**Engineering Mathematics - II**

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**Lecture - 01**

**Vector Functions**

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Hi, welcome to lectures on Engineering Mathematics II and this is a sequel course of Engineering Mathematics I. So, this is module number 1 on Vector Calculus and we will go through the vector functions in lecture 1.

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So, we will cover what are the vector functions in this lecture and their limit, continuity, and differentiability, also we will be talking about the gradient of a scalar function. So, these scalar functions are the functions that we have learned in calculus in the previous course.

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So, what are these vector functions of one variable? So, these are the functions that map a real number to a vector. So, we can define such functions by this vector here , vector is given by and t will vary from a to b. So, if for a given value of t, this vector will define a position vector of a point and as we vary the t there will be another point and so on. And all these, the collection of these points will form a curve in the space.

So, to define this vector function of a single variable, so, here the single variable is t. So, this factor the input is t which varies from a to b and the output is a vector whose components are for instance , and . So, this is the case of 3 dimensions. But in case of 2 dimensions, we can have like this vector equal to .There is no third component in this case. So, this is the situation in 2d plane.

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So, vector functions of one variable, these are the examples. So, for instance the equation of a straight line which passes through point A, whose position vector is given by the vector and the line is parallel to the vector . So, this is the situation here. We have a point A, whose position vector is given by this red vector and then there is another vector given which is here. So, we want to have a line which is parallel to this and passes through this point A.

So, we are interested to find the equation of this line. So let us consider a general point P here on the line whose position vector is given by this vector Then, since this direction of this line is , so, this segment here AP of this line can be described by the vector , some multiplication, some scalar multiplication to this vector , so that the magnitude of this vector will be adjusted to fit in this length AP. So, this is the vector AP which can be described by some t is a real number and this vector .

So, then we have this equation from the vector addition i.e., will give us this position vector . So that is this position vector can describe the equation or the position vector on this line, a general point given by this where t can vary from the set of real numbers. Another example where we can see the function of one variable. Consider four instance is , so the 2 components are and .

So, if you draw this curve in 2 dimensional plane, then this is basically the ellipse here, because this x component is , the y component here is . So, for instance, at t equal to 0, we have the 3 and this is 0. So (3, 0) point which is given here already. And then if t is for instance , so this will become zero and this will be . So that will be this point. So we are moving from this point in this direction and that is the orientation of the curve.

So in this vector setting, we are not only getting just a curve, but its orientation as well. So, for instance here this curve, the orientation is the clockwise orientation, which is described in the increasing direction of t. So, as we are increasing t we are moving in this clockwise direction and therefore, we call it the clockwise orientation of this curve. Another example in 3 dimensions for instance could be like .

So, this will be helix here, these () are the equation of the circle in 2 dimensions, but we have the third component also which will lift this curve in the direction of z. So, here we have a helix.

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Well, so, coming to the continuity and differentiability later. So, first, the limit and continuity of such functions. For limit, we can compute for such functions by computing the limit of each of its components. So, for instance, here , then the limit as t approaches to a, we can compute by computing the limit of these scalar functions or this function , and , as t approaches to a. So, naturally this limit exists, provided these all limits exist.

So, and about the continuity, so a vector valued function this is continuous at , if and only if each of its component function is continuous at . So, here if these , and these are continuous, then the given vector function will be also continuous. So for instance, if you want to discuss the continuity of this function, which is described by this .

So, what we observe here that each of its component whether it is t,1 or , they are continuous for all values of t. So, in that case, this function, the given function is continuous for all t in . If you want to discuss the continuity of this function which is given by , the second component is t in the third component is this . So, in this case we have a slightly different situation because this logarithmic function is defined only for positive values of t.

And there is another problem in this component which is not defined at t is equal to 2, so, we cannot discuss the continuity at t is equal to 2 and also for the negative values of t. Hence this function is continuous for all t except, I mean all t positive, except t is equal to 2.

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Coming back to the differentiability, a very important concept in vector calculus. So, the differentiability, the function is said to be differentiable, (the definition is parallel to what we have for the scalar functions) if this exists, then we call that the function is differentiable.

So, similar to the limit evaluation, differentiation of vector valued function can also be done component wise. That means, we can have the derivative of this vector function as the derivative of this x component plus this derivative of y component and derivative of the z component in this form . Coming back to the geometric interpretation, we have a curve which is described by this vector function .

So, this is the position vector of a point here at t and then this is the position vector at . So, this here vector will be the difference of the 2 vectors, that means the vector evaluated at and minus this. So, if we divide this difference by , the direction will not change, only the magnitude will change because is a scalar quantity, so, we can divide here by .

And then we are looking what will happen when this approaches to 0. So, naturally this line will approach to this point and it will become a tangent at this point here, which was described by this . So, this derivative here is exactly gives us the tangent, the equation of the tangent we can also get, but this is the tangent vector which we have denoted here.

And the direction of this tangent vector will be again in the direction of increasing values of t, because this was and this was . So when we have an increment here in , we are moving to this direction and the direction is given exactly by this one. But when approaches to 0, so this will become the tangent vector. So with the help of this the derivative, we can easily get what is the unit tangent vector.

So that means we can divide by this magnitude of this vector to get a unit tangent vector of a curve at a point P. So, that formula can be used to get the unit tangent vector for a given curve using just this derivative of the vector function.

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So, a nice application, which we already know, is how to find the arc length of a curve. So in this vector setting, we will see how this formula looks like. So let this curve be given by this vector function , where we have 3 components there, the , and z, this t again varies from a to b. So if we recall from the integral calculus, the parametric equation of the corresponding curve can be given by like x is equal to , y is equal to y and z is equal to z.

And the formula for the calculation of the length of this curve was given by this . Now in this vector setting what we should note here that if this curve is given by this equation, , then its magnitude can be evaluated by .

And this is precisely the integrant of this formula, which is used for the calculation of the arc length of a curve. So, in the vector setting, we can replace this formula by this formula that we can integrate from a to b, the magnitude of this , which is the length of the tangent vector, because is the tangent vector and its magnitude will be the length of the tangent vector. So if we integrate this length of the tangent vector over the given domain then we can get the arc length of a curve.

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Now, we will look into that how to get the tangent, the equation of the tangent of a curve at a point P. So, for instance, this is the curve given by this arc here. And then we are interested here to find the equation of this tangent line at this point P. We know how to get the tangent vector, but now we want to get the equation of this tangent line. So, suppose this is the vector here, the position vector of this point P.

And we take another general vector, this , that precisely will give us the equation of the tangent. So, we have vector at this and then, so this one P to this distance here, we can have this times the tangent vector, because this was the tangent vector. And we have multiplied here by to adjust the length accordingly, so that it covers from P to this general point.

So, there will be here, such that this and vector will become exactly this vector. And then if we just see the setting, that this vector will be . So, we get this equation of a point here, on this tangent line. So for different values of , we will be moving on this line. So that is precisely the equation of the tangent line. For instance, if we take the function here, , this will define the parabola because this is the t and then y component is.

So this is the equation of the parabola. So the tangent vector we can get just by differentiating this. So we have 1 here the derivative of this first component t and then , will give us . So we have the r derivative . So if we plot this, this is the equation for the parabola given by this, , vector r. And then at 2, if you want to get the tangent vector at this point 2 or the equation of the tangent line we want to get at this point t equal to 2.

So this is the position vector for this 2. And then we can also draw from this , the tangent vector at this point. And now to get the equation we have to just add the 2. So, we have the evaluated at 2 plus this , which is a real number times this again evaluated at 2. So, we can just evaluate the given vector at 2, which will give us here plus this and this prime again evaluated at 2. So, this is the equation of the tangent line, taking different values of we will be moving on the tangent line at this point 2.

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Now we will define that what is a gradient of a scalar function. So these scalar functions are nothing but the function of several variables which were studied in calculus. So let this be a function of x, y, z, such that it its derivative the partial derivatives exist. In that case the gradient of which is denoted by this is a vector quantity, which is defined by this expression .

So, gradient of , gradient of a scalar function is given by partial derivative of x in the ith component, then partial derivative of y and then partial derivative of in the direction of z axis. So, we have, this is a vector function again which we have just studied, so the gradient f is a vector function. And with the help of this or Del operator we can again define this for our convenience.

So, this del operator is defined as the partial . If we define this del by this vector operator, then this can be written as . So, this grad f will be this operated on . So, we will get exactly the defined here. So, this will be a convenient in future calculations to use this as this .

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Now, we will come to this, how to get the equation of the tangent plane and the normal line to a surface. So, suppose the surface S is given by is equal to . So, we can define a function here, the scalar function taking the difference, so can bring this z to the other side. So we have and note that the given surface here z is equal to this can be treated as the level surface, as the level surface of .

So, note that the level surface of a function is equal to, , are given by is equal to there putting just some constant there. So, if we take this constant precisely at the 0 here, then we will get the given equation of the surface. So, these level surfaces or level curves, we will also mention later, these are very useful for representing for instance the functions which have 3 variables.

So, if we have the W is equal to a function of x, y, z then the representation in a plane will be very difficult. So, with the help of these levels surfaces by putting some constant that means that we are drawing now for fixing the value here the C, the constant and then we are drawing this curve. So, that can be represented by surface which is easy to plot in a 3 dimensional space.

So, here the given surface we can also write down in this form . Now, moving further, so, we are having this for instance, if we take this function is equal to , in that case the level surfaces. So, by just putting this value equal to some constant, we will get the spheres which are centred at the origin.

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So, let this is a point on a surface and C be a curve which passes through this point and it lies on the surface. So, c be a curve on S, so the curve completely lies on the surface and it passes through the given point p. So, we will now find out that how to get the normal to the tangent at this point P and later on the equation of the tangent plane. So, the equation of the curve can be given by this vector valued function.

So, this is the curve given which is defined by this vector valued function, the equation of the surface we can take as a more general putting this c. So, , is the equation of the surface. So, the curve lies on the surface that means, these point for any as long as we are on the surface this curve lies on the surface, this will satisfy the given equation.

That means, this if we substitute these from this curve, this will satisfy the given equation for all t. So, for more genera l setting we can instead of 0 we can also work with c. So well, we have now this one we can differentiate both the sides. So taking the derivative left hand side, taking the derivative right hand side, so right hand side will become 0, so whether it is 0 or constant, the right hand side will be 0. Now we can apply the chain rule here.

So the chain rule says that the partial derivative of at x and then the derivative of x with respect to t, so this is , then we have , and , equal to 0 . So at this point or at any other point also it is a general point here, we have the setting here. So this expression we can also write in terms of this and the derivative of . So what is the derivative of ? The derivative of was .

So this was the , and then its dot product with this , which was defined as . So this is the gradient, if it is dot product will exactly give this equation . So we have written this equation in this form which is the and the dot product with this tangent vector .

So what is the situation now, that we have this which is the tangent vector and with this , the dot product is 0. That means this is perpendicular to this tangent vector. So, exactly this is the point here ,i.e., through this point P you take any curve and then this will point in the direction which is perpendicular to the tangent at this point. So, more precisely that this will be the normal to the tangent plane.

So, this can be used now. So if you want to find the normal vector to a surface , then we can just get this and divide by its magnitude.

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Getting through the equation of the tangent plane we can again consider that this is a surface and then at P we have computed this which points out in the direction of the normal to this tangent plane. So, we consider a general point here Q on the tangent plane now, and suppose this P has a position vector which is given by this and then we have a Q we have taken a general point this Q there, which can be given by this .

And now this difference of the two, we can get by this vector which will lie on the tangent plane because we have taken the Q also on the tangent plane and P is also a point on the tangent plane, so this PQ will be on the tangent plane. So, the second consideration here, so having this on the tangent plane what we realize that this line and also this which is normal to the tangent plane, so, the dot product of the two should be zero.

So, this is the line here and then the line vector this PQ and then we have the perpendicular which is . So, the dot product will give us 0. So, in that case, if you just put this dot product there, that means with this partial derivative plus this partial derivative and so on equals to zero, this is exactly the equation of the tangent plane because this Q was the general point on the plane now, which can be described by this formula, which is the equation of the tangent plane.

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Okay, so now we go through some examples. So find the unit normal to this surface here, we have given at this point (1, 1, 2). So note that this is like . So it is an equation of such figure here paraboloid. So we have the . If we get the gradient of this that means , so this is given by this at (1, 1, 2), we can also compute this so this vector will be .

So, we can find the unit normal vector and which we can divide by its length. So, this is the unit normal vector at the point (1, 1, 2). So, if we consider the situation here like one in the direction of x and y. So, there is some point this (1, 1, 2) somewhere there, and then at this we have this normal vector the equation is given by this one. So, we have the unit normal vector on this surface and this is in the outward direction.

So, there will be another normal which will be in the inner direction of this figure. So, we can get both the normals by just computing this , one will be pointing out in the outward direction, the other one will be pointing out in the inward direction. So, we have the other normal vector which can be just given by the . So putting we have this vector there. Okay.

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So, these are the references we have used for preparing this lecture. And to conclude this, so, we have gone through the vector valued functions, so that was a new concept which was not covered in the calculus. And second, the most important that , just getting the derivative of this, this is the tangent vector to the curve given by this . And the , another important concept we have covered which is defined by this expression, and most importantly, that this gives us the normal vector to the surface . So, thank you very much for your attention.