10

Asymptotes

10·1 Definition

A straight line at a fixed distance from the origin, is said to be an *asymptote to* an infinite branch of a curve, if the perpendicular distance of a point P on the curve from this straight line approaches zero, as the point P moves to infinity along the curve.

10·2 Determination of Asymptotes

The equation of a straight line not parallel to y-axis is of the form,

$$y=mx+c \qquad \text{...(1)}$$

Excluding at present the case of asymptotes parallel to y-axis, it is obvious from (1) that as x approaches infinity, m and c must both tend to finite limits for asymptotes to exist. Let p be the perpendicular distance of any point $P(x, y)$ on an infinite branch of a given curve from the line (1), then

$$p=\frac{y-mx-c}{\sqrt{1+m^2}}$$

If line (1) is to be an asymptote to a given curve, than as $x \to \infty$, $p \to 0$

$$\therefore \quad \underset{x\to\infty}{\text{Lt}} (y-mx-c)=0$$

or

$$\underset{x\to\infty}{\text{Lt}} (y-mx)=c \qquad \text{...(2)}$$

Also from (1), we have

$$\frac{y}{x}=m+\frac{c}{x}$$

Taking limits on both sides as $x \to \infty$, we have

$$\underset{x\to\infty}{\text{Lt}} \left(\frac{y}{x}\right)=\underset{x\to\infty}{\text{Lt}} \left(m+\frac{c}{x}\right)=m$$

$$\therefore \quad m=\underset{x\to\infty}{\text{Lt}} \left(\frac{y}{x}\right) \qquad \text{...(3)}$$

Thus from (3) and (2), we get the values of m and c and hence the equation of the asymptote.

Note. A given curve may have more than one infinite branches then it is possible that each branch may have separate asymptotes. Hence a given curve may have more than one asymptote.

Example 1. *Find the asymptotes of the curve*

$$x^3+y^3=3ax^2.$$

Sol. Here the equation of the given curve is

$$x^3+y^3=3ax^2 \quad ..(1)$$

In order to determine the asymptotes, we have to evaluate m and c given by $\underset{x\to\infty}{\text{Lt}}\ (y/x)$ and $\underset{x\to\infty}{\text{Lt}}\ (y-mx)$ respectively, then $y=mx+c$ will be the asymptote.

Dividing (1) by x^3, we have

$$1+\left(\frac{y}{x}\right)^3-\frac{3a}{x}=0$$

Taking limits as $x\to\infty$, we have

$$1+m^3=0 \qquad \left[\because \underset{x\to\infty}{\text{Lt}}\ (y/x)=m\right.$$

$$=(1+m)(1+m^2-m)=0$$

$\therefore$ $m=-1$, as the roots of $1+m^2-m=0$ are not real.

Now $c=\underset{x\to\infty}{\text{Lt}}\ (y-mx)$

$$=\underset{x\to\infty}{\text{Lt}}\ (y+x) \qquad [\because m=-1]$$

Let $y+x=K$, such that as $x\to\infty$, $K\to c$

Putting $y=(K-x)$ in (1), we have

$$x^3+(K-x)^3=3ax^2$$

or
$$3(K-a)x^2-3K^2x+K^3=0$$

Dividing throughout by x^2, we get

$$3(K-a)-\frac{3K^2}{x}+\frac{K^3}{x^2}=0$$

Taking limits as $x\to\infty$ and $K\to c$

$$3(c-a)=0$$

or
$$c=a$$

The asymptote to the curve is given by

$$y=mx+c$$

i.e.
$$y=-x+a$$

or
$$y+x=a$$

is the required asymptote.

Note. The method used to determine asymptotes in the above example is not convenient. The following methods are much easier and quicker to obtain the asymptotes.

10·3. The Asymptotes of the general Algebraic Curve

Let the equation to the curve be

$$(a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+\ldots+a_ny^n)+(b_0x^{n-1}+b_1x^{n-2}y$$
$$+b_2x^{n-3}y^2+\ldots+b_ny^{n-1})+(c_0x^{n-2}+c_1x^{n-3}y$$
$$+c_2x^{n-4}y^2+\ldots+c_ny^{n-2})+\ldots\ldots=0 \quad \ldots(1)$$

Further let $y=mx+c$ be an asymptote to the curve (1)

Since each expression in the brackets is homogeneous, (1) may be written as

$$x^n\phi_n\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)+x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right)+\ldots=0 \quad \ldots(2)$$

where $\phi_r\left(\frac{y}{x}\right)$ is an expression of rth degree in $\frac{y}{x}$

Dividing (2) by x^n, we get

$$\phi_n\left(\frac{y}{x}\right)+\frac{1}{x}\phi_{n-1}\left(\frac{y}{x}\right)+\frac{1}{x^2}\phi_{n-2}\left(\frac{y}{x}\right)+\ldots=0 \quad \ldots(3)$$

Now taking limits as $x\to\infty$

and $\qquad \underset{x\to\infty}{\mathrm{Lt}}\left(\frac{y}{x}\right)=m$, we have

$$\phi_n(m)=0, \quad \ldots(4)$$

which determines the slopes of the asymptotes.

Substituting $y=mx+c$ in (2), we get

$$x^n\phi_n\left(m+\frac{c}{x}\right)+x^{n-1}\phi_{n-1}\left(m+\frac{c}{x}\right)$$
$$+x^{n-2}\phi_{n-2}\left(m+\frac{c}{x}\right)+\ldots=0$$

Expanding $\phi_n\left(m+\frac{c}{x}\right)$, $\phi_{n-1}\left(m+\frac{c}{x}\right)$ etc. by Taylor's theorem, we have

$$x^n\left[\phi_n(m)+\frac{c}{x}\phi_n'(m)+\frac{1}{2!}\frac{c^2}{x^2}\phi_n''(m)+\frac{1}{3!}\frac{c^3}{x^3}\phi_n'''(m)+\ldots\right.$$
$$+x^{n-1}\left[\phi_{n-1}(m)+\frac{c}{x}\phi'_{n-1}(m)+\frac{c^2}{2!\,x^2}\phi''_{n-1}(m)+\ldots\right]$$
$$+x^{n-2}\left[\phi_{n-2}(m)+\frac{c}{x}\phi'_{n-2}(m)+\frac{c^2}{2!\,x^2}\phi''_{n-2}(m)+\ldots\right]$$
$$+\ldots=0$$

Arranging the terms in descending powers of x,

$$x^n\phi_n(m)+x^{n-1}[c\phi_n'(m)+\phi_{(n-1)}(m)]$$
$$+x^{n-2}\left[\frac{c^2}{2!}\phi_n''(m)+c\phi'_{(n-1)}m+\phi_{(n-2)}(m)\right]$$

$$+x^{n-3}\left[\frac{c^3}{3\,!}\phi_n'''(m)+\frac{c^2}{2\,!}\phi''_{n-1}(m)+c\phi'_{n-2}(m)\right.$$
$$\left.+\phi_{n-3}(m)\right]+\ldots=0$$

Putting $\phi_n(m)=0$ by (4) and dividing by x^{n-1}, we get

$$[c\phi_n'(m)+\phi_{(n-1)}m]+\frac{1}{x}\left[\frac{c^2}{2\,!}\phi_n''(m)+c\phi'_{(n-1)}(m)+\phi_{n-2}(m)\right]$$
$$+\frac{1}{x^2}\left[\frac{c^3}{3\,!}\phi_n'''(m)+\frac{c^2}{2\,!}\phi''_{n-1}(m)+c\phi'_{n-2}(m)+\phi_{n-3}(m)\right]$$
$$+\ldots\ldots=0. \qquad \ldots(5)$$

Taking limits as $x\to\infty$, we have

$$c\phi'_n(m)+\phi_{n-1}(m)=0$$

or
$$c=-\frac{\phi_{n-1}(m)}{\phi_n'(m)}\ [\text{provided } \phi'_n(m)\neq 0] \qquad \ldots(6)$$

Now equation (4) is of the nth degree in m, giving n values of m say $m_1, m_2, \ldots, m_n$. The corresponding values of c say $c_1, c_2 \ldots, c_n$ are given by the equation (6). Hence the asymptotes are

$$y=m_1x+c_1,$$
$$y=m_2x+c_2, \text{ etc.}$$

10·4. Parallel Asymptotes

In case $\phi_n(m)=0$ has two equal roots say $m_1=m_2$, then $\phi'_n(m_1)$ and $\phi_{n-1}(m_1)$ both become zero.

Now
$$c=-\frac{\phi_{n-1}(m_1)}{\phi_n'(m_1)}.$$

Thus c takes the indeterminate form 0/0.

In this case c is obtained from equation (5) of the Art. 10·3 as

$$\frac{c^2}{2\,!}\phi_n''(m)+c\phi'_{n-1}(m)+\phi_{n-2}(m)=0 \qquad [\phi'_n(m)=\phi_{n-1}(m)=0]$$

Thus we obtain two values of c say c_1, c_2 corresponding to $m=m_1$ (a repeated root).

Hence the asymptotes are $y=m_1x+c_1$. $y=m_1x+c_2$, *i.e.* parallel asymptotes.

Similarly in case there are three parallel asymptotes, c is obtained by the equation

$$\frac{c^3}{3\,!}\phi_n'''(m)+\frac{c^2}{2\,!}\phi''_{n-1}(m)+c\phi'_{n-2}(m)+\phi_{n-3}(m)=0.$$

This being cubic in c will give three values of c, correspond-ng to three repeated values of m.

10·5. Working Rule

(1) Substitute $y=mx+c$ in the equation of the curve.
(2) Equate the coefficients of two highest powers of x to zero.
(3) These give m and c and hence the asymptotes.

Example 1. *Find the asymptotes of the curve,*

$$x^3+2x^2y-xy^2-2y^3+xy-y^2-1=0.$$

Sol. Putting $y=mx+c$ in the given equation, we have

$$x^3+2x^2(mx+c)-x(mx+c)^2-2(mx+c)^3$$
$$+x(mx+c)-(mx+c)^2-1=0$$

or $$(1+2m-m^2-2m^3)x^3+(2c-2mc-6m^2c+m-m^2)x^2+\ldots=0$$

Equating co-efficients of x^3 and x^2 to zero, we get

$$1+2m-m^2-2m^3=0=(1+2m)(m+1)(m-1) \quad \ldots(1)$$

and $$2c-2mc-6m^2c+m-m^2=0$$

or $$c=\frac{m^2-m}{2-2m-6m^2} \quad \ldots(2)$$

From (1), we have $m=-\frac{1}{2}, -1, 1$

From (2), we get

(*i*) when $m=-\frac{1}{2}$, $c=\frac{1}{2}$,
(*ii*) when $m=-1$, $c=-1$, and
(*iii*) when $m=1$, $c=0$.

Hence asymptotes, are

$$y=-\frac{1}{2}x+\frac{1}{2},$$

$$y=-x-1 \quad \text{and} \quad y=x$$

or $$2y+x=1,$$
$$y+x+1=0 \quad \text{and} \quad y=x.$$

10·6. Shorter Method

The polynomials $\phi_n(m)$, $\phi_{n-1}(m)$, $\phi_{n-2}(m)$ etc. can be easily obtained by putting $x=1$ and $y=m$ in the nth degree, $(n-1)$th degree, $(n-2)$th degree, etc. terms respectively. The following examples show that this method gives asymptotes much quicker.

Example 1. *Find the asymptotes of the curve*

$$y^3-x^2y-2xy^2+2x^3-7xy+3y^2+2x^2+2x+2y+1=0$$

[*A.M.I.E. 1974, 75*]

Sol. Putting $x=1$ and $y=m$ in the third and second degree terms we get

$$\phi_3(m)=m^3-m-2m^2+2$$

and $$\phi_2(m)=-7m+3m^2+2$$

The slopes of asymptotes are given by

$$\phi_3(m)=m^3-m-2m^2+2=0$$

$\therefore$

or $$(m+1)(m-1)(m-2)=0$$

Now $$m=-1, 1, 2.$$

$$\phi_3'(m)=3m^2-4m-1$$

Again $$c=-\frac{\phi_2(m)}{\phi_3'(m)}$$

or $$c=-\frac{-7m+3m^2+2}{3m^2-4m-1}$$

(*i*) when $m=-1$, $c=-2$,

(*ii*) when $m=1$, $c=-1$ and

(*iii*) when $m=2$, $c=0$

Hence the asymptotes are

$$y=-x-2, \quad y=x-1 \quad \text{and} \quad y=2x$$

or $y+x+2=0$, $\quad y+1=x$ and $y=2x$.

Example 2. *Find the asymptotes of the curve*

$$x^3+4x^2y+5xy^2+2y^3+2x^2+4xy+2y^2-x-9y+2=0$$

Sol. Putting $x=1$ and $y=m$ in the third degree terms, we get

$$\phi_3(m)=1+4m+5m^2+2m^3$$

Also $$\phi_2(m)=2+4m+2m^2$$

$$\phi_1(m)=-1-9m.$$

The slopes of asymptotes are given by

$$\phi_3(m) \quad 1+4m+5m^2+2m^3=0$$

or $$(2m+1)(m+1)^2=0$$

$\therefore$ $$m=-\tfrac{1}{2},\ m=-1,\ -1$$

Now when $m=-\frac{1}{2}$

$$c=-\frac{\phi_2(m)}{\phi_3'(m)}=-\frac{2+4m+2m^2}{4+10m+6m^2}=-1$$

Hence the asymptote corresponding to $m=-\frac{1}{2}$ is

$$y=-\tfrac{1}{2}x-1 \quad \text{or} \quad 2y+x+2=0$$

when $m=-1$, c is obtained from the following relation

$$\frac{c^2}{2!}\phi_3''(m)+c\phi_2'(m)+\phi_1(m)=0.$$

$$\left[\because \begin{array}{l}\phi_3'(m)=0\\ \phi_2(m)=0\end{array}\right]$$

$$\frac{c^2}{2}(10+12m)+c(4+4m)+(-9m-1)=0$$

$$c^2(5-6)+c.0+(9-1)=0 \qquad [\because m=-1]$$

$$c^2=8 \qquad \therefore \qquad c=\pm2\sqrt{2}$$

Thus the asymptotes are

$$y=-x+2\sqrt{2} \quad \text{and} \quad y=-x-2\sqrt{2}$$

or $$y+x=2\sqrt{2} \quad \text{and} \quad y+x+2\sqrt{2}=0$$

Hence asymptotes are

$$2y+x+2=0, \quad y+x=2\sqrt{2} \quad \text{and} \quad y+x+2\sqrt{2}=0.$$

Example 3. *Find the asymptotes of*

$$y^4-2xy^3+2x^3y-x^4-3x^3+3x^2y+3xy^2-3y^3-2x^2+2y^2-1=0.$$

Sol. Putting $x=1$, $y=m$ in the highest degree terms, we get

$$\phi_4(m)=m^4-2m^3+2m-1$$
$$\phi_3(m)=-3+3m+3m^2-3m^3$$
$$\phi_2(m)=-2+2m^2$$
$$\phi_1(m)=0$$
$$\phi_0(m)=-1.$$

The slopes of the asymptotes are given by

$$\phi_4(m)=m^4-2m^3+2m-1=0$$

or $$(m^4-1)-2m(m^2-1)=0$$

or $$(m^2-1)(m^2+1)-2m(m^2-1)=0$$

or $$(m^2-1)(m^2-2m+1)=0$$

or $$(m-1)(m+1)(m-1)^2=0$$

or $$(m-1)^3(m+1)=0$$

or $$m=1, 1, 1, -1.$$

When $m=1$ (thrice repeated root), c is obtained from the equation

$$\frac{c^3}{3!}\phi_4'''(m)+\frac{c^2}{2!}\phi_3''(m)+c\,\phi_2'(m)+\phi_1(m)=0.$$

or $$\frac{c^3}{6}(24m-12)+\frac{c^2}{2}(6-18m)+c(4m)+0=0$$

or $$c^3(4-2)+c^2(3-9)+4c=0 \quad (\because m=1)$$

or $$2c^3-6c^2+4c=0$$

or $$2c(c^2-3c+2)=0$$

or $$2c(c-1)(c-2)=0$$

$\therefore$ $c=0, 1, 2$

$$\phi_4'(m)=4m^3-6m^2+2$$
$$\phi_4''(m)=12m^2-12m$$
$$\phi_4'''(m)=24m-12$$

Also

$$\phi_3'(m)=3+6m-9m^2$$
$$\phi_3''(m)=6-18m$$

and

$$\phi_2'(m)=4m$$

Hence asymptotes are $y=x$, $y=x+1$, $y=x+2$

For $\quad m=-1$

$$c=-\frac{\phi_3(m)}{\phi'_4(m)}$$

$$=0 \text{ (for } m=-1)$$

Hence asymptote is $y=-x$ or $y+x=0$

Thus asymptotes are

$$y=x,\ y=x+1,\ y=x+2,\ y+x=0$$

EXERCISE 10 (*a*)

Find the asymptotes of the following curves:

1. $x^2y+xy^2+xy+y^2+3x=0.$ (*D.U. 1979*)
2. $y^3-6xy^2+11x^2y-6x^3+y+x=0.$ (*D.U. 1977*)
3. $y^3-xy^2-x^2y+x^3+x^2-y^2-1=0.$ (*D.U. 1976*)
4. $x^3+2x^2y-xy^2-2y^3+xy-y^2+1=0.$ (*D.U. 1975*)
5. $x^3+2x^2y+xy^2-x^2-xy+2=0.$ (*D.U. 1974*)
6. $3x^3+2x^2y-7xy^2+2y^3-14xy+7y^2+4x+5y=0.$
7. $y(x-y)^2=x+y.$
8. $x^2y-xy^2+xy+y^2+y-1=0.$
9. $y^3+xy^3+2xy^2+y+1=0.$
10. $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

10·7 Asymptotes Parallel to the Co-ordinate Axes

So far we have been excluding the case of the asymptotes parallel to y-axis. The reason being that the slope m of such asymptotes is infinite. Now we shall determine asymptotes parallel to y-axis.

The general equation of the curve of nth degree {equation (1) of art. 10·3}

arranged in descending powers of y is

$$y^n\,\phi(x)+y^{n-1}\,\phi_1(x)+y^{n-2}\,\phi_2(x)+\ldots=0 \qquad \ldots(1)$$

where $\quad \phi(x),\ \phi_1(x),\ \phi_2(x)$ etc. are polynomials in x.

Dividing (1) by y^n, we get

$$\phi(x)+(1/y)\,\phi_1(x)+(1/y^2)\,\phi_2(x)+\ldots=0 \qquad \ldots(2)$$

Now as $y\to\infty$, let $x\to k$.

From equation (2), we get

$$\phi(k)=0$$

Thus k is a root of $\phi(x)=0$

Let $\phi(x)=0$ have the roots k_1, k_2, k_3 etc. Then the asymptotes parallel to y-axis are

$$x=k_1,\ x=k_2,\ x=k_3 \text{ etc.}$$

Thus the asymptotes parallel to y-axis are obtained by equating to zero the coefficient of highest power of y (if not a constant) in the equation of the curve. However, if the coefficient of highest power of y is a constant or has imaginary factors, no asymptotes parallel to y-axis exist.

Now asymptotes parallel to x-axis are easily determined by the previous methods when $m=0$. However, it is convenient to determine these asymptotes separately, by the following rule.

The asymptotes parallel to x-axis are easily determined by equating to zero the coefficient of highest power of x (if not a constant) in the equation of the curve. However if the coefficient of highest power of x is a constant or has imaginary factors, no asymptotes parallel to x-axis exist.

Example 1. *Find the asymptotes parallel to the coordinate axes of the curve,*

$$y^2x-a^2(x+a)=0.$$

Sol. The equation of the curve can be written as

$$(y^2-a^2)\,x-a^3=0$$

Here highest power of x is one and its coefficient is y^2-a^2. So asymptotes parallel to x-axis are $y^2-a^2=0$ or $y=\pm a$.

Also highest power of y is 2 and its co-efficient is x. So asymptote parallel to y-axis $x=0$.

Hence the asymptotes parallel to co-ordinate axes are

$$y=\pm a \text{ and } x=0$$

EXERCISE 10 (*b*)

Find the asymptotes, which are parallel to either axis, of the following curves.

1. $x^2y^2-a^2(x^2+y^2)=0$ (*D.U. 1978*)
2. $(x^2-a^2)y^2=x^2(x^2-4a^2)$
3. $y^2(a^2+x^2)=x^2$
4. $x^2y^3+x^3y^2=x^3+y^3$.
5. $\dfrac{a^2}{x^2}+\dfrac{b^2}{y^2}=1$.

10·8. Important Deductions (*From Art. 10·3*)

(*i*) The maximum number of asymptotes of an algebraic curve of *n*th degree cannot exceed n.

We have seen that the slopes of the asymptotes of the curve which are not parallel to y-axis, are given by the roots of $\phi_n(m)=0$, which is of degree n and hence can't give more than n values of m

If there are asymptotes parallel to y axis, then it is obvious that degree of $\phi_n(m)$ will be less than n by at least the same number.

(*ii*) If some of the roots of $\phi_n(m)=0$, are imaginary, the asymptotes corresponding to these roots are called imaginary asymptotes.

(*iii*) There may be cases when even corresponding to real roots of $\phi_n(m)=0$, no asymptotes exist. For example, the parabola $y^2=4ax$ has no asymptotes even though $\phi_n(m)=0$ has real roots.

10·9. Other Methods of Determining Asymptotes

(1) The asymptotes of an algebraic curve are parallel to the lines obtained by equating to zero, the linear factors of the highes degree terms in its equation.

Let $(y-ax)$ be a non-repeated factor of the nth degree terms of the equation of the curve. The equation of the curve can be written as

$$(y-ax)\,F_{n-1}+U_{n-1}=0, \qquad \ldots(1)$$

where F_{n-1} is a polynomial of degree $(n-1)$ and U_{n-1} contains terms of various degrees, not higher than $(n-1)$.

It is evident that one of the roots of $\phi_n(m)=0$ is $m=a$ and hence an asymptote of the curve is

$$y-ax=c$$

where c is to be determined.

By definition we know $\underset{x\to\infty}{\text{Lt}}\,(y-ax)=c$, where (x, y) lie on (1).

$\therefore$ From (1), we have

$$y-ax=-\frac{U_{n-1}}{F_{n-1}}.$$

Taking limits on both sides as $x\to\infty$, we have

$$\underset{x\to\infty}{\text{Lt}}\,(y-ax)=-\underset{x\to\infty}{\text{Lt}}\,U_{n-1}/F_{n-1}$$

$$\therefore \qquad c=-\underset{x\to\infty}{\text{Lt}}\,U_{n-1}/F_{n-1}$$

Hence c is determined.

(2) Let the terms of the highest degree in the equation of the curve have a repeated factor $(y-ax)^2$. Then the equation of the curve can be written as

$$(y-ax)^2\,F_{n-2}+(y-ax)U_{n-2}+V_{n-2}=0 \qquad \ldots(1)$$

where F_{n-2} and U_{n-2} contain terms of degree $(n-2)$ and V_{n-2} contains terms of all degrees none of which is higher than $(n-2)$. It is obvious that $\phi_n(m)=0$ has a repeated root $m=a$.

Hence there are asymptotes of the curve

$$y-ax=c_1$$
$$y-ax=c_2.$$

The values of c_1 and c_2 are determined by the method given above, using the fact

$$\underset{x\to\infty}{\text{Lt}}\ y/x=a$$

The following examples illustrate the use of these methods.

Example 1. *Find the asymptotes of the curve*

$(x^2-y^2)(y^2-4x^2)-6x^3+5x^2y+3xy^2-2y^3-x^2+3xy-1=0.$

Sol. The curve has no asymyptotes parallel to the coordinate axes, as the co-efficients of highest powers of x and y are both constants.

Factorizing the highest degree terms, we get

$$(x-y)(x+y)(y-2x)(y+2x)-6x^3+5x^2y+3xy^2-2y^3-x^2+3xy-1=0 \quad \ldots(1)$$

Hence the curve has asymptotes parallel to the lines

$$x-y=0, \qquad x+y=0$$
$$y-2x=0 \quad \text{and} \quad y+2x=0.$$

Now from (1), we have

$$x-y=\frac{6x^3-5x^2y-3xy^2+2y^3+x^2-3xy+1}{(x+y)(y-2x)(y+2x)}$$

$$=\frac{6-5(y/x)-3(y/x)^2+2(y/x)^3+1/x-3(y/x)\,.\,1/x+1/x^3}{(1+y/x)(y/x-2)(y/x+2)}$$

[Dividing N^r and D^r by x^3]

Now taking limits on both sides as $x\to\infty$, then

$$\underset{x\to\infty}{\text{Lt}}\ (y-x)=c$$

or
$$\underset{x\to\infty}{\text{Lt}}\ (x-y)=-c$$

and
$$\underset{x\to\infty}{\text{Lt}}\ y/x=1, \text{ we get} \qquad (\because \text{ Slope of line is 1})$$

$$-c=\frac{6-5-3+2}{(1+1)(1-2)(1+2)}$$

$\therefore \quad c=0.$

Hence the asymptote corresponding to the factor $x-y$ is

$$x-y=0 \qquad \text{or} \qquad y=x$$

Similarly we can find the value of c corresponding to other factors. These values of c are -1, 0 and -1 corresponding to the factors $(y+x)$, $y-2x$ and $y+2x$ respectively.

Hence the asymptotes are

$$y=x, \qquad y+x+1=0.$$

$$y=2x \quad \text{and} \quad y+2x+1=0.$$

Example 2. *Find the asymptotes of*

$$(x-y)(x-2y)^2+x^2-3xy+2y^2-7=0 \qquad \text{...(1)}$$

Sol. There are no asymptotes parallel to the axes as the coefficients of highest powers of x and y are constant.

The equation of the curve may be written as

$$(x-y)(x-2y)^2+(x-y)(x-2y)-7=0 \qquad \text{...(2)}$$

The curve has asymptotes parallel to the lines $x-y=0$ and $x-2y=0$. As $(x-2y)$ is a repeated factor, there will be two asymptotes parallel to the line $x-2y=0$.

Now $$x-y=\frac{7-(x-y)(x-2y)}{(x-2y)^2}$$

$$=\frac{\frac{7}{x^2}-\left(1-\frac{y}{x}\right)\left(1-\frac{2y}{x}\right)}{\left(1-\frac{2y}{x}\right)^2} \qquad \text{.. (3)}$$

[Dividing N^r and D^r by x^2]

Now taking limits as $x\to\infty$ then

$$\underset{x\to\infty}{\text{Lt}}\ (y-x)=c$$

or $$\underset{x\to\infty}{\text{Lt}}\ (x-y)=-c$$

and $$\underset{x\to\infty}{\text{Lt}}\ y/x=1, \quad \text{we get}$$

$$y-x=0$$

$\therefore$ $y=x$ is an asymptote.

Now the curve has two asymptotes parallel to the line

$$x-2y=0$$

We have to find $\underset{x\to\infty}{\text{Lt}}\ (y-\tfrac{1}{2}x)=c$

or $$\underset{x\to\infty}{\text{Lt}}\ (x-2y)=-2c$$

Dividing (2), by $(x-y)$, we have

$$(x-2y)^2+(x-2y)-\frac{7}{x(1-y/x)}=0.$$

Taking limits as $x\to\infty$,

$$(x-2y)\to-2c \text{ and } y/x=\tfrac{1}{2}$$

$\therefore \qquad 4c^2-2c=0$

$$2c(2c-1)=0$$

$\therefore \qquad c=0$

$$c=\tfrac{1}{2}.$$

Hence the asymptotes are

$$y=\tfrac{1}{2}x+0$$

and $\qquad y=\tfrac{1}{2}x+\tfrac{1}{2}.$

or $\qquad x=2y$

or $\qquad x-2y+1=0.$

Thus the asymptotes are

$$y=x, \quad x=2y, \quad x-2y+1=0.$$

Note. It may be noted that the asymptotes of this curve in these examples could also be found by the previous method of Art. 10·3 or 10·6

10·10. Asymptotes by Inspection

If the equation of a curve of nth degree can be put in the form

$$u_n+u_{n-2}=0$$

where u_{n-2} is a polynomial of degree not higher than $(n-2)$ then every linear factor of u_n, when equated to zero will give an asymptote, provided that no two straight lines so obtained are parallel or coincident.

Example 1. *Find the asymptotes of*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Sol. Here the equation of the curve can be written as

$$u_n+u_{n-2}=0$$

where $\qquad u_n=\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}.$

$$u_{n-2}=-1.$$

The asymptotes are given by

$$u_n=0$$

$\therefore \qquad \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=0$

$\therefore \qquad y=\pm\dfrac{b}{a}x$

Hence asymptotes are

$$y=\frac{b}{a}x,$$

and $\qquad y=-\dfrac{b}{a}x.$

Example 2. *Find the asymptotes of the curve*

$$y^3-6xy^2+11x^2y-6x^3+y+x=0.$$

Sol. The given equation is of the form $u_n+u_{n-2}=0$

Hence asymptotes are given by

$$y^3-6xy^2+11x^2y-6x^3=0$$

or $$(y-x)(y-2x)(y-3x)=0$$

or $$y=x, \qquad y=2x,$$

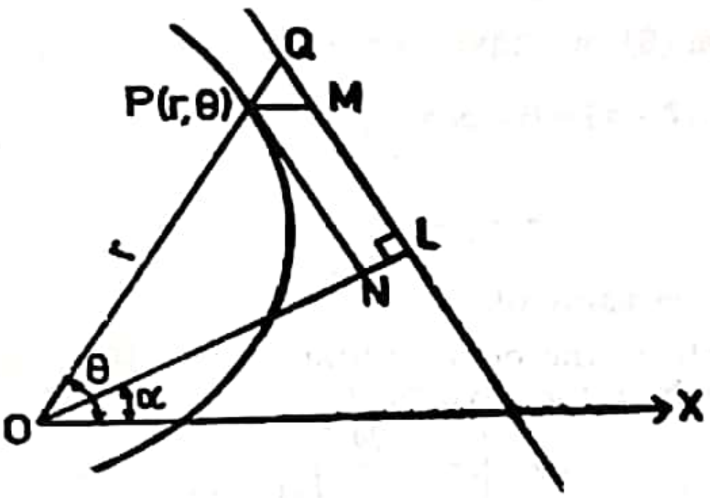
and $$y=3x.$$

10·11. Asymptotes of Polar Curves

Before proceeding with the determination of asymptotes of polar curves, we shall obtain the equation of a straight line in polar form.

Theorem 1. The polar equation of a straight line is

$$p=r\cos(\theta-\alpha)$$



where p is the perpendicular distance from the pole to the line and α the angle which this perpendicular makes with the initial line.

We know that the equation of a straight line can be written as

$$x\cos\alpha+y\sin\alpha=p \qquad ...(1)$$

where p is the length of the perpendicular from origin (pole) on the line and α the angle which this perpendicular makes with x-axis (initial line).

Putting $x=r\cos\theta$, $y=r\sin\theta$ in (1) we have the polar form of the line as

$$r(\cos\theta\cos\alpha+\sin\theta\sin\alpha)=p$$

$$r\cos(\theta-\alpha)=p$$

or $$p=r\cos(\theta-\alpha) \qquad ...(2)$$

Theorem 2. The line $r \sin(\theta-\gamma)=\frac{1}{f'(\gamma)}$ is an asymptote of the curve

$$\frac{1}{r}=f(\theta)$$

where γ is a root of the equation $f(\theta)=0$.

Let $P(r, \theta)$ be any point on the curve and OL a perpendicular on the line (2). Further let PM and PN be perpendiculars on the line (2) and OL respectively.

Now $$PM=LN=OL-ON$$

or $$PM=p-r\cos(\theta-\alpha) \qquad ...(3)$$

Now $r\to\infty$ as the point moves to infinity along the curve.

Let $\theta\to\alpha$, when $r\to\infty$

Now if line (2) is to be asymptote to the curve

$$\frac{1}{r}=f(\theta)$$

then $$PM\to 0$$

Hence from (3), we have as $r\to\infty$.

$$\underset{\theta\to\gamma}{\text{Lt}} \cos(\theta-\alpha)=0=\cos\frac{\pi}{2}$$

$\therefore$ $$\alpha=\gamma-\frac{\pi}{2}$$

This gives the value of α.

Again $p=OL$ is the polar subtangent to the curve at infinity, *i.e.* at the point where the asymptote touches the curve

$\therefore$ $$p=\left[r^2\frac{d\theta}{dr}\right]_{\text{at }\theta=\gamma} \qquad ...(4)$$

Let $$u=\frac{1}{r}=f(\theta)$$

$\therefore$ $$-\frac{1}{r^2}\frac{dr}{d\theta}=\frac{du}{d\theta}$$

Thus from (4), we get

$$p=\left[-\frac{d\theta}{du}\right]_{\text{at }\theta=\gamma}$$

Therefore the equation of asymptote is

$$\left[-\frac{d\theta}{du}\right]_{\text{at }\theta=\gamma}=r\cos\left(\theta-\gamma+\frac{\pi}{2}\right)$$

or $$-\frac{1}{f'(\gamma)}=-r\sin(\theta-\gamma)$$

Hence the equation of asymptote is

$$r \sin(\theta-\gamma)=\frac{1}{f'(\gamma)}.$$

Example 1. *Find the asymptotes of the curve*

$$r=a \tan\theta \qquad \text{(D.U. 1971, 76, 1979)}$$

Sol. Therefore $\frac{1}{r}=\frac{1}{a}\cot\theta=f(\theta)$ (say) ...(1)

Now $f(\theta)=0$

when $\cot\theta=0$

or $\theta=n\pi+\frac{\pi}{2}=\gamma$ (say)

Thus from (1), we have

$$f'(\theta)=-\frac{1}{a}\operatorname{cosec}^2\theta$$

$$\therefore \quad f'(\gamma)=-\frac{1}{a} \qquad \left[\because \operatorname{cosec}^2\left(n\pi+\frac{\pi}{2}\right)=1\right]$$

Now the asymptotes are $r\sin(\theta-\gamma)=\frac{1}{f'(\gamma)}$

when $\gamma=\frac{\pi}{2}$, the asymptote is

$$r\sin\left(\theta-\frac{\pi}{2}\right)=-a \text{ or } r\cos\theta=a$$

when $\gamma=\frac{3\pi}{2}$, the asymptote is

$$r\sin\left(\theta-\frac{3\pi}{2}\right)=-a \text{ or } r\cos\theta=-a.$$

Now when $\theta=\frac{5\pi}{2}$, the asymptote is

$$r\sin\left(\theta-\frac{5\pi}{2}\right)=-a$$

or $r\cos\theta=a$,

which is same as corresponding to

$$\gamma=\frac{\pi}{2}$$

Hence there are two asymptotes only

$$r\cos\theta=a$$

and $r\cos\theta=-a.$

Example 2. *Find the asymptotes of the curve $r\theta=a$.*

Sol. Here the curve is $r=a/\theta$

or $$\frac{1}{r}=\frac{1}{a}\theta$$

$\therefore$ $$\frac{1}{r}=\frac{1}{a}\theta=f(\theta) \quad \cdots (1)$$

Now $$f(\theta)=0$$

when $$\theta=0=\gamma$$

$$f'(\theta)=\frac{1}{a}$$

or $$f'(\gamma)=\frac{1}{a}$$

Equation of asymptote is

$$r \sin(\theta-\gamma)=\frac{1}{f'(\gamma)}$$

or $$r \sin\theta=a.$$

EXERCISE 10 (*c*)

Find the asymptotes of the following curves :

1. $r \sin 2\theta=a \cos 3\theta$
2. $r \sin n\theta=a$
3. $r=a \sec\theta+b \tan\theta$
4. $r=a \log\theta$
5. $r=\dfrac{a}{\frac{1}{2}-\cos\theta}$
6. $r=\dfrac{3a \sin\theta \cos\theta}{\sin^3\theta+\cos^3\theta}$

4. $\frac{3}{2}$ 5. $\frac{a}{2}$

7. $\frac{a}{4}$

Exercise 9 (c) (Page 233)

1. $\frac{a}{2}$ 2. $\frac{a}{2}$.

3. $\frac{2}{3}\sqrt{2\,ar}$ 4. $\frac{a^n}{(n+1)\,r^{n-1}}$

6. $\frac{a^2b^2}{p^3}$

Exercise 9 (d) (Page 240)

1. $\left(-\frac{11}{2}, \frac{16}{3}\right)$ 2. $(-2, 3)$

3. $\left\{\frac{a^2+b^2}{a^4}\alpha^3, -\frac{(a^2+b^2)}{b^4}\beta^3\right\}$

4. $\{\alpha+3\alpha^{1/3}\beta^{2/3}, \beta+3\alpha^{2/3}\beta^{1/3}\}$

5. $\left\{-4a(20+a^2), \quad b+\left(\frac{81+4a^2}{18}\right)\right\}$

6. $\left\{\frac{1}{e}, \frac{e^2-1}{e}\right\}$

Exercise 9 (e) (Page 246)

2. $\left\{\frac{2+\sqrt{2}}{2}, \frac{3+2\sqrt{2}}{2}e^{-(2+\sqrt{2})}\right\}$

and $\left\{\frac{2-\sqrt{2}}{2}, \frac{3-2\sqrt{2}}{2}e^{-(2-\sqrt{2})}\right\}$

3. $(0, 0), \left(\frac{\pi}{3}, \pi-\frac{3\sqrt{3}}{2}\right)$ for concavity

10. $\left(\pm\frac{1}{\sqrt{2}}, e^{-1/2}\right)$, interval $\frac{-1}{\sqrt{2}}$ to $\frac{1}{\sqrt{2}}$, convex outside this interval.

Exercise 10 (a) (Page 255)

1. $y+x=0, y=0, x+1=0$
2. $y=x+1, y+3=2x, y-3x=2$
3. $y=x, y=x+1, y+x=0$
4. $y=x, y+x+1=0, 2y+x=1$
5. $y=0, y+x=0, y+x=1$
6. $y-x+\frac{7}{6}=0, \; y-3x+\frac{3}{2}=0, \; 2y+x+\frac{5}{3}=0$
7. $y=0, y+x=0$
8. $y=0, x=1, y=x+2$

9. $y+x=\pm 1,\ y=0$ 10. $y=\pm\dfrac{b}{a}x$.

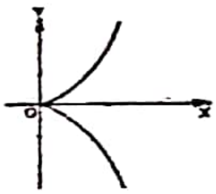
Exercise 10 (b) (Page 256)

1. $x=\pm\sqrt{a},\ y=\pm\sqrt{a}$ 2. $x=\pm a$
3. $y=\pm 1$ 4. $x=\pm 1,\ y=\pm 1$
5. $x=\pm a,\ y=\pm b$.

Exercise 10 (c) (Page 264)

1. $2r\sin\theta=a,\ 2\theta=\pi$
2. $r\sin\left(\theta-\dfrac{k\pi}{n}\right)=\dfrac{a}{n\cos k\pi}$, where k is an integer.
3. $r\cos\theta\pm b=a$ 4. $\theta=0$
5. $r\sin\left(\theta-\dfrac{\pi}{3}\right)=\dfrac{2a}{\sqrt{3}}$
6. $a+\sqrt{2}\,r\sin\left(\theta+\dfrac{\pi}{4}\right)=0$.

Exercise 11 (a) (Page 276)

1.


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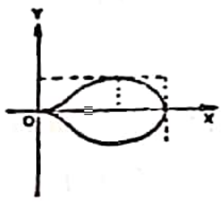

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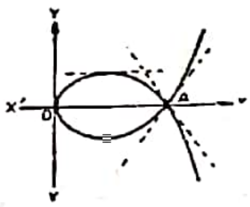

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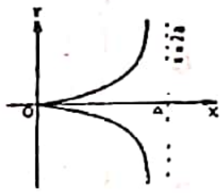

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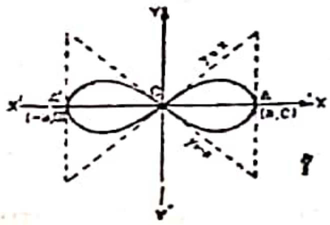

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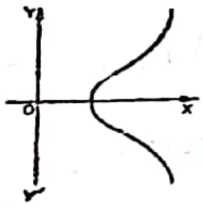

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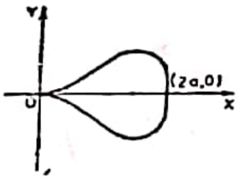

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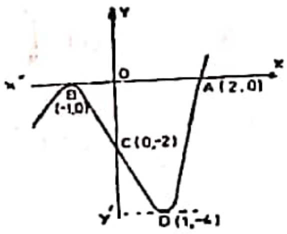

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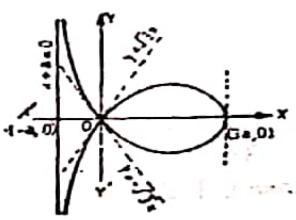

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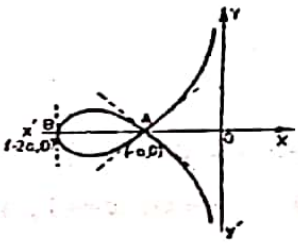

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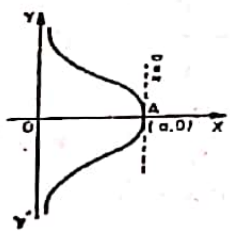

12.