CSC565 - Graph Theory | **North Carolina State University**
Fall 2019 | **Dept. of Computer Science**

Lecture 12: Oct 2, 2019

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12.1 Embeddings

12.1.1 Introduction

We recall that trees and forests have no K_3, which implies that they don't contain any cycles. Thus, we can say that trees and forests can be embedded into planes or we can obtain embedding of tree and forest graphs.

12.1.2 Understanding Embedding

In general, we can represent a graph G as:

$G = (V, E)$

The geometric realisation of G will be represented as:

$geom(G) \subseteq \mathbb{R}^n$

So, a continuous and injective mapping of this geometric realisation into a plane gives us an embedding of G, which can be denoted as:

$\underline{G} \subseteq \mathbb{R}^2$

Definition: Faces are connected components of $\mathbb{R}^2 \setminus \underline{G}$

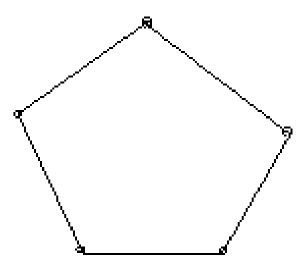

Figure 12.1: A C_5 graph

The embedding for C_5 graph as shown in 12.1 has 2 faces.

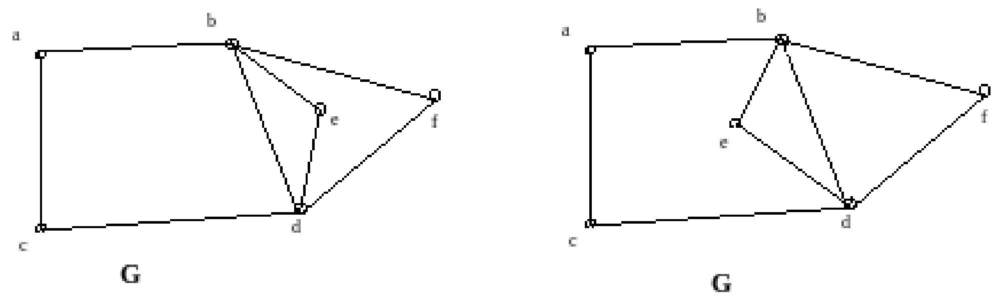

Figure 12.2: Example of faces on graphs

The number of faces are same for both the left and right graphs but some of the faces from the embedding of these 2 graphs are different as they are formed using different cycles.

12.1.3 Paths in graph and Topological spaces

A path in a topological space is a continuous function from $[0, 1]$ to space X, i.e., $f : [0, 1] \longrightarrow X$

A path in a graph is some graph $P_m = (\{0, ..., m\}, \{i - 1, i\} : i \in (1, ..., m))$

As we can see that these two definitions of paths are closely related, i.e. an embedding of a path graph can be seen as a mapping into the interval $[0, 1]$ (or can be seen as a path in linear space). The graph in 12.3 shows how a path graph P_m can be visualised as an embedding into a topological space.

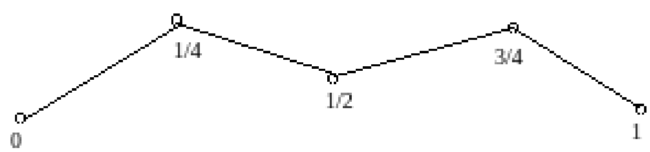

Figure 12.3: Embedding of P_m in topological space

12.1.4 Linear embeddings

Linear embedding maps straight lines to straight lines. It is defined as:

$M : \mathbb{R}^n \longrightarrow \mathbb{R}^2$ where M is a matrix, $M \in \mathbb{R}^{2xn}$

and if we define a point x as $x \in \mathbb{R}^n$

we get, $Mx \in \mathbb{R}^2$

$$\begin{bmatrix} \cdots & x_j & \cdots \\ \cdots & y_j & \cdots \end{bmatrix} b_j = \begin{bmatrix} x_j \\ y_j \end{bmatrix}$$

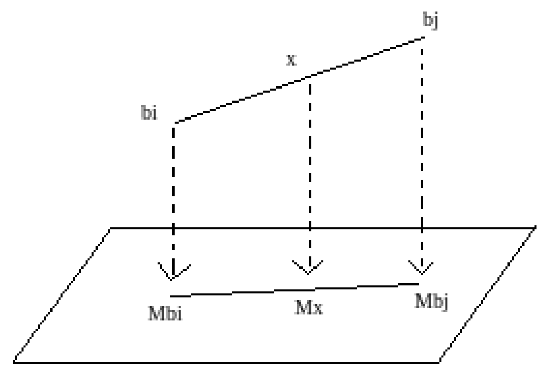

Figure 12.4: A linear embedding of edge on plane

From the figure 12.4, we understand that $x = \overline{b_i b_j}$

or, it an be written as a function of time as, $x = (1-t)b_i + tb_j$

Here, b_i and b_j are standard basis vectors. b_i is a vector with all 0's in (n-1) positions and 1 in i^{th} position. Consequently, b_j is formed in same way.

Equivalently, $Mx = M((1-t)b_i + tb_j)$

or, $Mx = (1-t)Mb_i + tMb_j$

12.1.5 Polygons

A simple polygon is a linear embedding of C_k (for some k).

Similarly, a polygon curve is a linear embedding of P_m.

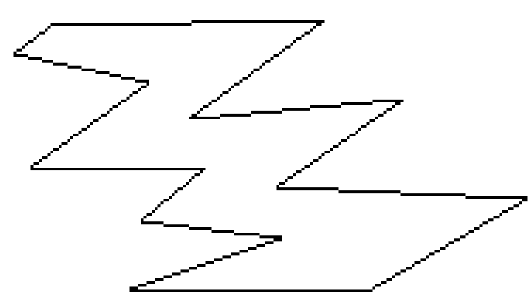

Figure 12.5: A simple polygon: a linear embedding of C_{14}

12.2 Jordan Curve Theorem

12.2.1 Definition

According to the (polygonal) Jordan Curve Theorem, every polygon P separates the plane into two pieces.

12.2.2 J(x)

In general, $J(x) =$ (No. of crossings with a ray) mod 2

$J(x, r) = |ray(x, r) \cap P| \% 2$ where $x \in \mathbb{R}^2 \setminus P$.

This says that if x is any point inside or outside of any simple polygon P, then J(x) gives the number of intersections with P of a ray from x in the direction of r mod 2.

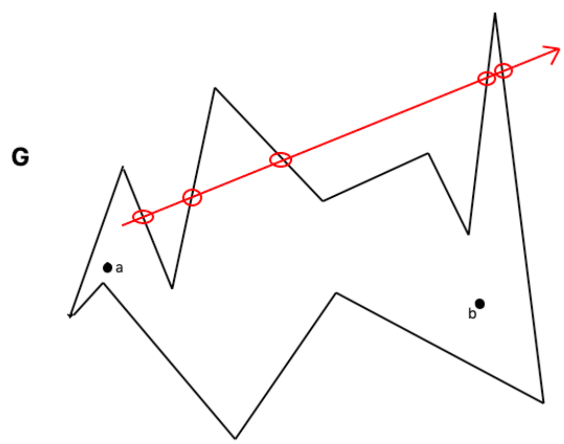

Figure 12.6: Significance of J(x)

In Figure 12.6, it can be noticed that the total number of intersections of the ray r with the polygon P is always odd (called odd parity) when the ray begins from inside of the polygon and eventually emerges outside of the polygon or when it starts outside and emerges inside. Similarly, the total number of intersections is always even (called even parity) when the ray stays inside or outside of the polygon relative to where it began.

12.2.3 Valid crossings

Figure 12.7 illustrates few examples of valid and invalid crossings:

In figure 12.7, (1), (4) and (5) are valid whereas (2) and (5) are invalid crossings.

12.2.4 Significance of the parity of number of crossings

If we rotate the ray, the parity of the number of crossings remain the same before and after the rotation and is independent of the choice of the ray.

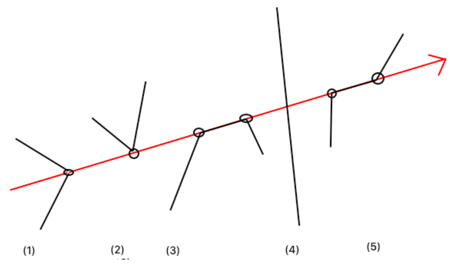

Figure 12.7: Valid Crossings

This can be illustrated by checking the parity of intersections of ray in different cases in Figure 12.7.

Case #	**Before crossing**	**After crossing**
1	1	1
2	0	2
3	2	0
4	1	1
5	1	1

As clear from the table, the parity remains the same in all the five cases of figure 12.7.

Theorem 12.1. *a and b are connected* $\iff J(a) = J(b)$

Proof. 1. a and b are connected $\implies J(a) = J(b)$
This can be proved by induction.
Say a, b are connected by a path of length m.
Base: one segment From figure 12.8, ray (a, r) \ray (x,r) = $\overline{ax}$

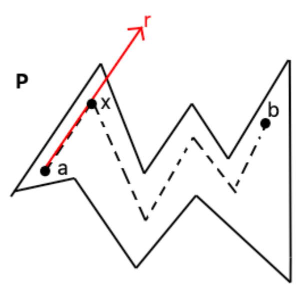

Figure 12.8: Proof by induction

$\implies J(a) = J(x)$

Induction step: Path x to b has length of m - 1.
$\implies J(x) = J(b)$

So, J(a) = J(b)

2. J(a) = J(b) $\implies$ a and b are connected

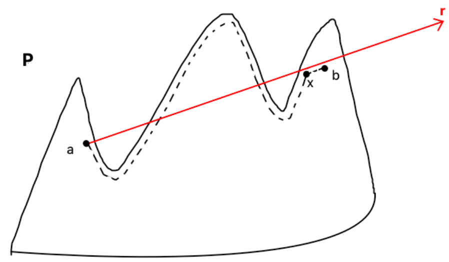

Figure 12.9: Path along the edge of P

J(x) = J(a)
$\implies J(x) = J(b)$, since J(a) = J(b)
So, $\overline{xb} \subseteq \mathbb{R}^2 \backslash$P $\square$

12.3 Face of graphs

12.3.1 Introduction

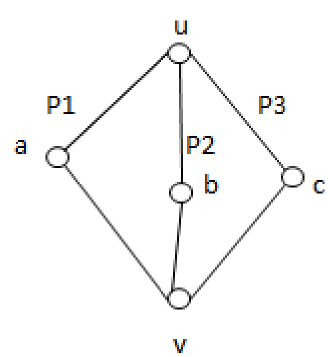

Figure 12.10: Three disjoint paths from u to v

In figure 12.11, there are 3 disjoint paths from u to v. Let us call them P_1, P_2, P_3. The $P_1 \cup P_2$, $P_2 \cup P_3$ and $P_1 \cup P_3$ are the three faces of the graph. If we take any pair of these paths, we get a simple closed curve having an inside and a outside. $P_1 \cup P_3$ is an embedding of a circle and the path P_2 has to be either inside or outside of it. If we want to consider the vertex in P_2 to be outside, one of the other vertices has to lie inside.

We draw two paths from a to c. One of them stays inside and the other outside. Together they form a Jordan curve. Now, any path that goes to u to v has to cross one of these paths. Let us draw a path P_2 on the outside which crosses

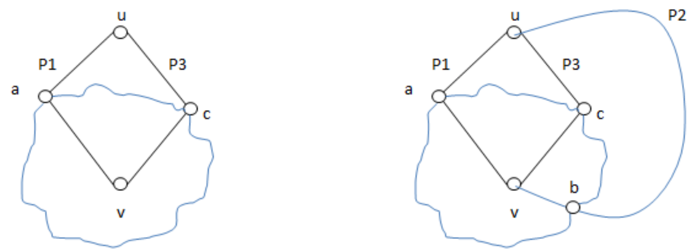

Figure 12.11: Identifying faces using Jordan curve

the Jordan curve at vertex B. We have now created a new Jordan curve with P_1 and P_2 which separates C from the outer face. This is how we show that we cannot draw a graph like $K_{2,3}$ with all the vertices on the outer surface. K_4 is also not outerplanar.

12.3.2 Euler's Formula for the plane

If G is an embedding then $e(G) = |V_G| - |E_G| + |F_G|$ where F_G is the number of faces in the embedding

Theorem 12.2. *$e(G) = 1 + |C_G|$ where C_G is the number of connected components of the graph*

Corollary 12.3. *Since trees do not have any cycles, they have only one face. Hence, every drawing of a tree is a outerplanar embedding. This is also true for a forest.*

Proof. We perform induction on the number of edges. We first remove an edge and add the edge back in and observe what happens.
We keep a track of 4 different things - size of edge set, size of vertex set, number of faces and number of components.
$|V_G| - |E_G| + |F_G| = 1 + |C_G|$

We do not change the number of faces but we are reducing the number of components by 1.

Let us remove edge e(u,v) of the following Tree. Now, we can get the following cases:

Case 1: Faces on each side of e were the same.

$|F'| = |F|$ i.e. when the edge e is removed, the number of faces do not change. This means that around the edge, either sides were already connected. There was already a path as shown in the figure.

The Jordan curve separates the inside from the outside. So, the number of connected components has to go up when we remove the edge.

So, there exists some polygon P separating the components.

After removing the edge, $|C'| = |C_G| + 1$

$1 + |C| = |V| - |E| + |F|$

$2 + |C_G| = |V_G| - |E - 1| + |F|$

Case 2: Different faces on each side of e

$|F'| = |F| - 1$

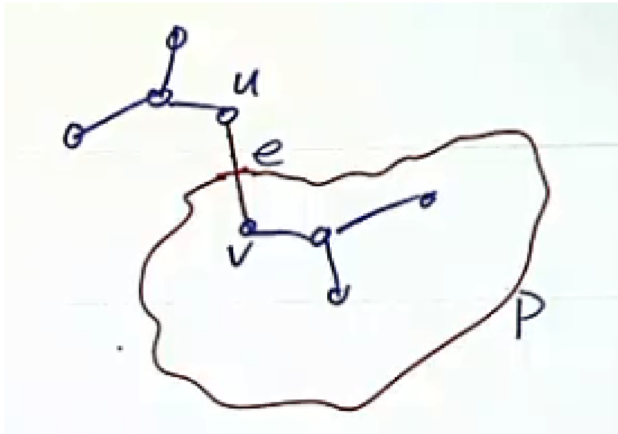

Figure 12.12:

In this case, e must have been part of a cycle surrounding one of the faces that separated it from the other one. If we look at that cycle, we find another path from u to v which does not include e. Since there was already a path from u to v, number of connected components does not change.

$|C'| = |C|$ Therefore, faces and edges go down by one and the equation holds in this case too. $\square$