Lecture 14

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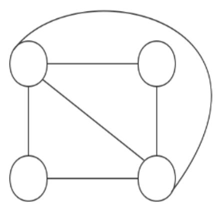
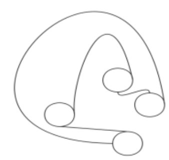
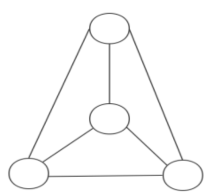
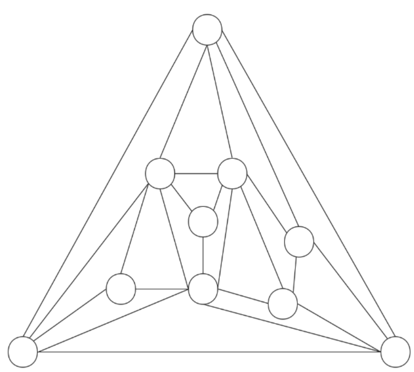
Theorem 1 (The Fary)**.** *If G is a planar, then there exists a linear embedding of G.*

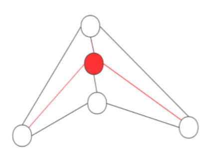
Proof. It suffices to consider maximal planar graphs.

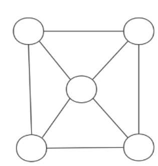












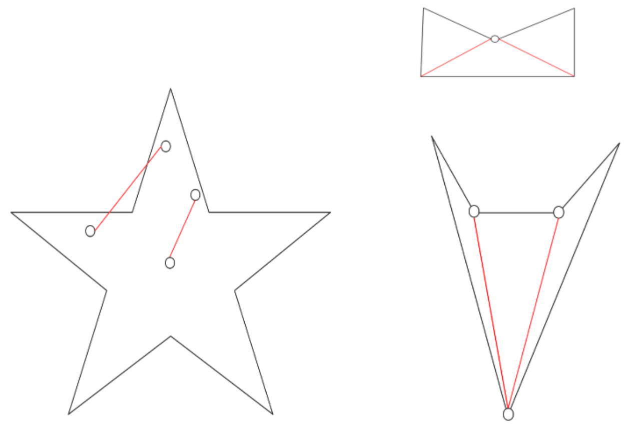
[star-shaped polygons]. Polygon P is a star-shaped if there exists

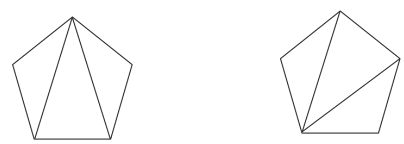
$$x \in int(P)$$

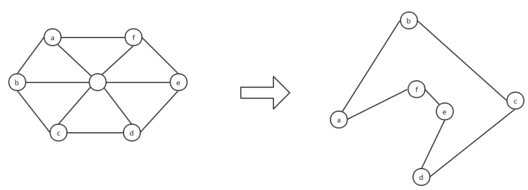
Such that

$$\bar{xy} \in P for\ all\ y \in P$$

Claim: All 5-gons are star-shaped







Since every planar graph has at least one vertex with $deg(v) \leq 5$. Let v be a vertex with $deg(v) \leq 5$. G' is $G \setminus v$ plus 2 edges to make it maximal. By induction, $\exists$ linear embedding of G'.

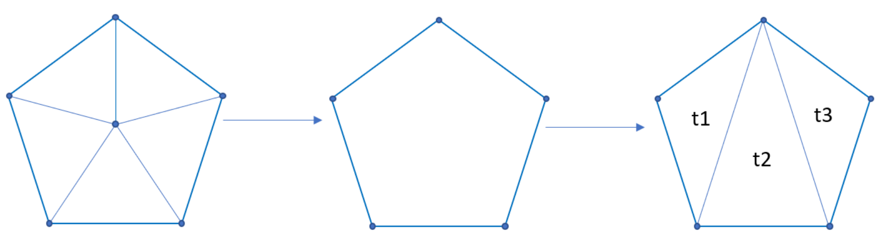


Figure 1: Remove a vertex and make the graph maximal.

The 3 triangles $t1, t2, t3$ are faces in the embedding. Their union is a k-gon for $k = deg(v)$. Since the k-gon is star-shaped($k \leq 5$), there is a point x such that straight line segments x to the k vertices stay inside the k-gon. Place v at point x to complete the embedding. $\square$

Theorem 2 (Steinitz Theorem)**.** *A graph is "polytopal" iff it is planar and 3-connected.*

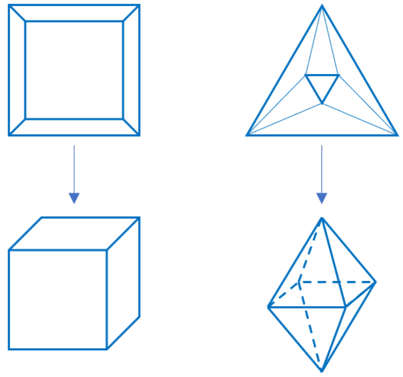


Figure 2: Examples of polytopals

Question 1. Minor means that some subgraph of C contracts to A. Contraction is a surjective relationship. Subgraph means that you have to remove some components or remain the graph unchanged. So $|V_A| \leq |V_C|\, and\, |E_A| \leq |E_C|$.

Question 2. A is a minor of B and B is a minor of A.From question 1 we can get

$$|V_A| \leq |V_B| \;\; |V_B| \leq |V_A|$$

The only case is that

$$|V_A| = |V_B|$$

We can also get that edges are equal too.If we contract or remove any edges, we can't get the same number of vertices and edges. A is a subgraph of B, and B is a subgraph of A. A and B must be isormorphic.

Question 3. Since all subdivisions or contractions are made of single subdivision or contraction, condition in figure 3 is enough. The reverse action of a subdivision is actually a contraction. So the topological minor is also a special case of minor.

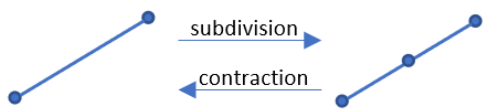


Figure 3: Question 3

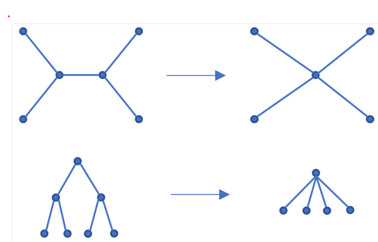


Figure 4: Examples of question 4

Question 4. Give an example of a pair of trees A and B such that $A \preceq B$, but A is not a topological minor of B. One way to think of it is to find a minor that can change the degree of vertices. The examples in figure 4 both have a vertex with degree of 4 by contract vertices.

Question 5. (Question 9 in homework3) If $N(u) \cap N(v) = \varnothing$, there is no common neighbour of u and v. In this way, the contraction of (u, v) can only delete one vertex and one edge, then $e(G) = e(H)$. The only way contraction could cause $e(G) \neq e(H)$ is u and v have at least one neighbour, as shown in figure 5.

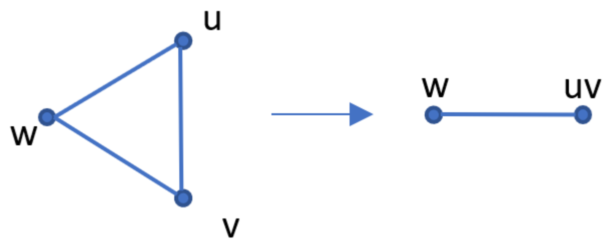


Figure 5: Example of question 9