**CSC 565: Graph Theory** | **North Carolina State University**
**Fall 2019** | **Computer Science**

# Lecture 9: Sep 23, 2019

*Lecturer: Don Sheehy* <`drsheehy@ncsu.edu`> *Scribe: Sharav Mathur, Kushal Batra, Nihal Narayanan*

## 1 The Torus Graph(s)

The *Torus Graph* $T_{j,k}$ is a $j \times k$ grid with extra edges connecting the leftmost and rightmost vertex in each row as well as the top and bottom vertices of each column.

**Question 1.** Prove that for any pair of vertices $a, b \in V_{T_{j,k}}$, there exists an isomorphism $f : T_{j,k} \to T_{j,k}$ such that $f(a) = b$.

*Proof.* Consider a vertex $a \epsilon T_{j,k}$ in the form $(m, n)$ where $m \epsilon (0, 1, ...j - 1)$ and $n \epsilon (0, 1, ...k - 1)$.

Its four adjacent vertices in the Torus graph would be :

$$((m + 1)\%j, n)$$
$$((m - 1)\%j, n)$$
$$(m, (n + 1)\%k)$$
$$(m, (n - 1)\%k)$$

Consider any other vertex $b \epsilon T_{j,k}$ in the form $((m + x)\%j, (n + y)\%k)$. Changing $x$ and $y$, we get every other vertex in $T_{j,k}$.

Consider the function $f$ that maps from $a$ to $b$ :

$$f(a) = b$$

then the following also holds true for its adjacent vertices :

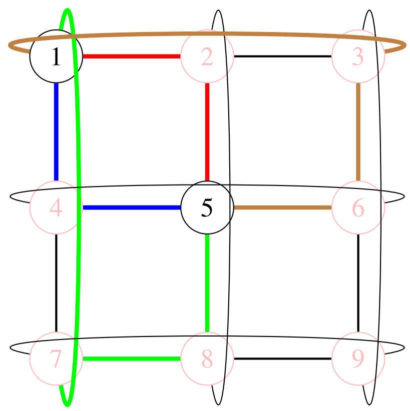
$$f(((m + 1)\%j, n)) = (((m + 1) + x)\%j, n + y) = (((m + x) + 1)\%j, n + y)$$
$$f(((m - 1)\%j, n)) = (((m - 1) + x)\%j, n + y) = (((m + x) - 1)\%j, n + y)$$
$$f((m, (n + 1)\%k)) = (m + x, ((n - 1) + y)\%k) = (m + x, ((n + y) - 1)\%k)$$
$$f((m, (n + 1)\%k)) = (m + x, ((n + 1) + y)\%k) = (m + x, ((n + y) + 1)\%k)$$

From above, we can see that there is a bijective mapping, which when applied to $a$'s adjacent vertices, gives us $b$'s adjacent vertices. Since b is any vertex in the Torus graph, $T_{j,k}$, we can conclude that for any pair of vertices $a, b \in V_{T_{j,k}}$, there exists an isomorphism $f : T_{j,k} \to T_{j,k}$ such that $f(a) = b$. □

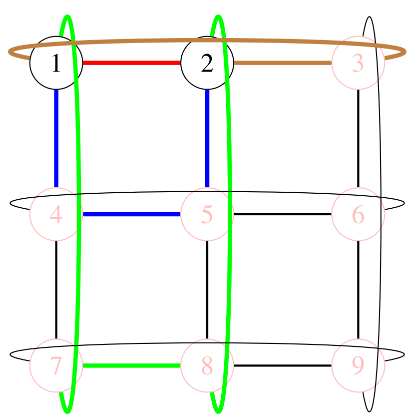
**Question 2.** Prove that $T_{3,3}$ is 4-connected

*Proof.* We need to show that for all $u, v \epsilon \ V_G$, there exist four disjoint paths from $u$ to $v$.
From the symmetry in the Isomorphism of the *Torus Graph* shown in the above question, we are left with just two cases to prove that it is 4-connected.

**Case 1.** The two vertices are adjacent to each other.




**Case 2.** The two vertices are diagonal to each other.



In **Case 1**, the four different colors show the four disjoint paths to reach from vertex 1 to vertex 5.
In **Case 2**, the four different colors show the four disjoint paths to reach from vertex 1 to vertex 2.

$\therefore$ From the above two diagrams it is easily observed that in both the cases, we are able to get four different paths that have no shared vertices (except the ends) from source to destination.

This is another way of defining that the *Torus Graph* $T_{3,3}$ is 4-connected. □

**Question 3.** Prove that if $j$ and $k$ are multiples of 3, then there exists a Morphism from $T_{j,k}$ to $T_{3,3}$.

*Proof.* Let $j = 3m$ and $k = 3n$.

Consider the mapping $f$:

$$f : T_{3m,3n} \to T_{3,3}$$
$$f_V((a, b)) = (a\%3, b\%3)$$

For the above mapping to be a Graph Morphism, we need to show that if there is an edge in $T_{3m,3n}$, then there is a corresponding edge in $T_{3,3}$

For example, consider a vertical edge, $E = \{(a, b), ((a + 1)\%3m, b) \epsilon T_{3m,3n}$

$$\begin{aligned} f_V((a + 1)\%3m, b)) &= (((a + 1)\%3m)\%3, b\%3) \\ &= ((a + 1)\%3, b\%3) \end{aligned}$$

We know that $((a + 1)\%3, b\%3) \epsilon T_{3,3}$

Similarly, we can also show this to be true for any horizontal edge.

$$\begin{aligned} f_V((a, (b + 1)\%3n)) &= (a\%3, ((b + 1)\%3n)\%3) \\ &= (a\%3, (b + 1)\%3) \end{aligned}$$

We know that $((a + 1)\%3, b\%3) \epsilon T_{3,3}$ and $(a\%3, (b + 1)\%3) \epsilon T_{3,3}$

$\implies$ There exists a morphism from $T_{j,k}$ to $T_{3,3}$.

□

**Question 4.** Prove that if $j \geq 3$ and $k \geq 3$, then $T_{j,k}$ contains a $T_{3,3}$ topological minor. Recall that this means there is a subgraph of $T_{j,k}$ that can be transformed into $T_{3,3}$ by contracting paths of degree 2 vertices (i.e. induced paths) into a single edge.

*Proof.* To prove the question, let us first take an example that satisfies the condition for $j \geq 3$ and $k \geq 3$.
The Graph $T_{4,6}$ looks as shown below:

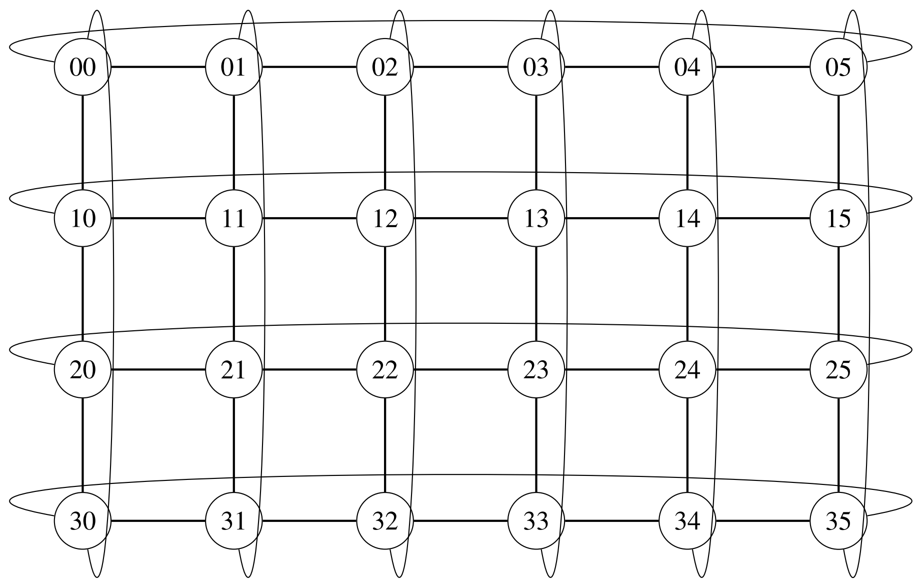



Figure 1: Torus Graph $T_{4,6}$

For $T_{4,6}$ to contain a $T_{3,3}$ topological minor, we first have to take a subgraph of $T_{4,6}$. This is done to obtain vertices of degree 2.

For any general $T_{j,k}$, we apply the following three operations to obtain our subgraph:

1. For $x \geq 0$ and $y \geq 3$,
   Remove edge between vertices (x,y) and (x+1%j,y)
2. For $x \geq 3$ and $y \geq 0$,
   Remove edges between vertices (x,y) and (x,y+1%k)
3. Remove all vertices (x,y) for $x \geq 3$ and $y \geq 3$.

This subgraph, $H$ now looks as shown below:

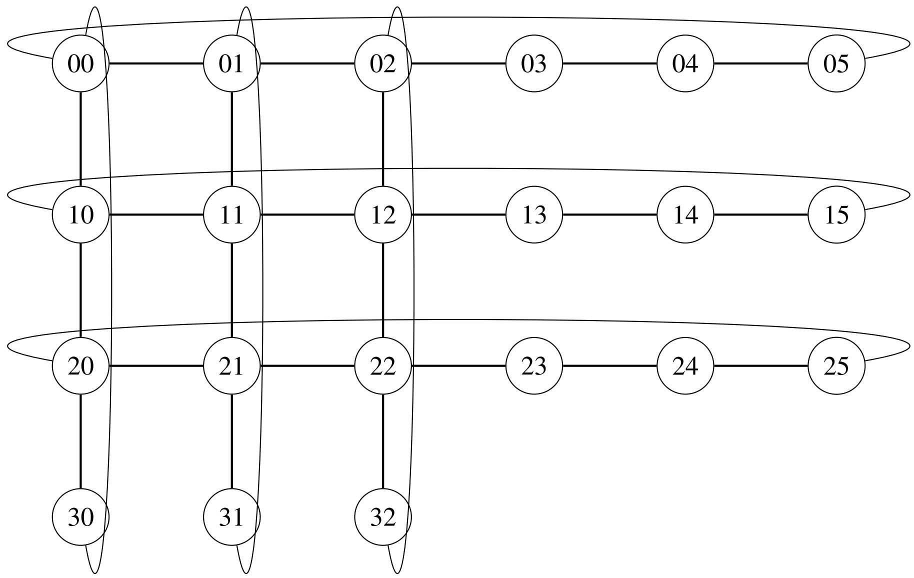



Figure 2: The above subgraph of $T_{4,6}$ is obtained by doing above operations

Now from the subgraph, we obtain the desired minor by contracting paths of degree 2 vertices into a single edge. This is done by applying the following 2 operations:

1. Contract Path $P_1 = \{(x, 2), (x, 3) \ldots (x, k-1), (x, 0)\}$ to the Edge $\{(x, 2), (x, 0)\}$, for x $\epsilon$ $\{0, 1, 2\}$
2. Contract Path $P_2 = \{(2, y), (3, y) \ldots (j-1, y), (0, y)\}$ to the Edge $\{(2, y), (0, y)\}$, for y $\epsilon$ $\{0, 1, 2\}$

Resulting graph is the Graph $T_{3,3}$.

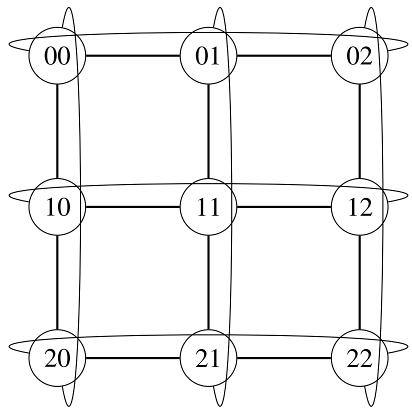



Figure 3: $T_{3,3}$ topological minor in $T_{j,k}$

$\implies$ Any Graph $T_{j,k}$ with $j \geq 3$ and $k \geq 3$ contains a $T_{3,3}$ minor □

## 2 Graph Products

**Question 5.** Show that $K_2 \times K_2 \cong C_4$.

*Proof.* In Graph theory, the Cartesian product $G \times H$ of graphs $G$ and $H$ is a graph such that the vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$; and two vertices $(u, u')$ and $(v, v')$ are adjacent in $G \times H$ if and only if either $u = v$ and $u'$ is adjacent to $v'$ in $H$, or $u' = v'$ and $u$ is adjacent to $v$ in $G$.

Without any loss of generality, shown below are two $K_2$ graphs.

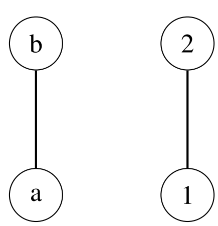



Figure 4: Two $K_2$ Graphs

Below is their Cartesian Product.

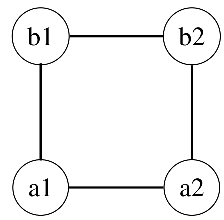



Figure 5: $C_4$ Graph obtained from $K_2$ X $K_2$

$\therefore$ We can see that the resulting graph is a $C_4$.

□

**Question 6.** Show that $K_2 \times K_3$ can be drawn in the plane without crossing.

*Proof.* Shown below are the two graphs $K_2$ and $K_3$ and their Cartesian product $K_2 \times K_3$

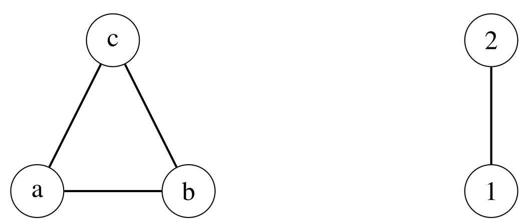



Figure 6: $C_3$ and $K_2$

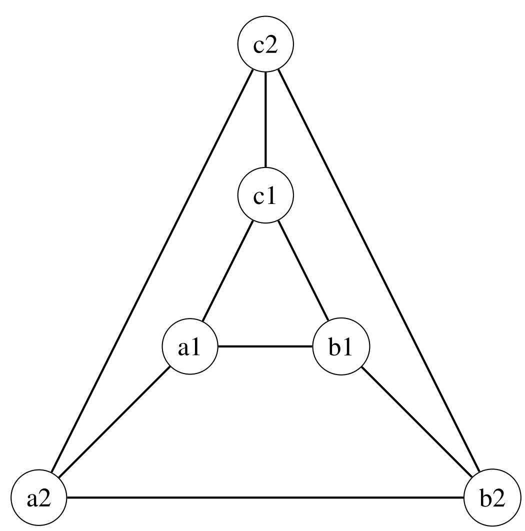



Figure 7: $C_3$ X $K_2$

The resulting graph can be drawn in the plane without crossing.

We can also see that this Graph does not contain any $K_{3,3}$ or $K_5$ minors or equivalently no $K_{3,3}$ or $K_5$ topological minors. From **Kurtowski's Theorem**, we have that the Graph is planar.

□

**Question 7.** Prove that if $B$ is connected, then $A$ is a minor of $A \times B$ by giving a surjective simplicial map with connected preimages.

*Proof.* Let us check the following mapping $f$ from $A \times B$ to $A$

$$f : A \times B \to A$$
$$f_V : (a, b) \to a$$
$$f_E : \{(a, b), (a, b')\} \to \{a, a\} = \{a\}$$
$$f_E : \{(a, b), (a', b)\} \to \{a, a'\} \epsilon E_A$$

To show that $A$ is a minor of $A \times B$, we need to show that firstly, the mapping above is surjective and secondly, that the pre-images are connected.

1. The mapping is surjective because under the mapping $f_V$, there are $|V_B|$ pre-images for every vertex $a \,\epsilon\, A$.
2. If we take the pre-image of any vertex $a \,\epsilon\, A$, then we get the set $\{(a, b) \mid b \,\epsilon\, V_B\}$. From the definition of the Cartesian product and the fact that $B$ is connected, we can conclude that the pre-image is also connected.

□

**Question 8.** Show that $C_j \times C_k \cong T_{j,k}$

*Proof.* Consider the cycles

$$C_j = (\{0, 1, ..j - 1\}, \{(m, (m + 1)\%j) \mid m \,\epsilon\, \{0, 1, ...j - 1\}\})$$
$$C_k = (\{0, 1, ...k - 1\}, \{(n, (n + 1)\%k) \mid n \,\epsilon\, \{0, 1, ...k - 1\}\})$$

In $C_j \times C_k$, we have from the definition of Cartesian product that,

$$\{(m, n), ((m + 1)\%j, n)\} \,\epsilon\, E_{C_j \times C_k}$$
$$\{(m, n), (m, (n + 1)\%k)\} \,\epsilon\, E_{C_j \times C_k}$$

where $m \,\epsilon\, \{0, 1, ... j - 1\}$ and $n \epsilon \{0, 1, ...k - 1\}$

For example, consider Graphs $C_3$ and $C_4$. Their Cartesian product is shown below

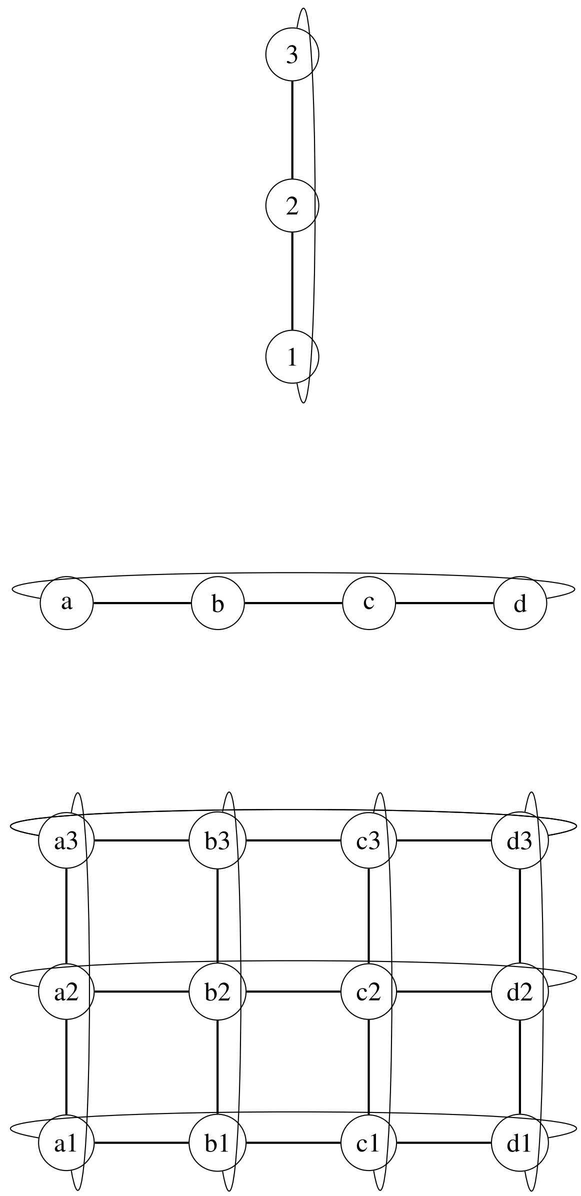



Figure 8: $C_3$ and $C_4$ gives $C_3$ X $C_4$

$\therefore$ Every vertex in $C_j \times C_k$ is connected to its four neighbours in a cyclic fashion which is nothing but the Torus graph $T_{j,k}$. □

## 3 Euler Walks

For all of the questions below, let $G$ be a graph with exactly four vertices $a, b, c, d$ of odd degree. Let $W$ be the edges of a walk from $a$ to $b$ that does not repeat any edges.

**Question 9.** Prove that $c$ and $d$ are connected even if we remove the edges of $W$. That is, $c$ and $d$ are connected in $G' = G \setminus W \setminus \{v \in (G \setminus W) \,|\, deg(v) = 0\}$.

*Proof.* The walk $W$ from $a$ to $b$ does not repeat any edges. In Graph $G'$, we remove the edges from G, which are part of the Walk and any isolated vertices that are left after removing it.

Also, for any vertex other than $a$ and $b$, when you enter the vertex you also have to leave it.
$\implies$ In $G'$, parity of all the degree of vertices remains the same, except the source and destination vertices.

To summarise :

1. Parity of degree of vertices $a$ and $b$ changes from odd to even.
2. Parity of degree of vertices of $c$ and $d$ remains odd.
3. Parity of degree for all other vertices remains even.

The Graph $G'$ may now have one or more connected components.
Let us consider two cases depending upon where $c$ and $d$ are located.

**Case 1.** If $c$ and $d$ are part of different (connected)components.

Let us assume that $c$ and $d$ are in fact not in the same connected component. We prove by contradiction using *Handshake Lemma* that this case is not possible.

**Lemma.** *Handshake Lemma*
Number of vertices of odd degree in a graph is even

If one connected component has vertex $c$ but not $d$, this would mean that $c$ is the only vertex of that connected component with odd degree, while all other vertices are of even degree.
*Handshake Lemma* clearly states that this is not possible.

**Case 2.** If $c$ and $d$ are part of the same (connected)component.

In this case, $c$ and $d$ are the only odd degree vertices of the component, while all other vertices have even degree.
By **Euler's Theorem**, there exists an Euler Walk from $c$ to $d$.
$\implies$ $c$ and $d$ are connected.

$\therefore$ Only Case 2 is possible.Vertices $c$ and $d$ are connected. $\square$

**Question 10.** Suppose that $W$ is the longest such walk and that $G$ is connected. Prove that in this case, $G'$ is connected.

*Proof.* Let us take Graph G as the following

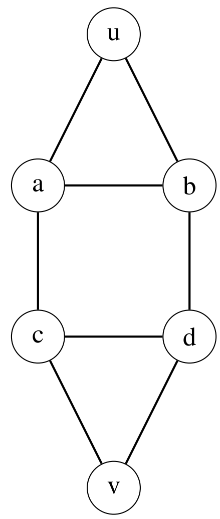



Figure 9: Graph $G$

The longest walk, $W$ from $a$ to $b$ can be denoted by the sequence of vertices $(a,b,u,a,c,v,d,b)$.

$G' = G \setminus W \setminus \{v \in (G \setminus W) \,|\, deg(v) = 0\}$

$\implies$ G' = {{c,d},(c,d)}

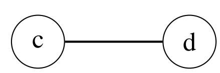



Figure 10: Graph $G'$

Every time, we do this operation of removing the longest walk, we will get only one connected component.
This is true because let us suppose after removing walk $W$ from $G$, we got more than 1 connected component.

1. First component would have both vertices $c$ and $d$ together. This was shown in the previous question using the *Handshake Lemma*.
2. Second, or any other, component would have all vertices with even degree.
   By **Euler's Theorem**, we know that a connected Graph with all even degree vertices has an *Euler Tour.* This is contradiction to the fact that we've chosen the longest walk. We could've added the Euler Tour to the existing walk and gotten an even longer walk.

$\implies$ There is only one connected component in $G'$ and therefore, $G'$ is a connected Graph.

□

**Question 11.** Prove that if $G$ is connected, then there is a pair of walk that don't repeat any edges, don't have any edges in common, and together, touch every edge of $G$.

*Proof.* It follows from the previous question that if $G$ is connected if we take the longest walk W from $a$ to $b$, then Graph $G'$ is still connected.

Again, by **Euler Theorem**, a connected graph with all even degree vertices except two vertices (source and destination) as odd degree vertices, has an *Euler Walk* from the source to destination. Let this be denoted by W2.

$\implies$ In Graph $G'$, as only $c$ and $d$ vertices have odd degree, while all other vertices have even degree, there exists an *Euler* from $c$ to $d$.

$\therefore$ Walks W and W2 don't repeat any edges, don't have any edges in common, and together, touch every edge of Graph $G$.

$\square$