Lecture 13

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1 Planar Graphs

Recall:

Definition 1 *Euler Invariant:*

$$|V_G| - |E_G| + |F_G| = 1 + |C_G|$$

Definition 2 *Outerplanar: A graph G is outerplanar if all vertices can be embedded on the outer face of the graph*

1.1 Maximal Planar Graphs

Definition 3 *Maximal: A planar graph G is maximal if no more edges can be added while keeping it planar*

Question: If G is planar and maximal with n vertices, how many edges does G have?

When approaching a question like this it is helpful to come up with a small example to discover patterns. Let's start with n = 5

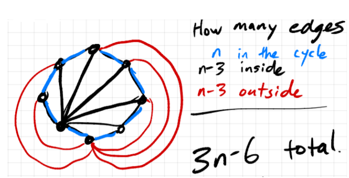

Figure 1: Maximal Planar Graph, n = 5

Lemma 1: If G is maximal and planar, then every face is a triangle, K_3

Proof: Suppose for contradiction that some face has greater than or equal to 4 vertices. Among these 4 every edge must be in G. Otherwise, we could add one by cutting though the face. So, the embedding contains and embedding of K_4 with all vertices on one face. This is a contradiction because K_4 is not outer planar.

Fact: Every finite planar graph has a vertex degree at most 5.
Proof: We know that the maximum number of edges in a planar graph is $3n - 6$. and the sum of degrees of all vertices is equal to twice the number of edges in the graph -

$$\sum_{v \in V} deg(v) = 2|E|$$

Substituting in the value of maximum number of edges possible, we get:

$$\sum_{v \in V} deg(v) = 2|E| \leq 6n - 12$$

Taking the average for all vertices, we see that

$$\frac{1}{n}(\sum_{v \in V} deg(v)) \leq 6 - \frac{12}{n}$$

which is clearly less than 5. Hence, proved.

Lemma 2: If G is 2-connected and $\underline{G}$ is an embedding of G, then every face of the embedding $\underline{G}$ is a cycle of G.
Proof: Suppose G has a face that's not a cycle. Then some vertex on the face is visited more than once.
Let us consider graph G' where $\{G' = G \backslash u\}$

$$|V'| = |V| - 1$$

$$|E'| = |E| - deg(u)$$

$$|F'| = |F| - deg^{\star}(u) + 1$$

where $deg^{\star}(u)$ is the number of faces touching u
Since the graph is 2-connected, we can use the Euler invariant

$$|V_G| - |E_G| + |F_G| = 1 + |C_G|$$

for both G and G' as follows:

$$|V_G| - |E_G| + |F_G| = |V'_G| - |E'_G| + |F'_G|$$

$$|V_G| - |E_G| + |F_G| = (|V_G| - 1) - (|E_G| - deg(u)) + (|F_G| - deg^\star(u) + 1)$$

$$deg(u) = deg^\star(u)$$

which is a contradiction. Hence, we can say that every face of the embedding is a cycle of the graph G.

Definition 4 *Non-separating: A subgraph $H \subseteq G$ is non-separating if G and $G \backslash H$ are connected.*

Theorem 1: If $G = (V, E)$ is planar and 3-connected then the faces of an embedding of G are the non-separating induced cycles of G.
Proof: Because G is 3-connected, it is also 2-connected and therefore the faces are simple cycles.
Suppose C is a non-separating induced cycle. Then, in any embedding C maps to a Jordan curve, so, in order to be non-separating all other vertices must be inside of outside of the the embedding of C. In either case, this implies C is a face.
Suppose C is a face. It is an induced cycle because any other edges between vertices in C would give a 2-vertex separator, contradicting the assumption that G is 3-connected.
If C separates G then there are vertices u, v such that every path from u to v contains a vertex of C.
By Menger's Theorem, there are 3 disjoint paths from u to v.

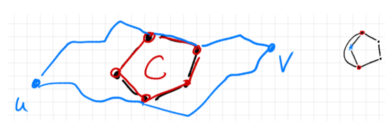

Figure 2: Application of Menger's Theorem

Two of these paths form a Jordan curve containing C. Note that there is no way to have another disjoint path that intersects C. Therefore, either C could not be a face or G is not 3-connected. The contradiction implies the theorem.

Corollary: Let G be a planar, 3-connected graph. Then the faces of any two embeddings or G are the same.