Measure Theory

Exam

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Definition 4.42. Let $\mathcal{A} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$ and let $f : \Omega \to \mathbb{R}$ be $\mathbb{P}$-integrable. Then a function $g : \Omega \to \mathbb{R}$ is called a *representative of the conditional expectation of f under the hypothesis* $\mathcal{A}$, iff it is $\mathbb{P}$-integrable, $\mathcal{A}$-*measurable*, and satisfies

$$\int_A f \, d\mathbb{P} = \int_A g \, d\mathbb{P}, \quad A \in \mathcal{A}. \tag{1.1}$$

Remark

Remark 4.46. Substituting $A = \Omega$ in (1.1), we obtain

$$\int_\Omega f \, d\mathbb{P} = \int_\Omega g \, d\mathbb{P} \tag{1.2}$$

$$\mathbb{E}[f] = \mathbb{E}\left[g\right] \tag{1.3}$$

$$\mathbb{E}[f] = \mathbb{E}\left[\mathbb{E}^{\mathcal{A}}\left[f\right]\right], \tag{1.4}$$

for all sub-σ-algebras $\mathcal{A} \subset \mathcal{F}$ and all $\mathbb{P}$-integrable $f : \Omega \to \mathbb{C}$.

Exercise 4.47

Exercise. Recall that $\{\emptyset, \Omega\}$ is a sub-σ-algebra of $\mathcal{F}$. Convince yourself of the validity of the formula

$$\mathbb{E}^{\{\emptyset,\Omega\}}[f] = \mathbb{E}[f], \quad \mathbb{P} - \text{a.s.}, \tag{1.5}$$

for every $\mathbb{P}$-integrable $f : \Omega \to \mathbb{R}$.

Solution. We start by defining $g := \mathbb{E}[f]$, then we have by the definition of the conditional expectation that g is a representative iff it is

- $\mathbb{P}$-integrable.
- $\mathcal{A}$-measurable.
- It satifies that

$$\int_A f \, d\mathbb{P} = \int_A g \, d\mathbb{P}, \quad A \in \{\emptyset, \Omega\}. \tag{1.6}$$

Exercise 4.47

We verify that it is in fact the case for both Ω and $\emptyset$,

$$\int_\Omega f \, d\mathbb{P} = \mathbb{E}[f] = \int_\Omega \mathbb{E}[f] \, d\mathbb{P} = \int_\Omega g \, d\mathbb{P}. \tag{1.7}$$

$$\int_\emptyset f \, d\mathbb{P} = 0 = \int_\emptyset g \, d\mathbb{P}. \tag{1.8}$$

This shows that $\mathbb{E}[f] = g = \mathbb{E}^{\{\emptyset,\Omega\}}[f] \ \mathbb{P}-$a.s.

Exercise 4.48

Exercise. *Linearity.* Let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{F}$, $\alpha, \beta \in \mathbb{R}$, and let $f, g : \Omega \to \mathbb{R}$ be $\mathbb{P}$-integrable. Show that

$$\mathbb{E}^{\mathcal{A}}[\alpha f + \beta g] = \alpha \mathbb{E}^{\mathcal{A}}[f] + \beta \mathbb{E}^{\mathcal{A}}[g], \quad \mathbb{P} - \text{a.s.} \tag{1.9}$$

Solution.

$$\begin{aligned}
&\int_A \mathbb{E}^{\mathcal{A}}\left[\alpha f + \beta g\right] \, d\mathbb{P} = \int_A \alpha f + \beta g \, d\mathbb{P} \\
&= \int_A \alpha f \, d\mathbb{P} + \int_A \beta g \, d\mathbb{P} = \alpha \int_A f \, d\mathbb{P} + \beta \int_A g \, d\mathbb{P} \\
&= \alpha \int_A \mathbb{E}^{\mathcal{A}}\left[f\right] \, d\mathbb{P} + \beta \int_A \mathbb{E}^{\mathcal{A}}\left[g\right] \, d\mathbb{P} = \int_A \alpha \mathbb{E}^{\mathcal{A}}\left[f\right] + \beta \mathbb{E}^{\mathcal{A}}\left[g\right] \, d\mathbb{P}.
\end{aligned}$$

Exercise 4.51

Exercise. Let $m, n \in \mathbb{N}$, $m < n$, and $X_1, \ldots, X_n : \Omega \to \mathbb{R}$ be independent $\mathbb{P}$-integrable random variables. Let

$$\mathcal{F}_m := \sigma(X_1, \ldots, X_m) \tag{1.10}$$

$$:= \sigma\left(\left\{X_i^{-1}(B) \,\middle|\, B \in \mathcal{B}(\mathbb{R}),\ i \in \{1, \ldots, m\}\right\}\right) \tag{1.11}$$

denote the σ-algebra induced by $X_1, \ldots, X_m$. Show that

$$\mathbb{E}^{\mathcal{F}_m}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{m} X_i + \sum_{j=m+1} \mathbb{E}\left[X_j\right], \tag{1.12}$$

$$\mathbb{E}^{\mathcal{F}_m}\left[\prod_{i=1}^{n} X_i\right] = \left(\prod_{i=1}^{m} X_i\right)\left(\prod_{j=m+1}^{n} \mathbb{E}[X_j]\right). \tag{1.13}$$

Solution. For the first part we have that

$$\text{for } i \leq m: \; X_i \in \mathcal{M}(\mathcal{F}^m) \Rightarrow \mathbb{E}\left[X_i | \mathcal{F}^m\right] = X_i \tag{1.14}$$

$$\text{for } j > m: \; X_j \perp \mathcal{F}^m \Rightarrow \mathbb{E}\left[X_j | \mathcal{F}^m\right] = \mathbb{E}[X_j]. \tag{1.15}$$

Thus we have that

$$\mathbb{E}^{\mathcal{F}^m}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{m} X_i + \sum_{j=m+1} \mathbb{E}\left[X_j\right]. \tag{1.16}$$

For the second part we have that, because $X_1, \dots, X_m \in \mathcal{M}(\mathcal{F}^m)$ and by independence

$$\mathbb{E}^{\mathcal{F}^m}\left[\prod_{i=1}^{n} X_i\right] = \left(\prod_{i=1}^{m} X_i\right) \mathbb{E}^{\mathcal{F}^m}\left[\prod_{j=m+1}^{n} X_j\right] \tag{1.17}$$

$$= \left(\prod_{i=1}^{m} X_i\right) \mathbb{E}\left[\prod_{j=m+1}^{n} X_j\right] \tag{1.18}$$

$$= \left(\prod_{i=1}^{m} X_i\right) \left(\prod_{j=m+1}^{n} \mathbb{E}[X_j]\right). \tag{1.19}$$

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- Defintion of Brownian motion
- Exercise 5.43
- Exercise 5.44

Definition 5.38. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $d \in \mathbb{N}$. Then a stochastic process $B = (B_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with target space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and parameter set $[0, \infty)$ is called a d-dimensional *standard Brownian motion* or, synonymously, *Wiener process* on $(\Omega, \mathcal{F}, \mathbb{P})$, iff it has the following four properties:

1. All its paths $B_\bullet(\omega)$, $\omega \in \Omega$, are continuous.
2. $\mathbb{P}(\{B_0 = 0\}) = 1$
3. B has independent and stationary increments.
4. For all $t > s \geq 0$, the distribution of $B_t - B_s$ is a d-dimensional centered Gaussian distribution with covariance matrix $(t-s)\mathbb{1}_d$.

Exercise. Compute the probabilities $\mathbb{P}(\{B_1 = 0\})$ and $\mathbb{P}(\{B_\bullet = f\})$ for $f \in C([0,\infty), \mathbb{R}^d)$.

Solution. For the first part we have

$$\begin{aligned}\mathbb{P}(\{B_1 = 0\}) &= \mathbb{P}(\{B_1 - B_0 = 0\}) \\ &= \int_{\{0\}} \frac{e^{-\|x\|^2/2(1-0)}}{(2\pi(1-0))^{d/2}}\,\mathrm{d}x = \int_{\{0\}} \frac{e^{-\|x\|^2/2}}{(2\pi)^{d/2}}\,\mathrm{d}x = 0.\end{aligned}$$

For the second part let us first look at the event itself

$$\{B_\bullet = f\} = \{\omega \in \Omega \mid \forall t \geq 0 : B_t(\omega) = f(t)\}. \tag{2.1}$$

Further we observe that

$$\{B_\bullet = f\} \subset \{B_1 = f(1)\}$$
$$\{\omega \in \Omega \mid \forall t \geq 0 \,:\, B_t(\omega) = f(t)\} \subset \{\omega \in \Omega \mid B_1(\omega) = f(1)\}.$$

Notice then that

$$\mathbb{P}(B_1 = f(1)) = 0 \quad \Rightarrow \quad \mathbb{P}(B_\bullet = f) = 0. \tag{2.2}$$

Exercise. Let B be a one-dimensional standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathbb{E}[\cdot]$ denote the expectation with respect to $\mathbb{P}$. Show that $\mathbb{E}[B_t] = 0$ and $\mathbb{E}[B_s B_t] = s \wedge t$ for all $s, t \geq 0$.

Solution. To show the first part observe that from 2. we have that

$$B_t = B_t - B_0. \tag{2.3}$$

It then follows from 4. that

$$B_t - B_0 \sim N(0, t). \tag{2.4}$$

Exercise 5.44

For the second part observe that from 3. B has independent increments. Without loss of generality assume that $s \leq t$. Then we can rewrite in the following way

$$\mathbb{E}[B_s B_t] = \mathbb{E}[B_s((B_t - B_s) + B_s)] \tag{2.5}$$
$$= \mathbb{E}[B_s(B_t - B_s)] + \mathbb{E}[B_s^2]. \tag{2.6}$$

Because $B_s = B_s - B_0$ from 2., we have that

$$\mathbb{E}[B_s(B_t - B_s)] = \mathbb{E}[(B_s - B_0)(B_t - B_s)] \tag{2.7}$$
$$= \mathbb{E}[(B_s - B_0)]\mathbb{E}[(B_t - B_s)] = 0. \tag{2.8}$$

Where the second equality follows from the two increments being independent. Finally we have

$$\mathbb{E}[B_s B_t] = \mathbb{E}[B_s^2] = s. \tag{2.9}$$

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Martingales & Quadratic Variation

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- Introduce the martingale definition
- Show that one-dimensional Brownian motion is a martingale
- Defintion of Quadratic Variation
- Find the Quadratic Variation of Brownian motion

Martingale definition

Let $(M_t)_{t \in I}$ be an adapted $\mathbb{R}$-valued stochastic process such that $M_t : \Omega \to \mathbb{R}$ is $\mathbb{P}$-integrable for every $t \in I$.

- M is called a $(\mathcal{F}_t)_{t \in I}$-martingale iff

$$\mathbb{E}^{\mathcal{F}_s}[M_t] = M_s, \quad \mathbb{P} - \text{a.s. for all } s, t \in I \text{ with } s \leq t. \tag{3.1}$$

Martingale definition

Let $(M_t)_{t\in I}$ be an adapted $\mathbb{R}$-valued stochastic process such that $M_t : \ \Omega \to \mathbb{R}$ is $\mathbb{P}$-integrable for every $t \in I$.

- M is called a $(\mathcal{F}_t)_{t\in I}$-submartingale iff

$$\mathbb{E}^{\mathcal{F}_s}[M_t] \geq M_s, \quad \mathbb{P} - \text{a.s. for all } s, t \in I \text{ with } s \leq t. \tag{3.2}$$

Martingale definition

Let $(M_t)_{t \in I}$ be an adapted $\mathbb{R}$-valued stochastic process such that $M_t: \ \Omega \to \mathbb{R}$ is $\mathbb{P}$-integrable for every $t \in I$.

- M is called a $(\mathcal{F}_t)_{t \in I}$-submartingale iff

$$\mathbb{E}^{\mathcal{F}_s}[M_t] \geq M_s, \quad \mathbb{P} - \text{a.s. for all } s, t \in I \text{ with } s \leq t. \tag{3.2}$$

- Analogously for supermartingale.

One-dimensional Brownian Motion

If B is a one-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion, then B is a $(\mathcal{F}_t)_{t\geq 0}$-martingale.
In particular, every one-dimensional standard Brownian motion B is a martingale with respect to its natural filtration $(\mathcal{B}_t^B)_{t\geq 0}$.

- Let us first recall that

$$\int_\Omega |B_t| \, d\mathbb{P} = \frac{1}{(2\pi t)^{1/2}} \int_{\mathbb{R}} |x| e^{-x^2/2t} \, dx < \infty, \tag{3.3}$$

$$\mathbb{E}[B_t] = \frac{1}{(2\pi t)^{1/2}} \int_{\mathbb{R}} x e^{-x^2/2t} \, dx = 0, \tag{3.4}$$

for every $t > 0$.

- Since $B_0 = 0$, $\mathbb{P}-$a.s., it follows in particular that B_t is $\mathbb{P}$-integrable with $\mathbb{E}[B_t] = 0$ for all $t \geq 0$.
- For all $0 \leq s \leq t < \infty$, we further observe that, $\mathbb{P}-$a.s.,

$$
\begin{aligned}
\mathbb{E}^{\mathcal{F}_s}[B_t] &= \mathbb{E}^{\mathcal{F}_s}[B_t - B_s] + \mathbb{E}^{\mathcal{F}_s}[B_s] && (3.5)\\
&= \mathbb{E}[B_t - B_s] + B_s && (3.6)\\
&= \mathbb{E}[B_t] - \mathbb{E}[B_s] + B_s && (3.7)\\
&= B_s, && (3.8)
\end{aligned}
$$

since the increment $B_t - B_s$ is $\mathcal{F}_s$-independent and B_s is $\mathcal{F}_s$-measurable.

Definition of Quadratic Variation

Theorem (and definition) 6.50. Assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfies the usual hypotheses and let M be a continuous local martingale. Then there exists an up to indishtinguishability unique finite variation process $\langle M\rangle = (\langle M\rangle_t)_{t\geq 0}$ such that $\langle M\rangle_0 = 0$, $\mathbb{P}$-a.s., and

$$M^2 - \langle M\rangle = \left(M_t^2 - \langle M\rangle_t\right)_{t\geq 0} \text{ is a local martingale.} \tag{3.9}$$

This process is called quadratic variation of M. The quadratic variation $\langle M\rangle$ can be chosen such that it is a non-drecreasing process. Furthermore,

$$\langle M^\tau\rangle = \langle M\rangle^\tau, \tag{3.10}$$

up to indishtinguishability for every stopping time $\tau : \Omega \to [0, \infty]$. Finally, again up to indishtinguishability,

$$\langle M\rangle = \langle M - M_0\rangle. \tag{3.11}$$

Exercise 6.51. Let $B = (B_t)_{t \geq 0}$ be a 1-dimensional standard Brownian motion. Show that

$$\langle B \rangle_t = t, \quad t \geq 0. \tag{3.12}$$

Solution. Define the continuous finite variation process $(t)_{t \geq 0}$. Note that, obviously, $\langle B \rangle_0 = 0$. Thus what is left to show is that $(B_t^2 - t)_{t \geq 0}$ is a local martingale.

Quadratic Variation of 1-dimensional standard Brownian motion

First of all note that $B_t^2 - t$ is $\mathbb{P}$-integrable and adapted. For $t > s$ we have that

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_s}\left[B_t^2 - t\right] &= \mathbb{E}^{\mathcal{F}_s}\left[(B_s + [B_t - B_s])^2\right] - t \\
&= \mathbb{E}^{\mathcal{F}_s}\left[B_s^2\right] + 2\mathbb{E}^{\mathcal{F}_s}[B_s(B_t - B_s)] + \mathbb{E}\left[(B_t - B_s)^2\right] - t \\
&= B_s^2 + 2B_s\mathbb{E}^{\mathcal{F}_s}[B_t - B_s] + \mathbb{E}\left[(B_t - B_s)^2\right] - t \\
&= B_s^2 + 2B_s\mathbb{E}[B_t - B_s] + \mathbb{E}\left[B_{t-s}^2\right] - t \\
&= B_s^2 + t - s - t \\
&= B_s^2 - s,
\end{aligned}$$

showing that $B_t^2 - t$ is in fact a martingale.

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- Theorem 7.4 (Approximation by bounded simple processes)
- 7.2.2 Extension of the stochastic integral by isometry

Definition 6.34 (Elementary stochastic integral)

Defintion 6.34. Let $Y = (Y_t)_{t\geq 0}$ be some progressively measurable process with target space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and let X be a predictable step process having the representation

$$X_t = \sum_{k=1}^{\infty} \xi_k \chi_{\{\tau_{k-1} < t \leq \tau_k\}}, \;\; t \geq 0. \tag{4.1}$$

for a non decreasing series of stopping times and some $\mathcal{F}_{\tau_{k-1}}$-measurable $\xi_k \,:\, \Omega \to \mathbb{R}$. Then the stochastic integral with integrand X with respect to the intregrator Y is the stochastic process

$$\int X\, dY := \left(\int_0^t X_s\, dY_s \right)_{t\geq 0} \tag{4.2}$$

defined by

$$\int_0^t X_s\, dY_s := \sum_{k=1}^{\infty} \xi_k \left(Y_{\tau_k \wedge t} - Y_{\tau_{k-1}\wedge t} \right). \tag{4.3}$$

Theorem 7.4. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfyisng the ususal hypotheses and M a continuous square-integrable martingale. Let X be a progressively measurable stochastic process satisfying

$$\mathbb{E}\left[\int_0^t X_s^2 \, d\langle M\rangle_s\right] < \infty, \;\; t \geq 0. \tag{4.4}$$

Then there exists a sequence $(X^{[n]})_{n\in\mathbb{N}}$ of bounded simple processes such that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^t (X_s^{[n]} - X_s)^2 \, d\langle M\rangle_s\right] = 0. \tag{4.5}$$

Let M be a continuous square integrable martingale and X a progressively measurable process satisfying

$$\mathbb{E}\left[\int_0^t X_s^2 \, d\langle M\rangle_s\right] < \infty, \;\; t \geq 0.$$

Theorem 7.4 then gives a sequence $(X^{[n]})_{n\in\mathbb{N}}$ of bounded predictable step processes satisfying

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^t (X_s^{[n]} - X_s)^2 \, d\langle M\rangle_s\right] = 0.$$

By the Minkowski inequalities we get that

$$\mathbb{E}\left[\int_0^t (X_s^{[m]} - X_s^{[n]})^2 \, d\langle M\rangle_s\right]^{1/2}$$
$$\leq \mathbb{E}\left[\int_0^t (X_s^{[m]} - X_s)^2 \, d\langle M\rangle_s\right]^{1/2} + \mathbb{E}\left[\int_0^t (X_s - X_s^{[n]})^2 \, d\langle M\rangle_s\right]^{1/2}.$$

Taking limits entails that

$$\mathbb{E}\left[\int_0^t (X_s^{[m]} - X_s^{[n]})^2 \, d\langle M\rangle_s\right] \xrightarrow{m,n\to\infty} 0. \tag{4.6}$$

Lemma 7.3. Let X be a bounded step process and $M = (M_t)_{t\geq 0}$ a continuous square-integrable martingale. Then we have the isometry

$$\mathbb{E}\left[\left(\int_0^t X_s\,dM_s\right)^2\right] = \mathbb{E}\left[\int_0^t X_s^2\,d\langle M\rangle_s\right],\ \ t \geq 0. \tag{4.7}$$

Extension of the stochastic integral

By linearity and isometry (Lemma 7.3) of the elementary stochastic integral we can rewrite

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t (X_s^{[m]} - X_s^{[n]})^2 \, d\langle M\rangle_s\right] \\
&= \mathbb{E}\left[\left(\int_0^t X_s^{[m]} - X_s^{[n]} \, dM_s\right)^2\right] \\
&= \mathbb{E}\left[\left(\int_0^t X_s^{[m]} \, dM_s - \int_0^t X_s^{[n]} \, dM_s\right)^2\right].
\end{aligned}$$

Thus

$$\mathbb{E}\left[\left(\int_0^t X_s^{[m]} \, dM_s - \int_0^t X_s^{[n]} \, dM_s\right)^2\right] \xrightarrow{m,n\to\infty} 0 \tag{4.8}$$

Theorem 6.32. Consider the case $I = [0, \infty)$ and assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfies the usual hypotheses. Let $M^{[n]} = (M_t^{[n]})_{t \geq 0}$ be a right-continuous, square-integrable martingale for every $n \in \mathbb{N}$. Assume that

$$\mathbb{E}\left[(M_t^{[p]} - M_t^{[n]})^2\right] \xrightarrow{n,p\to\infty} 0, \;\; t \geq 0. \tag{4.9}$$

Then there exists a càdlàg square-integrable martingale $M = (M_t)_{t \geq 0}$ such that

$$\lim_{n\to\infty} \mathbb{E}\left[(M_t^{[n]} - M_t)^2\right] = 0, \;\; t \geq 0. \tag{4.10}$$

Every other martingale M' having these properties is indishtinguishable from M.

Furthermore, there exists integers $1 \leq n_1 < n_2 < \ldots$ and some $\mathbb{P}$-zero set $N \in \mathcal{F}$ such that

$$\sup_{s\in[0,t]} \left\| M_s^{[n_j]}(\omega) - M_s(\omega) \right\| \xrightarrow{j\to\infty} 0, \;\; \omega \in \Omega \backslash N, \; t \geq 0. \tag{4.11}$$

In particular, if all $M^{[n]}$ with $n \in \mathbb{N}$ are continuous, then M can be chosen continuous as well.

Theorem 6.32 thus gives a continuous square integrable martingale $(I_t)_{t\geq 0}$ such that

$$\mathbb{E}\left[\left(\int_0^t X_s^{[m]}\, dM_s - I_t\right)^2\right] \xrightarrow{m\to\infty} 0, \tag{4.12}$$

and every martingale I' having these properties are indishtinguishable from I.

Extension of the stochastic integral

Next we can verify that I, does not depend on the choice of approximating sequence.

Let $(Y^{[n]})_{n\in\mathbb{N}}$ be another bounded predictable step process such that

$$\lim_{n\to\infty}\mathbb{E}\left[\int_0^t \left(Y_s^{[n]}-X_s\right)^2 \, d\langle M\rangle_s\right]=0. \tag{4.13}$$

Let $(J_t)_{t\geq 0}$ be a corresponding continuous square-integrable martingale satisfying

$$\mathbb{E}\left[\left(\int_0^t Y_s^{[n]}\,dM_s - J_t\right)^2\right] \xrightarrow{n\to\infty} 0 \tag{4.14}$$

Extension of the stochastic integral

For every $t \geq 0$, we then obtain

$$
\begin{aligned}
\mathbb{E}\left[(I_t - J_t)^2\right] &= \lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^t X_s^{[n]}\, dM_s - \int_0^t Y_s^{[n]}\, dM_s\right)^2\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^t X_s^{[n]} - Y_s^{[n]}\, dM_s\right)^2\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\int_0^t \left(X_s^{[n]} - Y_s^{[n]}\right)^2\, d\langle M\rangle_s\right] = 0,
\end{aligned}
$$

where we used isometry and

$$
\lim_{n\to\infty} \mathbb{E}\left[\int_0^t (X_s^{[n]} - X_s)^2\, d\langle M\rangle_s\right] = 0, \quad \lim_{n\to\infty} \mathbb{E}\left[\int_0^t \left(Y_s^{[n]} - X_s\right)^2\, d\langle M\rangle_s\right] = 0.
$$

The processes $(I_t)_{t\geq 0}$ and $(J_t)_{t\geq 0}$ are thus modification of each other and therfore indishtinguishable as they are both continuous.

Definition 7.8. Any choice of the up to indishtinguishability well-defined and unique continuous square-integrable martingale $(I_t)_{t\geq 0}$ introduced in the preceeding argumentation is called the stochastic integral of X with respect to M and we write

$$\int_0^t X_s \, dM_s := I_t, \; t \geq 0, \quad \int X \, dM := \left(\int_0^t X_s \, dM_s \right)_{t \geq 0}.$$

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- Definition of semi martingales
- Itô's formula for semi martingales
- Exercise 8 (Part of Example 4)

Denote by $\mathcal{M}_{c,loc}$ the linear space of $(\mathcal{F}_t)_{t\geq 0}$-continuous local martingales.

Denote by $\mathcal{A}_{loc}$ the space of adapted processes whose paths are continuous and locally of bounded variation.

Let $X = (X_t)_{t\geq 0}$ be a stochastic process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We say that X is a continuous semimartingale if X can be decomposed as

$$X_t = X_0 + M_t + A_t, \;\; t \geq 0, \tag{5.1}$$

where $M \in \mathcal{M}_{c,loc}$, $A \in \mathcal{A}_{loc}$. We will denote by $\mathcal{SM}^c$ the linear space of continuous semimartingales.

Theorem 7. Let $X = (X^{(1)}, \dots, X^{(d)})$, where $X^{(i)} \in \mathcal{SM}^c$, for all $i = 1, \dots, d$. If $f : \mathbb{R}^d \to \mathbb{R}$ is twice continuously differentiable and $X = (X^{(1)}, \dots, X^{(d)})$, then

$$f(X_t) = f(X_0) + \sum_{i=1}^{d} \int_0^t \partial_{x_i} f(X_s)\, dX_s^{(i)} + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \partial_{x_i}\partial_{x_j} f(X_s)\, d\langle X^{(i)}, X^{(j)}\rangle_s.$$

In the case $d = 1$ we have

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle X\rangle_s.$$

Example 4 (Complex Exponential Process). Let $M \in \mathcal{M}_{c,loc}$ starting at 0 and define for $t \geq 0$

$$N_t^{(1)} := \cos(M_t) e^{\frac{1}{2}\langle M \rangle_t}, \; N_t^{(2)} := \sin(M_t) e^{\frac{1}{2}\langle M \rangle_t}. \tag{5.2}$$

Verify that

$$N_t^{(2)} = \int_0^t N_s^{(1)} \, dM_s, \; t \geq 0. \tag{5.3}$$

We will start by observing that Itô's formula applied to the function $f(x, y) = xy$ results in the following integration by parts formula

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s + \langle X, Y \rangle_t, \ \forall X, Y \in \mathcal{SM}^c,$$

as $\frac{1}{2}\left(\int_0^t 1 \ d\langle X, Y \rangle_s + \int_0^t 1 \ d\langle X, Y \rangle_s\right) = \langle X, Y \rangle_t$.

Thus we get that

$$N_t^{(2)} = \sin(M_t) e^{\frac{1}{2}\langle M \rangle_t} \tag{5.4}$$

$$= \int_0^t \sin(M_s) \, d\left(e^{\frac{1}{2}\langle M \rangle_s}\right) + \int_0^t e^{\frac{1}{2}\langle M \rangle_s} \, d\left(\sin(M_s)\right). \tag{5.5}$$

since $\sin(M_0) = \sin(0) = 0$, and the fact that $e^{\frac{1}{2}\langle M \rangle_s}$ have no martingale part.

Exercise 8

Applying Itô's formula again ($d = 1$) we get

$$\begin{aligned} e^{\frac{1}{2}\langle M\rangle_t} &= 1 + \int_0^t \partial_{\langle M\rangle} e^{\frac{1}{2}\langle M\rangle_s}\, d\langle M\rangle_s + \frac{1}{2}\int_0^t \partial^2_{\langle M\rangle} e^{\frac{1}{2}\langle M\rangle_s}\, d\langle\langle M\rangle\rangle_s \\ &= 1 + \frac{1}{2}\int_0^t e^{\frac{1}{2}\langle M\rangle_s}\, d\langle M\rangle_s \qquad (5.6) \end{aligned}$$

$$\begin{aligned} \sin(M_t) &= \int_0^t \partial_M \sin(M_s)\, dM_s + \frac{1}{2}\int_0^t \partial^2_M \sin(M_s)\, d\langle M\rangle_s \qquad (5.7) \\ &= \int_0^t \cos(M_s)\, dM_s - \frac{1}{2}\int_0^t \sin(M_s)\, d\langle M\rangle_s. \qquad (5.8) \end{aligned}$$

Exercise 8

Using now that

$$X_t = X_0 + \int_0^t H_s \, dY_s$$
$$dX_s = H_s \, dY_s,$$

which gives that

$$\int_0^t G_s \, dX_s = \int_0^t G_s H_s \, dY_s.$$

We are able to "rewrite"

$$d\left(e^{\frac{1}{2}\langle M \rangle_s}\right) = \frac{1}{2} e^{\frac{1}{2}\langle M \rangle_s} \, d\langle M \rangle_s,$$
$$d\left(\sin(M_s)\right) = \cos(M_s) \, dM_s - \frac{1}{2}\sin(M_s) \, d\langle M \rangle_s.$$

Exercise 8

Now inserting what we just found

$$
\begin{aligned}
N_t^{(2)} &= \sin(M_t)e^{\frac{1}{2}\langle M\rangle_t} \\
&= \int_0^t \sin(M_s)\, d\left(e^{\frac{1}{2}\langle M\rangle_s}\right) + \int_0^t e^{\frac{1}{2}\langle M\rangle_s}\, d\left(\sin(M_s)\right) \\
&= \frac{1}{2}\int_0^t \sin(M_s)e^{\frac{1}{2}\langle M\rangle_s}\, d\langle M\rangle_s + \int_0^t \cos(M_s)e^{\frac{1}{2}\langle M\rangle_s}\, dM_s \\
&\qquad - \frac{1}{2}\int_0^t \sin(M_s)e^{\frac{1}{2}\langle M\rangle_s}\, d\langle M\rangle_s \\
&= \int_0^t \cos(M_s)e^{\frac{1}{2}\langle M\rangle_s}\, dM_s \\
&= \int_0^t N_t^{(1)}\, dM_s.
\end{aligned}
$$

which concludes the exercise.

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Definition of the exponential martingale

Definition 6. The process

$$\mathcal{E}(X)_t := \exp\left\{X_t - X_0 - \frac{1}{2}\langle X\rangle_t\right\}, \quad t \geq 0,$$

is called the exponential semimartingale, when $X \in \mathcal{SM}^c$, and the exponential local martingale, when $X \in \mathcal{M}_{c,loc}$.

Novikov's Condition

Theorem 12 (Novikov). Let $M \in \mathcal{M}_{c,loc}$. If for every $t \geq 0$

$$\mathbb{E}(e^{\frac{1}{2}\langle M\rangle_t}) < \infty,$$

then $\mathcal{E}(M)_t$ is a true martingale with $\mathbb{E}(\mathcal{E}(M)_t) = 1$, for all $t \geq 0$.

Introduction to Girsanov

Fix $0 < T < +\infty$ and consider $(\Omega, \mathscr{F}_T, (\mathscr{F}_t)_{0\leq t\leq T}, \mathbb{P})$. Let ξ be a non-negative r.v. on $(\Omega, \mathscr{F}_T)$ with $\mathbb{E}(\xi) = 1$. Then the measure

$$\mathbb{Q}(A) := \mathbb{E}_{\mathbb{P}}(\chi_A \xi), \; A \in \mathscr{F}_T,$$

is a probability measure which is absolutely continuous w.r.t. $\mathbb{P}$. We will write

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \xi, \text{ on } \mathscr{F}_T.$$

With $\mathbb{Q}$ as above we have

$$\mathbb{E}_{\mathbb{Q}}[X] = \mathbb{E}_{\mathbb{P}}[X\xi] = \int_{\Omega} X(\omega)\xi(\omega)\; d\mathbb{P}(\omega),$$

for an abitrary stochastic process $X \in \mathscr{F}_T$.

Theorem 14 (Girsanov). Let $(B_t)_{t\geq 0}$ be a Brownian motion on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, and H a progressively measurable process such that

$$\mathbb{E}_{\mathbb{P}}(e^{\frac{1}{2}\langle H\cdot B\rangle_T}) = \mathbb{E}_{\mathbb{P}}(e^{\frac{1}{2}\int_0^T H_s^2 \mathrm{d}s}) < \infty.$$

Then the process defined as

$$\tilde{B}_t := B_t - \int_0^t H_s \mathrm{d}s, \quad 0 \leq t \leq T,$$

is a Brownian motion on $(\Omega, \mathscr{F}_T, (\mathscr{F}_t)_{0\leq t\leq T}, \mathbb{Q})$, where

$$d\mathbb{Q} = \mathcal{E}(H\cdot B)_T \; d\mathbb{P} = exp\left\{\int_0^T H_s \; dB_s - \frac{1}{2}\int_0^T H_s^2 \; ds\right\} \; d\mathbb{P}.$$

Furthermore $\mathbb{Q} \sim \mathbb{P}$.

Black & Scholes Example

- Consider the Black and Scholes model

$$\mathrm{d}A_t = rA_t\mathrm{d}t, \quad A_0 = 1, \quad \frac{\mathrm{d}S_t}{S_t} = \mu\mathrm{d}t + \sigma\mathrm{d}B_t, \quad S_0 = s_0 > 0.$$

- We have seen that

$$\begin{aligned} A_t &= e^{rt}, \\ S_t &= s_0\ e^{(\mu-\frac{\sigma^2}{2})t+\sigma B_t} \quad t \geq 0. \end{aligned}$$

Black & Scholes Example

- Itô's formula can be simplified to

$$\begin{aligned}\tilde{S}_t &= e^{-rt} S_t \\ &= s_0 + \int_0^t \partial_t \tilde{S}_s \, \mathrm{d}s + \int_0^t \partial_{B_t} \tilde{S}_s \, \mathrm{d}B_s + \frac{1}{2} \int_0^t \partial^2_{B_t} \tilde{S}_s \, ds\end{aligned}$$

because the remaining terms have integrator $\langle B_s, s \rangle$ and $\langle s, s \rangle$ which both equal zero.

- We can then find the derivatives

$$\begin{aligned}\tilde{S}_t &= e^{-rt} s_0 \; e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t} \\ \partial_t \tilde{S}_t &= \left\{ \mu - r - \frac{\sigma^2}{2} \right\} \tilde{S}_t \\ \partial_{B_t} \tilde{S}_t &= \sigma \tilde{S}_t \\ \partial^2_{B_t} \tilde{S}_t &= \sigma^2 \tilde{S}_t\end{aligned}$$

Black & Scholes Example

Now rewriting

$$\tilde{S}_t = s_0 + \int_0^t \sigma \tilde{S}_s \, dB_s - \int_0^t (r - \mu) \tilde{S}_s \, ds \tag{6.1}$$

$$= s_0 + \int_0^t \sigma \tilde{S}_s \, d\tilde{B}_s, \tag{6.2}$$

where

$$\tilde{B}_t := B_t - \frac{(r - \mu)}{\sigma} t, \;\; t \geq 0. \tag{6.3}$$

Which follows from

$$\tilde{S}_t = s_0 + \int_0^t \sigma \tilde{S}_s \, d\left(B_s - \frac{r - \mu}{\sigma} s\right) \tag{6.4}$$

$$= s_0 + \int_0^t \sigma \tilde{S}_s \, dB_s - \int_0^t (r - \mu) \tilde{S}_t \, ds. \tag{6.5}$$

Black & Scholes Example

Applying Girsanov's Theorem to

$$\tilde{B}_t := B_t - \int_0^t H_s \, ds, \tag{6.6}$$

where $H_t := \frac{r-\mu}{\sigma}$ denotes the market price of risk.

Black & Scholes Example

Note first that, for all $t \geq 0$

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t H_s^2\, ds\right)\right] = \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \frac{(r-\mu)^2}{\sigma^2}\, ds\right)\right] < \infty.$$

Novikov's Condition and Girsanov's Theorem imply that the process $\tilde{B}$ is a Brownian motion w.r.t. to the measure

$$d\mathbb{Q} = \exp\left(\int_0^T \frac{r-\mu}{\sigma}\, dB_s - \frac{1}{2}\int_0^T \left(\frac{r-\mu}{\sigma}\right)^2 ds\right) d\mathbb{P}. \tag{6.7}$$

This further gives that $\mathbb{Q}$ is a local martingale measure as the discounted price process is a local martingale w.r.t. $\mathbb{Q}$

$$\tilde{S}_t = s_0 + \int_0^t \sigma \tilde{S}_s\, d\tilde{B}_s, \tag{6.8}$$

because the stochastic integral w.r.t. to a Brownian motion are in general local martingales.

Black & Scholes Example

Definition

A probability measure $\mathbb{Q}$ on $(\Omega, \mathscr{F}_T)$ is said to be a local martingale measure if the discounted price process $(\tilde{P}_t)_{0 \le t \le T}$ is a $\mathbb{Q}$ local martingale.

In our case the discounted price process is

$$\tilde{P}_t = \frac{1}{A_t}\left(A_t, S_t\right) = \left(1, \tilde{S}_t\right).$$

We only need to check that $\tilde{S}_t$ is a martingale, as 1 is clearly a martingale w.r.t. any probability measure.

The First Fundamental Theorem of Asset Pricing

The market M is arbitrage free, if there exists a local martingale measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$.

Thus we have shown that the Black & Scholes model is arbitrage free.