CSC 565: Graph Theory
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North Carolina State University
Computer Science

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1 Pre-class logistics

2 The Basics

Graphs are the most important abstraction in computation (after numbers and sets)

- They describe binary relations (i.e. sets of pairs of things)
- As the name implies, we often draw graphs

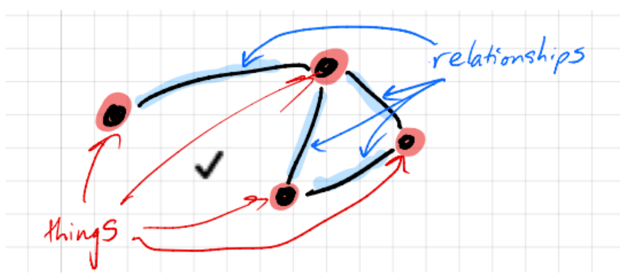


Figure 1: The drawing is not the graph. It's only a picture.

- Graphs are everywhere
  - Circuits
  - Networks
  - Roadmaps
  - Data Structures
  - ... and other less obvious settings as well

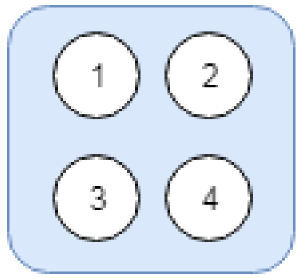
3 Introduction: What is graph?

When we talk about graphs, we could first start with some of the basic building component of it:

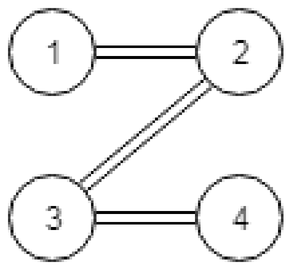
- Number

1, 2

- Set



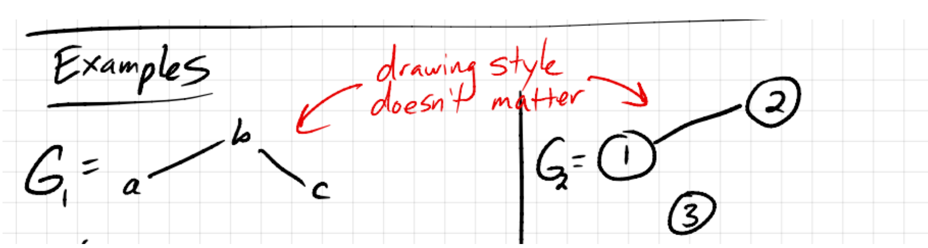
- Graph



4 Definitions

- A **graph** is a pair (V,E) where V is any set and E is a set of 2-elements subset ("unordered pairs") of V
- The set V is called the **vertex set** and its elements are called **vertices**
- The set E is called the **edge set** and its elements are called **edges**
- We often write *G = (V,E)* to assign a name (G) to a graph (V,E)
- We may also write *V(G)* and *E(G)* to refer to the vertex and edge sets of some given graph G

4.1 Examples



G_1:

$V(G_1) = \{a, b, c\}$

$E(G_1) = \{\{a, b\}, \{b, c\}\}$

G_2**:**

$V(G_2) = \{1, 2, 3\}$

$E(G_2) = \{\{1, 2\}\}$

Notation: It's easier to write (a,b) instead of $\{a,b\}$*. In this case, it's assumed (a,b) = (b,a).*

- Two vertices u,v are **adjacent** if $(u, v)\epsilon E$
- An edge e and a vertex v are **incident** if $v\epsilon e$ (i.e. $e = (u, v)$ for some $u\epsilon V$
- The number of edges incident to a vertex v is called a degree of v, and is written deg(v)

4.2 Exercise

Prove

$$\sum_{v\epsilon V} deg(v) = 2|E|$$

for any graph (V,E)

5 Graph Questions

1. Is it connected? (i.e. is it all one piece)
2. Does the graph have any cycles?
3. What is the shortest path form one vertex to another?
4. Can we assign a small number of colors to the vertices so that no two adjacent vertices have the same color?
5. Can we draw the graph so that no two edges cross?
6. Is one graph "equal" to another (allowing the vertices to be relabelled)
7. Does one graph contain another graph (or its equivalent)?
8. How quickly will a random walk on a graph mix?
9. How many spanning trees (minimally connected subgraphs using all the vertices) does a graph contain?

6 Different Perspective on Graphs

Combinatorics, Computation, Geometry, Topology, Algebra. As much as possible, we will try to represent these different perspectives as **categories** and our change of perspective as **functions**. I will tell you what these words mean.

6.1 Sets and Functions

This should all be review I will use all these concepts, definitions, and notation will reckless abandonment. The definition of a graph depends on the notion of a set.

You should know:

1. What is a set?

   Elements, Membership, Empty Set (Ø), Cardinality

2. Set Relations and Elements

   $a \epsilon S$ "a is in S" or "a is an element of S"

   $A \subset B$, $A \subseteq B$, $A = B$

3. Set Operations

   Union: $A \cup B$

   Intersection: $A \cap B$

   Difference: $A \setminus B$

   Complement: $\hat{A}$

   Cartesian Product: AXB

   $\bigcup_{i=1}^{n} A_i$, $\bigcap_{i=1}^{n} A_i$

4. Notation $\{1, 2, 3\}$

   (Sub) Set Builder: $\{x \epsilon \mathbb{R} | x \geq 2\}$

   Predicate: $x \geq 2$

5. Functions $f : A \Rightarrow B$

   domain, range, injective, surjective, bijective, inverse, preimage, composition

6.2 The Category of Sets

- Set fuctions: $f : A \Rightarrow B$ or $A \xrightarrow{f} B$

  A is the **domain** or **source**

  B is the **range** or **target**

- Functions can be **composed**

  $A \xrightarrow{f} B \xrightarrow{g} C$

  $A \xrightarrow{g \circ f} C$

  $x \epsilon A : (g \circ f)(x) = g(f(x)) \epsilon C$

- **Inclusion** (as a function)

  If $a \subseteq B$ there exists a unique injection $f : A \Rightarrow B$ such that for all $x \epsilon A : f(x) = x$

- Identity Functions

  For any set A there is a unique fucntion $id_A : A \Rightarrow A$ such that for all $x \epsilon A \quad id_A(x) = x$

- Let A,B be sets and $f : A \Rightarrow B$

    Image $imf = \{f(x) : x\epsilon A\} \subseteq B$

- Let $S \subseteq A$

    Restriction $f_{\setminus S} : S \Rightarrow B$ $f_{\setminus S}(x) = f(x)$ (for all $x\epsilon S$)

    Image of a set $f(s) = imf_{\setminus S} = \{f(x) : x\epsilon S\}$

    (*Note: This is an abuse of notation and I'm not sorry.*)

    Preimage $T \subseteq B$ $f^{-1}(T) = \{x\epsilon A : f(x)\epsilon T\}$

    (*Another abuse of notation.* f^{-1} *could also be an inverse.*)

    Inverse If f is bijective then there is a unique function $f^{-1} : B \Rightarrow A$ such that $f \circ f^{-1} = id_B$ and $f^{-1} \circ f = id_A$