If $x(t)$ is a time function, then the laplace transform of the function is defined as -

$$L\left[x(t)\right] = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

where $s$ is a complex variable and it is given by,

$$s = \sigma + j\omega$$

$$\text{Transfer function} = \frac{\text{Laplace Transform of output}}{\text{Laplace Transform of input}}$$

with zero initial conditions

## Mechanical Translational Systems

The model of a mechanical translational system can be obtained by using three basic elements mass, spring and dash-pot.

The weight of the mechanical system is represented by the element mass and is assumed to be concentrated at the center of the body. The elastic deformation of the body can be represented by a spring. The friction existing in rotating mechanical system can be represented by the dash-pot. The dash pot is a piston moving inside a cylinder filled with viscous fluid.

When a force is applied to a translational mechanical system, it is opposed by opposing forces due to mass, friction and elasticity of the system. The force acting on a mechanical body are governed by Newton's second law of motion. For translational systems it states that the sum of applied forces ~~is equal to the sum of opposing~~ forces acting on a body is zero.

## Symbols used in mechanical translational systems:

$x$ = displacement, m.

$v = \frac{dx}{dt}$ = velocity, m/sec.

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ = acceleration, m/sec$^2$.

$f$ = Applied force, N (newton)

$f_m$ = opposing force offered by mass of the body, N.

$f_K$ = opposing force offered by elasticity of the body (spring), N.

$f_b$ = opposing force offered by friction of the body (dash-pot), N.

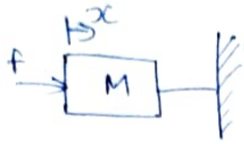
$M$ = mass, Kg.

$K$ = stiffness of spring, N/m.

$B$ = Viscous friction co-efficient, N-sec/m

Force Balance Equations of Idealized elements:-

(i) Ideal Mass




Let $f$ = applied force

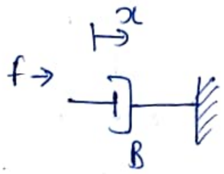
$f_m$ = opposing force due to mass

Here $f_m \propto \frac{d^2x}{dt^2}$ or $f_m = M\frac{d^2x}{dt^2}$

By Newton's second Law,

$$\boxed{f = f_m = M\frac{d^2x}{dt^2}}$$

(ii) Ideal frictional Element (dash-pot)



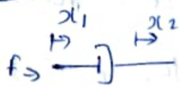

$f$ = applied force

$f_b$ = opposing force due to friction

$f_b \propto \frac{dx}{dt}$

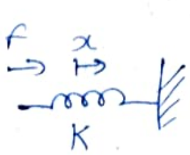
$f_b = B\frac{dx}{dt}$

$$\therefore \boxed{f = f_b = B\frac{dx}{dt}}$$




$f = B\frac{d}{dt}(x_1 - x_2)$

(iii) Ideal elastic element (Spring)




$f$ = applied force

$f_K$ = opposing force due to elasticity

$f_K \propto x$ or $f_K = Kx$

By newtons 2nd law, $\boxed{f = f_K = Kx}$

when the spring has displacement at both ends as


$$f_K \propto (x_1 - x_2)$$

$$f_K = K(x_1 - x_2)$$

$$\boxed{f = f_K = K(x_1 - x_2)}$$

In mechanical translational system, the differential equations governing the system are obtained by writing force balance equations at nodes in the system. The nodes are meeting points of elements. Generally the nodes are mass elements in the system.

The mass has to move in the direction of the applied force. Hence displacement, velocity & acceleration of the mass will be in the direction of applied force. If there is no applied force then the displacement, velocity and acceleration of the mass will be in a direction opposite to that of opposing force.

Note:-

Laplace transform of $x(t) = L[x(t)] = X(s)$

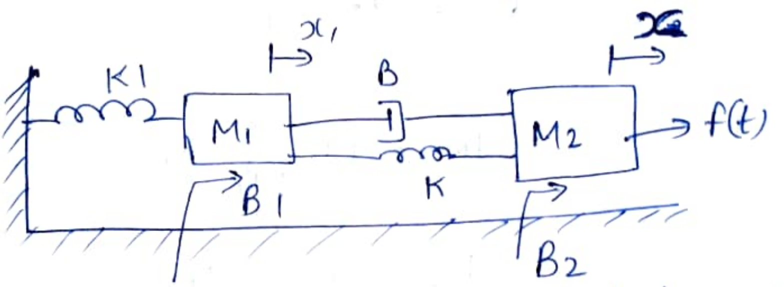
Laplace transform of $\frac{d}{dt}x(t) = L\left(\frac{d}{dt}x(t)\right) = sX(s)$

with zero initial conditions

Laplace transform of $\frac{d^2x(t)}{dt^2} = L\frac{d^2x(t)}{dt^2} = s^2X(s)$

with zero initial conditions

Q) eg: write the differential equations governing the mechanical system shown below and determine the transfer function.




In the given system, applied force f(t) is the input and displacement x is the output.

Let, Laplace transform of $f(t) = L(f(t)) = F(s)$

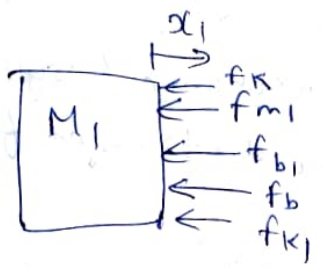
Laplace transform of $x = L\{x\} = X(s)$

Laplace transform of $x_1 = L\{x_1\}$
$= X_1(s)$

Hence required transfer function is $\frac{X(s)}{F(s)}$

The system has two nodes & they are mass $M_1$ & $M_2$. The differential equations governing the system are given by force balance equations at these nodes.

Let the displacement of mass $M_1$ be $x_1$. The free body diagram of mass $M_1$ is shown below




$$f_{m1} = M_1 \frac{d^2 x_1}{dt^2}$$

$$f_{b1} = B_1 \frac{d x_1}{dt}$$

$$f_b = B \frac{d}{dt}(x_1 - x)$$

$$f_{K1} = K_1 x_1$$

$$f_K = K(x_1 - x)$$

By newton's 2nd law

$$f_{m1} + f_{b1} + f_b + f_{K1} + f_K = 0$$

$$\therefore \quad M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B \frac{d}{dt}(x_1 - x) +$$

$$K_1 x_1 + K(x_1 - x) = 0 \qquad \text{—(1)—}$$

On taking laplace transform of above equation with zero initial conditions we get,

$$M_1 s^2 X_1(s) + B_1 s X_1(s) + Bs\left[X_1(s) - X(s)\right]$$

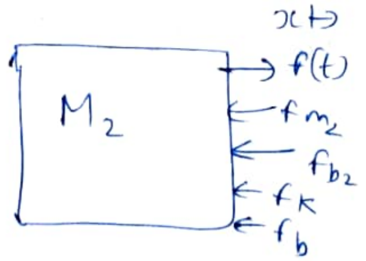
$$+ K_1 X_1(s) + K\left[X_1(s) - X(s)\right] = 0$$

$$\cancel{X_1(s)\left[M_1 s^2 + sB_1 + sB - sB X_1\right.}$$

$$X_1(s)\left[M_1 s^2 + (B_1 + B)s + (K_1 + K)\right] - X(s)\left[Bs + K\right] = 0$$

$$\therefore \quad X_1(s) = \frac{X(s)\,(Bs + K)}{M_1 s^2 + (B_1 + B)s + (K_1 + K)}$$

The free body diagram of mass $M_2$ is shown below



$$f_{m_2} = M_2 \frac{d^2x}{dt^2}$$

$$f_{b_2} = B_2 \frac{dx}{dt}$$

$$f_b = B \frac{d}{dt}(x - x_1)$$

$$f_k = K(x - x_1)$$

By Newton's second law,

$$f_{m_2} + f_{b_2} + f_b + f_k = f(t)$$

$$M_2 \frac{d^2x}{dt^2} + B_2 \frac{dx}{dt} + B \frac{d}{dt}(x - x_1) + K(x - x_1) = f(t) \quad -(2)-$$

On taking laplace transform of above eqn

$$M_2 s^2 X(s) + B_2 s X(s) + Bs\left(X(s) - X_1(s)\right) + K\left[X(s) - X_1(s)\right] = F(s)$$

$$X(s)\left[M_2 s^2 + (B_2 + B)s + K\right] - X_1(s)\left[Bs + K\right] = F(s)$$

substituting value of $X_1(s)$ in above equation, we get

$$X(S)\left[M_2 s^2 + (B_2+B)s + K\right] - X(S)\frac{(Bs+K)(Bs+K)}{M_1 s^2 + (B_1+B)s + (K_1+K)} = F(S)$$

$$\cancel{X(s)\left[\frac{(M_2 s^2 + (B_2+B)s + K)(M_1 s^2 + (B_1+B)s + (K_1+K))}{M_1 s^2 + (B_1+B)s + (K_1+K)}\right] = F(s)}$$

$$X(S)\,\frac{\left\{(M_2 s^2 + (B_2+B)s + K)(M_1 s^2 + (B_1+B)s + K_1 + K) - (Bs+K)^2\right\}}{M_1 s^2 + (B_1+B)s + (K_1+K)} = F(S)$$

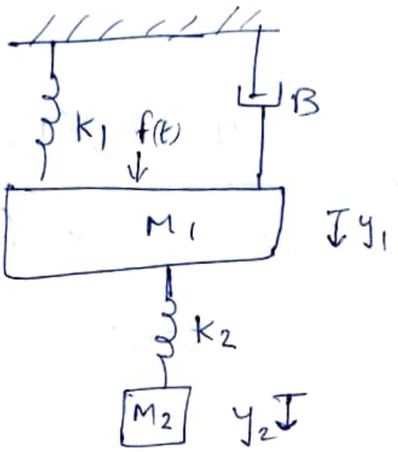
$$\frac{F(S)}{X(S)} = \frac{M_1 s^2 + (B_1+B)s + (K_1+K)}{\left[M_1 s^2 + (B_1+B)s + (K_1+K)\right]\left[M_2 s^2 + (B_2+B)s + K\right] - (Bs+K)^2}$$

↳ Transfer function

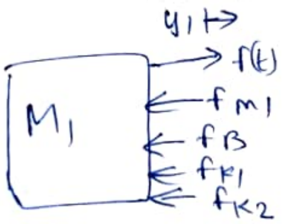
$$\frac{X(S)}{F(S)}$$

① & ② are the differential equations governing the system.

Q) Determine the transfer function $\frac{Y_2(S)}{F(S)}$ of the system shown below




Free body diagram of mass $M_1$




$$f_{m_1} = M_1 \frac{d^2}{dt^2} y_1$$

$$f_B = B \frac{d}{dt} y_1$$

$$f_{K_1} = K_1 y_1$$

$$f_{K_2} = K_2 (y_1 - y_2)$$

By Newton's 2nd law,

$$f_{m_1} + f_B + f_{K_1} + f_{K_2} = f(t)$$

$$M_1 \frac{d^2}{dt^2} y_1 + B \frac{d y_1}{dt} + K_1 y_1 + K_2 (y_1 - y_2) = f(t)$$

—(1)—

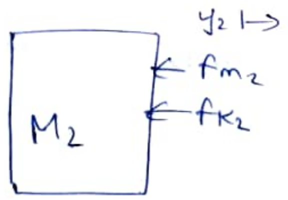
taking laplace transform,

$$M_1 s^2 Y_1(s) + B s Y_1(s) + K_1 Y_1(s) + K_2 (Y_1(s) - Y_2(s)) = F(s)$$

$$Y_1(s) \left[ M_1 s^2 + B(s) + (K_1 + K_2) \right] - Y_2(s) K_2 = F(s)$$

—(2)—

Free body diagram of mass M2




$$f_{m_2} = M_2 \frac{d^2}{dt^2} y_2$$

$$f_{K_2} = K_2 (y_2 - y_1)$$

By newton's 2nd law,

$$f_{m_2} + f_{K_2} = 0$$

$$M_2 \frac{d^2}{dt^2} y_2 + K_2 (y_2 - y_1) = 0$$

—(3)—

taking laplace transform

$$M_2 s^2 Y_2(s) + K_2 (Y_2(s) - Y_1(s)) = 0$$

$$Y_1(s) = Y_2(s) \left( \frac{M_2 s^2 + K_2}{K_2} \right)$$

-(4)-

substituting (4) in (2) we get

$$Y_2(s) \frac{(M_2 s^2 + K_2)}{K_2} [M_1 s^2 + Bs + (K_1 + K_2)] - Y_2(s) K_2 = F(s)$$

$$Y_2(s) \left[ \frac{(M_2 s^2 + K_2)(M_1 s^2 + Bs + (K_1 + K_2)) - K_2^2}{K_2} \right] = F(s)$$

$$\frac{Y_2(s)}{F(s)} = \frac{K_2}{[M_1 s^2 + Bs + (K_1 + K_2)][M_2 s^2 + K_2] - K_2^2}$$

Transfer function