i \in \{1, \dots, n\}$, we are given a measurable space $(\Omega_i, \mathcal{A}_i)$.*

- *The smallest σ-algebra on $\times_{i=1}^n \Omega_i$ containing*

$$\mathcal{A}_1 * \cdots * \mathcal{A}_n := \{A_1 \times \cdots \times A_n \mid A_1 \in \mathcal{A}_1, \dots, A_n \in \mathcal{A}_n\} \subset \mathcal{P}\left(\bigtimes_{i=1}^n \Omega_i\right)$$

  is called the product σ-algebra defined by measns of $\mathcal{A}_1, \dots, \mathcal{A}_n$. It is denoted by

$$\bigotimes_{i=1}^n \mathcal{A}_i := \sigma\left(\mathcal{A}_1 * \cdots * \mathcal{A}_n\right).$$

- *Let Γ be a set and let $f_i : \Gamma \to \Omega_i$ be an arbitrary map for every $i \in \{1, \dots, n\}$. Then the smallest σ-algebra on Γ turning all maps $f_1, \dots, f_n$ into measurable maps, i.e.,*

$$\sigma(f_1, \dots, f_n) := \sigma\left(\left\{f_i^{-1}(A_i) \mid A_i \in \mathcal{A}_i, i \in 1, \dots, n\right\}\right),$$

  is called the initial σ-algebra generated by $f_1, \dots, f_n$.

1 Betingede forventningsværdier

2 Processer med uafhængige of stationære tilvækst, specielt standard brownske bevægelser

3 Martingaler og kvadratisk variation

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- afsnit 5.8.1
  - def 5.101
  - eks 5.103
  - def 5.104
  - eks 5.105
  - sæt 5.110
- afsnit 5.8.2
  - def 5.112
  - bem 5.114
  - bem 5.115
  - sæt 5.118
  - sæt 5.120
- afsnit 5.8.3
  - def 5.122
  - bem 5.123
  - sæt 5.125
  - sæt (med def) 5.126

3.1 Martingales

3.1.1 One-dimensional Brownian Motion

Let B be a one-dimensional $(\mathcal{F}_t)_{t\geqslant 0}$-Brownian motion. Then B is a $(\mathcal{F}_t)_{t\geqslant 0}$ martingale. In particular, every one-dimensional standard Brownian motion B is a martingale with respect to its natural filtration $(\mathcal{B}_t^B)_{t\geqslant 0}$. By virtue of Remark 6.7 we can further conclude that every one-dimensional standard Brownian motion B is a martingale on the standard filtered probability space $(\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}_t^B})_{t\geqslant 0}, \tilde{\mathbb{P}})$ obtained by completing $(\Omega, \mathcal{F}, (\mathcal{F}_t^B)_{t\geqslant 0}, \mathbb{P})$.

Let us first recall that

$$\int_\Omega |B_t| \, d\mathbb{P} = \frac{1}{(2\pi t)^{1/2}} \int_\mathbb{R} |x| e^{-x^2/2t} \, dx < \infty, \mathbb{E}[B_t] = \frac{1}{(2\pi t)^{1/2}} \int_\mathbb{R} x e^{-x^2/2t} \, dx, \tag{3.1}$$

for every $t > 0$. Since $B_0 = 0$, $\mathbb{P} - a.s.$, it follows in particular that B_t is $\mathbb{P}$-integrable with $\mathbb{E}[B_t] = 0$ for all $t \geqslant 0$. For all $0 \leqslant s \leqslant t < \infty$, we further observe that, $\mathbb{P} - a.s.$,

$$\mathbb{E}^{\mathcal{F}_s}[B_t] = \mathbb{E}^{\mathcal{F}_s}[B_t - B_s] + \mathbb{E}^{\mathcal{F}_s} = \mathbb{E}[B_t - B_s] + B_s = \mathbb{E}[B_t] - \mathbb{E}[B_s] + B_s, \tag{3.2}$$

since the increment $B_t - B_s$ is $\mathcal{F}_s$-independent and B_s is $\mathcal{F}_s$-measurable.

4 Itô-formlen

5 Girsanov-transformationen