Day-II Session-III

Linear Algebra for Data Science

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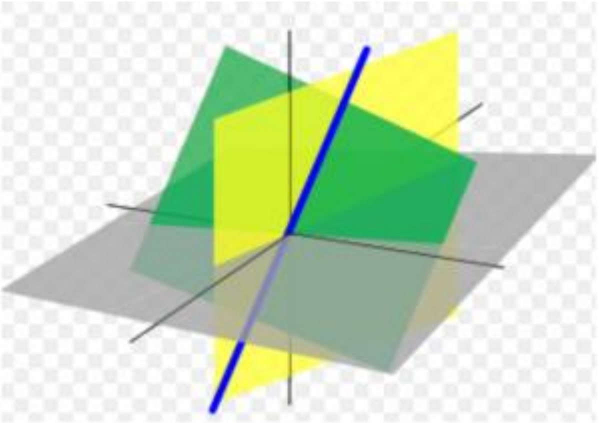
Linear Algebra

- Linear algebra is the branch of mathematics concerning linear equations such as

$$a_1x_1 + a_2x_2 \ldots\ldots + a_nx_n = b$$

  - By default, Vectors are denoted as Column Vectors.
  - In vector notation we say $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x}=b$

$$[a_1, a_2, \ldots a_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = b$$

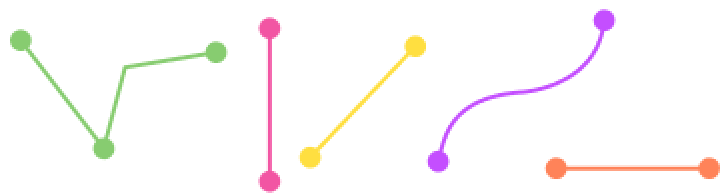


  - Called a linear transformation of $\boldsymbol{x}$

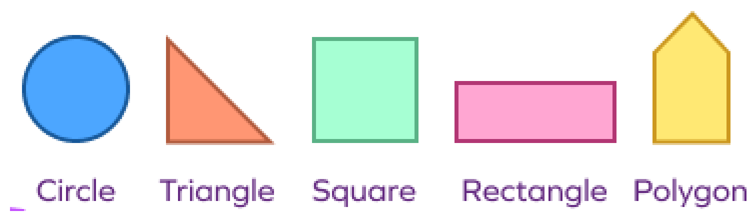
- **Linear algebra is fundamental to geometry, for defining objects such as lines, planes, rotations.**

Why Linear Algebra?

- Linear Algebra provides us the mathematical tool to understand in lower dimensions (2-D/3-D) and generalize for higher dimensions (n-D).
- 0-Dimensional : . (dot)
- 1-Dimensional



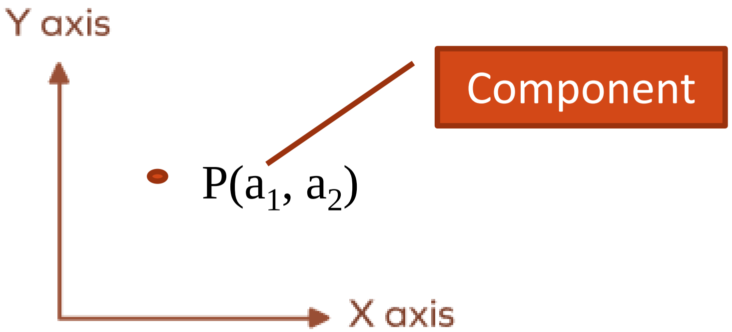
- 2-Dimensional



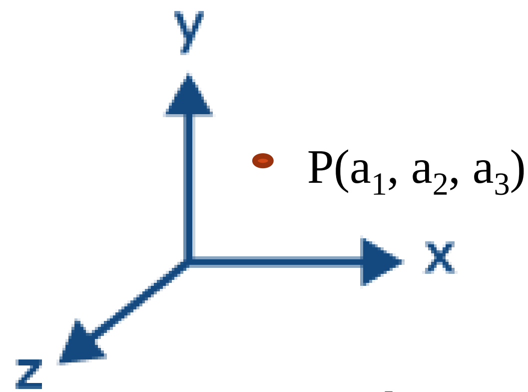
- 3-Dimensional

- n-Dimensional: Hypersphere, Hyperplane, Hypercube,…

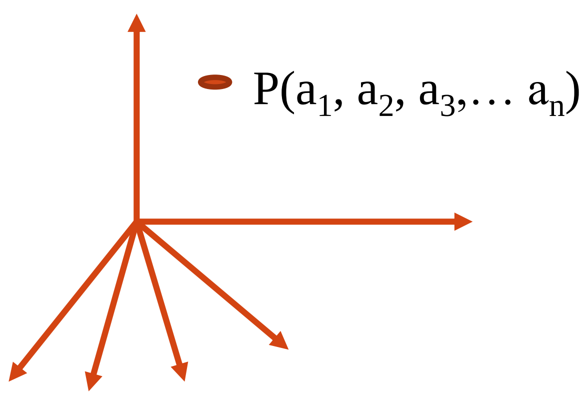
Point (Vector)



2-Dimensional
Space

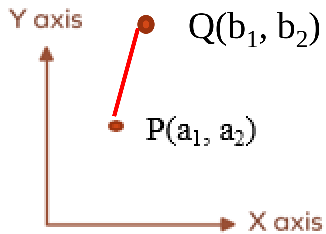


3-Dimensional
Space



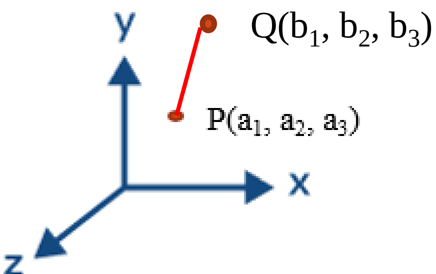
n-Dimensional
Space

Distance between two Points



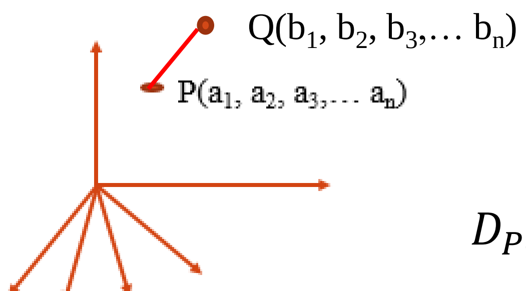
2-Dimensional Space

$$D_{PQ} = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$



3-Dimensional Space

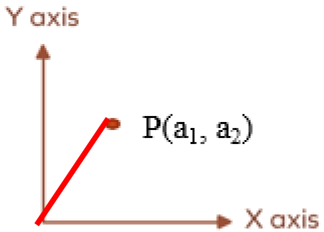
$$D_{PQ} = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$$



n -Dimensional Space

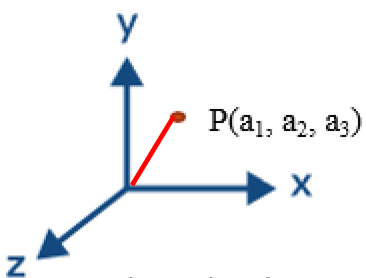
$$D_{PQ} = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2} = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$$

Distance of a Point from Origin



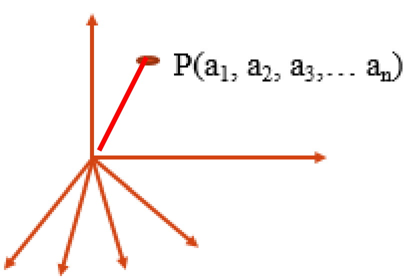
2-Dimensional Space

$$D = |P| = \sqrt{(a_1 - 0)^2 + (a_2 - 0)^2} = \sqrt{a_1^2 + a_2^2}$$



3-Dimensional Space

$$D = |P| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$



n -Dimensional Space

$$D = |P| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2} = \sqrt{\sum_{i=1}^{n} a_i^2}$$

Vector Operations

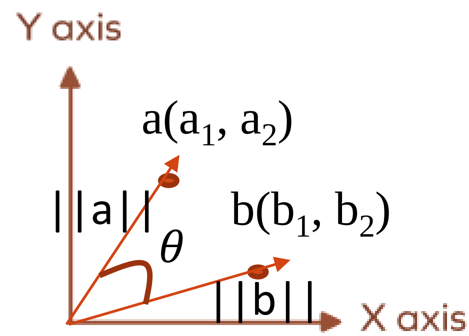
- $a = [a_1, a_2, \ldots, a_n]$
- $b = [b_1, b_2, \ldots, b_n]$
- Addition: $a + b = [a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n]$
- Subtraction: $a - b = [a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n]$
- Multiplication:
  - Dot Product:$a.b = [a_1 b_1 + a_2 b_2 + \cdots, a_n b_n]$

$$\mathrm{a.b} = [a_1, a_2, \ldots a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a^T b$$

  - Cross Product (Not much used in Data Science)

Vector Operations

- Dot Product: (Visualize in 2-D)
  - $a.b = \|a\|\|b\|\cos(\theta)$ *[***Geometry Definition***]*
  - $Where\ \|a\| = \sqrt{a_1^2 + a_2^2}$=*distance of a from Origin*
  - $a.b = a_1 b_1 + a_2 b_2$ **[Algebra Definition]**



- The angle between two vectors=$\theta = \cos^{-1}\left(\frac{a.b}{\|a\|\|b\|}\right)$

$$\theta = \cos^{-1}\left(\frac{a_1 b_1 + a_2 b_2}{\|a\|\|b\|}\right)$$

When $\theta = 90^o$ ➔ $\cos(90) = 0$ ➔ *a.b=0*

Vector Operations

- Dot Product: (In n-D)
  - $a.b = \|a\|\|b\|\cos(\theta)$
  - $Where\ \|a\| = \sqrt{\sum_{i=1}^{n} a_i^2}$ *=distance of a from Origin*
  - $a.b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i$
  - The angle between two vectors=$\theta = \cos^{-1}\left(\frac{a.b}{\|a\|\|b\|}\right)$

$$\theta = \cos^{-1}\left(\frac{\sum_{i=1}^{n} a_i b_i}{\|a\|\|b\|}\right)$$

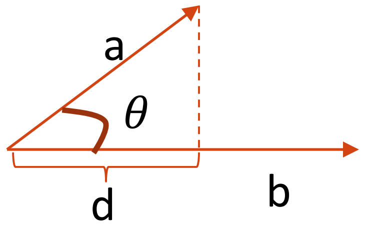
When $\theta = 90^o$ ➔ $\cos(90) = 0$ ➔ *a.b=0*

$$a.a = a_1 a_1 + a_2 a_2 + \cdots + a_n a_n = \sum_{i=1}^{n} a_i^2 = \|a\|^2$$

Dot product between two same vectors =(distance from Origin)²

Projection

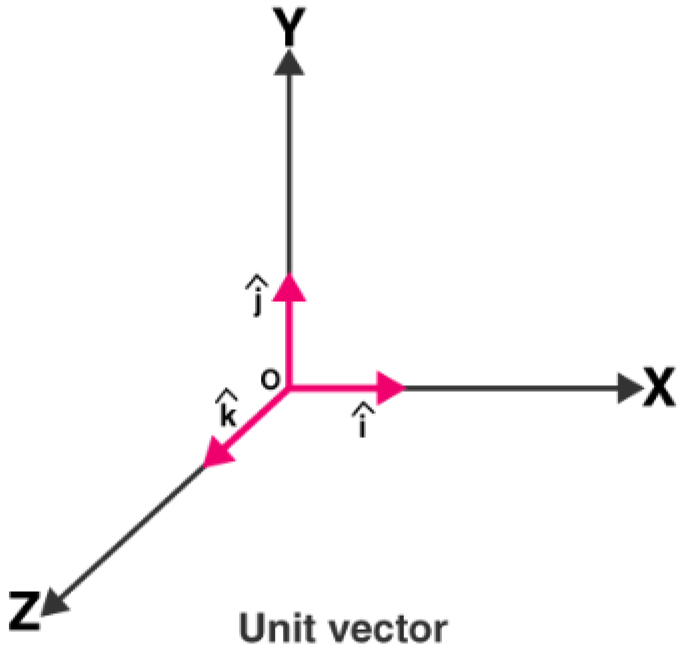
- $\cos(\theta) = \frac{p}{h} = \frac{d}{\|a\|}$
- $Projection\ of\ a\ on\ b\ i.e.\ d = \|a\|\cos(\theta)$



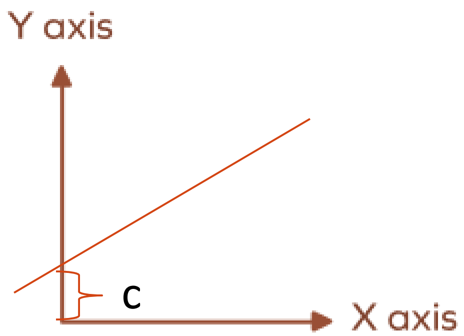
- $d = \frac{a.b}{\|b\|} = \frac{\|a\|\|b\|\cos(\theta)}{\|b\|} = \|a\|\cos(\theta)$

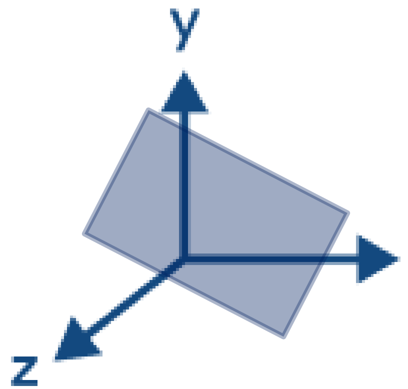
Unit Vector

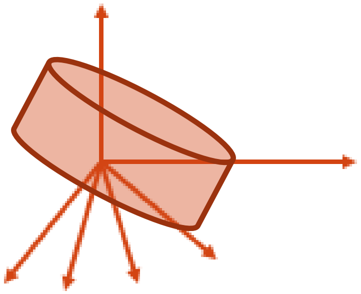
- A vector is a quantity that has both magnitude, as well as direction.
- A vector that has a magnitude of 1 is a **unit vector**. It is also known as **Direction Vector**.
- Unit vector $\hat{a} = \dfrac{a}{\|a\|}$



Line (2-D), Plane (3-D) & Hyperplane(n-D)







- 2-D:Line: $y = mx + c$

$$ax + by + c = 0$$

$$w_1x_1 + w_2x_2 + w_0 = 0$$

Note: *When* $w_0 = 0$, *the line passes through the origin.*

- 3-D: Plane:

$$w_1x_1 + w_2x_2 + w_3x_3 + w_0 = 0$$

- n-D:Hyperplane

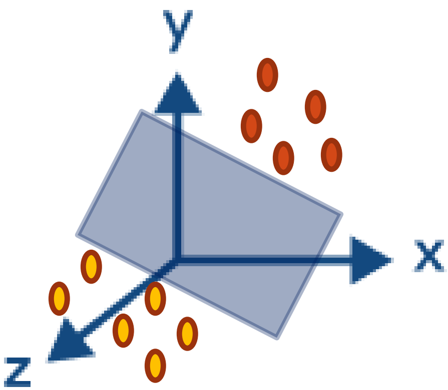
$$w_1x_1 + \cdots + w_nx_n + w_0 = 0$$

$$w.x + w_0 = 0$$

When $w_0 = 0$, the hyperplane passes through origin.

Line (2-D), Plane (3-D) & Hyperplane(n-D)

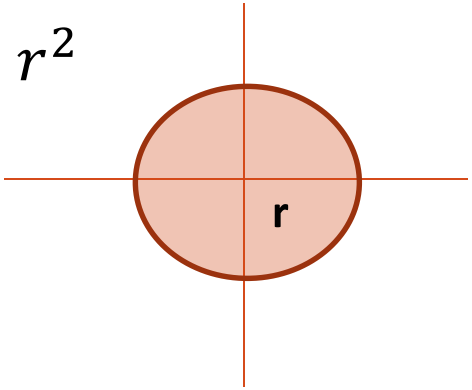
- A plane breaks a plane into two half-spaces one above the plane and one below the plane



- If the dot product of plane (w) and point (p) is positive then the point p is lying in the same half plane as that of w otherwise it is lying in the other half plane (i.e. opposite direction to w)

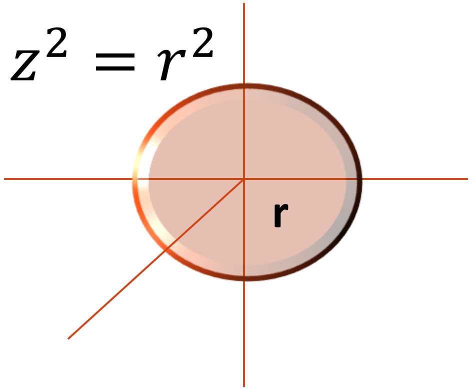
Circle (2-D), Sphere (3-D) & Hypersphere(n-D)

- 2-D (Circle)
- Equation of a circle centred at origin: $x^2 + y^2 = r^2$
- A point $p(x_1,x_2)$ lies
  - Inside the circle if $x_1^2 + x_2^2 < r^2$
  - Outside the circle if $x_1^2 + x_2^2 > r^2$
  - On the circle if $x_1^2 + x_2^2 = r^2$



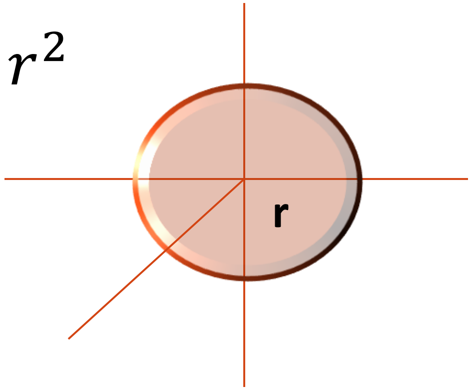
Circle (2-D), Sphere (3-D) & Hypersphere(n-D)

- 3-D (Sphere)
- Equation of a circle centred at origin: $x^2 + y^2 + z^2 = r^2$
- A point p(x_1,x_2, x_3) lies
  - Inside the circle if $x_1^2 + x_2^2 + x_3^2 < r^2$
  - Outside the circle if $x_1^2 + x_2^2 + x_3^2 > r^2$
  - On the circle if $x_1^2 + x_2^2 + x_3^2 = r^2$



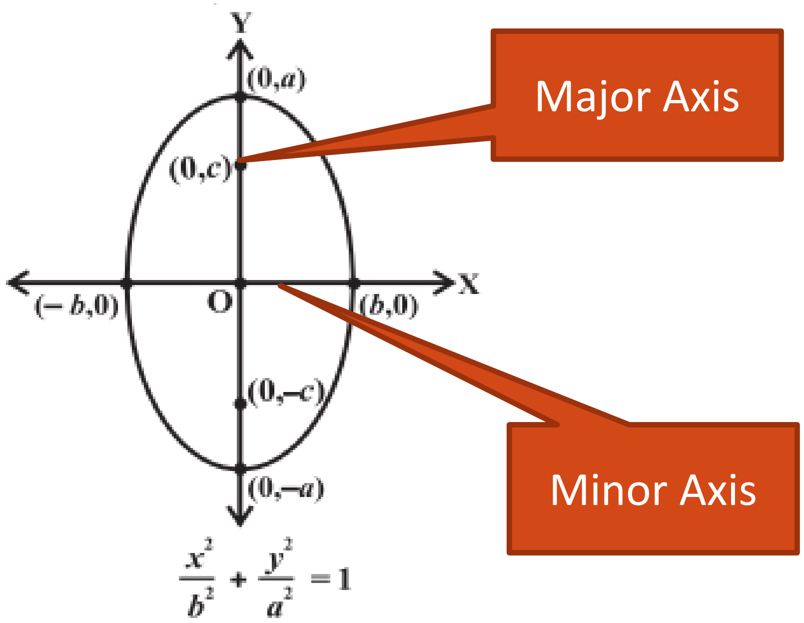
Circle (2-D), Sphere (3-D) & Hypersphere(n-D)

- n-D (Hypersphere)
- Equation of a circle centred at origin: $\sum_{i=1}^{n} x_i^2 = r^2$
- A point $p(x_1, x_2, x_3 \ldots x_n)$ lies
  - Inside the circle if $\sum_{i=1}^{n} x_i^2 < r^2$
  - Outside the circle if $\sum_{i=1}^{n} x_i^2 > r^2$
  - On the circle if $\sum_{i=1}^{n} x_i^2 = r^2$



Ellipse, Ellipsoid, Hyperellipsoid

- *Ellipse*
  - $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ point p(x,y) lies on the ellipse
  - $\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$ point p(x,y) lies inside ellipse
  - $\frac{x^2}{a^2} + \frac{y^2}{b^2} > 1$ point p(x,y) lies outside the ellipse
- *Ellipsoid(3-D)*
  - $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- *Hyperellipsoid(n-D)*
  - *Similarly*



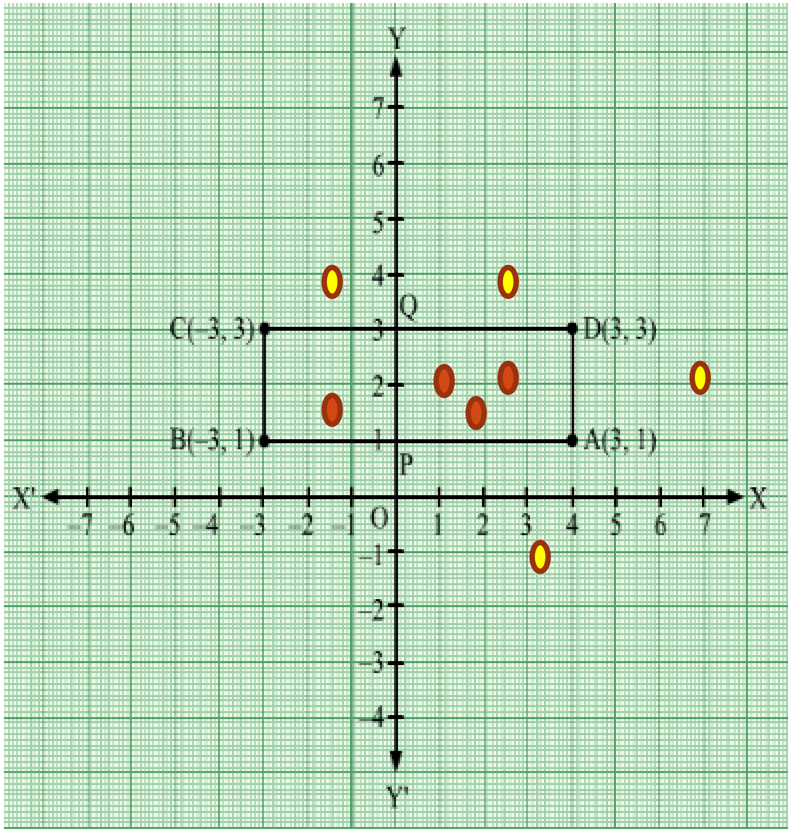
Rectangle, Hyperrectangle, Square, Cube, Hypercube

- If x>=-3 && x<=3

  If y>=1 && y<=3

  then point p(x,y)

  lies within the rectangle

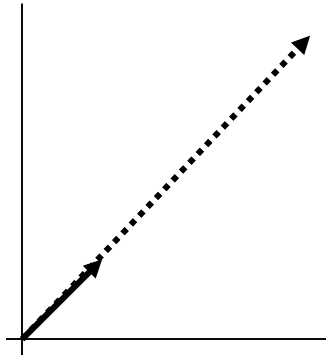


- **Eigenvectors & eigenvalues**

Introduction to Linear Algebra
by Mark Goldman, and Mily Mackevicius

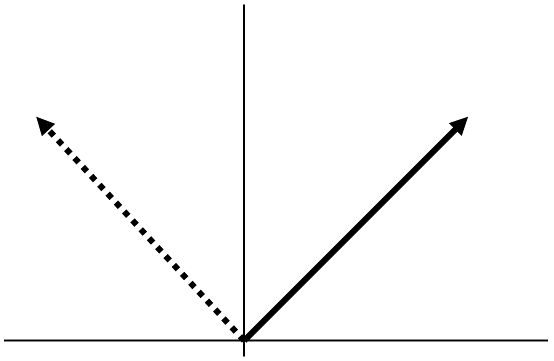
Matrices as linear transformations

$$\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$



(stretching)

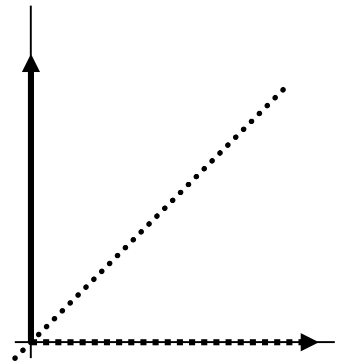
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



(rotation)

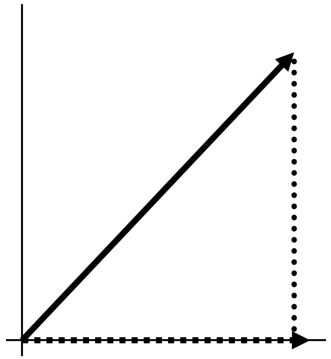
Matrices as linear transformations

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



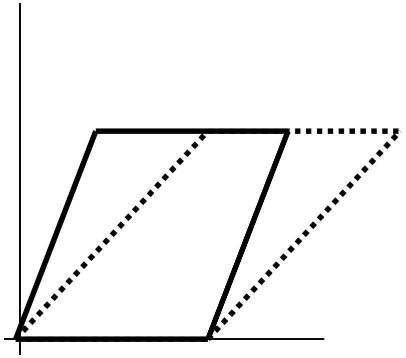
(reflection)

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



(projection)

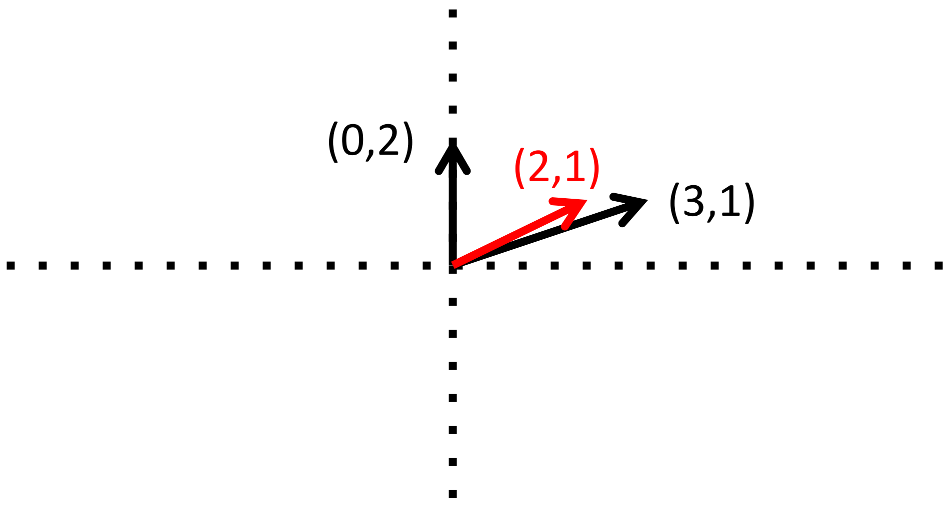
$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + cy \\ y \end{pmatrix}$$



(shearing)

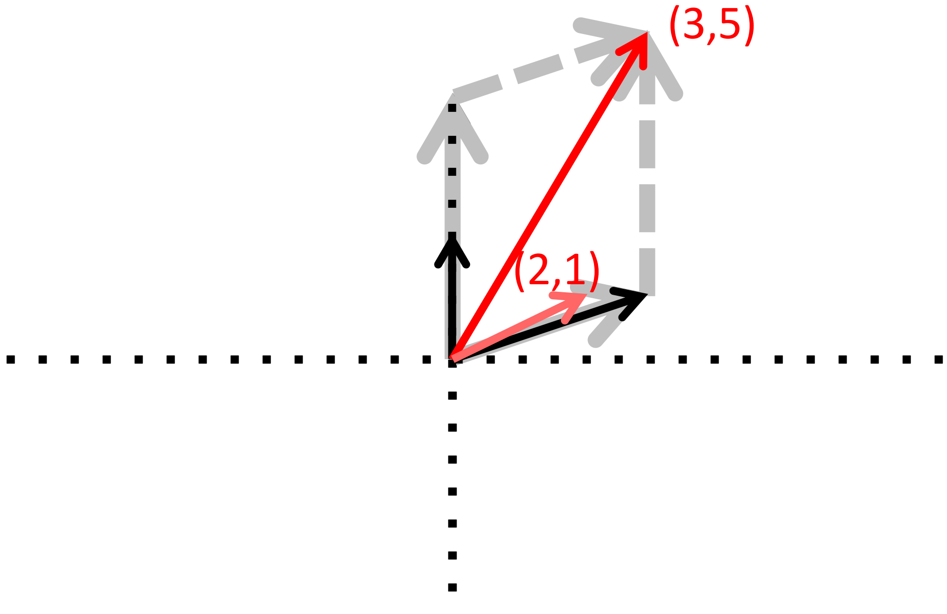
What do matrices do to vectors?

$$\overset{\overleftrightarrow{M}}{\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} =$$



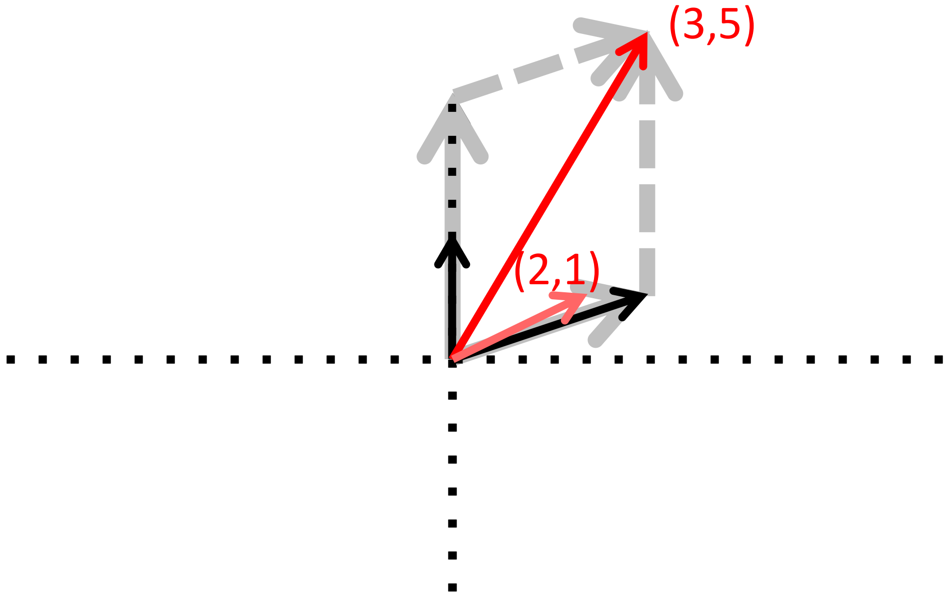
Recall

$$\overset{M}{\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$



What do matrices do to vectors?

$$\overbrace{\begin{pmatrix}0 & 3\\ 2 & 1\end{pmatrix}}^{M}\begin{pmatrix}2\\ 1\end{pmatrix} = 2\begin{pmatrix}0\\ 2\end{pmatrix} + 1\begin{pmatrix}3\\ 1\end{pmatrix} = \begin{pmatrix}0\\ 4\end{pmatrix} + \begin{pmatrix}3\\ 1\end{pmatrix} = \begin{pmatrix}3\\ 5\end{pmatrix}$$



- The new vector is:
  1) **rotated**
  2) **scaled**

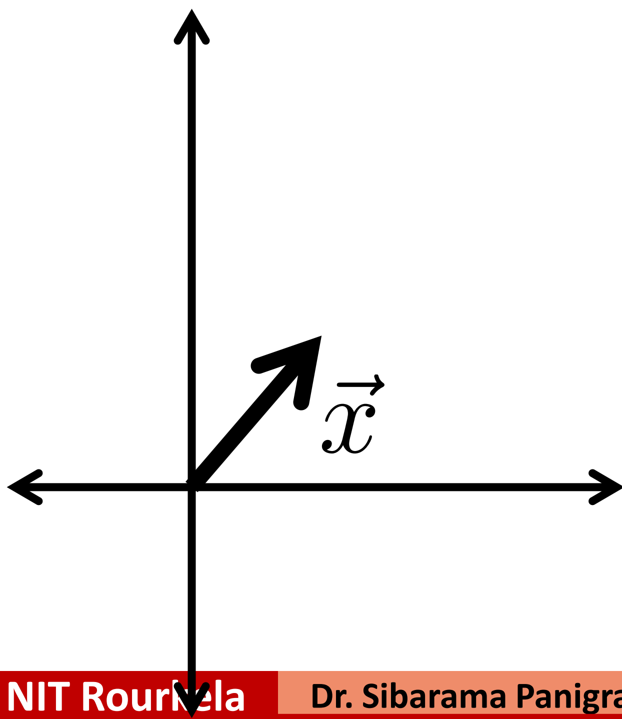
Are there any special vectors that **only get scaled**?

$$M$$

$$\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$$

Are there any special vectors that **only get scaled**?

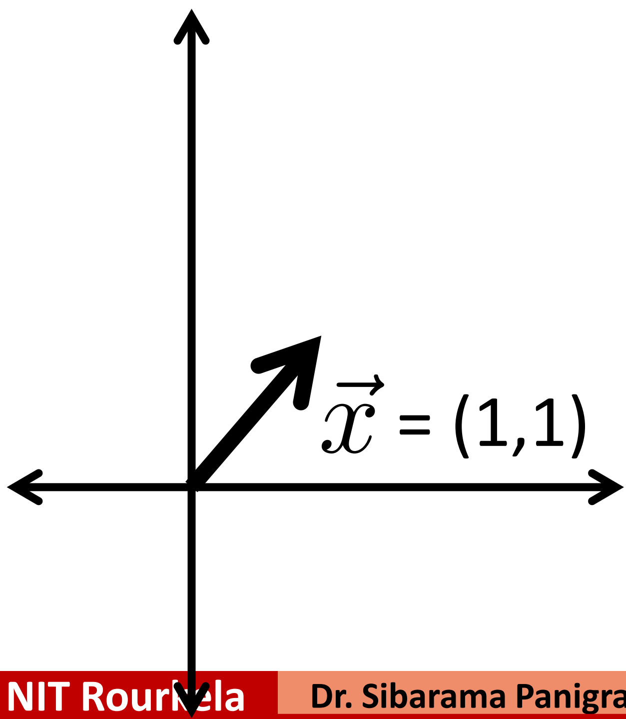
$$\underbrace{\begin{pmatrix}0 & 3\\ 2 & 1\end{pmatrix}}_{M}\begin{pmatrix}1\\ 1\end{pmatrix}$$



Try (1,1)

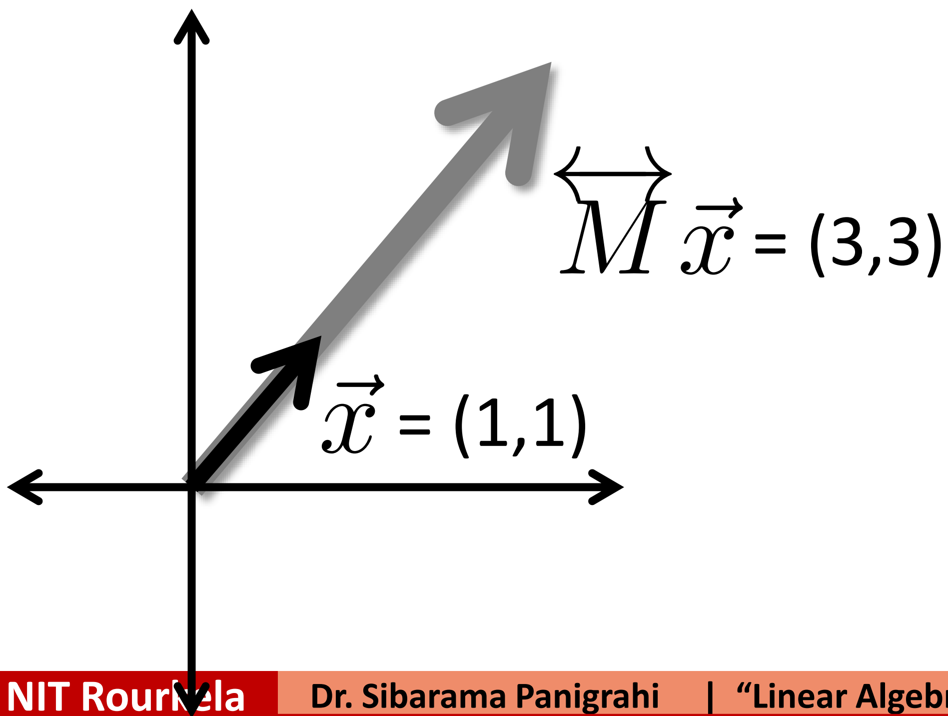
Are there any special vectors that **only get scaled**?

$$\overbrace{\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}}^{M} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$



Are there any special vectors that **only get scaled**?

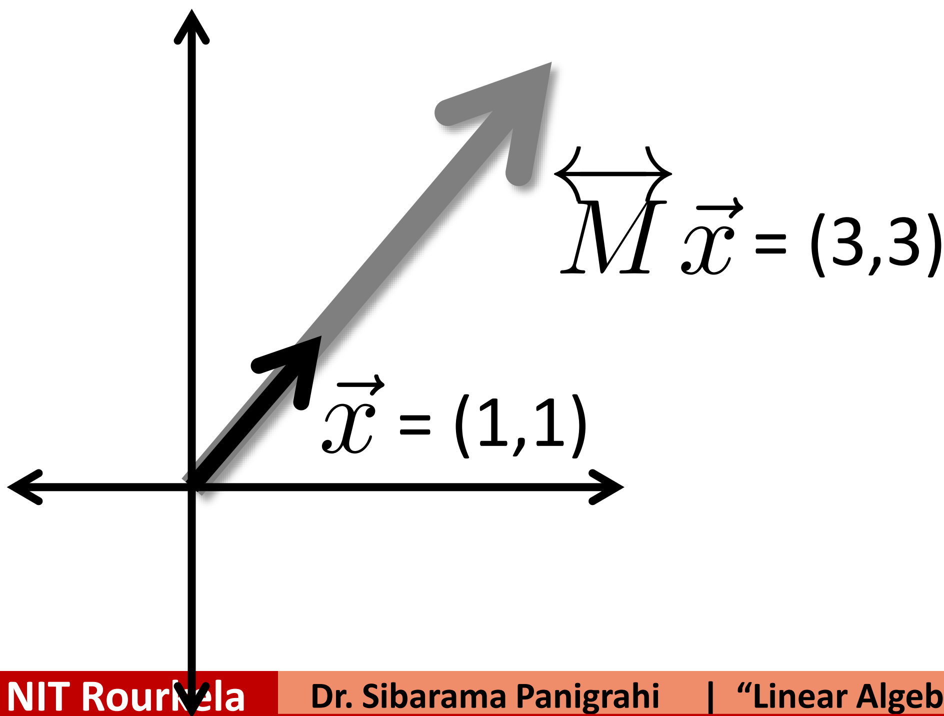
$$\overset{\overleftrightarrow{M}}{\begin{pmatrix}0 & 3\\ 2 & 1\end{pmatrix}}\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}0\\ 2\end{pmatrix} + \begin{pmatrix}3\\ 1\end{pmatrix} = \begin{pmatrix}3\\ 3\end{pmatrix} = 3\begin{pmatrix}1\\ 1\end{pmatrix}$$



Are there any special vectors that **only get scaled**?

$$\overleftrightarrow{M}$$

$$\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



- For this special vector, multiplying by M is like multiplying by a scalar.
- (1,1) is called an **eigenvector** of M
- 3 (the scaling factor) is called the **eigenvalue** associated with this eigenvector

Are there any other eigenvectors?

- Yes! The easiest way to find is with Python's eig()

```
import numpy as np
from numpy.linalg import eig
```

```
a = np.array([[0, 2],
              [2, 3]])
w,v=eig(a)
print('E-value:', w)
print('E-vector', v)
```

```
E-value: [-1.  4.]
E-vector [[-0.89442719 -0.4472136 ]
 [ 0.4472136  -0.89442719]]
```

• Vector space

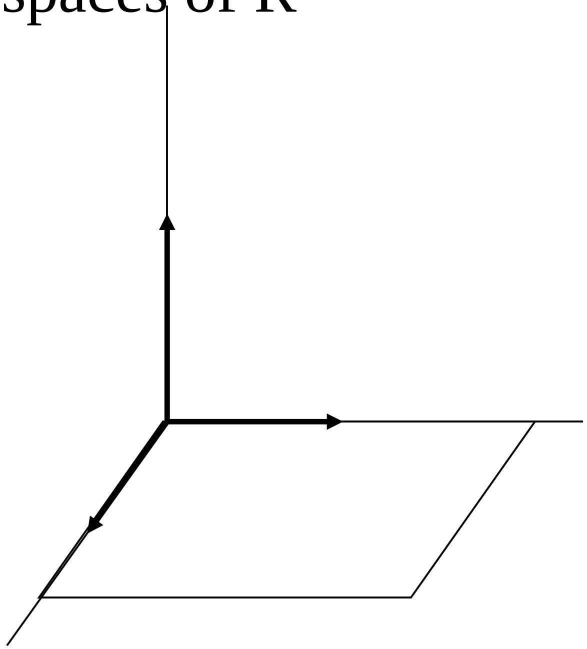
- 10-725 – Optimization 1/16/08 Recitation
- Joseph Bradley

Vector spaces

- Formally, a *vector space* is a set of vectors which is closed under addition and multiplication by real numbers.
- A *subspace* is a subset of a vector space which is a vector space itself, e.g. the plane z=0 is a subspace of R^3 (It is essentially R^2.).
- We'll be looking at R^n and subspaces of R^n

Our notion of planes in R^3 may be extended to *hyperplanes* in R^n(of dimension n-1)

Note: subspaces must include the origin (zero vector).

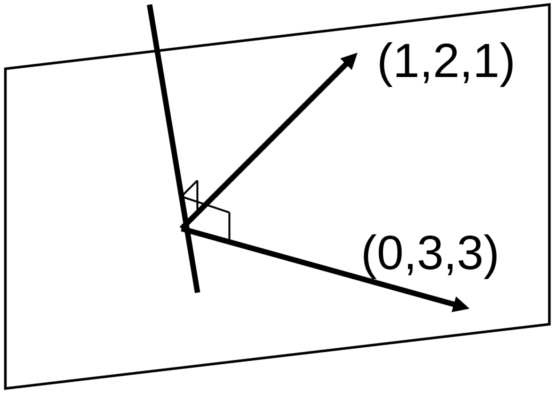


Linear system & subspaces

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$u \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + v \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

- Linear systems define certain subspaces
- **Ax = b is solvable** ***iff b may be written*** as a linear combination of the columns of A.
- The set of possible vectors b forms a subspace called the *column space of A.*



Linear system & subspaces

The set of solutions to Ax = 0 forms a subspace called the *null space* of A.

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

→ Null space: {(0,0)}

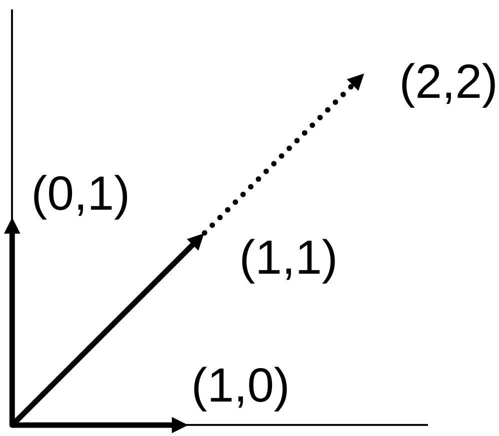
$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 5 \\ 1 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

→ Null space: {(c, c, -c)}

Linear independence and basis

- Vectors $v_1,\ldots,v_k$ are linearly independent if $c_1v_1+\ldots+c_kv_k = 0$ implies $c_1=\ldots=c_k=0$

  i.e. the nullspace is the origin



$$\begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

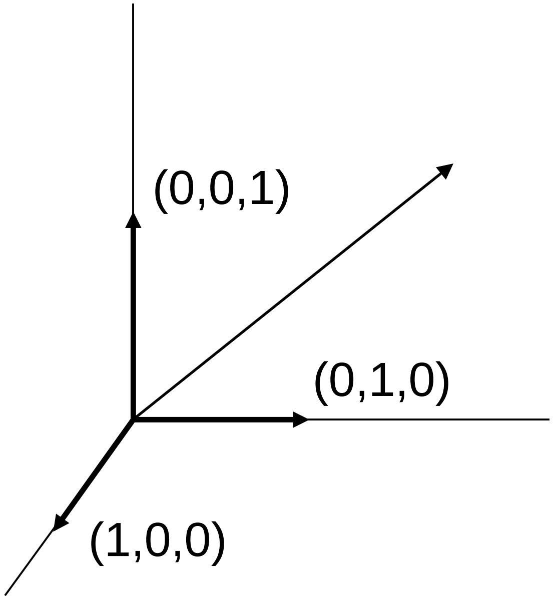
$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

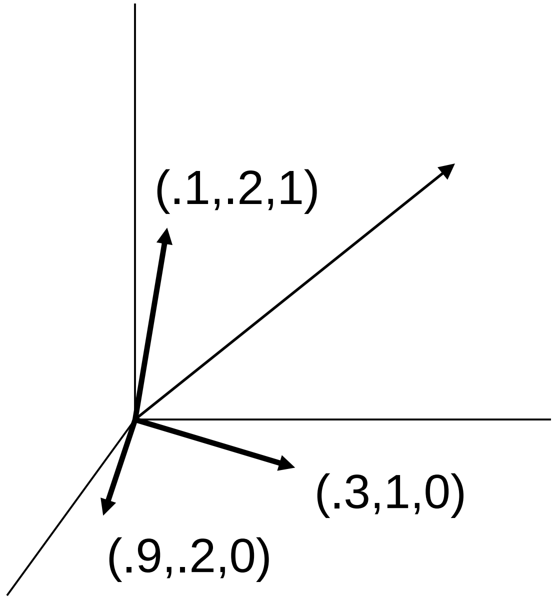
Recall nullspace contained only (u,v)=(0,0).
i.e. the columns are linearly independent.

Linear independence and basis

- If all vectors in a vector space may be expressed as linear combinations of $v_1,\ldots,v_k$, then $v_1,\ldots,v_k$ *span* the space.

$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\end{pmatrix} \qquad \begin{pmatrix}2\\2\\2\end{pmatrix} = 1.57\begin{pmatrix}.9\\.2\\0\end{pmatrix} + 1.29\begin{pmatrix}.3\\1\\0\end{pmatrix} + 2\begin{pmatrix}.1\\.2\\1\end{pmatrix}$$

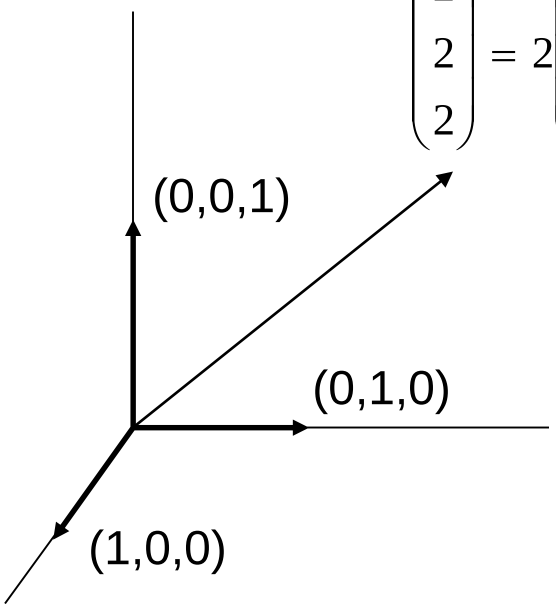




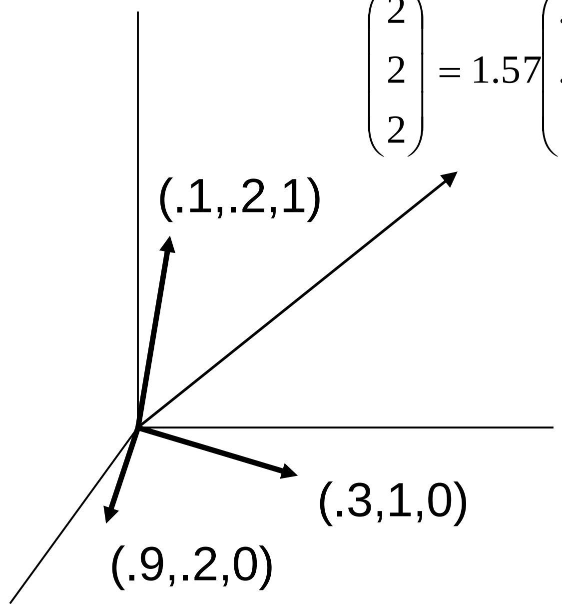
Linear independence and basis

- A *basis* is a set of linearly independent vectors which span the space.
- The *dimension* of a space is the # of "degrees of freedom" of the space; it is the number of vectors in any basis for the space.
- A basis is a maximal set of linearly independent vectors and a minimal set of spanning vectors.

$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\end{pmatrix}$$



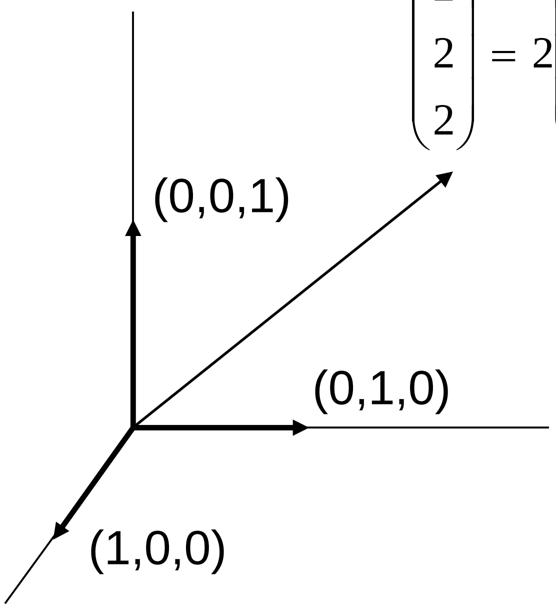
$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 1.57\begin{pmatrix}.9\\.2\\0\end{pmatrix} + 1.29\begin{pmatrix}.3\\1\\0\end{pmatrix} + 2\begin{pmatrix}.1\\.2\\1\end{pmatrix}$$



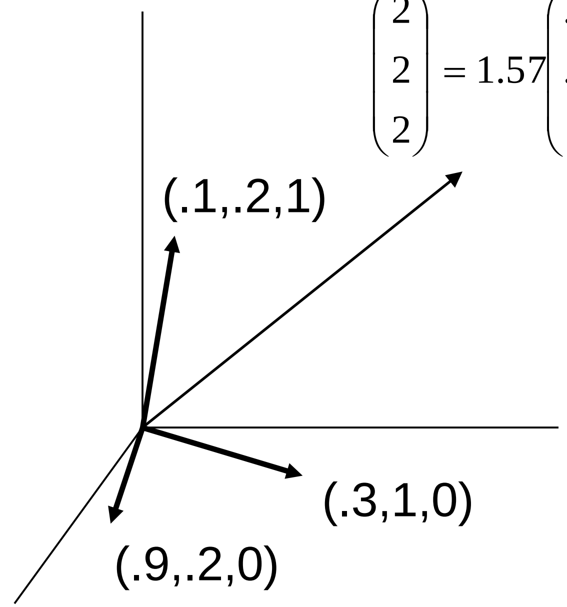
Linear independence and basis

- Two vectors are *orthogonal* if their dot product is 0.
- An *orthogonal basis* consists of orthogonal vectors.
- An *orthonormal basis* consists of orthogonal vectors of unit length.

$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\end{pmatrix}$$



$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 1.57\begin{pmatrix}.9\\.2\\0\end{pmatrix} + 1.29\begin{pmatrix}.3\\1\\0\end{pmatrix} + 2\begin{pmatrix}.1\\.2\\1\end{pmatrix}$$



About subspaces

- The *rank* of A is the dimension of the column space of A.
- It also equals the dimension of the *row space* of A (the subspace of vectors which may be written as linear combinations of the rows of A).

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 3 \end{pmatrix}$$

(1,3) = (2,3) – (1,0)

Only 2 linearly independent rows, so rank = 2.

About subspaces

Fundamental Theorem of Linear Algebra:

If A is m x n with rank r,

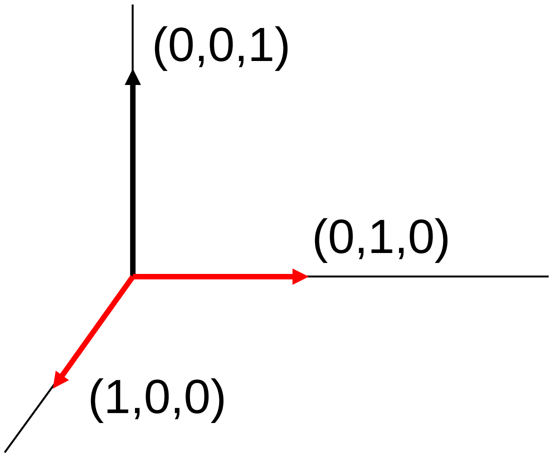
Column space(A) has dimension r

Nullspace(A) has dimension n-r (= *nullity* of A)

Row space(A) = Column space(A^T) has dimension r

Left nullspace(A) = Nullspace(A^T) has dimension m - r

Rank-Nullity Theorem: rank + nullity = n



$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

m = 3

n = 2

r = 2

Non-square matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{pmatrix}$$

m = 3
n = 2
r = 2
System Ax=b may not have a solution (x has 2 variables but 3 constraints).

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

m = 2
n = 3
r = 2
System Ax=b is underdetermined (x has 3 variables and 2 constraints).

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Basis transformations

- Before talking about basis transformations, we need to recall matrix inversion and projections.

Matrix inversion

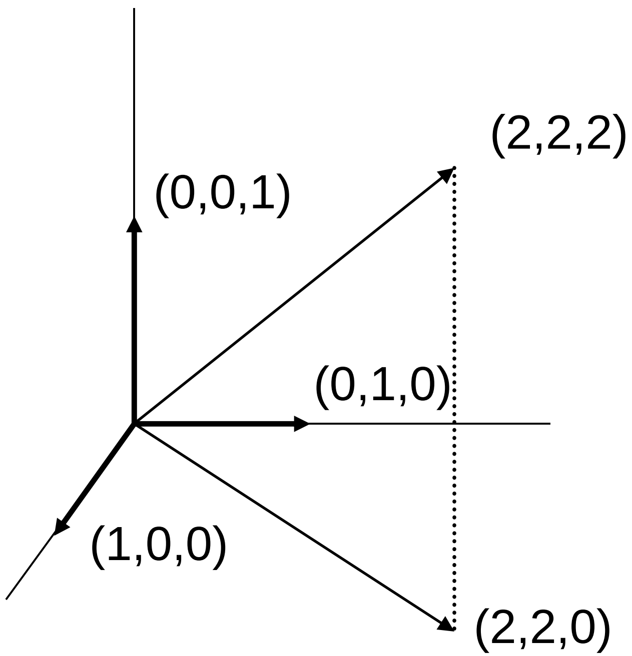
- To solve $Ax=b$, we can write a closed-form solution if we can find a matrix A^{-1}

  s.t. $AA^{-1} = A^{-1}A = I$ (identity matrix)
- Then $Ax=b$ iff $x=A^{-1}b$:

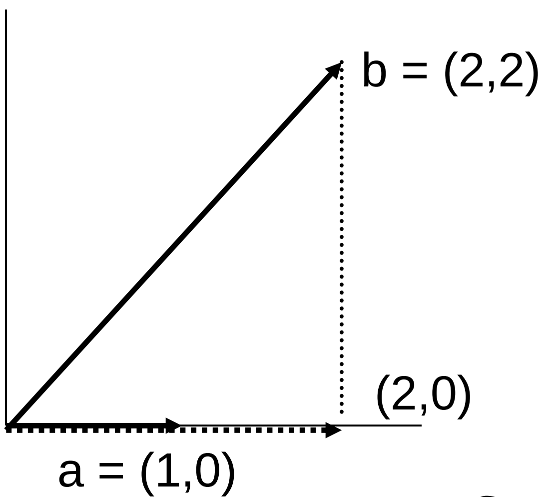
  $x = Ix = A^{-1}Ax = A^{-1}b$
- A is *non-singular* iff A^{-1} exists iff $Ax=b$ has a unique solution.
- Note: If A^{-1}, B^{-1} exist, then $(AB)^{-1} = B^{-1}A^{-1}$,

  and $(A^T)^{-1} = (A^{-1})^T$

Projections



$$\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

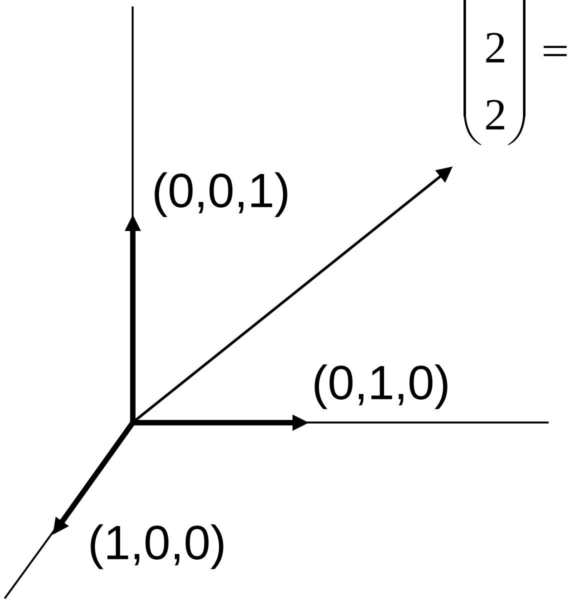


$$c = \frac{a^T b}{a^T a} a = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

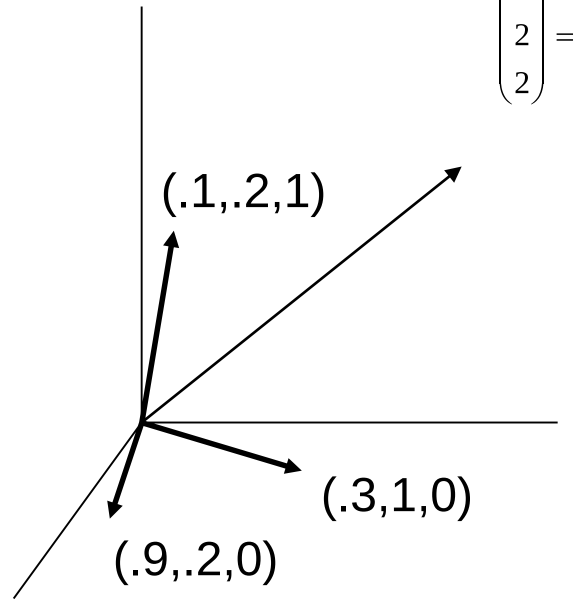
Basis transformations

We may write v=(2,2,2) in terms of an alternate basis:

$$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$



$$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} .9 & .3 & .1 \\ .2 & 1 & .2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1.57 \\ 1.29 \\ 2 \end{pmatrix}$$



Components of (1.57,1.29,2) are projections of v onto new basis vectors, normalized so new v still has same length.

Determinants

- If det(A) = 0, then A is singular.
- If det(A) ≠ 0, then A is invertible.
- To compute:
  - Simple example:
  - Python: det(A)

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Determinants

- m-by-n matrix A is *rank-deficient* if it has rank $r < m$ ($\leq n$)
- Thm: rank(A) < r iff

  det(A) = 0 for all t-by-t submatrices,

  $r \leq t \leq m$

- **Thank You**