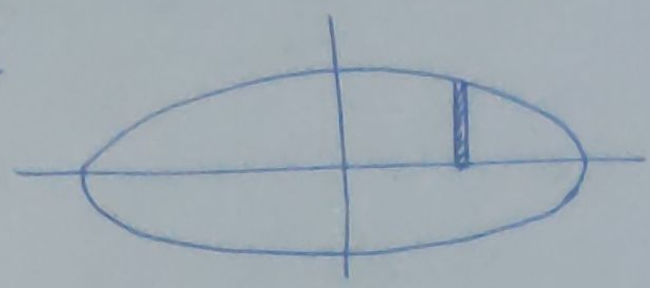
Q1
Ex 4(e)

Find by double integration, the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Soln



$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$$

$$\therefore y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

$$\text{Area} = 4 \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx\, dy$$

$$= 4 \int_0^a \left[y \right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx = 4 \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\, dx.$$

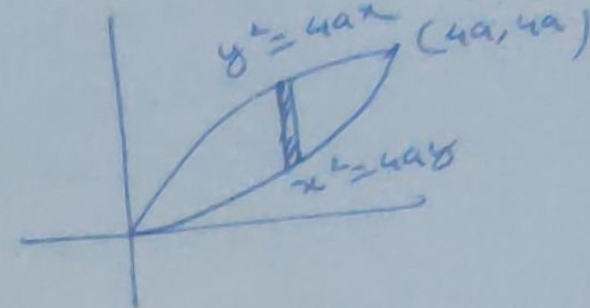
$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]_0^a$$

$$= \frac{4b}{a}\left[\frac{a}{2}\sqrt{a^2-a^2} + \frac{a^2}{2}\sin^{-1}\frac{a}{a} \right]$$

$$= \frac{4b}{a}\left[\frac{a^2}{2}\sin^{-1}(1) \right] = \frac{4ba^2}{2a} \times \frac{\pi}{2} = \pi ab$$

$\frac{2}{4(e)}$ Show by double integration that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$

Sol:-



$y^2 = 4ax \;;\; x^2 = 4ay$

$y^2 = 4a \cdot 2\sqrt{ay}$

$y^4 = 16a^2 \times 4 \times ay$

$y^4 = 64a^3 y \Rightarrow y^4 - 64a^3 y = 0$

$\therefore y(y^3 - 64a^3) = 0 \Rightarrow y = 0 \;;\; y = 4a$

$\therefore$ Points of intersection $(0,0)$ & $(4a, 4a)$

$$\text{Area} = \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\, dx = \int_0^{4a} \left[y \right]_{x^2/4a}^{2\sqrt{ax}}$$

$$= \int_0^{4a} \left(2\sqrt{a}\sqrt{x} - \frac{x^2}{4a} \right) dx$$

$$= \left[2\sqrt{a} \times \frac{2}{3} x^{3/2} - \frac{1}{4a} \times \frac{x^3}{3} \right]_0^{4a}$$

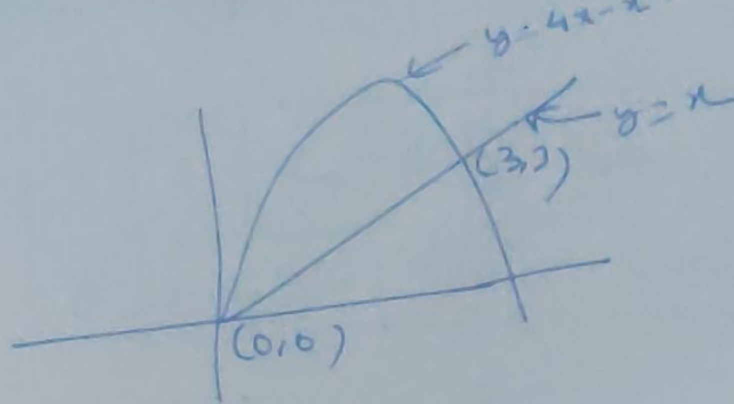
$$= \frac{4a^{1/2}}{3} \times (4a)^{3/2} - \frac{1}{12a}(4a)^3$$

$$= \frac{4a^{1/2}}{3} \times 8a^{3/2} - \frac{1}{12a} \times 64a^3$$

$$= \frac{32}{3}a^2 - \frac{32}{6}a^2 = \frac{32a^2 - 16a^2}{3} = \frac{16}{3}a^2$$

Q = 3
4(e)

Find the area lying between the parabola $y = 4x - x^2$ and the line $y - x = 0$.



$y = 4x - x^2$

$y - x = 0$

$\therefore x = 4x - x^2$

$3x - x^2 = 0$

$x(3 - x) = 0$

$x = 0, \; x = 3$

$y = 0, \; y = 3$

$$\therefore \text{Area} = \int_0^3 \int_x^{4x-x^2} dy\, dx$$

$$= \int_0^3 [y]_x^{4x-x^2} dx = \int_0^3 (4x - x^2 - x)\, dx$$

$$= \int_0^3 (3x - x^2)\, dx = \left[3 \times \frac{x^2}{2} - \frac{x^3}{3}\right]_0^3$$

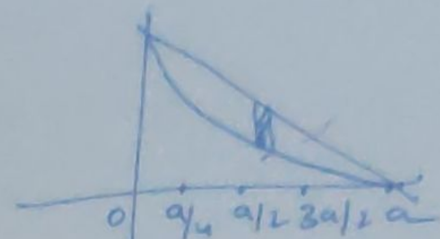
$$= \frac{3}{2} \times 9 - \frac{1}{3} \times 27$$

$$= \frac{27}{2} - \frac{27}{3} = 27 \times \frac{1}{6} = \frac{9}{2}$$ Ans

4
Ex 4(c)

Find by double integration the area of the region enclosed by the curves $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and $x + y = a$

x	0	$a/4$	$a/2$	$3a/2$	a
y	a	$a/4$	$.098a$	$.04a$	0



$$\text{Area} = \int_0^a \int_{(\sqrt{a}-\sqrt{x})^2}^{a-x} dy\, dx$$

$$= \int_0^a \left[y \right]_{(\sqrt{a}-\sqrt{x})^2}^{a-x} dx$$

$$= \int_0^a \left[(a-x) - (\sqrt{a}-\sqrt{x})^2 \right] dx$$

$$= \int_0^a \left[(a-x) - (a + x - 2\sqrt{ax}) \right] dx$$

$$= \int_0^a (\cancel{a} - x - \cancel{a} - x + 2\sqrt{ax})\, dx$$

$$= \int_0^a (2\sqrt{a}\sqrt{x} - 2x)\, dx$$

$$= \left[2\sqrt{a} \times x^{3/2} \times \frac{2}{3} - 2 \times \frac{x^2}{2} \right]_0^a$$

$$= \frac{4}{3}\sqrt{a} \times a^{3/2} - a^2$$

$$= \frac{4}{3} a^2 - a^2 = \frac{a^2}{3}$$

Q-6
Ex 4(e)

Changing to Polar co-ordinates, find the area bounded by the curves $x^2+y^2=2x$, $x^2+y^2=4x$, $y=x$, and $y=0$.

Sol:- Comparing eq. $x^2+y^2-2x=0$ ① by

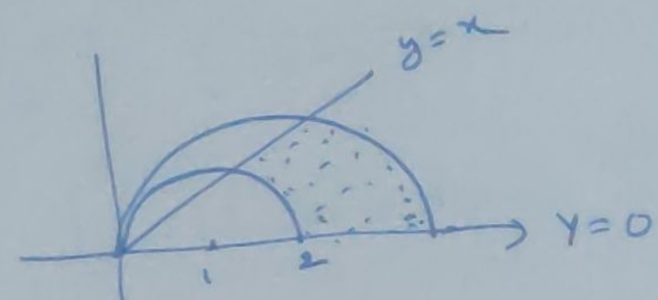
$$x^2+y^2+2gx+2fy+c=0$$

$2g=-2 \Rightarrow g=-1$, $f=0$

centre of the circle is $(-g,-f)$

$\therefore$ centre of circle ① is $(1,0)$

Similarly centre of circle $x^2+y^2=4x$ is $(2,0)$



Changing to polar the eq. $x^2+y^2=2x$ becomes

$$r^2(\cos^2\theta+\sin^2\theta)=2r\cos\theta$$

How?
By substitution
$x=r\cos\theta$
$y=r\sin\theta$.

$$\therefore r^2=2r\cos\theta$$

$$\boxed{r=2\cos\theta}$$

Similarly $x^2+y^2=4x$ becomes, $r^2=4r\cos\theta$

i.e. $r=\boxed{4\cos\theta}$

Area is Cartesian is $\iint dx\,dy$.

In polar $\displaystyle\int_{\theta=0}^{\pi/4}\int_{2\cos\theta}^{4\cos\theta} r\,dr\,d\theta = \int_0^{\pi/4}\left[\frac{r^2}{2}\right]_{2\cos\theta}^{4\cos\theta} d\theta$

why

$\because y=x$ is the line making angle $\pi/4$ with x axis

$$=\frac{1}{2}\int_0^{\pi/4}(16\cos^2\theta-4\cos^2\theta)\,d\theta$$

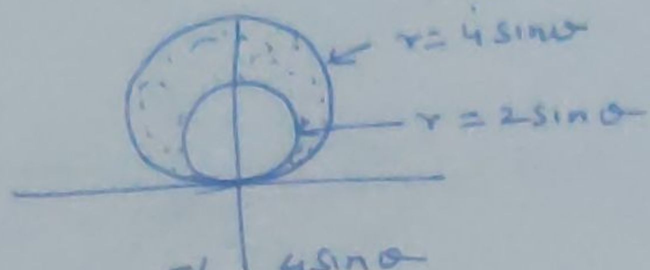
$$=\frac{1}{2}\int_0^{\pi/4}12\cos^2\theta\,d\theta = 6\int_0^{\pi/4}\cos^2\theta\,d\theta.$$

$$=6\int_0^{\pi/4}\frac{(1+\cos 2\theta)}{2}\,d\theta = 3\left\{\theta+\frac{\sin 2\theta}{2}\right\}_0^{\pi/4}$$

$* \; 3\left\{\frac{\pi}{4}+\frac{1}{2}\sin\frac{\pi}{2}\right\}$

$=3\left(\frac{\pi}{4}+\frac{1}{2}\right)$ Ans

Q7/Ex4(e) Find the area bounded by the circles $r = 2\sin\theta$, $r = 4\sin\theta$.



Sol:-

$$\text{Area} = 2\int_0^{\pi/2}\int_{2\sin\theta}^{4\sin\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{2}{2}\int_0^{\pi/2}(16\sin^2\theta - 4\sin^2\theta)\,d\theta$$

$$= \int_0^{\pi/2} 12\sin^2\theta\,d\theta$$

$$= 12\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$= 12\left[\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\frac{1}{2}}{2\Gamma\left(\frac{2+0+2}{2}\right)}\right]$$

$$= 12\left[\frac{\frac{1}{2}\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{2\Gamma(2)}\right]$$

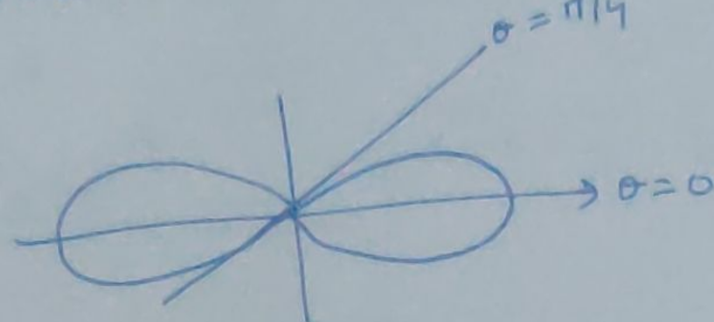
$$= \frac{12\left[\frac{1}{2}\times\pi\right]}{2} = 12\times\frac{\pi}{2}\times\frac{1}{2}$$

$$= 3\pi$$

Q=8
Ex 4(e)

Find by double integration the area of one loop of the leminscate of Bernoulli $r^2 = a^2 \cos 2\theta$.

Sol:-



Area of one loop of $r^2 = a^2 \cos 2\theta$

$$2 \int_0^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} r \, dr \, d\theta.$$

$$= 2 \int_0^{\pi/4} \left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \frac{2}{2} \int_0^{\pi/4} a^2 \cos 2\theta \, d\theta$$

$$= a^2 \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} = \frac{a^2}{2}\left[\sin 2 \times \frac{\pi}{4} - 0\right]$$

$$= \frac{a^2}{2} \sin \frac{\pi}{2} = \frac{a^2}{2}$$

Ans

(9) Find by double Integration the volume of the sphere $x^2+y^2+z^2=4$.

Ex 4(e)

Sol:- The desired volume

$$= 2\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4-x^2-y^2}\,dy\,dx.$$

$$= 2\times 2\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} \sqrt{4-x^2-y^2}\,dy\,dx.$$

$$= 4\int_{-2}^{2}\left[\frac{y}{2}\sqrt{4-x^2-y^2}+\frac{1}{2}(4-x^2)\sin^{-1}\frac{y}{\sqrt{4-x^2}}\right]_0^{\sqrt{4-x^2}} dx$$

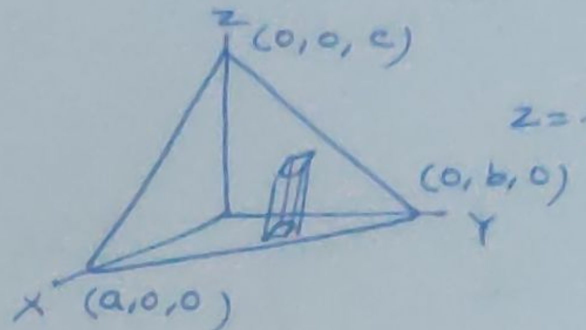
$$= 4\int_{-2}^{2}\frac{1}{2}(4-x^2)\left(\sin^{-1}(1)-\sin^{-1}(0)\right) dx.$$

$$= 4\times\frac{1}{2}\times 2\int_{0}^{2}(4-x^2)\frac{\pi}{2}\,dx = 4\times\frac{\pi}{2}\times\left[4x-\frac{x^3}{3}\right]_0^2$$

$$= 2\pi\times\left[4\times 2-\frac{8}{3}\right] = 2\pi\times 8\left[1-\frac{1}{3}\right]$$

$$= 16\pi\times\frac{2}{3} = \frac{32}{3}\pi$$

$\frac{Q11}{Ex\,4(e)}$ Find the volume bounded by the co-ordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.



$z = f(x,y) = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

The equation of the surface under which the region whose volume is required may be written in the form i.e. $z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

Hence the volume of the region

$$V = \iint_R c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dx\,dy.$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\,dx.$$

$$= c\int_0^a \left[y - \frac{xy}{a} - \frac{y^2}{2b}\right]_0^{b(1-\frac{x}{a})} dx$$

$$= c\int_0^a \left[b\left(1-\frac{x}{a}\right) - \frac{x}{a}b\left(1-\frac{x}{a}\right) - \frac{b^2}{2b}\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= c\left[bx - \frac{b}{a}\frac{x^2}{2} - \frac{b}{a}\frac{x^2}{2} + \frac{b}{a^2}\frac{x^3}{3} - \frac{b}{2}x\right.$$

$$\left. - \frac{b}{6a^2}\frac{x^3}{\cancel{2}} + \frac{b}{a}\cdot\frac{x^2}{2}\right]_0^a$$

$$= c\left\{\cancel{b}a - \frac{b}{\cancel{2}a}a^2 - \cancel{\frac{b}{2a}a^2} + \frac{b}{3a^2}a^3 - \cancel{\frac{b}{2}a}\right.$$

$$\left. - \frac{b}{6a^2}\times a^3 \cancel{+ \frac{1}{4a}c\cdot a^3} + \frac{b\cancel{a}}{2a\,\cancel{2a}}\cdot a^2\right] = \frac{ab}{3} - \frac{ab}{6}$$

$$= ab/6$$

$\therefore \boxed{\frac{abc}{6}}$

(12) Find the vol of the solid bounded by
Ex 4(c) the paraboloid $y^2+z^2=4x$ and ~~a~~ the plane $x=5$.

Sol

$$\text{Vol.} = \int_0^5 \int_{-2\sqrt{x}}^{2\sqrt{x}} \int_{-\sqrt{4x-y^2}}^{\sqrt{4x-y^2}} dz\, dy\, dx$$

$$= 4\int_0^5 \int_0^{2\sqrt{x}} \int_0^{\sqrt{4x-y^2}} dz\, dy\, dx$$

$$= 4\int_0^5 \int_0^{2\sqrt{x}} \Big[z\Big]_0^{\sqrt{4x-y^2}} dy\, dx$$

$$= 4\int_0^5 \int_0^{2\sqrt{x}} \sqrt{4x-y^2}\, dy\, dx.$$

$$= 4\int_0^5 \left[\frac{y}{2}\sqrt{4x-y^2} + \frac{4x}{2}\sin^{-1}\frac{y}{2\sqrt{x}}\right]_0^{2\sqrt{x}} dx.$$

$$= 4\int_0^5 \left[0 + 2x\sin^{-1}(1)\right] dx$$

$$= 4\int_0^5 \cancel{2}x \times \frac{\pi}{\cancel{2}}\, dx$$

$$= 4\pi\left[\frac{x^2}{2}\right]_0^5 = 2\pi \times 25$$

$$= 50\pi$$

$$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

Q=13
Ex 4(e) Compute the volume of the region bounded by the surface $z = 4 - x^2 - y^2$ and xy plane.

Sol:-

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{4-x^2-y^2} dx\,dy\,dz.$$

$$= 4\int_{0}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{0}^{4-x^2-y^2} dx\,dy\,dz.$$

$$= 4\int_{0}^{2} \int_{0}^{\sqrt{4-x^2}} (4-x^2-y^2)\,dx\,dy.$$

Now we have to evaluate this integral over the circle $x^2+y^2=4$.

Changing to polar co-ordinates

$$= 4\int_{0}^{\pi/2} \int_{0}^{2} (4-r^2)\,r\,dr\,d\theta.$$

$$= 4\int_{0}^{\pi/2} \int_{0}^{2} (4r-r^3)\,dr\,d\theta$$

$$= 4\int_{0}^{\pi/2} \left[4 \times \frac{r^2}{2} - \frac{r^4}{4}\right]_0^2 d\theta$$

$$= 4\int_{0}^{\pi/2} \left[2r^2 - \frac{1}{4}r^4\right]_0^2 d\theta$$

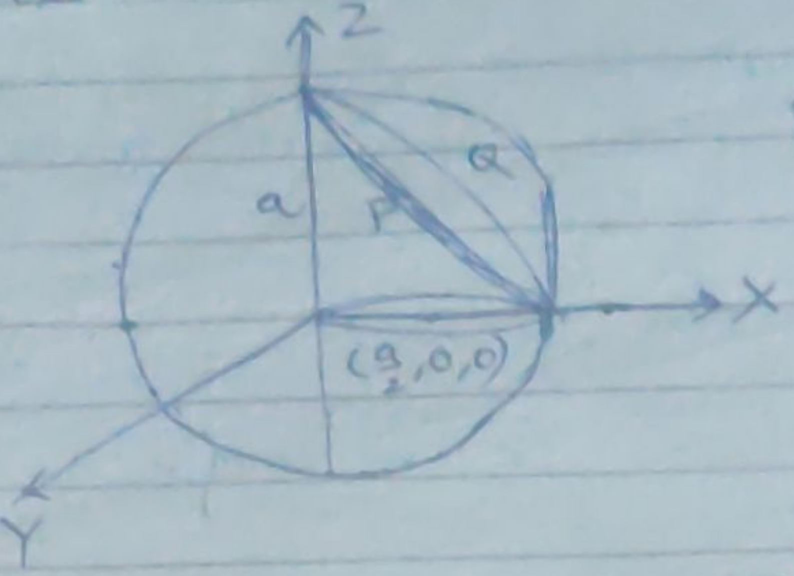
$$= 4\int_{0}^{\pi/2} 2\times 4 - \frac{1}{4}\times 16\Big]\, d\theta$$

$$= 4\int_{0}^{\pi/2} [8-4]\,d\theta = 4\int_{0}^{\pi/2} 4\,d\theta = 16 \times \frac{\pi}{2}$$

$$= 8\pi$$

Q14/Ex 4(c) **Ist Method**

Find the volume cut off from the sphere $x^2+y^2+z^2=a^2$ by the cylinder $x^2+y^2=ax$



{Note: Only the upper half of the vol. has been shown in this}

$$V = \int_0^a \int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dx\,dy\,dz$$

$$= 2\times 2\times \int_0^a \int_0^{\sqrt{ax-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dx\,dy\,dz$$

$$= 2\times 2\times \int_0^a \int_0^{\sqrt{ax-x^2}} \Big[z\Big]_0^{\sqrt{a^2-x^2-y^2}} dx\,dy$$

$$= 4\int_0^a \int_0^{\sqrt{ax-x^2}} \sqrt{a^2-x^2-y^2}\; dx\,dy$$

Q14/Ex-4(e) Contd.

Now transforming to polar co-ordinates

$$V = 4\int_0^{\pi/2}\int_0^{a\cos\theta} \sqrt{a^2 - r^2\cos^2\theta - r^2\sin^2\theta}\; r\,dr\,d\theta$$

$$** \quad = 4\int_0^{\pi/2}\int_0^{a\cos\theta} \sqrt{a^2-r^2}\; r\,dr\,d\theta$$

Now

$$* \quad \int_0^{a\cos\theta} \sqrt{a^2-r^2}\; r\,dr$$

Put $a^2 - r^2 = t^2$ — when $r = 0$, $t = a$

$-2r\,dr = 2t\,dt$

$r\,dr = -t\,dt$ — when $r = a\cos\theta$, $t = a\sin\theta$

$$\therefore * = -\int_0^{a\sin\theta} t.t\,dt$$

$$= -\int_a^{a\sin\theta} t^2\,dt = -\left[\frac{t^3}{3}\right]_a^{a\sin\theta}$$

$$= -\frac{1}{3}\left[a^3\sin^3\theta - a^3\right] = \frac{a^3}{3}\left(1-\sin^3\theta\right)$$

$$\therefore ** = 4\int_0^{\pi/2} \frac{a^3}{3}\left(1-\sin^3\theta\right)d\theta$$

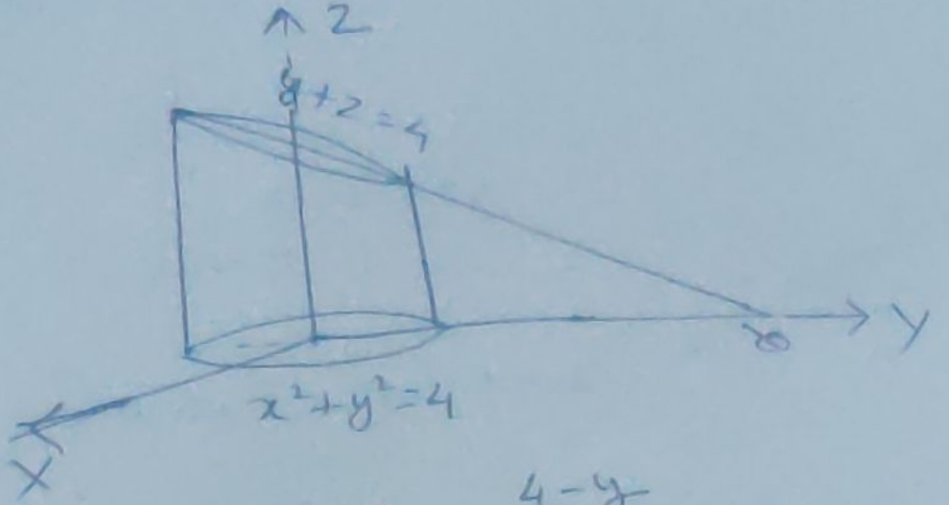
$$= \frac{4}{3}a^3\left[\{\theta\}_0^{\pi/2} - \frac{\Gamma(2)\,\Gamma(\tfrac{1}{2})}{2\,\Gamma\left(\frac{3+0+2}{2}\right)}\right]$$

$$= \frac{4}{3}a^3\left\{\frac{\pi}{2} - \frac{\sqrt{\pi}}{2\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}\right\} = \frac{4a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)$$

(15) Find the volume bounded by the cylinder

ex 4(c) $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$

Sol



$$V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-y} dz\, dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[z\right]_{0}^{4-y} dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\, dy\, dx$$

$$= \int_{-2}^{2} \left[4y - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \left\{4\sqrt{4-x^2} - \cancel{\frac{1}{2}(4-x^2)} + 4\sqrt{4-x^2} + \cancel{\frac{1}{2}(4-x^2)}\right] dx$$

$$= \int_{-2}^{2} 8\sqrt{4-x^2}\, dx.$$

$$= 8\left\{\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{-2}^{2}$$

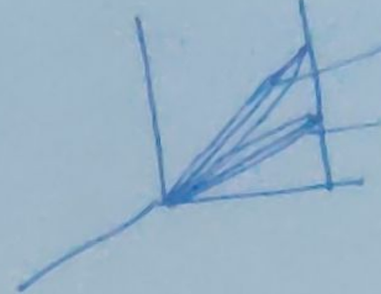
$$= 8\{\cancel{2\times\frac{\pi}{2} - 2\times\frac{3\pi}{2}}\quad 8\{2\sin^{-1}(1) - 2\sin^{-1}(-1)\} = 16\pi$$

Q16
Ex 4(c)

Find the volume of the cylinder $x^2+y^2=2ax$ which is intercepted by the planes $z=x\tan\alpha$ and $z=x\tan\beta$.

Sol:-

$$V = \iiint_{z=x\tan\beta}^{z=x\tan\alpha} dx\,dy\,dz.$$



$$= \iint dx\,dy\,(x\tan\alpha - x\tan\beta)$$

taken over the circle $x^2+y^2=2ax$.

y limits are from $y=-\sqrt{2ax-x^2}$ to $+\sqrt{2ax-x^2}$

and those for x are 0 to $2a$.

$$V = (\tan\alpha - \tan\beta)\int_0^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} x\,dx\,dy.$$

Transforming to polar-co-ordinates.

$$V = (\tan\alpha-\tan\beta)\int_{-\pi/2}^{\pi/2}\int_0^{2a\cos\theta} r\cos\theta . r\,dr\,d\theta.$$

$$= 2(\tan\alpha-\tan\beta)\int_0^{\pi/2}\cos\theta\int_0^{2a\cos\theta} r^2\,dr$$

$$= \frac{2}{3}(\tan\alpha-\tan\beta)\int_0^{\pi/2}\cos\theta\,d\theta\,\left[r^3\right]_0^{2a\cos\theta}$$

$$= \frac{2}{3}(\tan\alpha-\tan\beta)\int_0^{\pi/2} 8a^3\cos^3\theta . \cos\theta\,d\theta$$

$$= \frac{16a^3}{3}(\tan\alpha-\tan\beta)\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$$= \frac{16a^3}{3}(\tan\alpha-\tan\beta)\frac{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}{2\Gamma(3)}$$

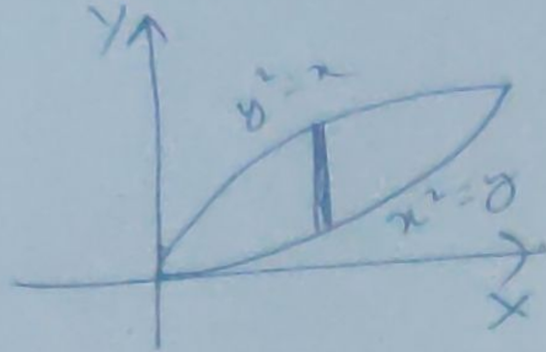
$$= \frac{16a^3}{3}(\tan\alpha-\tan\beta)\frac{\sqrt{\pi}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}{}$$

$* = \frac{16a^3}{3}(\tan\alpha-\tan\beta).\frac{\pi\times3}{16}$

$= \pi a^3(\tan\alpha-\tan\beta)$

(17)
4(c)

Find the vol. ~~common~~ of the cylindrical column standing on the area common to the parabolas $y^2 = x$, $x^2 = y$ and cut off by the surface $z = 12 + y - x^2$.



$$V = \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{12+y-x^2} dz\, dy\, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \left[z \right]_0^{12+y-x^2} dy\, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (12 + y - x^2)\, dy\, dx$$

$$= \int_0^1 \left[(12 - x^2) y + \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left[(12 - x^2)\sqrt{x} + \frac{x}{2} - (12 - x^2) x^2 - \frac{x^4}{2} \right] dx$$

$$= \left[12 \frac{x^{3/2}}{3/2} - \frac{2}{7} x^{7/2} + \frac{1}{2} \times \frac{x^2}{2} - 12 \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^5}{10} \right]_0^1 = 12 \times \frac{2}{3} - \frac{2}{7} + \frac{1}{4} - 4 + \frac{1}{5}$$