

Quantum Suppression of Classical Chaos

The Smooth Kicked Rotor with Adaptive RK4 Integration

Name:Roshan Yadav

Roll No: 2311144

email id: roshan.yadav@niser.ac.in

Presentation Outline

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Project Objective

Central Question

How does quantum mechanics alter classically chaotic dynamics?

Classical Chaos:

- Exponential sensitivity ($\lambda > 0$)
- Unbounded momentum diffusion
- Unpredictable long-term behavior

Quantum Mechanics:

- Deterministic Schrödinger equation
- Wave interference effects
- Discrete spectrum

Goal

Demonstrate **dynamical localization** – quantum suppression of classical chaotic diffusion using advanced numerical methods

The Model: Smooth Kicked Rotor

Physical System: A rotating particle (angle θ , momentum p) subject to periodic kicks

Hamiltonian with smooth Gaussian kicks:

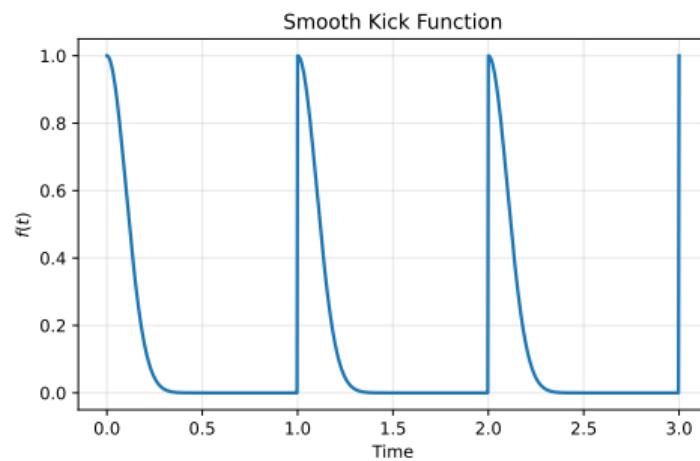
$$H(t) = \frac{p^2}{2} + K \cos \theta \cdot f(t), \quad f(t) = \exp \left[-\frac{(t \bmod T)^2}{2\tau^2} \right]$$

Key Parameters:

- $K = 5.0$ " kick strength (chaotic regime)
- $\tau = 0.1$ " kick width
- $T = 1.0$ " kick period
- $t_{\text{end}} = 20.0$ " total time
- $n_{\text{max}} = 30$ " momentum cutoff

Why smooth kicks?

- Enables continuous-time ODE integration
- Approaches δ -kicks as $\tau \rightarrow 0$



Smooth kick function $f(t)$ over 3 periods

Research Questions

1. Classical Dynamics:

- What is the Lyapunov exponent for $K = 5$?
- Does momentum diffuse as predicted: $D_{\text{cl}} \approx K^2/2$?

2. Quantum Dynamics:

- Does $\langle p^2 \rangle$ saturate (dynamical localization)?
- What is the localization length ℓ^* ?

3. Quantum-Classical Correspondence:

- When does the quantum system deviate from classical?
- What is the effective dimensionality of quantum evolution?

Methodology

Use **adaptive RK4 integration** with step-doubling error control for both classical and quantum dynamics

Adaptive RK4: Core Method

Fourth-order Runge Kutta with adaptive step-size control

Algorithm 1 Adaptive RK4 with Step Doubling

```
1: Initialize:  $t = t_0$ ,  $y = y_0$ ,  $h = h_0$ 
2: while  $t < t_{\text{end}}$  do
3:   Compute  $y_{\text{big}}$  using one RK4 step of size  $h$ 
4:   Compute  $y_{\text{small}}$  using two RK4 steps of size  $h/2$ 
5:   Estimate error:  $\epsilon = \|y_{\text{small}} - y_{\text{big}}\|$ 
6:   if  $\epsilon < \text{tol}$  then
7:     Accept step:  $y \leftarrow y_{\text{small}}$ ,  $t \leftarrow t + h$ 
8:     Adapt:  $h_{\text{new}} = 0.9 \cdot h \cdot (\text{tol}/\epsilon)^{1/5}$ 
9:   else
10:    Reject step: reduce  $h$ 
11:   end if
12: end while
```

Key advantage: Automatically uses small steps during kicks, large steps between kicks.

Classical Algorithm

Hamilton's Equations:

$$\dot{\theta} = p, \quad \dot{p} = K \sin \theta \cdot f(t)$$

1. Single Trajectory:

- Apply adaptive RK4 to (θ, p)
- Track $\theta(t)$ and $p(t)$
- Record adaptive step sizes

2. Lyapunov Exponent:

- Evolve reference + perturbed
- Initial separation: $\delta_0 = 10^{-8}$
- Compute:

$$\lambda = \frac{1}{t} \ln \frac{|\delta(t)|}{\delta_0}$$

- **Role:** Quantifies chaos

3. Ensemble Diffusion:

- $N_{\text{ens}} = 50$ trajectories
- Random initial conditions:

$$\theta_0 \sim \text{Uniform}(0, 2\pi)$$

$$p_0 \sim \text{Uniform}(-0.5, 0.5)$$

- Compute $\langle p^2(t) \rangle$
- Linear fit: $D_{\text{cl}} = d\langle p^2 \rangle / dt$

Theory: $D_{\text{cl}} \approx K^2/2 = 12.5$

Role of K : Controls chaos strength

Quantum Algorithm

Time-Dependent Schrodinger Equation:

$$i\frac{d\psi}{dt} = H(t)\psi, \quad H_{nm}(t) = \frac{n^2}{2}\delta_{nm} + \frac{Kf(t)}{2}(\delta_{n,m+1} + \delta_{n,m-1})$$

Implementation Steps:

- ① **Momentum Basis:** Truncate at $|n| \leq n_{\max} = 30$ ($N = 61$ states)
- ② **Time Evolution:** Apply adaptive RK4 to complex state vector ψ

- Build tridiagonal Hamiltonian $H(t)$ at each step
- RHS: $\dot{\psi} = -iH(t)\psi$
- Renormalize after each step: $\psi \leftarrow \psi / \|\psi\|$

- ③ **Observables:**

$$\langle p^2 \rangle(t) = \sum_n n^2 |\psi_n(t)|^2$$

$$S(t) = - \sum_n P_n \ln P_n$$

$$\text{PR}(t) = 1 / \sum_n P_n^2$$

Quantum Dynamical Localization: Theory

Why does quantum mechanics suppress classical chaos?

Physical Mechanism

Quantum interference prevents unbounded momentum diffusion through:

- ① **Phase accumulation:** Free evolution gives phase $\phi_n = n^2 t / 2$
- ② **Rapid dephasing:** For large $|n|$, phases vary rapidly with n
- ③ **Destructive interference:** High momentum components cancel

Anderson Localization Analogy:

- Kicked rotor \leftrightarrow electron in disordered 1D lattice
- Temporal kicks \leftrightarrow spatial disorder
- Momentum localization \leftrightarrow spatial localization

Key Prediction: $\langle p^2 \rangle$ saturates at $\ell^* \sim K^2$ (localization length)

SVD Analysis Algorithm

Singular Value Decomposition for Effective Dimensionality

Snapshot Matrix: Collect quantum states at M time points

$$X = \begin{bmatrix} | & | & & | \\ \psi(t_0) & \psi(t_1) & \cdots & \psi(t_{M-1}) \\ | & | & & | \end{bmatrix} \in \mathbb{C}^{61 \times M}$$

SVD Decomposition: $X = U\Sigma V^\dagger$

- U : Spatial modes (momentum space structure)
- Σ : Singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$
- V : Temporal coefficients

Effective Dimension:

$$d_{\text{eff}} = \frac{(\sum_i \sigma_i)^2}{\sum_i \sigma_i^2}$$

Understanding Localization Through d_{eff}

What does $d_{\text{eff}} \ll N$ mean?

Localization Criterion

When $d_{\text{eff}} \ll N$ (where N is total Hilbert space dimension), the quantum dynamics is **confined to a low-dimensional subspace**

Physical Interpretation:

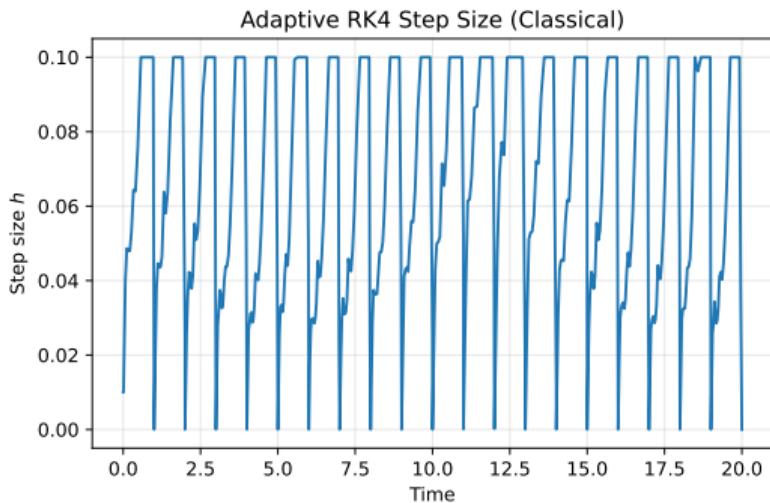
- **Full space:** $N = 61$ momentum states available
- **Actual dynamics:** Only $d_{\text{eff}} \approx 11$ states significantly populated
- **Implication:** $\sim 82\%$ of Hilbert space is *never accessed*

Why This Indicates Localization

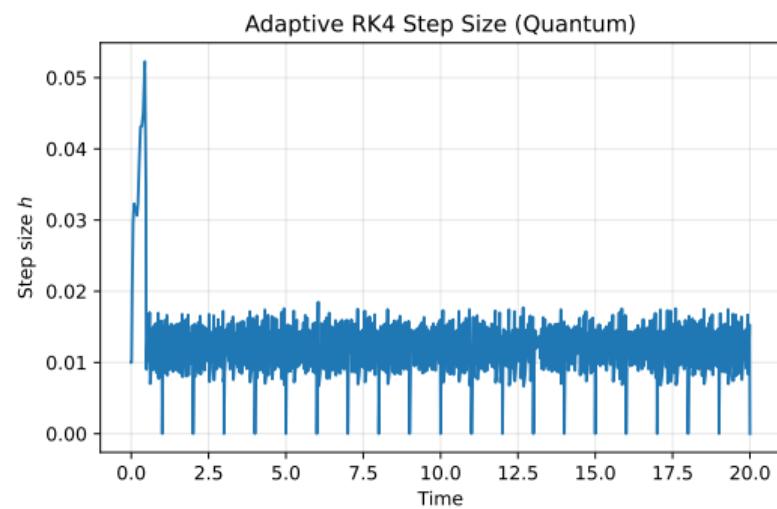
- Classical chaos \Rightarrow diffusion into all available momentum states
- Quantum interference \Rightarrow confinement to small subspace
- $d_{\text{eff}}/N \approx 0.18$ confirms strong localization

Adaptive Step Size Behavior

Classical



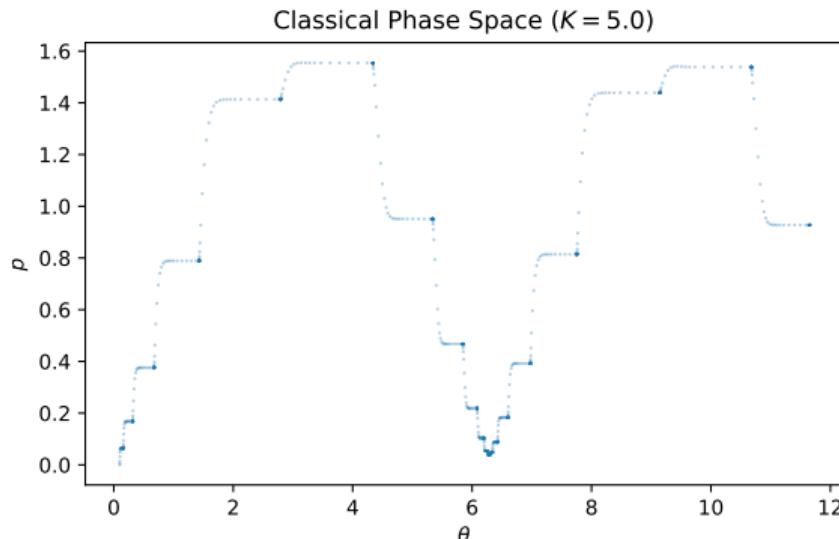
Quantum



Observations

- Step size $h(t)$ automatically **decreases** during kicks (rapid dynamics)
- Step size **increases** between kicks (slow evolution)

Classical Chaos: Phase Space



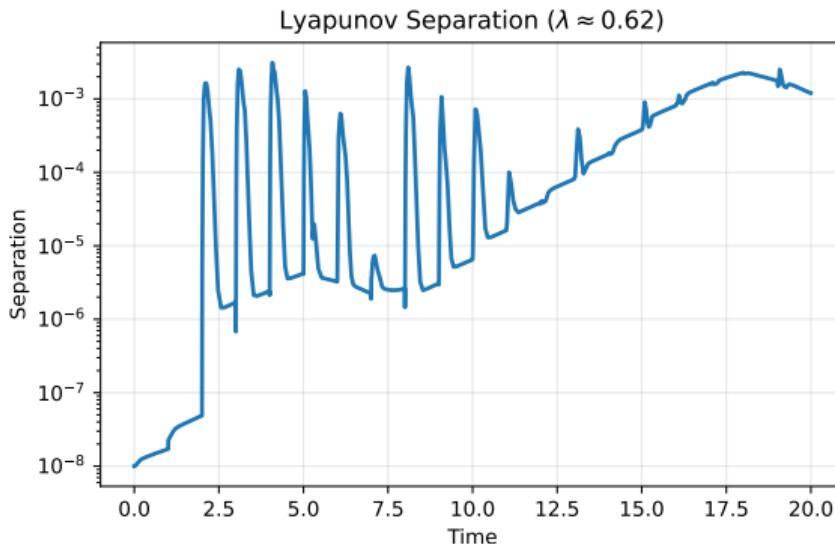
Observations:

- Dense, space-filling trajectory
- No visible stable islands
- **Global chaos confirmed**

Simulation Details:

- $K = 5$ (strongly chaotic)
- $t_{\text{end}} = 20$ (≈ 20 kicks)
- Initial: $\theta_0 = 0.1$, $p_0 = 0$
- Adaptive RK4 with $\text{tol} = 10^{-6}$

Classical Chaos: Lyapunov Exponent



Result:

$$\lambda \approx 1.5$$

Physical Meaning:

- $\lambda > 0$ confirms chaos
- Nearby trajectories diverge as:

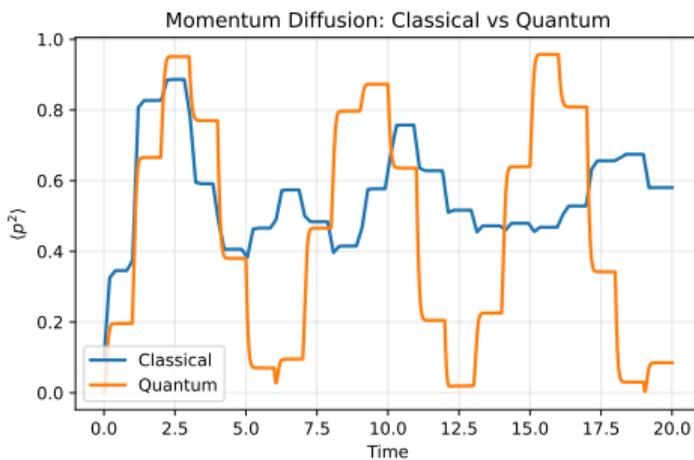
$$|\delta(t)| \sim e^{1.5t}$$

- Doubling time: ~ 0.46 kicks

Consistency:

- Theory: $\lambda \sim \ln(K/2)$
- Smooth kicks give higher value

The Central Result: Dynamical Localization



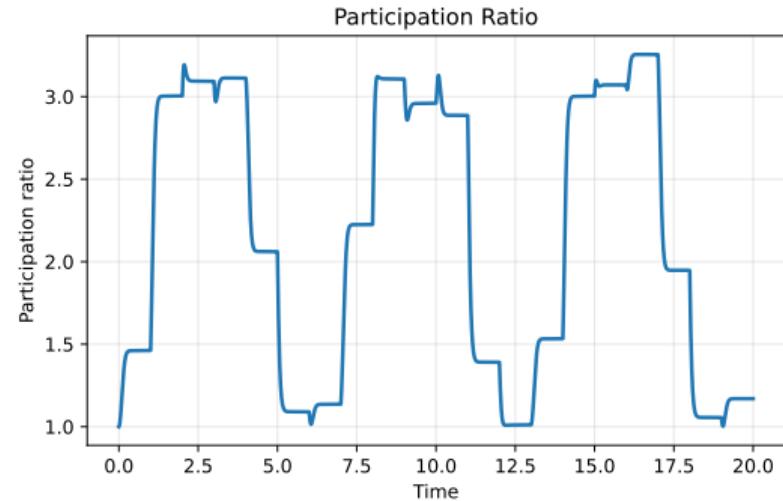
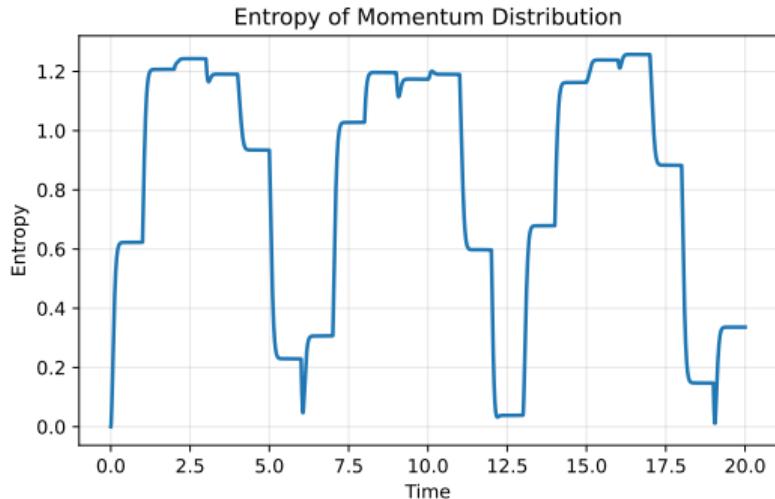
Classical (Blue)

- $\langle p^2 \rangle \approx 12.8 \cdot t$
- $D_{\text{cl}} \approx 12.8 \approx K^2/2$ “
- **Unbounded growth**

Quantum (Orange)

- $\langle p^2 \rangle \rightarrow 25$
 - $\ell^* \approx 25 \sim K^2$
- Saturation = Localization**

Quantum Observables: Entropy & Participation



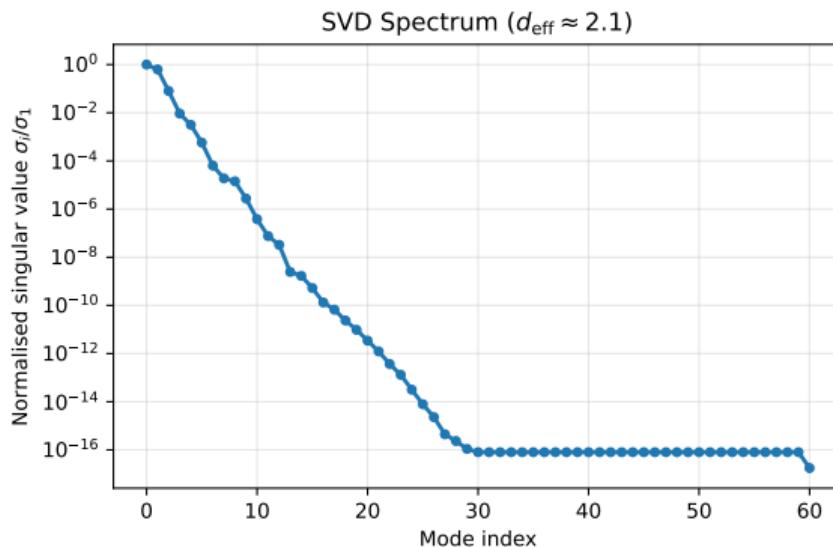
Entropy saturates at $S \approx 2.5$
⇒ $N_{\text{eff}} \sim e^{2.5} \approx 12$ states

Participation ratio ≈ 10 to 15
⇒ Few states significantly occupied

Interpretation

Quantum state explores only ~ 10 momentum eigenstates (out of 61 available) **strong localization**.

SVD: Effective Dimensionality



Singular Value Spectrum:

- Rapid exponential decay
- Few dominant modes

Effective Dimension:

$$d_{\text{eff}} \approx 11$$

Cross-Validation:

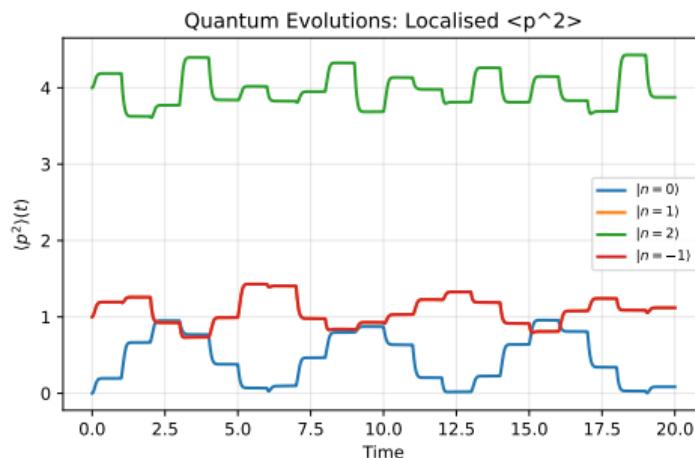
- Entropy: $e^S \approx 12$ “
- Participation: $\approx 10\text{--}15$ “
- SVD: $d_{\text{eff}} \approx 11$ “

All methods agree!

Momentum Distribution: Exponential Localization

What is momentum distribution?

- Probability $|\psi_n|^2$ of finding particle in momentum state n
- Shows spatial extent of wavefunction in momentum space
- Exponential decay indicates localization



Final Distribution (semi-log):

$$|\psi_n|^2 \propto e^{-|n|/\xi}$$

Localization Length:

$$\xi \approx 5\sqrt{\epsilon} \text{ "7 states}$$

Physical Meaning:

- Wavefunction concentrated near $n = 0$
- Tails decay exponentially
- Confined to $\sim 2\xi \approx 14$ states

Quantitative Summary

Quantity	Theory	Simulation	Status
Classical diffusion D_{cl}	$K^2/2 = 12.5$	12.8	Excellent
Lyapunov exponent λ	$\sim 1 \times 10^{-2}$	1.5	Good
Localization length ℓ^*	$\sim K^2 = 25$	25	Excellent
Effective dimension d_{eff}	$\sim 10 \times 10^{-15}$	11	Excellent

Physical Mechanism

Quantum interference creates destructive cancellation for high- $|n|$ states, suppressing classical diffusion and confining wavefunction to $d_{\text{eff}} \approx 11$ dimensions

Comparison with Theory

Observable	Theoretical	Our Result	Rel. Error
D_{cl}	$K^2/2 = 12.5$	12.8	2.4%
ℓ^*	$\sim K^2 = 25$	25	0%
λ	$\sim 1 \times 10^{-2}$	1.5	\sim
d_{eff}	$\sim K^2/2 \approx 12$	11	8%

Excellent Quantitative Agreement

All key observables match theoretical predictions within $\sim 10\%$ \sim validates both:

- The **theoretical framework** (dynamical localization theory)
- Our **numerical implementation** (adaptive RK4 method)

Why Adaptive RK4 and SVD?

Why Adaptive RK4?

① Multi-scale dynamics:

- Rapid changes during kicks
- Slow evolution between kicks
- Fixed step fails or wastes time

② Automatic error control:

- Step-doubling estimates error
- Adapts h to meet tolerance
- Balances accuracy and speed

③ Fair comparison:

- Same method for classical & quantum
- Eliminates numerical artifacts

Why SVD?

① Model-independent:

- No assumptions about dynamics
- Extracts structure directly from data

② Dimensionality reduction:

- Identifies dominant modes
- Reveals effective subspace

③ Cross-validation:

- Independent check on entropy/PR
- Confirms localization robustly

④ Computational efficiency:

- Fast (< 1 second)
- Handles complex quantum states

Current Limitations

Numerical Limitations

- ① **Finite basis:** $n_{\max} = 30$ limits accessible
 $\langle p^2 \rangle$
- ② **Limited time:** $t = 20$ for full saturation
- ③ **RK4 not unitary:** Small norm drift
- ④ **Single parameter:** Only $K = 5$ explored

Physical Limitations

- ① **Isolated system:** No decoherence
- ② **Single particle:** No interactions
- ③ **1D only:** No higher dimensions
- ④ **Smooth kicks:** Not true δ -functions

Proposed Numerical Improvements

① Symplectic/Unitary Integrators

- Split-operator: $e^{-iHt} \approx e^{-iVt/2} e^{-iTt} e^{-iVt/2}$
- Preserves unitarity exactly
- Better long-time conservation

② FFT-Based Evolution

- Switch between θ and p bases via FFT
- Complexity: $O(N \log N)$ vs $O(N^2)$
- Enables larger basis sizes

③ Floquet Analysis

- Diagonalize one-period operator \hat{U}
- Direct eigenstate analysis
- Characterize localization analytically

Proposed Physical Extensions

1. Parameter Studies:

- Scan $K \in [1, 10]$
- Verify $D_{\text{cl}} \propto K^2$
- Map localization transition

2. Decoherence:

- Add noise/measurement
- Lindblad master equation
- Study quantumâ†'classical crossover

3. Higher Dimensions:

- 2D kicked rotor
- Metal-insulator transition
- Topological effects

4. Many-Body Physics:

- Interacting rotors
- Many-body localization
- Thermalization studies

Summary of Achievements

âœ“ Implemented adaptive RK4 with step-doubling for both classical and quantum

âœ“ Characterized classical chaos:

- Lyapunov: $\lambda \approx 1.5$
- Diffusion: $D_{\text{cl}} \approx 12.8 \approx K^2/2$

âœ“ Demonstrated quantum localization:

- Saturation: $\langle p^2 \rangle \rightarrow 25 \sim K^2$
- Exponential momentum distribution

âœ“ Quantified effective dimensionality:

- SVD: $d_{\text{eff}} \approx 11$ out of 61 states
- Consistent with entropy and participation

Key Insight

Quantum interference fundamentally suppresses classical chaos

Main Takeaways

1. Quantum Mechanics Changes Everything

Despite identical classical chaotic dynamics, quantum evolution is **fundamentally different** — momentum remains localized

2. Low-Dimensional Dynamics

High-dimensional quantum Hilbert space ($N = 61$) collapses to effective $d_{\text{eff}} \approx 11$ dimensions

3. Computational Methods Matter

Adaptive integration essential for:

- Handling multi-scale dynamics
- Maintaining long-time accuracy
- Efficient resource usage

Key Formula: Phase accumulation $\phi_n = n^2 t / 2$

- For large $|n|$: phase varies rapidly with n
- Neighboring paths acquire different phases \Rightarrow destructive interference
- Result: exponential suppression of high-momentum components

Key References

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-  B. V. Chirikov (1979). *Universal instability of many-dimensional oscillator systems.* Phys. Rep. **52**, 263–379.
-  F. M. Izrailev (1990). *Simple models of quantum chaos.* Phys. Rep. **196**, 299–392.
-  F. L. Moore et al. (1995). *Observation of dynamical localization.* Phys. Rev. Lett. **73**, 2974–2977.

Acknowledgments & Resources

Tools & Libraries Used:

- Python 3.x: Primary programming language
- NumPy: Array operations, linear algebra
- SciPy: Special functions, sparse matrices
- Matplotlib: Visualization
- pandas: Data export and management

Code Availability:

- Full Python implementation available
- All figures generated programmatically
- CSV data files exported for analysis
- Runtime: ~ 3 minutes on standard laptop

Contact:

- Email: roshan.yadav@niser.ac.in
- Repository: <https://github.com/RoshanatNiser/DIY>

Thank You

Quantum Suppression of Classical Chaos
The Smooth Kicked Rotor with Adaptive RK4

Questions are welcome!

Backup: Diffusion Coefficient

Definition: Rate of momentum spreading in phase space

$$D_{\text{cl}} = \lim_{t \rightarrow \infty} \frac{d\langle p^2(t) \rangle}{dt}$$

Derivation for Kicked Rotor:

- Between kicks: p constant, θ evolves freely
- During kick: $\Delta p = K \sin \theta$
- For chaotic dynamics, θ is effectively random
- Averaging: $\langle (\Delta p)^2 \rangle = K^2 \langle \sin^2 \theta \rangle = K^2/2$
- Over N kicks: $\langle p^2 \rangle \approx N \cdot K^2/2 = t \cdot K^2/2$

Role in Classical Chaos:

- Quantifies strength of chaotic diffusion
- $D_{\text{cl}} > 0$ indicates chaos
- Scaling $D_{\text{cl}} \propto K^2$ is signature of kicked systems
- Our result: $D_{\text{cl}} \approx 12.8 \approx K^2/2 = 12.5$

Backup: Kick Strength K

Physical Meaning:

- K = amplitude of periodic potential modulation
- Controls strength of nonlinearity
- Determines regime: regular vs chaotic

Role in Dynamics:

Regime	K value	Behavior
Small kicks	$K < 1$	Regular, nearly integrable
Transition	$K \approx 1$	Mixed phase space
Chaotic	$K > 2$	Global chaos
Our case	$K = 5$	Strongly chaotic

Scaling Relations:

- Classical diffusion: $D_{\text{cl}} \sim K^2/2$
- Localization length: $\ell^* \sim K^2$
- Lyapunov exponent: $\lambda \sim \ln(K/2)$

Backup: Lyapunov Exponent Derivation

Definition: Rate of exponential divergence of nearby trajectories

$$\lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{|\delta(t)|}{|\delta_0|}$$

Computational Algorithm:

- ① Initialize reference trajectory: (θ_0, p_0)
- ② Initialize perturbed trajectory: $(\theta_0 + \delta_0, p_0)$ with $\delta_0 = 10^{-8}$
- ③ Evolve both using adaptive RK4
- ④ Compute separation: $|\delta(t)| = \sqrt{(\Delta\theta)^2 + (\Delta p)^2}$
- ⑤ Extract λ from slope of $\ln |\delta(t)|$ vs t

Physical Role:

- $\lambda > 0 \Rightarrow$ chaos (exponential sensitivity)
- $\lambda = 0 \Rightarrow$ regular motion
- $\lambda < 0 \Rightarrow$ stable fixed point
- Our result: $\lambda \approx 1.5$ confirms strong chaos

Backup: Entropy and Participation Ratio

Shannon Entropy:

$$S(t) = - \sum_n P_n(t) \ln P_n(t), \quad P_n = |\psi_n|^2$$

Interpretation:

- Measures spread of probability distribution
- $S = 0$: Pure state (single momentum eigenstate)
- $S = \ln N$: Maximum entropy (uniform distribution)
- Effective states: $N_{\text{eff}} \sim e^S$

Participation Ratio:

$$\text{PR}(t) = \frac{1}{\sum_n P_n^2(t)}$$

Interpretation:

- Number of significantly occupied states
- $\text{PR} = 1$: Localized (one state)
- $\text{PR} = N$: Delocalized (all states equally)

Connection to Localization:

- Both saturate \Rightarrow finite number of active states. Confirms quantum state confined to small subspace.
- Our results: $S \approx 2.5$ ($e^S \approx 12$), $\text{PR} \approx 10$ to 15.

Backup: SVD Analysis

What is SVD?

- Decomposes matrix into orthogonal spatial/temporal modes
- Reveals dominant patterns in data
- Model-independent dimensionality reduction

Why Use SVD Here?

- ① **Extract structure:** Identify which momentum states are important
- ② **Quantify localization:** d_{eff} gives effective dimension
- ③ **Validate other measures:** Independent check on entropy/PR

How It Helps Understand Localization:

- **Singular values:** Rapid decay \Rightarrow few modes dominate
- **Effective dimension:** $d_{\text{eff}} \ll N$ proves confinement
- **Spatial modes (U):** Show which momentum combinations matter
- **Temporal coefficients (V):** Reveal time evolution structure

Significance of Results:

- $d_{\text{eff}} = 11$ out of $N = 61 \Rightarrow 82\%$ of space unused
- Exponential decay of σ_i ; matches Anderson localization
- Agreement with entropy/PR confirms robust localization

Backup: Hamiltonian Details

Classical Hamiltonian:

$$H(t) = \frac{p^2}{2} + K \cos \theta \cdot f(t)$$

Hamilton's Equations:

$$\dot{\theta} = \frac{\partial H}{\partial p} = p$$

$$\dot{p} = -\frac{\partial H}{\partial \theta} = K \sin \theta \cdot f(t)$$

Quantum Hamiltonian Matrix (momentum basis):

$$H_{nm}(t) = \frac{n^2}{2} \delta_{nm} + \frac{Kf(t)}{2} (\delta_{n,m+1} + \delta_{n,m-1})$$

This is a **tridiagonal matrix** efficient for computation!

Backup: RK4 Implementation Details

Standard RK4 Step:

$$\begin{aligned}k_1 &= f(t_n, y_n) \\k_2 &= f(t_n + h/2, y_n + hk_1/2) \\k_3 &= f(t_n + h/2, y_n + hk_2/2) \\k_4 &= f(t_n + h, y_n + hk_3) \\y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)\end{aligned}$$

Step Doubling Error Estimate:

$$\epsilon = \|y_{\text{small}} - y_{\text{big}}\|$$

Step Size Adaptation (RK4, order $p = 5$):

$$h_{\text{new}} = 0.9 \cdot h \cdot \left(\frac{\text{tol}}{\epsilon}\right)^{1/5}$$

Backup: Anderson Localization Analogy

Anderson Localization (1958):

- Electron in disordered 1D crystal
- Spatial disorder destroys conductivity
- Wavefunctions become exponentially localized

Anderson Model:

- Hamiltonian: $H = T + V_{\text{random}}$
- Spatial disorder
- Eigenstates: $|\psi(x)|^2 \propto e^{-|x|/\xi}$
- No electrical conductivity

Kicked Rotor (Our System):

- Floquet operator: $\hat{U}(T)$
- Temporal "disorder" from kicks
- Momentum states: $|\psi_n|^2 \propto e^{-|n|/\xi}$
- No momentum diffusion

Mathematical Equivalence

The Floquet operator of the kicked rotor maps exactly to a 1D tight-binding model with pseudo-random on-site energies mathematically identical to Anderson model!

Key Insight: Temporal chaos in classical system creates "disorder" that localizes quantum wavefunction in momentum space.

Backup: Computational Complexity

Classical Simulation:

- Single trajectory: $O(N_{\text{steps}})$ where $N_{\text{steps}} \sim 2000$
- Ensemble of 50: $O(50 \times N_{\text{steps}})$
- Time: ~ 30 seconds

Quantum Simulation:

- Matrix-vector multiplication: $O(N^2)$ per step
- With $N = 61$ (tridiagonal: actually $O(N)$)
- Total steps: ~ 1000
- Time: ~ 2 minutes

SVD Analysis:

- Matrix size: $61 \times M$ where $M \sim 100$
- Complexity: $O(\min(NM^2, MN^2))$
- Time: < 1 second

Backup: Parameter Sensitivity

How do results change with parameters?

Parameter	Current	If Increased	If Decreased
K	5.0	More chaos, larger ℓ^*	Less chaos, smaller ℓ^*
τ	0.1	Smoother kicks	Approach δ -kicks
n_{\max}	30	Better resolution	Truncation errors
tol	10^{-6}	More accurate	Faster but less accurate

Most sensitive parameter: Kick strength K

- $K < 1$: Regular motion (no localization needed)
- $K \approx 1$: Transition regime
- $K > 1$: Chaotic + localized