Lecture 2

Scribed by: Yuhan Chen, Xianpeng Liu

September 19, 2019

1 Graph Isomorphisms

Definition 1. Graphs A and B are isomorphic iff $\mathbb{E}$ bijection $V_A \leftrightarrow V_B$ that maps edges to edges.

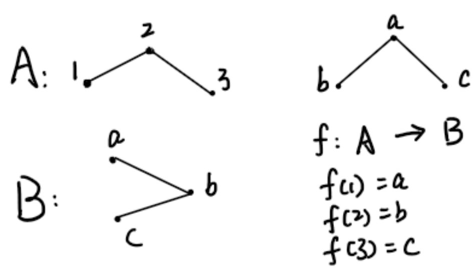
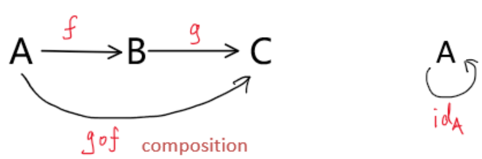


Figure 1: Example for isomorphisms



$$(gf)(a) \;=\; g(f(a))$$
$$id_A(a) \;=\; a$$

Inclusions $A \cong B$ iff $\exists f : A \to B,\ g : B \to A \quad s.t.\ gf = id_A \quad fg = id_B$

2 The Category Set

Theorem 1. The set $\mathbb{Z}_+$ of positive integers is not isomorphic to the set X of sets of positive integers.

Proof. Suppose the set $\mathbb{Z}_+$ of positive integers is isomorphic to the set X of sets of positive integers. which means $\mathbb{Z}_+ \underset{g}{\overset{f}{\rightleftarrows}} X$, $fg = id_X$, $gf = id_{\mathbb{Z}_+}$, Let

$$S = \{i \in \mathbb{Z}_+ : \ i \notin f(i)\}$$

$$S = id_X(S) = (fg)(S) = f(g(S))$$

Is $g(S) \in S$, let $i = g(S)$. Then we can get $g(S) \in S$ iff $g(S) \notin S$, which is impossible. □

	1	2	3	...
f(1)	1	0	1	...
f(2)	0	0	1	...
f(3)	1	1	0	...
...	...	...	...	... i

Figure 2: Example for set S

3 Graph Morphisms

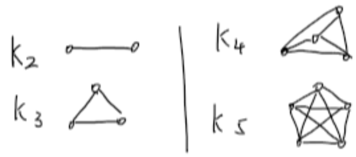
Definition 2. A morphism is $G \to H$ is a function $f : V_G \to V_H$

$$s.t. \forall (u, v) \in E_G$$

we have $(f(u), f(v)) \in E_H$

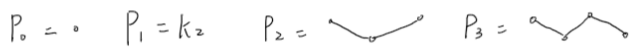
4 Some Special Graphs

(1). <u>Clique</u> or <u>Complete Graph</u> on n vertices K_n



$$K_n = \left(\{1,2,\dots n\}, \binom{\{1,2,\dots,n\}}{2}\right)$$

(2). <u>Path</u> P_m with m edges, "length of P_m" is the number of edges (m)

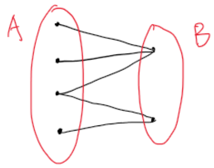


$$P_m = (\{0,\dots,m\}, \{(i-1,i) \mid i \in [m]\})$$

(3). <u>Cycle</u>

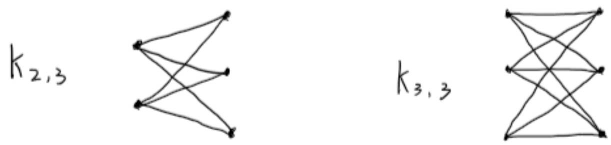
$$C_3 = K_3 \qquad C_n = (\{0,\dots,n-1\}, \{(i,(i+1)\%n) \mid i \in \{0,\dots,n-1\}\}) \quad n \geq 3$$

(4). <u>Bipartite Graphs</u>, $G = (A \cup B, E)$, where $E \in A \times B$ and $S \cap B = \oslash$



(5). Complete Bipartite Graphs, $K_{K\,a,b}$. All edges from every vertex of A to every vertex of B.

$$K_{a,b} = ([a] \sqcup [b]\,, [a] \times [b])$$



Claim:Graph with at least two vertices is bipartite iff you can map it's morphism to a graph into K_2

5 Subgraphs

Definition 3. $G = (V, E)$, $G' = (V', E')$, G is subgraph of G'($G \subseteq G'$) iff $V \subseteq V'$ and $E \subseteq E'$

Edits

$$\begin{aligned} G + e \; or \; G \cup e &= (V,\; E \cup e) \\ G + v \; or \; G \cup v &= (V \cup v,\; E) \\ G \setminus v \; or \; G - v &= (V \setminus \{v\}\,,\; E \setminus (\{v\} \times V)) \end{aligned}$$

6 Induced Subgraphs

Definition 4. $G = (V, E)$, G' is an induced subgraph (on V') iff $G' = (V', E \cap (V' \times V')$, $V'V$, $G' = G \setminus (V \setminus V')$