Lecture 11: Sept 30, 2019

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11.1 Subdivision

11.1.1 Introduction

The two graphs G and H shown in fig 11.1 are not isomorphic. This is because the number of edges and the number of vertices are not same, which is an invariant for finding isomorphism among graphs.

Figure 11.1: Two graphs G and H

From figure 11.1, the relation between graphs these graphs can be extended to geometric realisation as:

$$\begin{array}{ccc} G & \cong & H \\ \downarrow & & \downarrow \\ \text{geom(G)} & \cong & \text{geom(H)} \end{array}$$

On the other hand, if we take the geometric realization of these two graphs, we find that those are homeomorphic i.e. there exists a continuous mapping from all the vertices in geometric realization of G to all the vertices in geometric realization of H in the topological space.

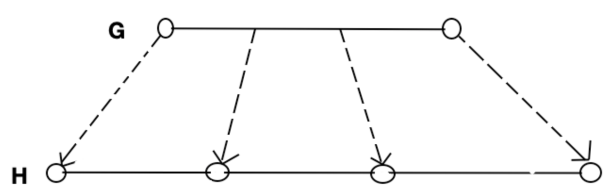

Figure 11.2: Homeomorphism b/w geom(G) and geom(H)

11.1.2 Understanding Subdivision

H is a subdivision of G if it is formed from G by subdividing zero or more number of edges.

If we can take one edge of a graph and keep on subdividing it, the resultant graph will be a subdivision of the initial graph. Geometric realizations for the initial graph and the final graph (after subdivision) will be homeomorphic to each other.

If we recall, the Eulerian characteristic for graphs is:

$e(G) = |V_G| - |E_G|$

There is an interesting fact for subdivision with regard to Eulerian Characteristic.

Fact: If G is a subdivision of H, then

$$e(G) = e(H)$$

The reason behind such characteristic is that if we subdivide an edge once, the number of edges and the number of vertices both increase by one. Similarly, if we contract an edge once, the number of edge and the number of vertices both decrease by one.

11.2 Topological Minor

11.2.1 Definition

G is a topological minor of H, if H contains a sub graph which is a subdivision of G.

Example 1 : Consider the following two graphs G and H.

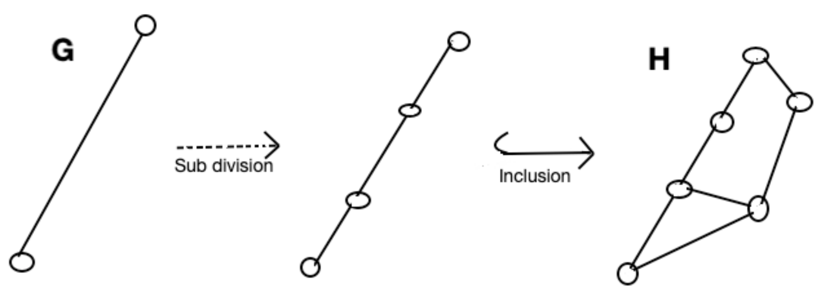

Figure 11.3: Graph G is a topological minor of Graph H

In figure 11.3, an intermediate graph is formed by subdividing the graph G and then including that into a larger graph H through inclusion (the intermediate graph becomes a subgraph of H). Thus, we can say that graph G is a topological minor of graph H.

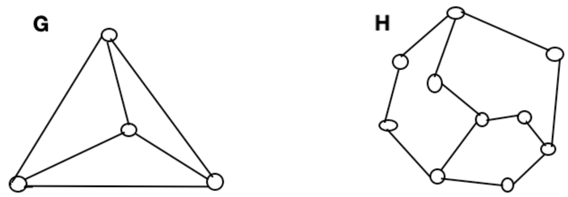

Figure 11.4: G is a sub division of H.

Example 2 : Consider the following two graphs G (K_4) and H.

In figure 11.4, H is formed by subdividing G, i.e., the edges of G are sub divided to form H.

Example 3: Consider the following two graphs G (K_4) and H.

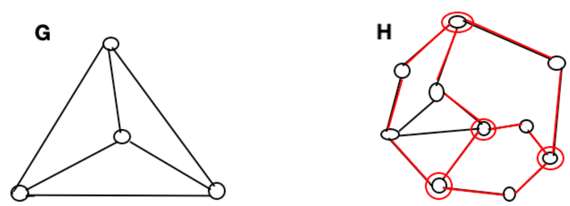

Figure 11.5: Graph G (K_4) is a topological minor of Graph H

In figure 11.5, H is not a subdivision of G, but G is still a topological minor of H.

The graph highlighted with red edges and red vertices is actually a subgraph of H and was formed by subdividing graph G. Thus, by definition, we can say that graph G is a topological minor of graph H.

Subdivision and inclusion follow **partial ordering** in case of topological minor.

For a relation to be a Partial order, it has to be:

- Anti-Symmetric
- Reflexive
- Transitive

11.2.2 Transitivity in topological minors

Topological transitivity is denoted by $\preceq_T$ is transitive.
Transitivity here means that if A $\preceq_T$ B and B $\preceq_T$ C, then A $\preceq_T$ C.

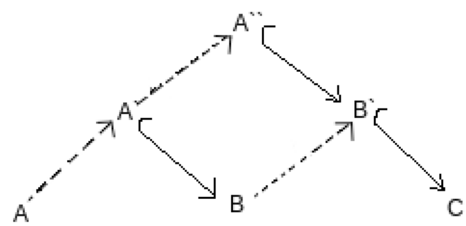

Figure 11.6: Transitivity in topological minor

11.3 Contraction

11.3.1 Introduction

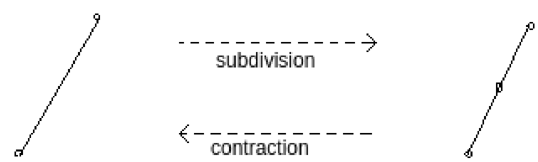

Figure 11.7: Special case of contraction

Figure 11.7 shows a special case of contraction, as this contraction can actually be considered as reverse of subdivision which is not true always.

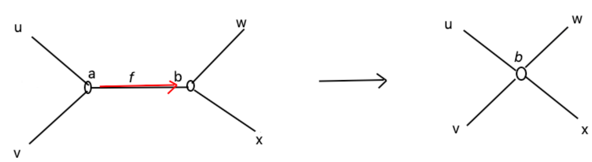

Figure 11.8: Contraction

Contraction can be understood more like a function rather than an operation. From figure-11.8, the function f can be defined as:

$f_v(a) = b$ and $f_s(\sigma) = \{f_v(u) : u \in \sigma\}$

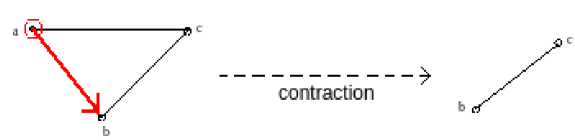

Figure 11.9: General case of contraction

11.3.2 Definition

A contraction is a surjective simplicial map such that pre-images of connected sub graphs are connected. Also, G is a contraction of H if there is a contraction H$\rightarrow$ G.

From this definition, it follows that the contraction of a contraction is also a contraction.

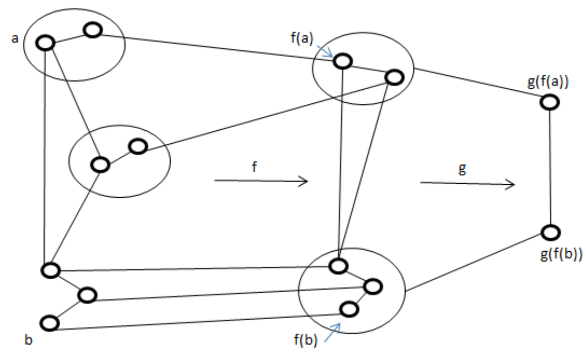

Figure 11.10: Graph H is a sub division of Graph G

In figure 11.10, we have two contraction maps , f and g. We want to find a path from a to b using the fact that they are connected in the contraction g of f.

Next, we compose a path from f(a) to f(b) using the fact that the set of vertices in the pre-images are connected and taking the path between these the sets which contain f(a) and f(b) respectively. Extending this to the first graph, we can find the path between vertices a and b in similar fashion.

11.3.3 Minor

G is a minor of H if G is a contraction of a sub graph of H

$G \preceq H \iff$ G is a minor of H

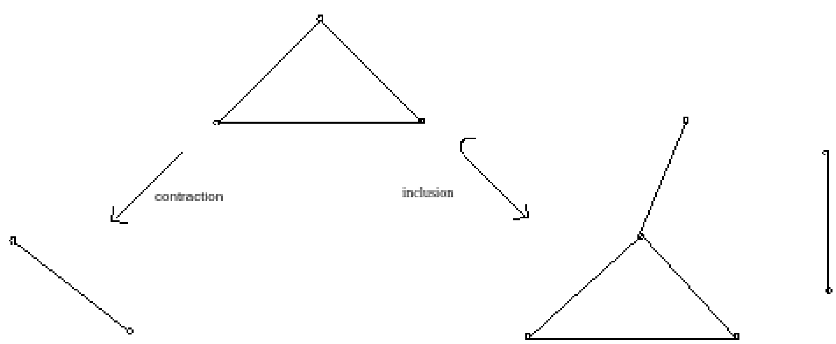

Figure 11.11: Minor in graphs

Claim: $\preceq$ is transitive

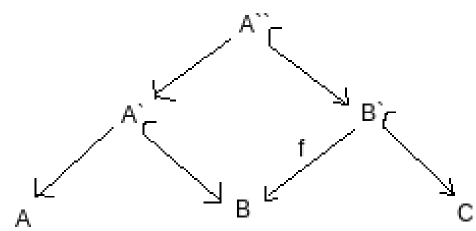

Figure 11.12: Transitivity in minors

An interesting fact to be noted from the figure 11.12 is that A‘ is a minor of B‘.

Claim: If G is a topological minor of H, then G is a minor of H.

However, the converse does not necessarily hold.

Topological minors are subsets of all minors of a graph.

Theorem 11.1. *$geom(G) \cong geom(H)$ iff there exists some common subdivision, i.e. a graph X such that X is a subdivision of G and H.*

Proof. Euler characteristic is a topological invariant.
If X is a subdivision of G, then

$$e(X) = e(G) \tag{11.1}$$

Similarly, if X is a subdivision of H then

$$e(X) = e(H) \tag{11.2}$$

By eqn 11. and eqn 11.2,
e(G) = e(H)
$\implies geom(G) \cong geom(H)$ □