HW-03 ROSHAN MAKHIJANI

Q1a) To prove: i) $p(a,b) \neq p(a)p(b)$

ii) $p(a,b|c) = p(a|c)\,p(b|c)$

i) $p(a=1) = 0.192 + 0.064 + 0.048 + 0.096 = 0.4$

$p(a=0) = 1 - p(a=1) = 0.6$.

$p(b=1) = 0.048 + 0.216 + 0.048 + 0.096 = 0.408$

$p(b=0) = 1 - p(b=1) = 0.592$

$p(a=0)\,p(b=0) = 0.6 \times 0.592 = 0.3552 \longrightarrow \mathrm{I}$.

$p(a=0)\,p(b=1) = 0.6 \times 0.408 = 0.2448 \longrightarrow \mathrm{II}$

$p(a=1)\,p(b=0) = 0.4 \times 0.592 = 0.2368 \longrightarrow \mathrm{III}$

$p(a=1)\,p(b=1) = 0.4 \times 0.408 = 0.1632 \longrightarrow \mathrm{IV}$

$p(a=0, b=0) = 0.192 + 0.144$

$= 0.336 \longrightarrow \mathrm{V}$

As we see comparing I and V

$p(a=0, b=0) \neq p(a=0)\,p(b=0)$

Similarly

$p(a=0, b=1) \neq p(a=0)\,p(b=1)$

$p(a=1, b=0) \neq p(a=1)\,p(b=0)$

$p(a=1, b=1) \neq p(a=1)\,p(b=1)$

$\therefore$ a and b are not independent.

Q1) a)(ii) Following are the tables.

a	b	c	P(a,b,c)	P(a,b|c)
0	0	0	0.192	0.4
0	1	0	0.048	0.1
1	0	0	0.192	0.4
1	1	0	0.048	0.1
0	0	1	0.144	0.2769
0	1	1	0.216	0.4153
1	0	1	0.064	0.1231
1	1	1	0.096	0.1847

a	c	P(a|c)
0	0	0.5
1	0	0.5
0	1	0.6923
1	1	0.3077

b	c	P(b|c)
0	0	0.8
1	0	0.2
0	1	0.4
1	1	0.6

i) $P(a=0, b=0 \mid c=0) = 0.4$

$P(a=0 \mid c=0) = 0.5$ $\quad P(b=0 \mid c=0) = 0.8$ $\quad (0.5 \times 0.8) = 0.4$

$\therefore P(ab \mid c) = P(a \mid c) P(b \mid c)$

ii) $P(a=0, b=1 \mid c=0) = 0.1$

$P(a=0 \mid c=0) = 0.5$ $\quad P(b=1 \mid c=0) = 0.2$ $\quad (0.5 \times 0.2) = 0.1$

$\therefore P(ab \mid c) = P(a \mid c) P(b \mid c)$

iii) $P(a=1, b=0 \mid c=0) = 0.4$

$P(a=1 \mid c=0) = 0.5$ $\quad P(b=0 \mid c=0) = 0.8$ $\quad (0.5 \times 0.8) = 0.4$

$\therefore P(ab \mid c) = P(a \mid c) P(b \mid c)$

iv) $P(a=1, b=1 | c=0) = 0.1$

$P(a=1|c=0) = 0.5 \quad P(b=1|c=0) = 0.2 \quad (0.5 \times 0.2) = 0.1$

$\therefore P(ab|c) = P(a|c)P(b|c)$

v) $P(a=0, b=0 | c=1) = 0.2769$

$P(a=0|c=1) = 0.6923 \quad P(b=0|c=1) = 0.4 \quad (0.6923 \times 0.4) = 0.2769$

$\therefore P(ab|c) = P(a|c)P(b|c)$

vi) $P(a=0, b=1 | c=1) = 0.4153$.

$P(a=0|c=1) = 0.6923 \quad P(b=1|c=1) = 0.6 \quad (0.6923 \times 0.6) = 0.4153$

$\therefore P(ab|c) = P(a|c)P(b|c)$

vii) $P(a=1, b=0 | c=1) = 0.1231$

$P(a=1|c=1) = 0.3077 \quad P(b=0|c=1) = 0.4 \quad (0.3077 \times 0.4) = 0.1231$

$\therefore P(ab|c) = P(a|c)P(b|c)$

viii) $P(a=1, b=1 | c=1) = 0.1847$

$P(a=1|c=1) = 0.3077 \quad P(b=1|c=1) = 0.6 \quad (0.3077 \times 0.6) = 0.18462$

(approximately equal due to decimals)

$\therefore P(ab|c) \approx P(a|c)P(b|c)$

Hence all cases the equation $P(ab|c) = P(a|c)P(b|c)$ holds true. $\therefore$ a and b are independent given c.

Q1 b) From Q1) a) $p(a=1) = 0.4$ $p(a=0) = 0.6$.

b	c	b|c
0	0	0.8
0	1	0.4
1	0	0.2
1	1	0.6

a	c	a|c
0	0	0.4
0	1	0.6
1	0	~~0.4~~ 0.6
1	1	~~0.6~~ 0.4

$p(a)\,p(b|c)\,p(c|a)$ for all 8 cases is as follows.

→ when $a=0, b=0, c=0$

From the tables above

$0.4 \times 0.8 \times 0.4 = 0.192$

→ when $a=0, b=0, c=1$

$0.6 \times 0.6 \times 0.4 = 0.144$

→ when $a=0$ $b=1$ $c=0$

$0.6 \times 0.4 \times 0.2 = 0.048$

→ when $a=0$ $b=1$ $c=1$

$0.6 \times 0.6 \times 0.6 = 0.216$.

→ when $a=1$ $b=0$ $c=0$

$0.4 \times 0.6 \times 0.8 = 0.192$

→ when $a=1$ $b=0$ $c=1$

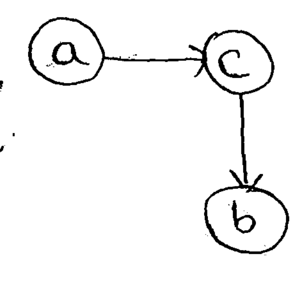
$0.4 \times 0.4 \times 0.4 = 0.064$

→ when $a=1$ $b=1$ $c=0$ $= 0.048$

~~0.1x 0.6x~~ $0.4 \times 0.6 \times 0.2$

→ when $a=1$ $b=1$ $c=1$

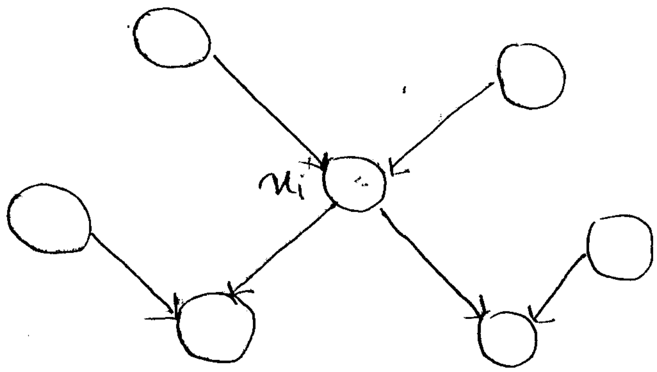
$0.4 \times 0.4 \times 0.6 = 0.096$.



Bayes Network.

Comparing these computed values with the corresponding values in the joint distribution table we see that $p(a,b,c) = p(a)\,(p(b|c))\,p(c|a)$

Q2)



Markov Blanket

In order to apply d-separation lets consider all paths from given node to all possible nodes outside the Markov blanket

Following are the possibilities

i) Paths via parent nodes. It will have either

$\bigcirc \rightarrow \bigcirc \rightarrow \bigcirc x_i$ or $\bigcirc \leftarrow \bigcirc \rightarrow \bigcirc x_i$

and by d-separation given the parent node these 2 nodes are independent

ii) Path via child nodes; that do not have co-parents (siblings)

$x_i \bigcirc \rightarrow \bigcirc \rightarrow \bigcirc$ Again by d-separation, this is allowed and characteristic of independence given the child node

iii) Passes through child-node and sibling

a) for child node it will be as above $\bigcirc \rightarrow \bigcirc \rightarrow \bigcirc$ by d-separation this is allowed

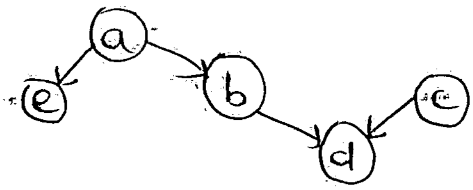
b) for sibling

$\bigcirc \rightarrow \bigcirc \rightarrow \bigcirc$ or $\bigcirc \leftarrow \bigcirc \rightarrow \bigcirc$

again by d-separation they are independant given, the child and the sibling.

∴ all possible paths leaving node x_i will be blocked by Markov blanket and so the distribution of x_i, conditioned on variables in the Markov blanket, will be independent of all of the remaining variables in the graph.

Q3 a)



b and c are independent of each other in the given network.

b) In the given network following are conditionally independent of each other given a third variable.

i) $(e \perp b \mid a)$

ii) $(a \perp d \mid b)$

Q4 a) Given $P(a)$, $P(b|a)$, $P(c|b)$

$P(a,b,c) = P(b|ac)\,P(ac)$

$= P(ac|b)\,P(b)$

Now from given N/W and applying d-separation

$\therefore (a \perp c)|b)$

$\therefore P(ac|b)\,P(b)$

$= P(a|b)\,P(c|b)\,P(b)$

$= P(a|b)\,p(b)\,P(c|b)$

Also $P(a|b)P(b) = P(a)\,P(b|a)$

$\therefore P(a|b)\,P(b)\,P(c|b)$

$= P(a)P(b|a)P(c|b) = P(a,b,c)$

Q4) b) For a and c to be independent

$P(c|a) = P(c)$

Now $P(c|a) = \dfrac{P(a)\,P(b|a)\,P(c|b)}{P(b|ca)}$

$P(c) = \dfrac{P(a)\,P(b|a)\,P(c|b)}{P(a|bc)\,P(b|c)}$

To prove independence the 2 equations should be equal

i.e $P(b|ca) = P(a|bc)\,P(b|c)$

these 2 are not equal

$\therefore$ c and are not independent

Q4c) ~~P(b|a~~, $P(c|a,b) = \dfrac{P(a,b,c)}{P(a,b)} = \dfrac{\cancel{P(a)}\,\cancel{P(b|a)}\,P(c|b)}{\cancel{P(a)}\,\cancel{P(b|a)}} = P(c|b)$

$\therefore$ a ~~c~~ is independent of ~~@~~ c given b.

Q5) a) Given:- $P(a), p(c), P(b|ac)$

Joint probability is from the network graph and d-separation.

$$P(a,b,c) = P(b|ac)\,P(a|c)\,P(c)$$

Now a and c are independent

$$\therefore \quad P(a,b,c) = P(b|ac)\,P(a)\,P(c)$$

Q5) b) In order to prove independence

To prove: $P(c|a)$ should be equal to $P(a)$

$$P(a,b,c) = P(b|ca)\,P(c|a)\,P(a)$$

$$P(c|a) = \frac{P(a,b,c)}{P(b|ca)\,P(a)}$$

Now from (a) $P(a,b,c) = P(a)P(c)P(b|ac)$

$$\therefore P(c|a) = \frac{\cancel{P(a)}\,P(c)\,\cancel{P(b|ac)}}{\cancel{P(b|ca)}\,\cancel{P(a)}} = P(c)$$

$\therefore$ a and c are independent of each other.

Q5) c) To prove $P(c|ab) = P(c|b)$

$$P(a,b,c) = P(c|ab)\,P(a|b)\,P(b)$$

$$\therefore P(c|ab) = \frac{P(a,b,c)}{P(a|b)P(b)} = \frac{P(a)P(c)P(b|ac)}{P(a|b)P(b)} \neq P(c|b)$$

$\therefore$ given b, a and c cease to be independent.

Q6) a) Only (ii) and (iii) are asserted by the graph.

For (iii) we have to consider the Markov blanket of M

b) $P(b, i, \neg m, g, j) =$

$$P(b)\,P(\neg m)\,P(i \mid b, \neg m)\,P(g \mid b, i, \neg m)\,P(j \mid g)$$

$$= 0.9 \times 0.9 \times 0.5 \times 0.8 \times 0.9 = \cancel{0.2196}\ 0.2916.$$

c) $P(J \mid b, i, m)$ as B, I, M are given and fixed also we can treat G as having a prior of 0.9

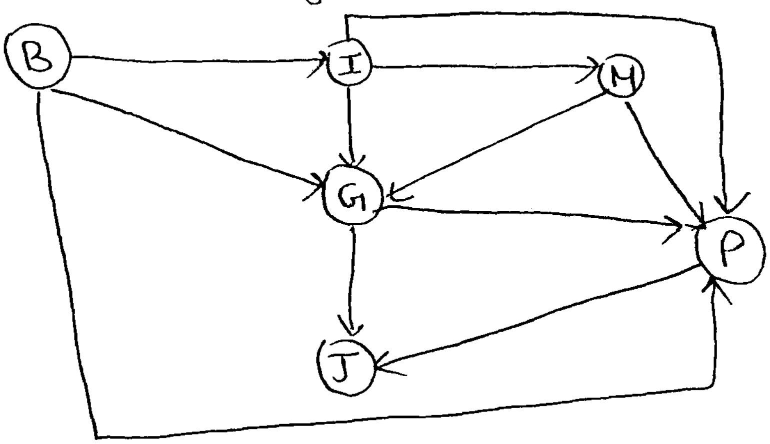
$$P(J \mid b, i, m) = \alpha \sum_g P(J, g) = \alpha\{P(J, g) + P(J, \neg g)\}$$

$$= \alpha[\{P(j, g) + P(\neg j, g)\} + \{P(j, \neg g) + P(\neg j, \neg g)\}].$$

$$= \alpha[(0.81, 0.09) + (0, 0.1)] = [0.81, 0.19]$$

So the probability of going to jail is 0.81.

e)



Presidential pardon is dependant on whether or not the person is guilty and whether or not he is indicted. Also if he broke the law or not and whether he faced a politically motivated prosecutor matter in P. ∴ all these are parents of P. Whether he/she goes to jail is dependent on whether or not the accused gets a presidential pardon. ∴ J is a child of P.

Q7) a) Lets assume q_1 and q_2 are in detailed balance

$$\therefore \pi(x)\left(\alpha q_1(x \to x') + (1-\alpha) q_2(x \to x')\right)$$

$$= \alpha \pi(x) q_1(x \to x') + (1-\alpha)\pi(x) q_2(x \to x')$$

$$= \alpha \pi(x) q_1(x' \to x) + (1-\alpha)\pi(x) q_2(x' \to x)$$

$$= \pi(x)\left(\alpha q_1(x' \to x) + (1-\alpha) q_2(x' \to x)\right)$$

b) $$(q_1 \circ q_2)(x \to x') = \sum_{x''} q_1(x \to x'') q_2(x'' \to x')$$

If q_1 and q_2 both have π as their stationary distribution

$$\sum_x \pi(x)(q_1 \circ q_2)(x \to x') = \sum_x \pi(x) \sum_{x''} q_1(x \to x'') q_2(x'' \to x')$$

$$= \sum_{x''} q_2(x'' \to x') \sum_x \pi(x) q_1(x \to x'')$$

$$= \sum_{x''} q_2(x'' \to x') \pi(x'')$$

$$= \pi(x')$$