CSC 565: Graph Theory **North Carolina State University**
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1 Simplicial Map

Definition 1. Let X and Y be simplicial complexes. A simplicial map $f : X \to Y$ is a pair of functions:

$$f_v : V_X \to V_Y \text{ and } f_S : S_X \to S_Y$$

where $f_S(\sigma) = \{f_V(u) : u \in \sigma\}$

So, to get a simplicial map $f' : simp(G) \to simp(H)$ from a graph morphism $f = (f_0, f_1)$ where $f_0 : V_G \to V_H$ and $f_1 : E_G \to E_H$, we simply define $f'_V = f_0$ and $f'_S(\sigma) = \{f'_V(u) : u \in \sigma\} = \{f_0(u) : u \in \sigma\}$ Note f'_S maps vertices to vertices and edges to edges.

Example 1. In the following example, let G be the graph on the right and H on the left. The morphism can be defined as

$$\begin{cases} f(A) = C \\ f(B) = D \end{cases}$$

The simplicial map can be defined as $\hat{f} = (\hat{f_V}, \hat{f_S})$ where

$$\begin{cases} \hat{f_V}(A) = C \\ \hat{f_V}(B) = D \end{cases}$$

and $\hat{f_S}$ can be expressed as

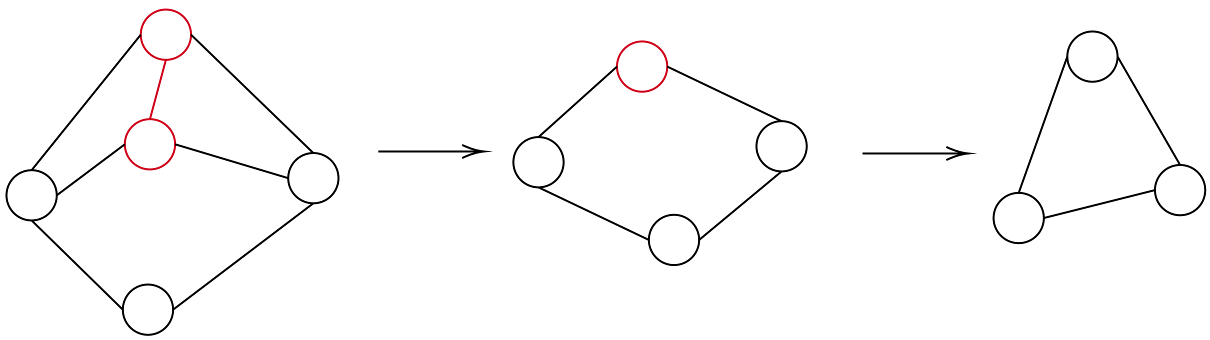
$$\begin{cases} \hat{f_S}(\emptyset) = \emptyset \\ \hat{f_S}(\{A\}) = \{C\} \\ \hat{f_S}(\{B\}) = \{D\} \\ \hat{f_S}(\{A, B\}) = \{C, D\} \end{cases}$$

Definition 2. A graph G is planer $\iff$ G has no $K_{3,3}$ or K_5 minor $\iff$ G has no $K_{3,3}$ or K_5 topological minor

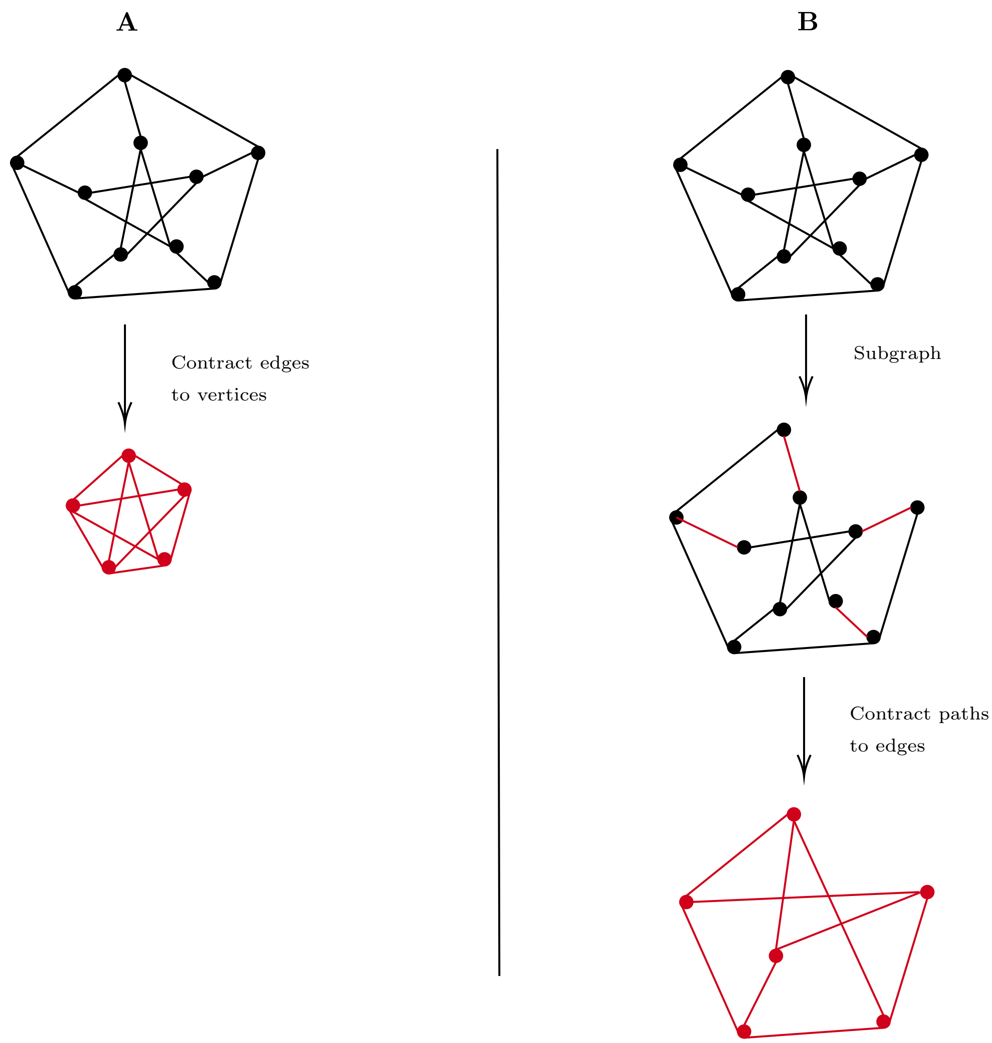
Definition 3. A is a *minor* of B if there exists a surjective map $\text{sim}(B) \to \text{sim}(A)$ such that preimages are connected, i.e., $\forall v \in V_a$.

Note : What does connected mean for a simplicial complex?
In this case $f^{-1}(v)$ is a set of vertices and edges. Every edge in $f^{-1}(v)$ has both ends in $f^{-1}(v)$ and so it can naturally be treated as a graph.

Example 2. Equivalently, we form a minor by contracting edges as can be seen in the diagram below.



Example 3. Consider the set of graphs A and B. Both represent Peterson graphs being reduced to their minors and topological minors respectively. For the set of graphs in A, edges are contracted to show the existence of K_5 in the minor. For the case of graph set B, $K_{3,3}$ is shown to exist in the topological minor of the Peterson graph and hence, the original graph can't be planar (Definition 2).



<u>An "Euler" Invariant</u>
Two general graph invariants for isomorphism are :

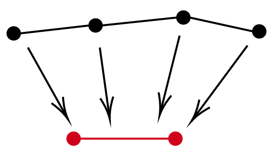
$$c(G) = |V_G| + |E_G|$$

$$e(G) = |V_G| - |E_G|$$

Both are clearly invariants and are easy to compute. However, they capture very different information. In particular, e(G) will not distinguish between topologically equivalent graphs and hence is a topological invariant.

Theorem 1. *If geomG≡geomH, then* $e(G) = e(H)$.

Proof idea. For topologically equivalent graphs, the only difference possible is for some path in one graph to be replaced by a single edge in the other. Each such operation removes k vertices and k edges. Thus, e(G) remains unchanged. □



2 Path Connectivity in Top

Definition 4. Let $X \subset \mathbb{R}^2$ and $a, b \in X$. We say a and b are *path-connected* if there exists a (continuous) map $f : [0, 1] \to \mathbb{R}^2$ such that $f(0) = a$ and $f(1) = b$.
We say X is path-connected if a and b are connected $\forall a, b \in X$.

Exercise: Prove that path-connected is an equivalence relation.

Theorem 2. *A graph G is connected iff geom(G) is path-connected.*