CHAPTER

39

INFINITE SERIES

39.1 SEQUENCE

A *sequence* is a succession of numbers or terms formed according to some definite rule. The nth term in a sequence is denoted by u_n.

For example, if $u_n = 2n + 1$.

By giving different values of n in u_n, we get different terms of the sequence.

Thus, $u_1 = 3$, $u_2 = 5$, $u_3 = 7$, ...

A sequence having unlimited number of terms is known as an *infinite sequence*.

39.2 LIMIT

If a sequence tends to a limit l, then we write $\lim\limits_{n \to \infty} (u_n) = l$

39.3 CONVERGENT SEQUENCE

If the limit of a sequence is finite, the sequence is *convergent*. If the limit of a sequence does not tend to a finite number, the sequence is said to be *divergent*.

e.g., $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots, \frac{1}{n^2} + \cdots$. is a convergent sequence.

$3, 5, 7, \ldots, (2n + 1), \ldots$ is a divergent sequence.

39.4 BOUNDED SEQUENCE

$u_1, u_2, u_3 \ldots, u_n \ldots$ is a bounded sequence if $u_n < k$ for every n.

39.5 MONOTONIC SEQUENCE

The sequence is either increasing or decreasing, such sequences are called *monotonic.*

e.g., 1, 4, 7, 10, ... is a monotonic sequence.

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is also a monotonic sequenece.

$1, -1, 1, -1, 1, \ldots$ is not a monotonic sequence.

A sequence which is monotonic and bounded is a convergent sequence.

EXERCISE 39.1

Determine the general term of each of the following sequence. Prove that the following sequences are convergent.

1. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ **Ans.** $\frac{1}{2^n}$

2. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ **Ans.** $\frac{n}{n+1}$

3. $1, -1, 1, -1, \ldots$ **Ans.** $(-1)^{n-1}$

4. $\frac{1^2}{1!}, \frac{2^2}{2!}, \frac{3^2}{3!}, \frac{4^2}{4!}, \frac{5^2}{5!}, \ldots$ **Ans.** $\frac{n^2}{n!}$

Which of the following sequences are convergent ?

5. $u_n = \dfrac{n+1}{n}$ **Ans.** Convergent

6. $u_n = 3n$ **Ans.** Divergent

7. $u_n = n^2$ **Ans.** Divergent

8. $u_n = \dfrac{1}{n}$ **Ans.** Convergent

39.6 REMEMBER THE FOLLOWING LIMITS

(*i*) $\lim\limits_{n \to \infty} x^n = 0$ if $x < 1$ and $\lim\limits_{n \to \infty} x^n = \infty$ if $x > 1$

(*ii*) $\lim\limits_{n \to \infty} \dfrac{x^n}{n!} = 0$ for all values of x

(*iii*) $\lim\limits_{n \to \infty} \dfrac{\log n}{n} = 0$

(*iv*) $\lim\limits_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n = e$

(*v*) $\lim\limits_{n \to \infty} (n)^{1/n} = 1$

(*vi*) $\lim\limits_{n \to \infty} [n!]^{1/n} = \infty$

(*vii*) $\lim\limits_{n \to \infty} \left[\dfrac{(n!)}{n}\right]^{1/n} = \dfrac{1}{e}$

(*viii*) $\lim\limits_{n \to \infty} n\, x^n = 0$ if $x < 1$

(*ix*) $\lim\limits_{n \to \infty} n^h = \infty$

(*x*) $\lim\limits_{n \to \infty} \dfrac{1}{n^h} = 0$

(*xi*) $\lim\limits_{x \to \infty} \left[\dfrac{a^x - 1}{x}\right] = \log a$ or $\lim\limits_{n \to \infty} \dfrac{a^{1/n} - 1}{1/n} = \log a$

(*xii*) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

(*xiii*) $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1$

39.7 SERIES

A *series* is the sum of a sequence.

Let $u_1, u_2, u_3, \ldots\ldots\ldots, u_n, \ldots\ldots\ldots$ be a given sequence. Then, the expression $u_1 + u_2 + u_3 + \ldots\ldots\ldots + u_n + \ldots\ldots\ldots$ is called the series associated with the given sequence.

For example, $1 + 3 + 5 + 7 + \ldots$ is a series.

If the number of terms of a series is limited, the series is called *finite*. When the number of terms of a series are unlimited, it is called an *infinite series.*

$$u_1 + u_2 + u_3 + u_4 + \ldots + u_n + \ldots \infty$$

is called an infinite series and it is denoted by $\sum\limits_{n=1}^{\infty} u_n$ or $\Sigma\, u_n$. The sum of the first n terms of a series is denoted by S_n.

39.8 CONVERGENT, DIVERGENT AND OSCILLATORY SERIES

Consider the infinite series

$$\Sigma\, u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots \infty$$

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n$$

Three cases arise:

(*i*) If S_n tends to a finite number as $n \to \infty$, the series $\Sigma\, u_n$ is said to be *convergent.*

(*ii*) If S_n tends to infinity as $n \to \infty$, the series $\Sigma\, u_n$ is said to be *divergent.*

(*iii*) If S_n does not tend to a unique limit, finite or infinite, the series $\Sigma\, u_n$ is called *oscillatory.*

39.9 PROPERTIES OF INFINITE SERIES

1. The nature of an infinite series does not change:
   (*i*) by multiplication of all terms by a constant k.
   (*ii*) by addition or deletion of a finite number of terms.
2. If two series $\Sigma\, u_n$ and $\Sigma\, v_n$ are convergent, then $\Sigma\,(u_n + v_n)$ is also convergent.

Example 1. *Examine the nature of the series* $1 + 2 + 3 + 4 + \ldots + n + \ldots \infty$.

Solution. Let
$$S_n = 1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2} \qquad \text{[Series in A.P.]}$$

Since
$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{n(n+1)}{2} \Rightarrow \infty$$

Hence, this series is divergent. **Ans.**

Example 2. *Test the convergence of the series* $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \infty$

Solution. Let
$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \infty \qquad \text{[Series in G.P.]}$$
$$= \frac{1}{1 - \frac{1}{2}} = 2 \qquad \left(S_n = \frac{a}{1-r}\right)$$
$$\lim_{n \to \infty} S_n = 2$$

Hence, the series is convergent. **Ans.**

Example 3. *Prove that the following series:*

$\frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots$ *is convergent and find its sum.* (*M.U. 2008*)

Solution. Here,
$$u_n = \frac{n+1}{(n+2)!} = \frac{n+2-1}{(n+2)!} = \frac{n+2}{(n+2)!} - \frac{1}{(n+2)!}$$
$$= \frac{1}{(n+1)!} - \frac{1}{(n+2)!}$$
$$S_n = \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \left(\frac{1}{4!} - \frac{1}{5!}\right) + \ldots$$
$$+ \left(\frac{1}{(n+1)!} - \frac{1}{(n+2)!}\right) = \frac{1}{2!} - \frac{1}{(n+2)!}$$
$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left[\frac{1}{2!} - \frac{1}{(n+2)!}\right] = \frac{1}{2}$$

$\therefore\ \Sigma\, u_n$ converges and its limit is $\frac{1}{2}$. **Ans.**

Example 4. *Discuss the nature of the series* $2 - 2 + 2 - 2 + 2 - \ldots \infty$.

Solution. Let
$$S_n = 2 - 2 + 2 - 2 + 2 - \ldots \infty$$
$$= 0 \text{ if } n \text{ is even}$$
$$= 2 \text{ if } n \text{ is odd.}$$

Hence, S_n does not tend to a unique limit, and, therefore, the given series is oscillatory. **Ans.**

EXERCISE 39.2

Discuss the nature of the following series:

1. $1 + 4 + 7 + 10 + \ldots \infty$ **Ans.** Divergent
2. $1 + \frac{5}{4} + \frac{6}{4} + \frac{7}{4} + \ldots \infty$ **Ans.** Divergent
3. $6 - 5 - 1 + 6 - 5 - 1 + 6 - 5 - 1 + \ldots \infty$ **Ans.** Oscillatory
4. $3 + \frac{3}{2} + \frac{3}{2^2} + \ldots \infty$ **Ans.** Convergent
5. $1^2 + 2^2 + 3^2 + 4^2 + \ldots \infty$ **Ans.** Divergent
6. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots \infty$ **Ans.** Convergent
7. $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots \infty$ **Ans.** Convergent
8. $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots \infty$ **Ans.** Convergent
9. $\log 3 + \log \frac{4}{3} + \log \frac{5}{4} + \ldots \infty$ **Ans.** Divergent
10. $\sum \log \frac{n}{n+1}$ **Ans.** Divergent
11. $\Sigma (\sqrt{n+1} - \sqrt{n})$ **Ans.** Divergent
12. $\sum \frac{1}{n(n+2)}$ **Ans.** Convergent
13. $\sum \frac{1}{n(n+1)(n+2)(n+3)}$ **Ans.** Convergent
14. $\sum \frac{n}{(n+1)(n+2)(n+3)}$ **Ans.** Convergent
15. $\sum \frac{2n+1}{n^2(n+1)^2}$ **Ans.** Convergent

39.10 PROPERTIES OF GEOMETRIC SERIES

The series $1 + r + r^2 + r^3 + \ldots \infty$ is

(*i*) convergent if $|r| < 1$ (*ii*) divergent if $r \geq 1$ (*iii*) oscillatory if $r \leq -1$.

Proof. $$S_n = 1 + r + r^2 + \ldots + r^{n-1} = \frac{1 - r^n}{1 - r}$$

(*i*) When $|r| < 1$, $$\lim_{n \to \infty} r^n = 0$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1 - r^n}{1 - r} = \frac{1 - 0}{1 - r} = \frac{1}{1 - r}$$

Hence, the series is convergent.

(*ii*) (*a*) When $r > 1$, $\lim_{n \to \infty} r^n = \infty$ $\therefore$ $\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{r^n - 1}{r - 1} \Rightarrow \infty$

Hence, the series is divergent.

(*b*) When $r = 1$, the series becomes $1 + 1 + 1 + 1 + \ldots \infty$

$$S_n = 1 + 1 + 1 + 1 + \ldots = n$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} n = \infty$$

Hence, the series is divergent.

(*iii*) (*a*) When $r = -1$, the series becomes $1 - 1 + 1 - 1 + 1 - \ldots \infty$

$$S_n = 0 \text{ if } n \text{ is even}$$
$$= 1 \text{ if } n \text{ is odd}$$

Hence, the series is oscillatory.

(*b*) When $r < -1$, let $r = -k$ where $k > 1$.

$$r^n = (-k)^n = (-1)^n k^n$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1 - r^n}{1 - r} = \lim_{n \to \infty} \frac{1 - (-1)^n k^n}{1 - (-k)}$$

$$= +\infty \text{ if } n \text{ is odd}$$

$$= -\infty \text{ if } n \text{ is even}$$

Hence, the series is oscillatory. **Proved.**

EXERCISE 39.3

Test the nature of the following series :

1. $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots \infty$ **Ans.** Convergent **2.** $1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \ldots \infty$ **Ans.** Convergent

3. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots \infty$ **Ans.** Convergent **4.** $1 - 2 + 4 - 8 + \ldots \infty$ **Ans.** Oscillatory

5. $2 + 3 + \frac{9}{2} + \frac{27}{4} + \ldots \infty$ **Ans.** Divergent **6.** $1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^3 + \ldots \infty$ **Ans.** Divergent

7. State, which one of the alternatives in the following is correct:

The series $1 - 1 + 1 - 1 + \ldots$ is

(*i*) Convergent with its sum equal to 0. (*ii*) Convergent with its sum equal to 1.

(*iii*) Divergent. (*iv*) Oscillatory. **Ans.** Oscillatory series

39.11 POSITIVE TERM SERIES

If all terms after few negative terms in an infinite series are positive, such a series is a positive term series.

e.g., $-10 - 6 - 1 + 5 + 12 + 20 + \ldots$ is a positive term series.

By omitting the negative terms, the nature of a positive term series remains unchanged.

39.12 NECESSARY CONDITIONS FOR CONVERGENT SERIES

For every convergent series Σu_n.

$$\lim_{n \to \infty} u_n = 0$$

Solution. Let $S_n = u_1 + u_2 + u_3 + \ldots + u_n$

$$\lim_{n \to \infty} S_n = k \qquad \text{(a finite quantity)}$$

Also $\lim_{n \to \infty} S_{n-1} = k$ (a finite quantity)

$$S_n = S_{n-1} + u_n$$

$$u_n = S_n - S_{n-1}$$

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} [S_n - S_{n-1}] = 0$$

$$\lim_{n \to \infty} u_n = 0$$

Corollary. Converse of the above theorem is not true.

e.g., $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}} + \ldots \infty$ is divergent.

$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}}$$

$$> \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots + \frac{1}{\sqrt{n}}$$

$$> \frac{n}{\sqrt{n}} > \sqrt{n}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \sqrt{n} = \infty$$

Thus, the series is divergent, although $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$

So $\lim_{n \to \infty} u_n = 0$ is a necessary condition but not a sufficient condition for convergence.

Note: 1. Test for divergence

If $\lim_{n \to \infty} u_n \neq 0$***, the series*** $\Sigma\, u_n$ ***must be divergent.***

2. ***To determine the nature of a series we have to find*** S_n***. Since it is not possible to find*** S_n ***for every series, we have to device tests for convergence without involving*** S_n***.***

39.13 CAUCHY'S FUNDAMENTAL TEST FOR DIVERGENCE

If $\lim_{n \to \infty} u_n \neq 0$, the series is divergent.

Example 5. *Test for convergence of the series* $1 + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots + \frac{n}{n+1} + \ldots \infty$

Solution. Here, $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n}} = 1 \neq 0$

Hence, by **Cauchy's Fundamental Test** for divergence, the series is divergent. **Ans.**

Example 6. *Test for convergence the series* $1 + \frac{3}{5} + \frac{8}{10} + \frac{15}{17} + \ldots + \frac{2^n - 1}{2^n + 1} + \ldots \infty$

Solution. Here, $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{2^n - 1}{2^n + 1} = \lim_{n \to \infty} \frac{1 - \frac{1}{2^n}}{1 + \frac{1}{2^n}} = 1 \neq 0$

Hence, by **Cauchy's Fundamental Test** for divergence the series is divergent. **Ans.**

Example 7. Test the convergence of the following series:

$$\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{3}{8}} + \cdots + \sqrt{\frac{n}{2(n+1)}} + \cdots$$ (*M.D.U., 2000*)

Solution. Here, we have

$$\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{3}{8}} + \cdots + \sqrt{\frac{n}{2(n+1)}} + \cdots$$

$$u_n = \sqrt{\frac{n}{2(n+1)}} = \sqrt{\frac{1}{2\left(1 + \frac{1}{n}\right)}}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{2\left(1 + \frac{1}{n}\right)}} = \frac{1}{\sqrt{2}} \neq 0$$

$\Rightarrow \Sigma\, u_n$ does not converge.

The given series is a series of + ve terms,

Hence by Cauchy fundamental test for divergence, the series is divergent. **Ans.**

EXERCISE 39.4

Examine for convergence:

1. $\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{5}}+\frac{4}{\sqrt{17}}+\ldots+\frac{2^n}{\sqrt{4^n+1}}+\ldots\infty$ **Ans.** Divergent

2. $\sum_{n=1}^{\infty}\frac{n}{n+1}$ **Ans.** Divergent

3. $\sum_{n=1}^{\infty}\sqrt{\frac{n}{n+1}}$ **Ans.** Divergent

4. $\sum \cos\frac{1}{n}$ **Ans.** Divergent

5. $1+\frac{1}{2}+2+\frac{1}{3}+3+\frac{1}{4}+4+\ldots$ **Ans.** Divergent

6. $\Sigma\,(6-n^2)$ **Ans.** Divergent

7. $\Sigma\,(-2^n)$ **Ans.** Divergent

8. $\Sigma\,3^{n+1}$ **Ans.** Divergent

39.14 *p*-SERIES

The series $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots\infty$ *is* (*i*) *convergent if* $p>1$ (*ii*) *Divergent if* $p\le 1$.

(*MDU, Dec. 2010*)

Solution. Case 1: ($p > 1$)

The given series can be grouped as

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\left(\frac{1}{8^p}+\frac{1}{9^p}+\frac{1}{10^p}+\frac{1}{11^p}+\frac{1}{12^p}+\frac{1}{13^p}+\frac{1}{14^p}+\frac{1}{15^p}\right)+\ldots$$

Now

$$\frac{1}{1^p}=1 \qquad \ldots(1)$$

$$\frac{1}{2^p}+\frac{1}{3^p}<\frac{1}{2^p}+\frac{1}{2^p}=\frac{2}{2^p} \qquad \ldots(2)$$

$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}=\frac{4}{4^p} \qquad \ldots(3)$$

$$\frac{1}{8^p}+\frac{1}{9^p}+\ldots+\frac{1}{15^p}<\frac{1}{8^p}+\frac{1}{8^p}+\ldots+\frac{1}{8^p}=\frac{8}{8^p} \qquad \ldots(4)$$

On adding (1), (2), (3) and (4), we get:

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\left(\frac{1}{8^p}+\frac{1}{9^p}+\ldots+\frac{1}{15^p}\right)+\ldots$$

$$<\frac{1}{1^p}+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\ldots$$

$$<1+\left(\frac{1}{2}\right)^{p-1}+\left(\frac{1}{2}\right)^{2p-2}+\left(\frac{1}{2}\right)^{3p-3}+\ldots$$

$$<\frac{1}{1-\left(\frac{1}{2}\right)^{p-1}} \qquad \left[\text{G.P., } r=\left(\frac{1}{2}\right)^{p-1},\ S=\frac{1}{1-r}\right]$$

$<$ Finite number if $p > 1$

Hence, the given series is convergent when $p > 1$.

Case 2: $p = 1$

When $p = 1$, the given series becomes

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\ldots+\frac{1}{16}\right)+\ldots$$

$$1+\frac{1}{2} = 1+\frac{1}{2} \qquad \ldots(1)$$

$$\frac{1}{3}+\frac{1}{4} > \frac{1}{4}+\frac{1}{4}=\frac{1}{2} \qquad \ldots(2)$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2} \qquad \ldots(3)$$

$$\frac{1}{9}+\frac{1}{10}+\ldots+\frac{1}{16} > \frac{1}{16}+\frac{1}{16}+\ldots+\frac{1}{16}=\frac{8}{16}=\frac{1}{2} \qquad \ldots(4)$$

On adding (1), (2), (3) and (4), we get

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\ldots+\frac{1}{16}\right)+\ldots$$

$$> 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$$

$$> 1+\frac{n}{2} \quad (n\to\infty)$$

$$> \infty$$

Hence, the given series is divergent when $p = 1$.

Case 3: $p < 1$

$$\frac{1}{2^p} > \frac{1}{2}, \qquad \frac{1}{3^p} > \frac{1}{3}, \qquad \frac{1}{4^p} > \frac{1}{4} \text{ and so on}$$

Therefore, $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots > 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$

$$> \text{divergent series } (p=1) \qquad \text{[From Case 2]}$$

$$\left[\text{As the series on R.H.S. } \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\right) \text{ is divergent}\right]$$

Hence, the given series is divergent when $p < 1$.

39.15 COMPARISON TEST

If two positive terms $\Sigma\, u_n$ and $\Sigma\, v_n$ be such that

$\lim\limits_{n\to\infty}\frac{u_n}{v_n}=k$ **(finite number), then both series converge or diverge together.**

Proof. By definition of limit there exists a positive number ε, however small, such that

$$\left|\frac{u_n}{v_n}-k\right| < \varepsilon \text{ for } n > m \qquad i.e., -\varepsilon < \frac{u_n}{v_n}-k < +\varepsilon$$

$$k-\varepsilon < \frac{u_n}{v_n} < k+\varepsilon \text{ for } n > m$$

Ignoring the first m terms of both series, we have

$$k - \varepsilon < \frac{u_n}{v_n} < k + \varepsilon \text{ for all } n. \quad \text{...(1)}$$

Case 1. $\Sigma\, v_n$ is convergent, then

$$\lim_{n \to \infty} (v_1 + v_2 + \ldots v_n) = h \text{ (say)} \qquad \text{where } h \text{ is a finite number.}$$

From (1), $u_n < (k + \varepsilon)\, v_n$ for all n.

$$\lim_{n \to \infty} (u_1 + u_2 + \ldots + u_n) < (k + \varepsilon) \lim_{n \to \infty} (v_1 + v_2 + \ldots + v_n) = (k + \varepsilon)h$$

Hence, $\Sigma\, u_n$ is also convergent.

Case 2. $\Sigma\, v_n$ is divergent, then

$$\lim_{n \to \infty} (v_1 + v_2 + \ldots + v_n) \to \infty \quad \text{...(2)}$$

Now from (1) $\qquad k - \varepsilon < \frac{u_n}{v_n}$

$$u_n > (k - \varepsilon) v_n \text{ for all } n$$

$$\lim_{n \to \infty} (u_1 + u_2 + \ldots + u_n) > (k - \varepsilon) \lim_{n \to \infty} (v_1 + v_2 + \ldots + v_n)$$

From (2), $\lim_{n \to \infty} (u_1 + u_2 + \ldots + u_n) \to \infty$

Hence, $\Sigma\, u_n$ is also divergent.

Note. For testing the convergence of a series, this Comparison Test is very useful. We choose $\Sigma\, v_n$ (p-series) in such a way that

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \text{finite number.}$$

Then the nature of both the series is the same. The nature of $\Sigma\, v_n$ (p-series) is already known, so the nature of $\Sigma\, u_n$ is also known.

Example 8. *Test the series* $\sum_{n=1}^{\infty} \frac{1}{n + 10}$ *for convergence or divergence.*

Solution. Here, $\qquad u_n = \frac{1}{n + 10}$

Let $\qquad v_n = \frac{1}{n}$

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{n}{n + 10} = \lim_{n \to \infty} \frac{1}{1 + \frac{10}{n}} = 1 = \text{finite number.}$$

According to Comparison Test both series converge or diverge together, but $\Sigma\, v_n$ is divergent as $p = 1$.

$\therefore\quad \Sigma\, u_n$ is also divergent. **Ans.**

Example 9. *Test the convergence of the following series:*

$$\frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \cdots \qquad \textit{(M.D.U., 2000)}$$

Solution. Here, we have

$$\frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \cdots$$

$$u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}\left[1+\sqrt{1+\frac{1}{n}}\right]}$$

Let us compare Σu_n with Σv_n, where

$$v_n = \frac{1}{\sqrt{n}}$$

$$\frac{u_n}{v_n} = \frac{1}{\sqrt{n}\left[1+\sqrt{1+\frac{1}{n}}\right]} \cdot \frac{\sqrt{n}}{1} = \frac{1}{1+\sqrt{1+\frac{1}{n}}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{1+\sqrt{1+\frac{1}{n}}} = \frac{1}{1+1} = \frac{1}{2}$$

Which is finite and non-zero.

$\therefore \Sigma u_n$ and Σv_n, converge or diverge together since Σv_n, $= \sum \frac{1}{n^{\frac{1}{2}}}$ is of the form $\sum \frac{1}{n^p}$.

$$P = \frac{1}{2} < 1$$

$\therefore\ \Sigma v_n$ is divergent $\Rightarrow \Sigma u_n$ is also divergent. **Ans.**

Example 10. *Examine the convergence of the series:*

$$\sum\left(\sqrt[3]{n^3+1}-n\right)$$ (*M.D.U. 2003*)

Solution. Here, we have $\sum\left(\sqrt[3]{n^3+1}-n\right)$

$$u_n = (n^3+1)^{\frac{1}{3}} - n = \left[n^3\left(1+\frac{1}{n^3}\right)\right]^{\frac{1}{3}} - n$$

$$= n\left[\left(1+\frac{1}{n^3}\right)^{\frac{1}{3}} - 1\right] = n\left[1+\frac{1}{3}\cdot\frac{1}{n^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{n^3}\right)^2 + \ldots - 1\right]$$

$$= \frac{n}{n^3}\left[\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{n^3}\ldots\right] = \frac{1}{n^2}\left[\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{n^3}\ldots\right]$$

Let $$v_n = \frac{1}{n^2}$$

$$\frac{u_n}{v_n} = \frac{1}{n^2}\left[\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{n^3}\ldots\right]\cdot\frac{n^2}{1} = \left[\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{n^3}\ldots\right]$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{n^3}\ldots\right) = \frac{1}{3}$$

which is finite and non-zero.

$\therefore\ \sum u_n$ and $\sum v_n$ converge or diverge together.

Since $\sum v_n = \sum \frac{1}{n^2}$ is of the form $\sum \frac{1}{n^p}$ with $p = 2 > 1$

$\therefore \quad \sum v_n$ is convergent $\Rightarrow \sum u_n$ is convergent. **Ans.**

Example 11. *Test the convergence of the following series*

$$\frac{1}{1+2^{-1}}+\frac{2}{1+2^{-2}}+\frac{3}{1+2^{-3}}+\cdots$$ (*M.D.U., 2000*)

Solution. Here, we have

$$\frac{1}{1+2^{-1}}+\frac{2}{1+2^{-2}}+\frac{3}{1+2^{-3}}+\cdots$$

Here $$u_n = \frac{n}{1+2^{-n}} = \frac{n}{1+\frac{1}{2^n}}$$

Let $$u_n = n$$

Let us compare $\sum u_n$ with $\sum v_n$,

$$\frac{u_n}{v_n} = \frac{n}{1+\frac{1}{2^n}}\cdot\frac{1}{n} = \frac{1}{1+\frac{1}{2^n}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{2^n}} = \frac{1}{1+0} = 1$$

Which is finite and non-zero.

$\therefore \quad \sum u_n$ and $\sum v_n$, converge or diverge together since $\sum v_n = \sum \frac{1}{n}$ is of the form $\sum \frac{1}{n^p}$ with $p = 1$.

$\therefore \quad \sum v_n$ is divergent $\Rightarrow \sum u_n$ is also divergent. **Ans.**

Example 12. *Examine the convergence of the series* $\frac{\sqrt{2}-1}{3^3-1}+\frac{\sqrt{3}-1}{4^3-1}+\frac{\sqrt{4}-1}{5^3-1}+\cdots$

(*M.D.U., 2000*)

Solution. Here, we have

$$\frac{\sqrt{2}-1}{3^3-1}+\frac{\sqrt{3}-1}{4^3-1}+\frac{\sqrt{4}-1}{5^3-1}+\cdots$$

Here $$u_n = \frac{\sqrt{n+1}-1}{(n+2)^3-1} = \frac{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}\right)}{n^3\left[\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right]} = \frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{n^{\frac{5}{2}}\left[\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right]}$$

Let $$v_n = \frac{1}{n^{\frac{5}{2}}}$$

Let us compare $\sum u_n$ with $\sum v_n$,

$$\frac{u_n}{v_n} = \frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{n^{\frac{5}{2}}\left[\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right]}\times\frac{n^{\frac{5}{2}}}{1} = \frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{\left[\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right]}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{\left[\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right]} = \frac{\sqrt{1+0}-0}{(1-0)^3-0} = 1$$

Which is finite and non-zero.

$\therefore \sum u_n$ and $\sum v_n$, converge or diverge together since $\sum v_n$, $= \sum \frac{1}{n^{\frac{5}{2}}}$ is of the form

$\sum \frac{1}{n^p}$. where $P = \frac{5}{2} > 1$.

$\therefore\ \sum v_n$ is convergent $\Rightarrow \sum u_n$ is convergent. **Ans.**

Example 13. *Test the convergence and divergence of the following series.*

$$\sum_{n=1}^{\infty} \frac{2n^2+3n}{5+n^5}$$ (*Gujarat, I Semester, Jan. 2009*)

Solution. Here, $$u_n = \frac{2n^2+3n}{5+n^5} = \frac{n^2\left(2+\frac{3}{n}\right)}{n^5\left(\frac{5}{n^5}+1\right)} = \frac{1}{n^3}\,\frac{2+\frac{3}{n}}{\frac{5}{n^5}+1}$$

Let $$v_n = \frac{1}{n^3}$$

By Comparison Test

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{n^3\left(2+\frac{3}{n}\right)}{n^3\left(\frac{5}{n^5}+1\right)} = \lim_{n\to\infty}\frac{2+\frac{3}{n}}{\frac{5}{n^5}+1} = 2 = \text{Finite number.}$$

According to comparison test both series converge or diverge together but $\Sigma\, v_n$ is convergent as $p = 2$.

Hence, the given series is convergent. **Ans.**

Example 14. *Test the following series for convergence* $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots$

Solution. Given series is $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots$

Here $$u_n = \frac{n+1}{n^p} = \frac{1+\frac{1}{n}}{n^{p-1}}$$

Let $$v_n = \frac{1}{n^{p-1}} \quad \therefore \quad \frac{u_n}{v_n} = \frac{1+\frac{1}{n}}{n^{p-1}}\times\frac{n^{p-1}}{1} = 1+\frac{1}{n}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = 1$$

Therefore, both the series are either convergent or divergent.

But Σv_n is convergent if $p-1>1$, *i.e.*, if $p>2$ (P series)

and is divergent if $p-1\le 1$, *i.e.*, if $p\le 2$

$\therefore$ The given series is convergent if $p>2$ and divergent if $p\le 2$. **Ans.**

EXERCISE 39.5

Examine the convergence or divergence of the following series:

1. $2 + \frac{3}{2}.\frac{1}{4} + \frac{4}{3}.\frac{1}{4^2} + \frac{5}{4}.\frac{1}{4^3} + \dots \infty$ **Ans.** Convergent
2. $1 + \frac{1.2}{1.3} + \frac{1.2.3}{1.3.5} + \frac{1.2.3.4}{1.3.5.7} + \dots \infty$ **Ans.** Convergent
3. $\frac{1}{1.2} + \frac{2}{3.4} + \frac{3}{5.6} + \dots \infty$ **Ans.** Divergent
4. $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots\dots\dots \infty$ **Ans.** Convergent (*M.D. University, Dec. 2004*)
5. $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \dots \infty$ **Ans.** Convergent
6. $\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \dots$ **Ans.** Convergent (*M.D. University, 2001*)
7. $\frac{1}{3} + \frac{2!}{3^2} + \frac{3!}{3^3} + \dots \infty$ **Ans.** Convergent
8. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n+1}}$ **Ans.** Divergent
9. $\sum_{n=1}^{\infty} \frac{2n^3 + 5}{4n^5 + 1}$ **Ans.** Convergent
10. $\sum_{n=1}^{\infty} \frac{a^n}{x^n + n^a}$ **Ans.** If $x > a$, convergent; if $x \le a$, Divergent
11. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$ **Ans.** Convergent
12. $\sum_{n=1}^{\infty} \sqrt{(n^2+1)} - n$ **Ans.** Divergent
13. $\sum_{n=1}^{\infty} \left[\sqrt{(n^4+1)} - \sqrt{(n^4-1)}\right]$ **Ans.** Convergent
14. $\sum_{n=1}^{\infty} \frac{2^n + 1}{3^n + n}$ **Ans.** Convergent
15. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ **Ans.** Convergent
16. $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$ **Ans.** Convergent

39.16 D'ALEMBERT'S RATIO TEST

Statement. ***If $\Sigma\, u_n$ is a positive term series such that $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = k$ then***

(i) the series is convergent if $k < 1$ (ii) the series is divergent if $k > 1$

Solution.

Case I. When $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = k < 1$

By definition of a limit, we can find a number r (< 1) such that

$$\frac{u_{n+1}}{u_n} < r \text{ for all } n \ge m \qquad \left[\frac{u_2}{u_1} < r, \frac{u_3}{u_2} < r, \frac{u_4}{u_3} < r \dots\right]$$

Omitting the first m terms, let the series be

$$u_1 + u_2 + u_3 + u_4 + \dots \infty$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \frac{u_4}{u_1} + \dots\right) = u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}.\frac{u_2}{u_1} + \frac{u_4}{u_3}.\frac{u_3}{u_2}.\frac{u_2}{u_1} + \dots \infty\right)$$

$$< u_1\,(1 + r + r^2 + r^3 + \dots \infty) \qquad (r < 1)$$

$$= \frac{u_1}{1-r}, \text{ which is a finite quantity.}$$

Hence, $\Sigma\, u_n$ is convergent.

Case 2. When $\lim\limits_{n \to \infty} \frac{u_{n+1}}{u_n} = k > 1$

By definition of limit, we can find a number m such that $\frac{u_{n+1}}{u_n} \geq 1$ for all $n \geq m$

$$\frac{u_2}{u_1} > 1, \qquad \frac{u_3}{u_2} > 1, \qquad \frac{u_4}{u_3} > 1$$

Ignoring the first m terms, let the series be

$$u_1 + u_2 + u_3 + u_4 + \ldots \infty$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \frac{u_4}{u_1} + \ldots\right) = u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}.\frac{u_2}{u_1} + \frac{u_4}{u_3}.\frac{u_3}{u_2}.\frac{u_2}{u_1} + \ldots \infty\right)$$

$$\geq u_1\,(1 + 1 + 1 + 1 \ldots \text{ to } n \text{ terms}) = nu_1$$

$$\left[\because \lim_{n \to \infty} (u_1 + u_2 + \ldots + u_n) = nu_1\right]$$

$$\lim_{n \to \infty} S_n \geq \lim_{n \to \infty} nu_1 = \infty$$

Hence, $\Sigma\, u_n$ is divergent.

Note. When $\frac{u_{n+1}}{u_n} = 1 \quad (k = 1)$

The ratio test fails.

For Example. Consider the series whose n^{th} term $= \frac{1}{n}$

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n}} = 1 \qquad \ldots(1)$$

Consider the second series whose n^{th} term is $\frac{1}{n^2}$.

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n \to \infty} \left(\frac{n}{n+1}\right)^2 = 1 \qquad \ldots(2)$$

Thus, from (1) and (2) in both cases $\lim\limits_{n \to \infty} \frac{u_{n+1}}{u_n} = 1$

But we know that the first series is divergent as $p = 1$.
The second series is convergent as $p = 2$.

Hence, when $\lim\limits_{n \to \infty} \frac{u_{n+1}}{u_n} = 1$, the series may be convergent or divergent.

Thus, ratio test fails when $k = 1$.

Example 15. *Test for convergence of the series whose n^{th} term is* $\frac{n^2}{2^n}$.

Solution. Here, we have $u_n = \frac{n^2}{2^n}, \qquad u_{n+1} = \frac{(n+1)^2}{2^{n+1}}$

By D'Alembert's Test

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{(n+1)^2}{2^{n+1}}.\frac{2^n}{n^2} = \lim_{n \to \infty} \frac{1}{2}\left(1 + \frac{1}{n}\right)^2 = \frac{1}{2} < 1$$

Hence, the series is convergent by D'Alembert's Ratio Test. **Ans.**

Example 16. *Test for convergence the series whose* n^{th} *term is* $\dfrac{2^n}{n^3}$.

Solution. Here, we have $u_n = \dfrac{2^n}{n^3}$, $\quad u_{n+1} = \dfrac{2^{n+1}}{(n+1)^3}$

By D'Alembert's Ratio Test

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n} = \frac{2}{\left(1+\dfrac{1}{n}\right)^3} \quad \Rightarrow \quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2}{\left(1+\dfrac{1}{n}\right)^3} = 2 > 1$$

Hence, the series is divergent. **Ans.**

Example 17. *Discuss the convergence of the series*:

$$\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n. \qquad (x > 0)$$

(*M.D.University, Dec., 2001*)

Solution. Here, we have

$$u_n = \sqrt{\frac{n}{n^2+1}}\, x^n$$

$$\therefore \qquad u_{n+1} = \sqrt{\frac{n+1}{(n+1)^2+1}} . x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \sqrt{\frac{n}{n+1}} . \sqrt{\frac{n^2+2n+2}{n^2+1}} . \frac{1}{x} = \sqrt{\frac{1}{1+\dfrac{1}{n}} . \frac{\left(1+\dfrac{2}{n}+\dfrac{2}{n^2}\right)}{\left(1+\dfrac{1}{n^2}\right)}} . \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+\dfrac{1}{n}} . \frac{\left(1+\dfrac{2}{n}+\dfrac{2}{n^2}\right)}{\left(1+\dfrac{1}{n^2}\right)}} . \frac{1}{x} = \frac{1}{x}$$

$\therefore$ By D' Alembert's Ratio Test, $\sum u_n$ converges if $\dfrac{1}{x} > 1$, *i.e.* $x < 1$ and diverges if $\dfrac{1}{x} < 1$ *i.e.*, $x > 1$.

When $x = 1$, the Ratio Test fails.

When $x = 1$, $u_n = \sqrt{\dfrac{n}{n^2+1}} = \sqrt{\dfrac{n}{n^2\left(1+\dfrac{1}{n^2}\right)}} = \dfrac{1}{\sqrt{n}} . \dfrac{1}{\sqrt{1+\dfrac{1}{n^2}}}$

$$v_n = \frac{1}{\sqrt{n}},$$

$$\frac{u_n}{v_n} = \frac{1}{\sqrt{n}} . \frac{1}{\sqrt{1+\dfrac{1}{n^2}}} . \frac{\sqrt{n}}{1} = \frac{1}{\sqrt{1+\dfrac{1}{n^2}}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\dfrac{1}{n^2}}} = 1$$

Which is finite and non-zero.

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

Since $\sum v_n = \sum \frac{1}{\sqrt{n}}$ is of the form $\sum \frac{1}{n^p}$ with $p = \frac{1}{2} < 1$.

$\sum v_n$ diverges $\Rightarrow \sum u_n$ diverges.

Hence, the given series $\sum u_n$ converges is $x < 1$ and diverges if $x \geq 1$. **Ans.**

EXERCISE 39.6

Test the convergence for series:

1. $\sum_{n=1}^{n} \frac{n^2}{3^n}$ **Ans.** Convergent

2. $\sum_{n=1}^{n} \frac{n!}{n^n}$ **Ans.** Convergent

3. $\left(\frac{1}{3}\right)^2 + \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3}{3.5.7}\right)^2 + \ldots \infty$ **Ans.** Convergent

4. $\frac{2}{1} + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + \ldots \infty$ **Ans.** Convergent

5. $\sum_{n=1}^{n} \frac{n!.2^n}{n^n}$ **Ans.** Convergent

6. $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n.3^n}$ **Ans.** Convergent if $x > 3$, Divergent if $x < 3$

7. Prove that, if $u_{n+1} = \frac{k}{1 + u_n}$, where $k > 0$, $u_1 > 0$, then the series Σu_n converges to the positive root of the equation $x^2 + x = k$.

39.17 RAABE'S TEST (HIGHER RATIO TEST)

> ***If Σu_n is a positive term series such that $\lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = k$, then***
> ***(i) the series is convergent if $k > 1$ (ii) the series is divergent if $k < 1$.***

Proof. Case I. $k > 1$

Let p be such that $k > p > 1$ and compare the given series Σu_n with $\sum \frac{1}{n^p}$ which is convergent as $p > 1$.

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} \quad \text{or} \quad \left(\frac{u_n}{u_{n+1}}\right) > \left(1 + \frac{1}{n}\right)^p > 1 + \frac{p}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2} + \ldots$$

(Binomial Theorem)

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + \frac{p(p-1)}{2!}\frac{1}{n} + \ldots .$$

If $$\lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > p$$

and $k > p$ which is true as $k > p > 1$; Σu_n is convergent when $k > 1$.

Case II. $k < 1$ Same steps as in Case I.

Notes:

1. Raabe's Test fails if $k = 1$
2. Raabe's Test is applied only when D'Alembert's Ratio Test fails.

Example 18. *Test the convergence for the series* $\frac{x}{1.2}+\frac{x^2}{3.4}+\frac{x^3}{5.6}+\frac{x^4}{7.8}+\ldots$ (*M.U. 2009*)

Solution. Here, $u_n = \frac{x^n}{(2n-1)\,2n}$ and $u_{n+1}=\frac{x^{n+1}}{(2n+1)\,(2n+2)}$

By D'Alembert's Test

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(2n+1)\,(2n+2)}\times\frac{(2n-1)2n}{x^n} = \frac{x\left(1-\frac{1}{2n}\right)}{\left(1+\frac{1}{2n}\right)\left(1+\frac{2}{2n}\right)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = x$$

(*i*) If $x<1$, $\Sigma\, u_n$ is convergent (*ii*) If $x>1$, $\Sigma\, u_n$ is divergent (*iii*) If $x=1$, Test fails.
Let us apply **Raabe's Test** when $x = 1$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(2n+1)(2n+2)}{2n(2n-1)}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(2n+1)(2n+2)-2n(2n-1)}{2n\,(2n-1)}\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(8n+2)}{(2n)\,(2n-1)}\right] = \lim_{n\to\infty}\frac{\left(2+\frac{2}{4n}\right)}{1\left(1-\frac{1}{2n}\right)} = 2$$

So the series is convergent.
Hence we can say that the given series is convergent if $x \le 1$ and divergent, if $x > 1$. **Ans.**

Example 19. *Test the following series for convergence* $\sum\frac{1}{\sqrt{n+1}-1}$.

Solution. Here, $u_n = \frac{1}{\sqrt{n+1}-1}$, $u_{n+1}=\frac{1}{\sqrt{n+2}-1}$

$$\frac{u_{n+1}}{u_n} = \frac{\sqrt{n+1}-1}{\sqrt{n+2}-1}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n}}-\frac{1}{n^2}}{\sqrt{1+\frac{2}{n}}-\frac{1}{n^2}} = 1$$

D'Alembert's test fails.

By Raabe's test.

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{\sqrt{n+2}-1}{\sqrt{n+1}-1}-1\right)$$

$$= \lim_{n\to\infty} n\left[\frac{\sqrt{n+2}-1-\sqrt{n+1}+1}{\sqrt{n+1}-1}\right] = \lim_{n\to\infty} n\left[\frac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{n+1}-1}\right]$$

$$= \lim_{n \to \infty} n \left[\frac{\sqrt{1+\frac{2}{n}} - \sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{1}{n}-\frac{1}{n^2}}} \right] = 0 < 1$$

Hence, $\Sigma\, u_n$ is divergent. **Ans.**

Example 20. *Discuss the convergence of the series:*

$$\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}+\ldots\,(x>0)$$ (*M.D.U. Dec., 2001*)

Solution. Here, we have

$$\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}+\ldots$$

Neglecting the first term, we have

$$u_n = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 5\ldots(2n)}\cdot\frac{x^{2n+1}}{2n+1}$$

and $$u_{n+1} = \frac{1\cdot 3\cdot 5\ldots(2n-1)\,(2n+1)}{2\cdot 4\cdot 5\ldots(2n)\,(2n+2)}\cdot\frac{x^{2n+3}}{2n+3}$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n-1)\,x^{2n+1}}{2n\,(2n+1)}\times\frac{2n\,(2n+2)\,(2n+3)}{(2n-1)\,(2n+1)\,x^{2n+3}} = \frac{(2n+2)\,(2n+3)}{(2n+1)\,(2n+1)}\,\frac{1}{x^2}$$

$$\therefore \quad = \frac{2n+2}{2n+1}\cdot\frac{2n+3}{2n+1}\cdot\frac{1}{x^2} = \frac{2n\left(1+\frac{1}{n}\right).2n\left(1+\frac{3}{2n}\right)}{2n\left(1+\frac{1}{2n}\right).2n\left(1+\frac{1}{2n}\right)}\cdot\frac{1}{x^2} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{2n}\right)^2}\cdot\frac{1}{x^2}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{2n}\right)^2}\cdot\frac{1}{x^2} = \frac{1}{x^2}$$

$\therefore$ Ratio Test, $\sum u_n$ is convergent if $\frac{1}{x^2} > 1$.

i.e.; $x^2 < 1$ and divergent if $\frac{1}{x^2} < 1$. *i.e.*, $x^2 > 1$.

If $x^2 = 1$, then Ratio Test fails.

Now Raabe's test

When $x^2 = 1$, we have $$\frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2} = \frac{4n^2+10n+6}{4n^2+4n+1}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{4n^2+10n+6}{4n^2+4n+1}-1\right)$$

$$= \lim_{n\to\infty}\frac{6n^2+5n}{4n^2+4n+1} = \lim_{n\to\infty}\frac{6+\frac{5}{n}}{4+\frac{4}{n}+\frac{1}{n^2}} = \frac{6}{4} = \frac{3}{2} > 1$$

$\therefore$ By Raabe's Test, the series converges.

Hence, $\sum u_n$ is convergent if $x^2 \le 1$ and divergent if $x^2 > 1$. **Ans.**

Example 21. *Test the following series for convergence*

$$\frac{1}{2}x + x^2 + \frac{9}{8}x^3 + x^4 + \frac{25}{32}x^5 + \ldots \infty$$

Solution. Here, $u = \dfrac{n^2.x^n}{2^n}, \quad u_{n+1} = \dfrac{(n+1)^2.x^{n+1}}{2^{n+1}}$

By D'Alembert's Test

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2 x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n^2 x^n} = \left(\frac{n+1}{n}\right)^2 \frac{x}{2}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 \frac{x}{2} = \frac{x}{2}$$

(*i*) If $\frac{x}{2} < 1$ or $x < 2$, then $\Sigma\, u_n$ is convergent. (*ii*) If $\frac{x}{2} > 1$ or $x > 2$, then $\Sigma\, u_n$ is divergent.

(*iii*) If $\frac{x}{2} = 1$ or $x = 2$, then the test fails.

Let us apply **Raabe's test**

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{n^2}{(n+1)^2}\frac{2}{2} - 1\right] = n\left[\frac{n^2 - n^2 - 2n - 1}{(n+1)^2}\right] = \frac{-2n^2 - n}{(n+1)^2}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{-2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)^2} = -2 < 1$$

Hence, $\Sigma\, u_n$ is divergent if $x \geq 2$, and convergent if $x < 2$. **Ans.**

Example 22. *Show that the series* $\dfrac{1}{x} + \dfrac{2!}{x(x+1)} + \dfrac{3!}{x(x+1)(x+2)} + \ldots$ *converges if* $x > 2$ *and diverges if* $x < 2$.

Solution. Here, $u_n = \dfrac{n!}{x(x+1)(x+2)\ldots(x+n-1)}$

$$u_{n+1} = \frac{(n+1)!}{x(x+1)(x+2)\ldots(x+n-1)(x+n)}$$

By D'Alembert's test

$$\frac{u_{n+1}}{u_n} = \frac{n+1}{(x+n)}, \qquad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1+\frac{1}{n}}{1+\frac{x}{n}} = 1$$

Test fails. Let us apply **Raabe's Test**.

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{x+n}{n+1} - 1\right) = \lim_{n\to\infty} n\left(\frac{x-1}{n+1}\right) = \lim_{n\to\infty} \frac{x-1}{1+\frac{1}{n}} = x - 1$$

If $x - 1 > 1$ or $x > 2$, then $\Sigma\, u_n$ is convergent.

If $x - 1 < 1$ or $x < 2$, then $\Sigma\, u_n$ is divergent. **Ans.**

Example 23. *Discuss the convergence of the series* $\dfrac{x^2}{2\log 2} + \dfrac{x^3}{3\log 3} + \dfrac{x^4}{4\log 4} + \ldots$

Solution. Here, we have $\dfrac{x^2}{2\log 2}+\dfrac{x^3}{3\log 3}+\dfrac{x^4}{4\log 4}+\ldots$

$$u_n=\frac{x^{n+1}}{(n+1)\log(n+1)},\qquad u_{n+1}=\frac{x^{n+2}}{(n+2)\log(n+2)}$$

By D'Alembert's Test

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{x^{n+2}}{(n+2)\log(n+2)}\times\frac{(n+1)\log(n+1)}{x^{n+1}}$$

$$=\lim_{n\to\infty}x\left(\frac{n+1}{n+2}\right)\frac{\log(n+1)}{\log(n+2)}$$

$$=\lim_{n\to\infty}x\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)\frac{\log n+\log\left(1+\frac{1}{n}\right)}{\log n+\log\left(1+\frac{2}{n}\right)}$$

$$=\lim_{n\to\infty}x\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)\left[\frac{\log n+\frac{1}{n}-\frac{1}{2}\frac{1}{n^2}+\ldots}{\log n+\frac{2}{n}-\frac{1}{2}\cdot\frac{4}{n^2}+\ldots}\right]$$

$$=\lim_{n\to\infty}x\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)\left[\frac{1+\frac{1}{n\log n}+\ldots}{1+\frac{2}{n\log n}+\ldots}\right]=x$$

(*i*) When $x<1$, the series is convergent (*ii*) When $x>1$, the series is divergent.

(*iii*) When $x=1$, the test fails.

Let us apply Raabe's Test

$$\frac{u_n}{u_{n+1}}=\left(\frac{n+2}{n+1}\right)\frac{\log(n+2)}{\log(n+1)}=\left(\frac{n+2}{n+1}\right)\frac{\log n+\log\left(1+\frac{2}{n}\right)}{\log n+\log\left(1+\frac{1}{n}\right)}$$

By D'Alembert's Test

$$=\left(\frac{n+2}{n+1}\right)\frac{\log n+\frac{2}{n}-\frac{1}{2}\cdot\frac{4}{n^2}+\ldots}{\log n+\frac{1}{n}-\frac{1}{2}\cdot\frac{1}{n^2}+\ldots}=\left(\frac{n+2}{n+1}\right)\frac{1+\frac{2}{n\log n}+\ldots}{1+\frac{1}{n\log n}+\ldots}$$

$$=\frac{n+2}{n+1}\left(1+\frac{2}{n\log n}\right)\left(1+\frac{1}{n\log n}\right)^{-1}=\frac{n+2}{n+1}\left(1+\frac{2}{n\log n}\right)\left(1-\frac{1}{n\log n}\right)$$

$$=\frac{n+2}{n+1}\left(1+\frac{2}{n\log n}-\frac{1}{n\log n}+\ldots\right)=\left(\frac{n+2}{n+1}\right)\left[1+\frac{1}{n\log n}\right]$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}=\left[\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right]\left[1+\frac{1}{n\log n}\right]=1+\frac{1}{n\log n}$$

$$n\left[\frac{u_n}{u_{n+1}}-1\right]=n\left[1+\frac{1}{n\log n}-1\right]=\frac{1}{\log n}=0<1$$

Thus the series is divergent when $x=1$.

Hence, the series converges if $x < 1$ and diverges if $x \geq 1$. **Ans.**

Example 24. *Test the series for convergence*

$$1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1).\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2).\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\ldots$$

Solution. $$u_n=\frac{\alpha(\alpha+1)(\alpha+2)\ldots[\alpha+(n-1)].\beta(\beta+1)\ldots[\beta+(n-1)]}{n!\,\gamma(\gamma+1)\ldots[\gamma+(n-1)]}x^n$$

$$u_{n+1}=\frac{\alpha(\alpha+1)(\alpha+2)\ldots[\alpha+(n-1)](\alpha+n).\beta(\beta+1)\ldots[\beta+(n-1)](\beta+n)}{(n+1)!\,\gamma(\gamma+1)\ldots[\gamma+(n-1)](\gamma+n)}x^{n+1}$$

By D'Alembert's Test

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}x=\lim_{n\to\infty}\frac{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}.x$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=x$$

(*i*) If $x < 1$, the series is convergent. (*ii*) If $x > 1$, the series is divergent.

(*iii*) If $x = 1$, the test fails.

Let us apply Raabe's Test

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left[\frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}-1\right]=n\left[\frac{n\gamma+n^2+\gamma+n-\alpha\beta-n\alpha-n\beta-n^2}{(\alpha+n)(\beta+n)}\right]$$

$$\lim_{n\to\infty}n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty}\frac{\gamma+\frac{\gamma}{n}+1-\frac{\alpha\beta}{n}-\alpha-\beta}{\left(\frac{\alpha}{n}+1\right)\left(\frac{\beta}{n}+1\right)}=\gamma+1-\alpha-\beta$$

(*i*) If $\gamma+1-\alpha-\beta > 1$ or $\gamma > \alpha+\beta$, then $\Sigma\, u_n$ is convergent.

(*ii*) If $\gamma+1-\alpha-\beta < 1$ or $\gamma < \alpha+\beta$, then $\Sigma\, u_n$ is divergent. **Ans.**\

EXERCISE 39.7

Determine the nature of the following series:

1. $1+\frac{2!}{2^2}+\frac{3!}{3^2}+\frac{4!}{4^2}+\ldots\infty$ **Ans.** Divergent
2. $\frac{1}{1}+\frac{1.3}{1.4}+\frac{1.3.5}{1.4.7}+\frac{1.3.5.7}{1.4.7.10}+\ldots\infty$ **Ans.** Convergent
3. $1+\frac{1+\alpha}{1+\beta}+\frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)}+\ldots\infty$ **Ans.** If $\beta-\alpha > 1$, convergent. If $\beta-\alpha \leq 1$, Divergent.
4. $\sum_{n=1}^{\infty}\frac{n^3}{e^n}$ **Ans.** Convergent
5. $x+\frac{2x^2}{2!}+\frac{3x^3}{3!}+\frac{4x^4}{4!}+\ldots\infty$ **Ans.** Convergent
6. $1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\ldots$ **Ans.** Divergent
7. $1+\frac{1}{2}x+\frac{1}{5}x^2+\frac{1}{10}x^3+\ldots$ **Ans.** Convergent if $-1 \leq x < 1$ and divergent if $|x| > 1$
8. $1+\frac{(1!)^2}{2!}x^2+\frac{(2!)^2}{4!}x^4+\frac{(3!)^2}{6!}x^6+\ldots\infty\ (x>0)$

   Ans. If $x^2 < 4$, convergent; and divergent if $x^2 \geq 4$

Find the values of x for which the following series converges:

9. $x^2 (\log 2)^q + x^3 (\log 3)^q + x^4 (\log 4)^q + \ldots$

Ans. If $x < 1$, convergent; and divergent if $x \geq 1$

10. $\sum_{n=0}^{\infty} \frac{(-1)^n n! x^n}{10^n}$ **12.** $\sum \frac{x^n}{2n(2n+1)}$ **Ans.** If $x \leq 1$, convergent; and if $x > 1$, divergent

11. $\sum \frac{1.2 \ldots n}{4.7 \ldots (3n+1)} x^n$ **Ans.** If $0 < x < 3$, convergent and divergent if $x \geq 3$.

12. $1 + \frac{(1!)^2}{2!} x + \frac{(2!)^2}{4!} x^2 + \frac{(3!)^2}{6!} x^3 + \ldots\ldots$ *(M.D.U., Dec. 2010)*

Ans. convergent if $x < 4$; divergent if $x \geq 4$.

39.18. GAUSS'S TEST

Statement. *If Σu_n is a positive term series such that*

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma}{n^2} \quad \textit{where } \alpha > 0$$

(i) if $\alpha > 1$, convergent if $\alpha < 1$, divergent, whatever β may be

(ii) if $\alpha = 1$ and $\begin{bmatrix} \beta > 1, \textit{ convergent} \\ \beta \leq 1, \textit{ divergent} \end{bmatrix}$

Example 25. *Test for convergence the series* $\frac{2^2.4^2}{3^2.5^2} + \frac{2^2.4^2.6^2}{3^2.5^2.7^2} + \frac{2^2.4^2.6^2.8^2}{3^2.5^2.7^2.9^2} + \ldots$

Solution. The given series is $\frac{2^2.4^2}{3^2.5^2} + \frac{2^2.4^2.6^2}{3^2.5^2.7^2} + \frac{2^2.4^2.6^2.8^2}{3^2.5^2.7^2.9^2} + \ldots$

$$+ \frac{2^2.4^2.6^2.8^2 \ldots (2n+2)^2}{3^2.5^2.7^2.9^2 \ldots (2n+3)^2} + \ldots \infty$$

$$u_{n+1} = \frac{2^2.4^2.6^2.8^2 \ldots (2n+2)^2 (2n+4)^2}{3^2.5^2.7^2.9^2 \ldots (2n+3)^2 (2n+5)^2}$$

By D'Alembert's Test

$$\frac{u_{n+1}}{u_n} = \frac{(2n+4)^2}{(2n+5)^2} = \frac{4n^2 + 16n + 16}{4n^2 + 20n + 25} = \frac{4 + \frac{16}{n} + \frac{16}{n^2}}{4 + \frac{20}{n} + \frac{25}{n^2}}$$

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{4 + \frac{16}{n} + \frac{16}{n^2}}{4 + \frac{20}{n} + \frac{25}{n^2}} = 1$$

D'Alembert's Test fails. Let us apply **Raabe's Test**.

$$\lim_{n \to \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = \lim_{n \to \infty} n \left(\frac{4n^2 + 20n + 25}{4n^2 + 16n + 16} - 1 \right)$$

$$= \lim_{n \to \infty} \left(\frac{4n^2 + 9n}{4n^2 + 16n + 16} \right) = \lim_{n \to \infty} \left[\frac{4 + \frac{9}{n}}{4 + \frac{16}{n} + \frac{16}{n^2}} \right] = 1, \text{ Raabe's Test fails}$$

Let us apply **Gauss's Test**

$$\frac{u_n}{u_{n+1}} = \frac{(2n+5)^2}{(2n+4)^2} = \frac{\left(1+\dfrac{5}{2n}\right)^2}{\left(1+\dfrac{2}{n}\right)^2} = \left(1+\frac{5}{n}+\frac{25}{4n^2}\right)\left(1+\frac{2}{n}\right)^{-2}$$

$$= \left(1+\frac{5}{n}+\frac{25}{4n^2}\right)\left(1-\frac{4}{n}+\frac{(-2)\times(-3)}{2!}\frac{4}{n^2}+\ldots\right) = \left(1+\frac{5}{n}+\frac{25}{4n^2}\right)\left(1-\frac{4}{n}+\frac{12}{n^2}+\ldots\right)$$

$$= 1-\frac{4}{n}+\frac{12}{n^2}+\frac{5}{n}-\frac{20}{n^2}+\frac{25}{4n^2}+\ldots = 1+\frac{1}{n}-\frac{7}{n^2} \qquad \left(\frac{u_n}{u_{n+1}} = \alpha+\frac{\beta}{n}+\frac{\gamma}{n^2}\right)$$

Hence, $\alpha = 1$, $\beta = 1$. Thus, the series is divergent. **Ans.**

39.19 CAUCHY'S INTEGRAL TEST

Statement. *A positive term series $f(1) + f(2) + f(3) + \ldots + f(n) + \ldots$ where $f(n)$ decreases as n increases, converges or diverges according to the integral*

$$\int_1^\infty f(x)\, dx$$

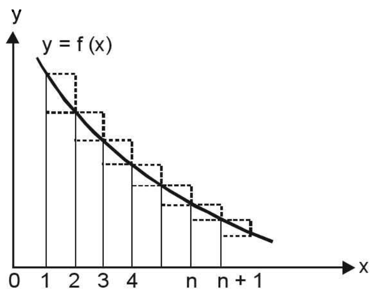
is finite or infinite.

Proof. In the figure, the area under the curve from $x = 1$ to $x = n + 1$ lies between the sum of the areas of small rectangles (small height) and sum of the areas of large rectangles (large height).

[$f(1), f(2)$ ... represent the height of the rectangles]

$$\Rightarrow \quad f(1)+f(2)+\ldots+f(n) \geq \int_1^{n+1} f(x)\,dx \geq f(2)+f(3)+\ldots+f(n+1)$$

$$S_n \geq \int_1^{n+1} f(x)\,dx \geq S_{n+1} - f(1)$$



As $n \to \infty$, from the second inequality that if the integral has a finite value then $\lim\limits_{n\to\infty} S_{n+1}$ is also finite, so $\Sigma f(n)$ is convergent.

Similarly, if the integral is infinite, then from the first inequality that $\lim\limits_{n\to\infty} S_n \to \infty$, so the series is divergent.

Example 26. *Apply the integral test to determine the convergence of the p-series*

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}+\ldots\infty$$

Solution. (*i*) When $p > 1$, $\quad f(x) = \dfrac{1}{x^p}$

$$\int_1^\infty f(x)\,dx = \lim_{m\to\infty}\int_1^m \frac{1}{x^p}\,dx = \lim_{m\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^m = \lim_{m\to\infty}\frac{1}{1-p}(m^{1-p}-1)$$

$$= \lim_{m\to\infty}\frac{1}{1-p}\left[\frac{1}{m^{p-1}}-1\right] = \frac{1}{p-1}, \text{ which is finite.}$$

By Cauchy's Integral Test, the series is convergent for $p > 1$.

(*ii*) When $p < 1$,

$$\int_1^{\infty} f(x)\,dx = \frac{1}{1-p}\left[\lim_{m\to\infty}(m^{1-p}-1)\right] \to \infty$$

Thus, the series is divergent, if $p < 1$.

(*iii*) When $p = 1$,

$$\int_1^{\infty} \frac{1}{x}\,dx = [\log x]_1^{\infty} \to \infty$$

Thus, the series is divergent.

Hence, $\Sigma \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$. **Ans.**

Example 27. *Examine the convergence of* $\sum_{n=2}^{\infty} \frac{1}{n\log n}$.

Solution. Here $f(x) = \frac{1}{x\log x}$

$$\int_2^{\infty} \frac{1}{x\log x}\,dx = \lim_{m\to\infty}\left[\log\log x\right]_2^m = \lim_{m\to\infty}[\log\log m - \log\log 2] \to \infty$$

By Cauchy's Integral Test the series is divergent. **Ans.**

Example 28. *Examine the convergence of* $\sum_{x=1}^{\infty} x\,e^{-x^2}$

Solution. Here $f(x) = x\,e^{-x^2}$

Now, $\int_1^{\infty} x\,e^{-x^2}\,dx = \lim_{m\to\infty}\left[\frac{e^{-x^2}}{-2}\right]_1^m = \lim_{m\to\infty}\left[\frac{e^{-m^2}}{-2}+\frac{e^{-1}}{2}\right] = \frac{e^{-1}}{2} = \frac{1}{2e}$, which is finite.

Hence, the given series is convergent. **Ans.**

EXERCISE 39.8

Examine the convergence:

1. $1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^2}{4^3} + \ldots \infty \quad (x > 0)$ **Ans.** Convergent
2. $\frac{2}{1^2}x + \frac{3^2}{2^3}x^2 + \frac{4^3}{3^4}x^3 + \ldots + \frac{(n+1)^n}{n^{n+1}}x^n + \ldots$ **Ans.** $x < 1$, convergent; $x \ge 1$, divergent
3. $1 + \frac{2^2}{3^2} + \frac{2^2 . 4^2}{3^2 . 5^2} + \frac{2^2 . 4^2 . 6^2}{3^2 . 5^2 . 7^2} + \ldots \infty$ **Ans.** Divergent
4. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ **Ans.** Divergent
5. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ **Ans.** Convergent
6. $\sum_{n=1}^{\infty} \frac{1}{n^n}$ **Ans.** Convergent
7. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ **Ans.** Convergent
8. $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^2}$ **Ans.** Convergent

39.20 CAUCHY'S ROOT TEST

Statement. *If* $\Sigma\, u_n$ *is positive term series such that* $\lim_{n\to\infty}(u_n)^{1/n} = k$, *then*

(*i*) *if* $k < 1$, *the series converges.* (*ii*) *if* $k > 1$, *the series diverges.*

Proof. By definition of limit

$$|(u_n)^{1/n} - k| < \varepsilon \text{ for } n > m$$

$$k - \varepsilon < (u_n)^{1/n} < k + \varepsilon \text{ for } n > m$$

(*i*)
$$k < 1$$
$$k + \varepsilon < r < 1$$
$$(u_n)^{1/n} < k \quad \Rightarrow u_n < k^n$$
$$u_1 + u_2 + \dots \infty < k + k^2 + \dots + k^n + \dots \infty$$
$$< \frac{1}{1-k} \text{ (a finite quantity)}$$

$\therefore$ The series is convergent.

(*ii*)
$$k > 1$$
$$k - \varepsilon > 1$$
$$(u_n)^{1/n} > k - \varepsilon > 1$$
$$u_n > 1$$
$$S_n = u_1 + u_2 + \dots u_n > n$$
$$\lim_{n \to \infty} S_n \to \infty$$

$\therefore$ The series is divergent.

(*iii*)
$$k = 1$$

If $\lim_{n \to \infty} (u_n)^{1/n} = 1$, the test fails.

For example,
$$\Sigma u_n = \sum \frac{1}{n^p}$$
$$\lim_{n \to \infty} (u_n)^{1/n} = \lim_{n \to \infty} \left(\frac{1}{n^p}\right)^{1/n} = \lim_{n \to \infty} \left(\frac{1}{n^{1/n}}\right)^{-p} = 1 \text{ for all } p,\ k = 1$$

But $\sum \frac{1}{n^p}$ is convergent for $p > 1$ and divergent for $p \leq 1$.

Thus, we cannot say whether Σu_n is convergent or divergent for $k = 1$.

Example 29. *Examine the convergence of the series* $\sum \frac{1}{\left(1 + \frac{1}{n}\right)^{n^2}}$ (*MDU, Dec. 2010*)

Solution. Here,
$$u_n = \frac{1}{\left(1 + \frac{1}{n}\right)^{n^2}} \Rightarrow (u_n)^{1/n} = \left[\frac{1}{\left(1 + \frac{1}{n}\right)^{n^2}}\right]^{\frac{1}{n}} = \frac{1}{\left(1 + \frac{1}{n}\right)^{n}}$$
$$\lim_{n \to \infty} (u_n)^{1/n} = \lim_{n \to \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^{n}} = \frac{1}{e} < 1$$

Hence, the given series is convergent. **Ans.**

Example 30. *Test the following series for convergence*
$$\sum_{n=1}^{\infty} \frac{(n+1)^n x^n}{n^{n+1}}$$
(*M.D.U. Dec., 2001*)

Solution. Here, we have
$$\sum_{n=1}^{\infty} \frac{(n+1)^n x^n}{n^{n+1}}$$

Here,
$$u_n = \frac{(n+1)^n x^n}{n^{n+1}} = \left[\frac{(n+1)x}{n}\right]^n \cdot \frac{1}{n}$$

$$\Rightarrow \qquad (u_n)^{\frac{1}{n}} = \frac{(n+1)x}{n} \cdot \frac{1}{n^{\frac{1}{n}}} = \left(1+\frac{1}{n}\right)x \cdot \frac{1}{n^{\frac{1}{n}}}$$

$$\Rightarrow \qquad \lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)x\right]\left[\lim_{n\to\infty}\frac{1}{n^{\frac{1}{n}}}\right]$$

$$= (1+0)\,x \cdot \frac{1}{1} = x \qquad \left[\because \lim_{n\to\infty} n^{\frac{1}{n}} = 1\right]$$

$\therefore$ By Cauchy's root test, $\sum u_n$ is convergent if $x < 1$ and divergent if $x > 1$. The test fails when $x = 1$.

When $x = 1$, $$u_n = \frac{(n+1)^n}{n^{n+1}} = \frac{1}{n} \cdot \frac{(n+1)^n}{n^n} = \frac{1}{n}\left(1+\frac{1}{n}\right)^n$$

Let $$v_n = \frac{1}{n},$$

$$\frac{u_n}{v_n} = \left(1+\frac{1}{n}\right)^n$$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, which is finite and non-zero.

$\therefore$ By camparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

Since $\sum v_n = \sum \frac{1}{n}$ is of the form $\sum \frac{1}{n^p}$ with $p = 1$,

$\sum v_n$ **divergnt** $\Rightarrow \sum u_n$ **also divergent**

Hence, $\sum u_n$ is convergent is $x < 1$ and divergent if $x \geq 1$. **Ans.**

Example 31. *Discuss the convergence of the following series:*

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \ldots \infty$$

Solution. Here, $$u_n = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{(n+1)}{n}\right]^{-n}$$

$$[u_n]^{1/n} = \left[\left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right]^{-n}\right]^{\frac{1}{n}} = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right]^{-1}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)\right]^{-1} = (e-1)^{-1} = \frac{1}{e-1} < 1$$

Hence, the given series is convergent. **Ans.**

EXERCISE 39.9

Discuss the convergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{1}{n^n}$ **Ans.** Convergent
2. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$ **Ans.** Divergent

3. $\sum_{n=1}^{\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ **Ans.** Convergent
4. $\sum_{n=1}^{\infty}\frac{1}{(\log n)^n}$ **Ans.** Convergent
5. $\Sigma\, n^{-k}$ **Ans.** If $k > 1$, convergent
6. $\sum_{n=2}^{\infty}\frac{1}{n(\log n)^k}$ **Ans.** If $k > 1$, convergent; and divergent if $k \le 1$.
7. $\Sigma\,(n \log n)^{-1}(\log \log n)^{-k}$ **Ans.** If $k > 1$, convergent; and divergent if $k \le 1$.
8. $\sum\left(1-\frac{1}{n}\right)^{n^2}$ **Ans.** Convergent
9. $\sum\frac{x^n}{n^n}$ **Ans.** Convergent
10. $(a+b)+(a^2+b^2)+(a^3+b^3)+\ldots$ **Ans.** Convergent if $a < 1, b < 1$; divergent if $a \ge 1, b \ge 1$

39.21 LOGARITHMIC TEST

If $\Sigma\, u_n$ is a positive term series such that $\lim_{n\to\infty}\left(n \log \frac{u_n}{u_{n+1}}\right) = k$

(*i*) If $k > 1$, then the series is convergent. (*ii*) If $k < 1$, then the series is divergent.

Proof. (*i*) If $k > 1$

Compare $\Sigma\, u_n$ with $\sum\frac{1}{n^p}$, if $k > p > 1$, then $\Sigma\, u_n$ converges.

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)^p}{n^p} = \left(1+\frac{1}{n}\right)^p \qquad \ldots(1)$$

Taking logarithm of both sides of (1), we have:

$$\log\frac{u_n}{u_{n+1}} > p\log\left(1+\frac{1}{n}\right) \qquad \left[\log(1+x) = x - \frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right]$$

if $$\log\frac{u_n}{u_{n+1}} > p\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\ldots\right)$$

if $$n\log\frac{u_n}{u_{n+1}} > p\left(1-\frac{1}{2n}+\frac{1}{3n^2}-\frac{1}{4n^3}+\ldots\right)$$

$$\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} > p$$

i.e., $k > p$ which is true as $k > p > 1$. $\left[\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = k\right]$

Hence, $\Sigma\, u_n$ is convergent.
When $p < 1$
Similarly, when $p < 1$, $\Sigma\, u_n$ is divergent.
When $p = 1$, the test fails.

Example 32. *Test the convergence of the series* $x + \frac{2^2.x^2}{2!}+\frac{3^3.x^3}{3!}+\frac{4^4.x^4}{4!}+\ldots\infty$ (*MDU, Dec. 2010*)

Solution. Here the series is $x + \frac{2^2.x^2}{2!}+\frac{3^3.x^3}{3!}+\frac{4^4.x^4}{4!}+\ldots+\frac{n^n.x^n}{n!}+\ldots\infty$

$$u_n = \frac{n^n.x^n}{n!} \quad\text{and}\quad u_{n+1} = \frac{(n+1)^{n+1}.x^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^n . x^n}{n!} \frac{(n+1)!}{(n+1)^{n+1} . x^{n+1}} = \frac{n^n}{(n+1)^n} \frac{1}{x} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \frac{1}{x}$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \frac{1}{x} = \frac{1}{e} . \frac{1}{x}$$

If $\frac{1}{e\,x} > 1$ or $x < \frac{1}{e}$, the series is convergent.

If $\frac{1}{e\,x} < 1$ or $\frac{1}{e} < x$, the series is divergent. If $\frac{1}{e\,x} = 1$ or $x = \frac{1}{e}$, the test fails.

$$\log \frac{u_n}{u_{n+1}} = \log \frac{1}{\left(1+\frac{1}{n}\right)^n} . e = \log e - \log \left(1+\frac{1}{n}\right)^n$$

$$= 1 - n \log \left(1+\frac{1}{n}\right) = 1 - n \left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \ldots\right]$$

$$= 1 - 1 + \frac{1}{2n} - \frac{1}{3n^2} + \ldots = \frac{1}{2n} - \frac{1}{3n^2}$$

$$\lim_{n \to \infty} n \log \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \left[\frac{1}{2} - \frac{1}{3n}\right] = \frac{1}{2} < 1.$$

Thus, the series is divergent. **Ans.**

Example 33. *Discuss the convergence of the series:*

$$1 + \frac{x}{2} + \frac{2!}{3^2} x^2 + \frac{3!}{4^3} x^3 + \frac{4!}{5^4} x^4 + \ldots\ldots\ldots \infty \; (x > 0)$$ *(M.D.University, I Semester, 2009)*

Solution. Here, we have

$$1 + \frac{x}{2} + \frac{2!}{3^2} x^2 + \frac{3!}{4^3} x^3 + \frac{4!}{5^4} x^4 + \ldots\ldots\ldots \infty$$

Neglecting the first term, we get

$$u_n = \frac{n!}{(n+1)^n} x^n \quad \text{and} \quad u_{n+1} = \frac{(n+1)!}{(n+2)^{n+1}} x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{n!}{(n+1)^n} . x^n . \frac{(n+2)^{n+1}}{(n+1)! . x^{n+1}} = \frac{(n+2)^{n+1}}{(n+1)^n . (n+1)} . \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n+2)^{n+1}}{(n+1)^n . (n+1)} . \frac{1}{x}$$

$$= \lim_{n\to\infty} \frac{n^{n+1}\left(1+\frac{2}{n}\right)^{n+1}}{n^n \left(1+\frac{1}{n}\right)^n . n\left(1+\frac{1}{n}\right)} . \frac{1}{x} = \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)^n \left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right)} . \frac{1}{x} \qquad \ldots(1)$$

$$= \frac{e^2}{e} . \frac{1}{x} = \frac{e}{x} . \left[\because \lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n = \lim_{n\to\infty} \left\{\left(1+\frac{a}{n}\right)^{\frac{n}{a}}\right\} = e\right.$$

$$\left[\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n = \lim_{n\to\infty} \left[\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right]^2 = e^2\right]$$

$\therefore$ By D' Alembert's ratio test, the series converges if $1 < \frac{e}{x}$ or if $x < e$ and diverges if $\frac{e}{x} < 1$ or if $e < x$.

If $x = e$, the ratio test fails, $\because \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$

Now when $x = e$

Putting the value of x in (1), we get

$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{2}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}} \cdot \frac{1}{e}$$

Since the expression $\frac{u_n}{u_{n+1}}$ involves the number e, so we do not apply Raabe's test but apply logarithmic test.

$$\therefore \quad \log \frac{u_n}{u_{n+1}} = (n+1)\log\left(1+\frac{2}{n}\right) - (n+1)\log\left(1+\frac{1}{n}\right) - \log e$$

$$= (n+1)\left[\log\left(1+\frac{2}{n}\right) - \log\left(1+\frac{1}{n}\right)\right] - 1$$

$$= (n+1)\left[\left(\frac{2}{n} - \frac{1}{2}\cdot\frac{4}{n^2} + \frac{1}{3}\cdot\frac{8}{n^3}\ldots\right) - \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \ldots\right)\right] - 1$$

$$= (n+1)\left[\frac{1}{n} - \frac{3}{2n^2} + \frac{7}{3n^3} - \ldots\right] - 1$$

$$= \left(1 - \frac{3}{2n}\right) + \left(\frac{1}{n} - \frac{3}{n^2} + \ldots\right) - 1 = \frac{1}{n} - \frac{3}{n^2} - \frac{3}{2n^2}$$

$$= 1 - \frac{1}{2n} - \frac{3}{2n^2} + \ldots - 1 = -\frac{1}{2n} - \frac{3}{2n^2}$$

$$\therefore \quad \lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n\left[-\frac{1}{2n} - \frac{3}{2n^2} + \ldots\right] = \lim_{n\to\infty}\left(-\frac{1}{2} - \frac{3}{2n} + \ldots\right) = -\frac{1}{2} < 1$$

$\therefore$ By log test, the series diverges.

Hence, the given series $\sum u_n$ converges if $x < e$ and diverges if $x \geq e$. **Ans.**

EXERCISE 39.10

Examine the convergence for the following series :

1. $\frac{1^2}{4^2} + \frac{5^2}{8^2} + \frac{9^2}{12^2} + \frac{13^2}{16^2} + \ldots\infty$ **Ans.** Convergent

2. $1 + \frac{1!}{2}x + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \ldots\infty$ **Ans.** If $x < e$, convergent and divergent if $x \geq e$

3. $\frac{1^2}{2^2} + \frac{1^2 . 3^2}{2^2 . 4^2}x + \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2}x^2 + \ldots\infty$ **Ans.** Convergent if $x < 1$, and divergent if $x \geq 1$

4. $\frac{a+x}{1!} + \frac{(a+2x)^2}{2!} + \frac{(a+3x)^3}{3!} + \ldots$ **Ans.** Convergent if $x < \frac{1}{e}$, divergent if $x \geq \frac{1}{e}$

39.22 DE MORGAN'S AND BERTRAND'S TEST

> **If Σu_n is a positive term series such that**
>
> $$\lim_{n \to \infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right] = k$$
>
> **then the series is convergent if $k > 1$ and divergent if $k < 1$.**

Example 34. *Test for convergence the series* $1^p + \left(\frac{1}{2}\right)^p + \left(\frac{1 \times 3}{2 \times 4}\right)^p + \left(\frac{1 \times 3 \times 5}{2 \times 4 \times 6}\right)^p + \ldots$

Solution. The given series is :

$$1^p + \left(\frac{1}{2}\right)^p + \left(\frac{1\times 3}{2\times 4}\right)^p + \left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)^p + \ldots$$

Here $$u_n = \left[\frac{1\times 3\times 5\times\ldots\times(2n-3)}{2\times 4\times 6\times\ldots\times(2n-2)}\right]^p$$

$\therefore$ $$u_{n+1} = \left[\frac{1\times 3\times 5\times\ldots(2n-3)(2n-1)}{2\times 4\times 6\times\ldots\times(2n-2)(2n)}\right]^p$$

$\therefore$ $$\frac{u_{n+1}}{u_n} = \left(\frac{2n-1}{2n}\right)^p = \left(1-\frac{1}{2n}\right)^p$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = 1$$

$\therefore$ **D'Alembert's Test fails.**

Now let us apply Raabe's Test.

Here $$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left[\left(1-\frac{1}{2n}\right)^{-p}-1\right] = n\left[1+\frac{p}{2n}+\frac{p(p+1)}{8n^2}+\ldots-1\right] = \frac{p}{2}+\frac{p(p+1)}{8n}+\ldots$$

$\therefore$ $$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right] = \frac{p}{2}$$

If $\frac{p}{2} > 1$, *i.e.*, $p > 2$, the series is convergent and divergent if $\frac{p}{2} < 1$, *i.e.*, $p < 2$.

This test fails if $\frac{p}{2} = 1$, *i.e.*, $p = 2$.

Now let us apply **De Morgan's Test**. When $p = 2$

$$n\left[\frac{u_n}{u_{n+1}}-1\right] = 1+\frac{3}{4n}+\ldots$$

Now, $$\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right]\log n = \lim_{n\to\infty}\left[1+\frac{3}{4n}+\ldots-1\right]\log n$$

$$= \lim_{n\to\infty}\frac{3}{4}\left[\frac{\log n}{n}-\ldots\right] = 0 < 1 \qquad \left[\lim_{x\to\infty}\frac{\log n}{n}=0\right]$$

$\therefore$ Σu_n is divergent when $p = 2$. **Ans.**

39.23 CAUCHY'S CONDENSATION TEST

If $\phi(n)$ is positive for all positive integral values of n and continually diminishes as n increases and if a be a positive integer greater than 1, then the two series $\Sigma\,\phi(n)$ and $\Sigma\,a^n\,\phi(a^n)$ are either both convergent or both divergent.

Example 35. *Show that the series*

$$1+\frac{1}{2(\log 2)^p}+\frac{1}{3(\log 3)^p}+\ldots+\frac{1}{n(\log n)^p}+\ldots$$

is convergent if $p > 1$ and divergent if $p = 1$ or $p < 1$.

Solution. We apply Cauchy's Condensation Test.

Here $$\phi(n) = \frac{1}{n(\log n)^p}$$

$\therefore$ nth term of the second series $\Sigma\,a^n\,\phi(a^n)$ is :

$$a^n\left[\frac{1}{a^n(\log a^n)^p}\right] \quad i.e., \frac{1}{(\log a^n)^p}\; i.e., \frac{1}{(n\log a)^p}\; i.e., \frac{1}{(\log a)^p}\times\frac{1}{n^p}$$

$\therefore$ The given series will be convergent or divergent if $\sum\left[\frac{1}{(\log a)^p}\times\frac{1}{n^p}\right]$ is convergent or divergent, i.e., if $\sum\frac{1}{n^p}$ is convergent or divergent.

But we know the $\sum\frac{1}{n^p}$ is convergent when $p > 1$ and divergent if $p = 1$ or < 1.

Hence, the given series is convergent if $p > 1$ and divergent if $p = 1$ or < 1. **Proved.**

39.24 ALTERNATING SERIES

A series in which the terms are alternately negative is called the alternating series.

e.g., $$u_1 - u_2 + u_3 - u_4 + \ldots\infty$$

39.25 LEIBNITZ'S RULE FOR CONVERGENCE OF AN ALTERNATING SERIES

(*i*) Each term is numerically less than its preceeding term.

(*ii*) $\lim\limits_{n\to\infty} u_n = 0$

Exmaple 36. *Discuss the convergence of the series* $\sum\limits_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}$.

Solution. The terms of the given series are alternately positive and negative;

(*i*) $$|u_n| = \frac{n}{n^2+1} \quad\text{and}\quad |u_{n+1}| = \frac{(n+1)}{(n+1)^2+1}$$

$$|u_n| - |u_{n+1}| = \frac{n}{n^2+1} - \frac{(n+1)}{(n+1)^2+1} = \frac{n(n+1)^2+n-(n+1)(n^2+1)}{(n^2+1)[(n+1)^2+1]}$$

$$= \frac{n^2+n-1}{(n^2+1)[(n+1)^2+1]} = +\text{ve}$$

and each term is numerically less than its preceeding term.

(*ii*) $$\lim_{n \to \infty} |u_n| = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n+\frac{1}{n}} = 0$$

Both conditions are satisfied.

Hence, by Leibnitz's rule, the given series is convergent. **Ans.**

Example 37. *Test the convergence of the series* $\frac{1}{6}-\frac{2}{11}+\frac{3}{16}-\frac{4}{21}+\frac{5}{26}-...\infty$

Solution. The terms of the given series are alternately positive and negative.

$$u_n = (-1)^{n-1} \frac{n}{5n+1}$$

$$|u_n| = \frac{n}{5n+1} \quad \text{and} \quad |u_{n+1}| = \frac{n+1}{5(n+1)+1}$$

(*i*) $$|u_n| - |u_{n+1}| = \frac{n}{5n+1} - \frac{n+1}{5(n+1)+1} = \frac{5n^2+6n-5n^2-5n-n-1}{(5n+1)(5n+6)}$$

$$= \frac{-1}{(5n+1)(5n+6)}$$

$$\therefore \quad |u_n| > |u_{n+1}|$$

Thus each term is not numerically less than its preceeding terms.

(*ii*) $$= \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{5n+1} = \lim_{n\to\infty} \frac{1}{5+\frac{1}{n}} = \frac{1}{5} \neq 0$$

$$\lim_{n \to \infty} u_n \neq 0$$

Both conditions for convergence are not satisfied.

Hence, the series is not convergent. It is oscillatory. **Ans.**

Example 38. *Test the following series for convergence and absolute convergence*:

$$1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+...$$ (*M.D.U. Dec., 2002*)

Solution. The given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^{n-1}.\frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{\frac{3}{2}}} = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

It is an alternating series.

Here, $$a_n = \frac{1}{n^{\frac{3}{2}}}$$

$$a_{n+1} = \frac{1}{(n+1)^{\frac{3}{2}}}$$

Since, $$\frac{1}{n^{\frac{3}{2}}} > \frac{1}{(n+1)^{\frac{3}{2}}} \ \forall\, n \qquad (\therefore a_n > a_{n+1} \ \forall\, n)$$

Also, $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{\frac{3}{2}}} = 0$$

$\therefore$ By Leibnitz's test, the series $\sum u_n$ is convergent.

$$|u_n| = \frac{1}{n^{\frac{3}{2}}}$$

Now $\sum |u_n| = \sum \frac{1}{n^{\frac{3}{2}}}$ is convergent $\left(\because p = \frac{3}{2} > 1\right)$

Hence, the series $\sum u_n$ is absolutely convergent. **Ans.**

Example 39. *Test the convergence of the series:*

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{2^n n^2}$$ *(M.D. University, I Semester, 2009)*

Solution. Here, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{2^n n^2} \quad ...(1)$$

Here, $$u_n = \frac{(-1)^n (x+1)^n}{2^n . n^2} \text{ and } u_{n+1} = \frac{(-1)^{n+1} (x+1)^{n+1}}{2^{n+1} . (n+1)^2}$$

$$\frac{|u_n|}{|u_{n+1}|} = \frac{|x+1|^n}{2^n . n^2} \cdot \frac{2^{n+1} . (n+1)^2}{|x+1|^{n+1}}$$

$$= 2\left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{|x+1|} = 2\left(1+\frac{1}{n}\right)^2 \cdot \frac{1}{|x+1|}$$

$$\therefore \quad \lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} 2\left(1+\frac{1}{n}\right)^2 \cdot \frac{1}{|x+1|} = \frac{2}{|x+1|}$$

$\therefore$ By ratio test, the series $\sum |u_n|$ is convergent if

i.e., $1 < \frac{2}{|x+1|}$ *i.e.,* if $|x+1| < 2$

i.e., if $-2 < x+1 < 2$ *i.e.,* if $-3 < x < 1$

Also $\sum |u_n|$ is divergent if $\frac{2}{|x+1|} < 1$

i.e., if $|x+1| > 2$ *i.e.,* if $x+1 > 2$ or $x+1 < -2$ *i.e.,* if $x > 1$ or $x < -3$.

Ratio test fails when $x = 1$ or -3.

When $x = 1$, $\sum u_n = \sum \frac{(-1)^n . 2^n}{2^n . n^2} = \sum \frac{(-1)^n}{n^2} = \sum (-1)^n . v_n$ From (1)

It is an alternating series.

Here, $$v_n = \frac{1}{n^2}, \quad v_{n+1} = \frac{1}{(n+1)^2}$$

Clearly $v_n > v_{n+1}$ $\forall n$

Also $$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$$

$\therefore$ By Leibnitz's test, $\sum u_n$ is convergent.

When $x = -3$, $\sum u_n = \sum \frac{(-1)^n . (-2)^n}{2^n . n^2} = \sum \frac{(-1)^{2n} 2^n}{2^n . n^2} = \sum \frac{1}{n^2}$

Which is convergent.

Hence, the given series is convergent if $-3 \le x \le 1$ and divergent if $x > 1$ or $x < -3$

EXERCISE 39.11

Discuss the convergence of the following series :

1. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots$ **Ans.** Convergent
2. $1-2x+3x^2-4x^3+\ldots\infty\ (x<1)$ **Ans.** Convergent
3. $\frac{x}{1+x}-\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}-\frac{x^4}{1+x^4}+\ldots\infty\ (0<x<1)$ **Ans.** Convergent
4. $\frac{1}{1^p}-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\ldots$ **Ans.** If $p>0$, convergent; oscillatory if $p<0$.
5. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{2n-1}$ **Ans.** Oscillatory
6. Show that the series $\frac{1}{x+1}-\frac{1}{x+2}+\frac{1}{x+3}-\frac{1}{x+4}+\ldots$ is convergent for all real values of x other than negative integers.
7. Prove that the series $x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots$ converges if $-1\ x\le 1$.

39.26 ALTERNATING CONVERGENT SERIES

There are two types of alternating convergent series :

(1) Absolutely convergent series. (2) Conditionally convergent series.
Absolutely convergent series. If $u_1+u_2+u_3+\ldots$ be such that $|u_1|+|u_2|+|u_3|+\ldots$ be convergent then $u_1+u_2+u_3+\ldots\infty$ is called absolutely convergent.

Example 40. *Show that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots$ is convergent but not absolutely convergent.*

Solution. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$

The terms of the series are alternately positive and negative.

(*i*) $|u_{n+1}|<|u_n|$ as $\frac{1}{n+1}<\frac{1}{n}$ (*ii*) $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{1}{n}=0$

Both conditions are satisfied. Hence, the given series is convergent.

But $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\infty$ is divergent since in *p-series*, $p=1$.

Hence, the given series is conditionally convergent. **Ans.**

Example 41. *What can you say about the series $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\infty$?*

Solution. $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$

$$|u_n|=\frac{1}{n^2},\qquad\text{and}\qquad |u_{n+1}|=\frac{1}{(n+1)^2}$$

(*i*) $|u_{n+1}|<|u_n|$ (*ii*) $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{1}{n^2}=0$

Thus, the given series is convergent by Leibnitz's rule.

And $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots.$ is also convergent since in p-series, $p=2>1$.

Both the conditions are satisfied.
Hence, the given series is absolutely convergent. **Ans.**

Example 42. *Discuss the series for convergence* $1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{3^3} + \frac{1}{2^2} - \frac{1}{3^5} + \frac{1}{2^3} - \frac{1}{3^7} + \ldots$

Solution. The given series is rewitten as

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots - \left(\frac{1}{3} + \frac{1}{3^3} + \frac{1}{3^5} + \frac{1}{3^7} + \ldots\right)$$

$$\lim_{n\to\infty} S_n = \frac{1}{1-\frac{1}{2}} - \frac{\frac{1}{3}}{1-\frac{1}{3^2}} = 2 - \frac{3}{8} = 1\frac{5}{8}$$

The given series is convergent.

Again $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{3} + \frac{1}{3^3} + \frac{1}{3^5} + \ldots$

$$\lim_{n\to\infty} S_n = \frac{1}{1-\frac{1}{2}} + \frac{\frac{1}{3}}{1-\frac{1}{3}} \qquad \left(\lim_{n\to\infty} S_n = \frac{a}{1-r}\right)$$

$$\lim_{n\to\infty} S_n = 2 + \frac{3}{8} = \frac{19}{8}$$

Both the conditions are satisfied.
This series is also convergent.
Hence, the given series is absolutely convergent. **Ans.**

Example 43. *Test the convergence and divergence of the series*

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \ldots\ldots\ldots$$ (*Gujarat, I Semester, Jan. 2009*)

Solution. The terms of the given series are alternately positive and negative and the given series is geometric infinite series.

(*i*) $$S = 5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \frac{80}{81} + \ldots\ldots\ldots$$

Here $a = 5$ and $r = -\frac{2}{3}$, $\quad S = \frac{a}{1-r}$

$$S = \frac{5}{1-\left(-\frac{2}{3}\right)} = \frac{5}{1+\frac{2}{3}} = \frac{5}{\frac{5}{3}} = 3$$

Sum of the series is finite.
Hence, the given series is convergent.

(*ii*) Again $5 + \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + \frac{80}{81} + \ldots$

This is also G.P.

Here, $a = 5$ and $r = \frac{2}{3}$

$$S = \frac{a}{1-r}, \qquad S = \frac{5}{1-\frac{2}{3}} = \frac{5}{\frac{1}{3}} = 15$$

Again sum of the positive terms is finite.
Thus the series is also convergent.
Both of the conditions are satisfied.
Hence, the given series is absolutely convergent. **Ans.**

Example 44. *Test the following series for convergence and divergence.*

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{n^2+n+1}\right)$$ (*Gujarat, I Semester, Jan. 2009*)

Solution. Let $u_n = \tan^{-1} \dfrac{1}{1+n(n+1)}$

$$u_n = \tan^{-1} \frac{(n+1)-n}{1+n(n+1)}$$

$$u_n = \tan^{-1}(n+1) - \tan^{-1}(n)$$

$$u_{n+1} = \tan^{-1}(n+2) - \tan^{-1}(n+1)$$

By D' Alembert's test

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\tan^{-1}(n+2)-\tan^{-1}(n+1)}{\tan^{-1}(n+1)-\tan^{-1} n} = \lim_{n\to\infty} \frac{\dfrac{1}{1+(n+2)^2} - \dfrac{1}{1+(n+1)^2}}{\dfrac{1}{1+(n+1)^2} - \dfrac{1}{1+n^2}}$$

[L'Hopital Rule]

$$= \lim_{n\to\infty} \frac{\dfrac{1+(n+1)^2-1-(n+2)^2}{[1+(n+2)^2]\,[1+(n+1)^2]}}{\dfrac{1+n^2-1-(n+1)^2}{[1+(n+1)^2]\,[1+n^2]}} = \lim_{n\to\infty} \frac{(2n+3)(-1)}{(2n+1)(-1)} \times \frac{1+n^2}{1+(n+2)^2}$$

$$= \lim_{n\to\infty} \frac{(2n+3)(1+n^2)}{(2n+1)[n^2+4n+5]} = \lim_{n\to\infty} \frac{\left(2+\dfrac{3}{n}\right)\left(\dfrac{1}{n^2}+1\right)}{\left(2+\dfrac{1}{n}\right)\left(1+\dfrac{4}{n}+\dfrac{5}{n^2}\right)} = \frac{2}{2} = 1$$

Test fails.
Let us apply **Raabe's Test.**

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(2n+1)(n^2+4n+5)}{(2n+3)(n^2+1)}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(2n+1)(n^2+4n+5)-(2n+3)(n^2+1)}{(2n+3)(n^2+1)}\right]$$

$$= \lim_{n\to\infty} n\left[\frac{2n^3+8n^2+10n + n^2+4n+5-2n^3-2n-3n^2-3}{(2n+3)\,(n^2+1)}\right]$$

$$= \lim_{n\to\infty} n\left[\frac{6n^2 + 12n + 2}{(2n+3)\,(n^2+1)}\right] = \lim_{n\to\infty} \frac{6+\dfrac{12}{n}+\dfrac{2}{n^2}}{\left(2+\dfrac{3}{n}\right)\left(1+\dfrac{1}{n^2}\right)} = \frac{6}{2} = 3 > 1$$

Hence, the given series is convergent. **Ans.**