Measure Theory

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1 Notation

1.1 Measure with densities

Theorem 1.1 (Measures with densities)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $f : \Omega \to [0, \infty]$ be measureable. Then*

$$(f \odot \mu)(A) := \int_A f \, d\mu = \int_\Omega \chi_A f \, d\mu, \quad A \in \mathcal{A}, \tag{1.1}$$

defines a new measure $f \odot \mu : (A) \to [0, \infty]$, called the measure with density f with respect to μ. For every $N \in \mathcal{A}$, the following implication holds,

$$\mu(N) = 0 \ \Rightarrow \ (f \odot \mu)(N) = 0.$$

1.2 Product σ-algrebras

Definition 1.1. *Let $n \in \mathbb{N}$, $n \geqslant 2$, and suppose that, for every $i \in \{1, \ldots, n\}$, we are given a measurable space $(\Omega_i, \mathcal{A}_i)$.*

1. *The smallest σ-algebra on $\times_{i=1}^n \Omega_i$ containing*

$$\mathcal{A}_1 * \cdots * \mathcal{A}_n := \{A_1 \times \cdots \times A_n \mid A_1 \in \mathcal{A}_1, \ldots, A_n \in \mathcal{A}_n\} \subset \mathcal{P}\left(\bigtimes_{i=1}^n \Omega_i\right)$$

   is called the product σ-algebra defined by measns of $\mathcal{A}_1, \ldots, \mathcal{A}_n$. It is denoted by

$$\bigotimes_{i=1}^n \mathcal{A}_i := \sigma\left(\mathcal{A}_1 * \cdots * \mathcal{A}_n\right).$$

2. *Let Γ be a set and let $f_i : \Gamma \to \Omega_i$ be an arbitrary map for every $i \in \{1, \ldots, n\}$. Then the smallest σ-algebra on Γ turning all maps $f_1, \ldots, f_n$ into measurable maps, i.e.,*

$$\sigma(f_1, \ldots, f_n) := \sigma\left(\left\{f_i^{-1}(A_i) \mid A_i \in \mathcal{A}_i, i \in 1, \ldots, n\right\}\right),$$

   is called the initial σ-algebra generated by $f_1, \ldots, f_n$.