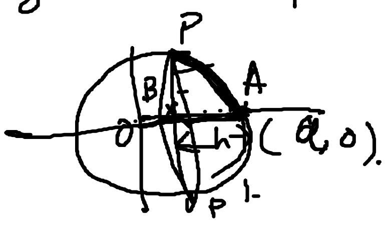
a) Find the volume of a spherical cap of height h cut off from a sphere of radius "a"



$$x^2+y^2=a^2$$

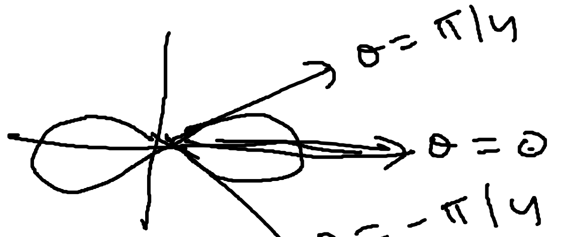
Required volume

$$= \pi \int_{a-h}^{a} y^2 \, dx. = \pi \int_{a-h}^{a} (a^2 - x^2) \, dx$$

$$= \pi \left[a^2 x - \frac{x^3}{3} \right]_{a-h}^{a} = \pi h^2 \left(a - \frac{h}{3} \right)$$

2) Find the volume of the solid formed by revolving one loop of the curve $r^2 = a^2 \cos 2\theta$. about the <u>initial line</u>.

$r^2 = a^2 \cos 2\theta$.



$$V = \frac{2}{3} \pi \int_0^{\pi/4} r^3 \sin\theta \, d\theta.$$

$$= \frac{2\pi}{3} \int_0^{\pi/4} a^3 (\cos 2\theta)^{3/2} \sin\theta \, d\theta.$$

$$= \frac{2\pi a^3}{3} \int_0^{\pi/4} (2\cos^2\theta - 1)^{3/2} \sin\theta \, d\theta.$$

$$= -\frac{2\pi a^3}{3\sqrt{2}} \int_{\sqrt{2}}^{1} (t^2 - 1)^{3/2} dt \quad \left\{ \sqrt{2}\cos\theta = t \right.$$

$$= \frac{2\pi a^3}{3\sqrt{2}} \int_1^{\sqrt{2}} (t^2 - 1)^{3/2} dt.$$

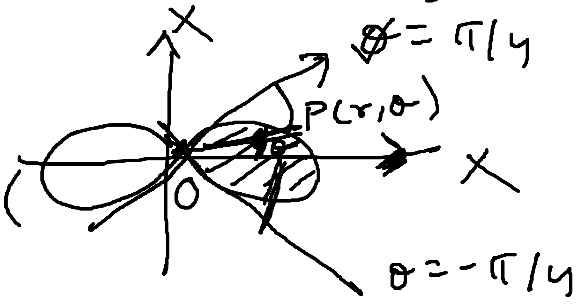
Putting $t = \sec\phi$.

$$V = \frac{\pi a^3 \sqrt{2}}{3} \int_0^{\pi/4} (\sec^2\phi - 1)^{3/2} \sec\phi \tan\phi \, d\phi$$

$$= \frac{\pi a^3 \sqrt{2}}{3} \int_0^{\pi/4} \tan^4\phi \sec\phi \, d\phi.$$

$$\boxed{\int_0^{\pi/4} \sec^n\phi \, d\theta}$$

Q) Find the volume of the solid generated by revolving the curve $r^2 = a^2 \cos 2\theta$ about the tangent at the pole



$V = 2 \times$ volume generated by one loop

$$= 2 \times \int_{-\pi/4}^{\pi/4} \frac{2}{3}\pi r^3 \sin\left(\theta + \frac{\pi}{4}\right) d\theta.$$

$$= \frac{4\pi}{3}\int_{-\pi/4}^{\pi/4} (a^2\cos 2\theta)^{3/2} \left\{\sin\theta \cos\frac{\pi}{4} + \cos\theta \sin\frac{\pi}{4}\right\} d\theta$$

$$= \frac{4\pi}{3} a^3 \int_{-\pi/4}^{\pi/4} (\cos 2\theta)^{3/2} \cdot \frac{1}{\sqrt{2}} (\sin\theta + \cos\theta)\, d\theta$$

$$= \frac{2\sqrt{2}\pi a^3}{3} \left[\int_{-\pi/4}^{\pi/4} (\cos 2\theta)^{3/2} \sin\theta\, d\theta + \int_{-\pi/4}^{\pi/4} (\cos 2\theta)^{3/2} \cos\theta\, d\theta\right]$$

$\leftarrow 0$

$$= \frac{2\sqrt{2}\pi a^3}{3} \times 2 \times \int_0^{\pi/4} (\cos 2\theta)^{3/2} \cos\theta\, d\theta.$$

$$= \frac{4\sqrt{2}}{3}\pi a^3 \int_0^{\pi/4} (1 - 2\sin^2\theta)^{3/2} \cos\theta\, d\theta.$$

Put $\sqrt{2}\sin\theta = \sin\phi$

$$V = \frac{4\sqrt{2}\pi a^3}{3} \times \frac{1}{\sqrt{2}} \int_0^{\pi/2} (1-\sin^2\phi)^{3/2} \cos\phi \, d\phi$$

$$V = \frac{4\pi a^3}{3} \int_0^{\pi/2} \cos^4\phi \, d\phi$$

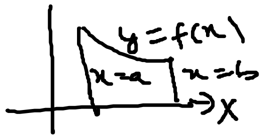
$$= \frac{4\pi a^3}{3} \times \frac{\Gamma 1/2 \; \Gamma 5/2}{2\,\Gamma \frac{0+4+2}{2}} = \frac{\frac{4\pi a^3}{3} \times \sqrt{\pi}\,\Gamma 5/2}{2 \times \Gamma 3}$$

$$= \frac{1}{4}\pi^2 a^3.$$

4(e) Jaggi & Mathur.

The volume of the solid generated by the revolution about the x axis, of the area bounded by the curve $y = f(x)$, the X-axis and the ordinates $x = a$, $x = b$.

$$V = \int_a^b \pi y^2 \, dx$$

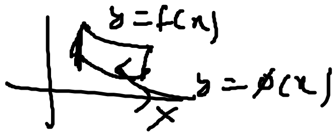


(2) $x = f(y)$, $y = c$, $y = d$. } About y-axis.

$$V = \int_c^d \pi x^2 \, dy.$$

(3) Revolution about x axis of the curve bounded by. $y = f(x)$, $y = \phi(x)$ and the ordinates $x = a$, $x = b$

$$V = \int_a^b \pi (y_2^2 - y_1^2) \, dx.$$



⇒ The vol. of the solid generated by the revolution about the x axis of the area bounded by the curve $x = f(t)$; $y = \phi(t)$

the x axis and the ordinates where $t = t_1$; $t = t_2$ is given by

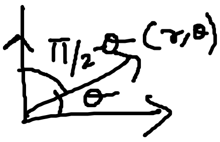
$$V = \int_{t_1}^{t_2} \pi y^2 \frac{dx}{dt} \cdot dt .$$

$\Rightarrow$ Revolution about the initial line of the area bounded by the curve $r = f(\theta)$ the radii vectors $\theta = \alpha$; $\theta = \beta$ is

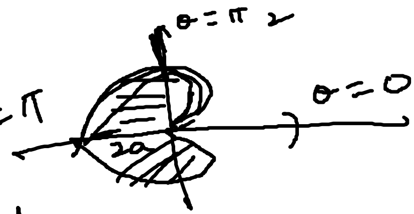
$$V = \frac{2}{3} \int_{\alpha}^{\beta} \pi r^3 \sin\theta \, d\theta .$$

$\Rightarrow$ About the line $\theta = \pi/2$

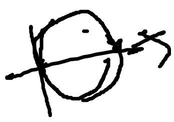
$$V = \frac{2}{3} \int_{\alpha}^{\beta} \pi r^3 \underline{\cos\theta} \, d\theta .$$



Q) Find the volume of the solid generated by the revolution of the cardiod $r = a(1-\cos\theta)$ about the initial line.



$$V = \frac{2}{3} \int_0^{\pi} \pi r^3 \sin\theta \, d\theta$$



$$= \frac{2}{3} \int_0^{\pi} \pi a^3 (1-\cos\theta)^3 \sin\theta \, d\theta .$$

$\cos\theta = t$

$$= -\frac{2}{3} \int_1^{-1} \pi a^3 (1-t)^3 \, dt$$

$$= -\frac{2}{3} \pi a^3 \int_1^{-1} (1 - t^3 + 3t^2 - 3t) \, dt .$$

$$= -\frac{2}{3} \pi a^3 \left[t - \frac{t^4}{4} + t^3 - \frac{3t^2}{2} \right]_1^{-1}$$

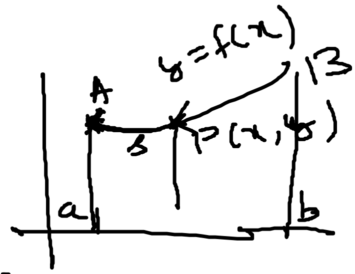
$$= \frac{8}{3} \pi a^3 .$$

Rectification

The length of the arc of the curve $y=f(x)$ between the points where $x=a$, $x=b$ is given by.

$$s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$$

$$\int_a^b \frac{ds}{dx}\,dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$$



$$[s]_a^b = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$\left\{\begin{array}{l}\text{value of } s \text{ at } B \\ -\text{ value of } s \text{ at } A\end{array}\right] = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$\overset{\frown}{AB} - 0$

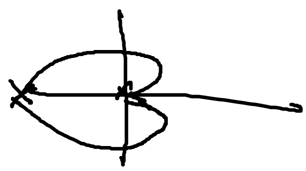
$x = f(y)$ $\qquad y=c,\ y=d$

$$\text{Length} = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy.$$

$$\text{Length (in polar form)} = \int_\alpha^\beta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Q) Find the perimeter of the cardiod.
$r = a(1-\cos\theta)$.

$$\text{Perimeter} = 2\int_0^\pi \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$



$$= 2\int_0^\pi \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\, d\theta.$$

$$= 2\int_0^\pi \sqrt{a^2(1+\cos^2\theta-2\cos\theta)+a^2\sin^2\theta}\, d\theta$$

$$= 8a$$

Find the perimeter of the loop of the curve.

$$3ay^2 = x^2(a-x)$$

$$6ay\frac{dy}{dx} = 2ax - 3x^2$$

$$\frac{dy}{dx} = \frac{x(2a-3x)}{6ay}.$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2(2a-3x)^2}{36a^2y^2}$$

$$= 1 + \frac{\cancel{x^2}(2a-3x)^2}{36a^2 \times \frac{\cancel{x^2}(a-x)}{3a}}$$

$$= 1 + \frac{(2a-3x)^2}{12a(a-x)}$$

$$= \frac{12a^2 - 12ax + 4a^2 + 9x^2 - 12ax}{12a(a-x)}$$

$$= \frac{16a^2 - 24ax + 9x^2}{12(a-x)}$$

$$= \frac{(4a-3x)^2}{12a(a-x)}$$

Required perimeter.

$$= 2\int_0^a \left\{\frac{(4a-3x)^2}{12a(a-x)}\right\}^{1/2} dx$$

$$= \frac{\cancel{2}}{\cancel{2}\sqrt{3a}}\int_0^a \frac{(4a-3x)\,dx}{\sqrt{a-x}}$$

$$= \frac{1}{\sqrt{3a}}\int_0^a \left\{\frac{3(a-x)+a}{\sqrt{a-x}}\right\} dx.$$

$$= \frac{1}{\sqrt{3a}} \left[3\int_0^a \sqrt{a-x}\, dx + a\int_0^a (a-x)^{-1/2} dx \right]$$

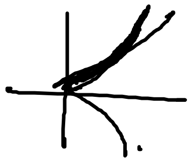
$$= \frac{1}{\sqrt{3a}} \left[\left\{ -2(a-x)^{3/2} \right\}_0^a + a\left\{ -2(a-x)^{1/2} \right\}_0^a \right]$$

$$= \frac{1}{\sqrt{3a}} \left[2a^{3/2} + 2a\, a^{1/2} \right]$$

$$= \frac{1}{\sqrt{3a}} \left[4a^{3/2} \right] = \frac{4a}{\sqrt{3}}$$

Q) Find the length of the arc of the semi cubical parabola $ay^2 = x^3$ from the origin to the point (a, a)

Sol. $ay^2 = x^3$



$$y = \frac{x^{3/2}}{a^{1/2}} \Rightarrow$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{a}} \cdot \frac{3}{2} x^{1/2} = \frac{3\sqrt{x}}{2\sqrt{a}}$$

Required length.

$$\int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^a \sqrt{1 + \frac{9x}{4a}}\, dx$$

$$= \frac{1}{2\sqrt{a}} \int_0^a (9x + 4a)^{1/2} dx$$

$$= \frac{1}{2\sqrt{a}} \left[\frac{(9x+4a)^{3/2}}{\frac{3}{2} \times 9} \right]_0^a$$

$$= \frac{1}{\cancel{2}\sqrt{a}} \times \frac{\cancel{2}}{27} \left[(13a)^{3/2} - (4a)^{3/2} \right]$$

$$= \frac{a^{3/2}}{27\sqrt{a}} \left[13\sqrt{13} - 8 \right]$$

$$= \frac{a}{27} \left[13\sqrt{13} - 8 \right]$$